Negative energy density states for the Dirac field in flat spacetime

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Negative energy densities in the Dirac field produced by state vectors that are the superposition of two single particle electron states are examined. I show that for such states the energy density of the field is not bounded from below and that the quantum inequalities derived for scalar fields are satisfied. I also show that it is not possible to produce negative energy densities in a scalar field using state vectors that are arbitrary superpositions of single particle states. [S0556-2821(98)05106-6]

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INTRODUCTION

Recent work on wormholes [1–3] and the ''warp drive'' [4] has generated interest in matter that violates the weak energy condition. Most discussions of such exotic matter occur within the context of quantum field theory and deal with bosonic fields [1,2,5,6,8,9] (see [10–12] for a classical discussion and [7] for a discussion of fermionic fields in a curved spacetime). Recently Ford and Roman [8,9] have shown that in flat spacetime the energy density of a massless scalar field and the electromagnetic field satisfy the quantum inequalities

$$\hat{\rho}\equiv\frac{t_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle :T_{00}:\rangle}{(t^2+t_0^2)}dt\geqslant-\frac{A}{2V}\sum_k \omega_k e^{-2\omega_k t_0} \tag{1}$$

in the finite volume case and

$$\hat{\rho}\geqslant-\frac{3A}{32\pi^2 t_0^4} \tag{2}$$

in the infinite volume case, where $\langle :T_{00}:\rangle$ is the expectation value of the normal ordered energy density, $A=1$ for a massless scalar field, and $A=2$ for the electromagnetic field. The quantity $\hat{\rho}$ samples $\langle :T_{00}:\rangle$ over a time of order t_0. For simplicity I will refer to $\langle :T_{00}:\rangle$ as the energy density of the field.

Unfortunately the methods used above to obtain general constraints on the energy densities cannot be applied to the Dirac equation. In this paper I will look at the negative energy densities that occur in states that are the superposition of two single particle electron states. For such states I show that the energy density is not bounded from below and that an observer at a fixed spatial point sees the energy density as a wave propagating by at the speed of light superimposed on a positive background. In certain regions of this wave the energy density is negative and in other regions it is positive. Thus the negative energy densities do not persist indefinitely. In fact, I will show that the negative energy density persists for a time that is inversely proportional to the minimum value of the energy density. I will also show that the quantum inequality (1) is satisfied for the states considered in this paper.

Finally I will show that, in contrast to the Dirac field, a scalar field cannot have negative energy densities for states that are arbitrary superpositions of single particle states.

Throughout this paper I will take $\hbar=c=1$ and the metric will be taken to have the signature $(-+++)$.

NEGATIVE ENERGY STATES FOR THE DIRAC FIELD

The Lagrangian for the Dirac field is

$$L=\frac{\imath}{2}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi-m\bar{\psi}\psi. \tag{3}$$

Since the canonical energy momentum tensor $\theta^{\mu\nu}$ is not symmetric the Belinfante tensor [13]

$$\begin{aligned}T^{\mu\nu}=\theta^{\mu\nu}-\frac{\imath}{2}\partial_{\alpha}\Bigg[&\frac{\partial L}{\partial(\partial_{\alpha}\psi^l)}(J^{\mu\nu})^l{}_m\psi^m+\frac{\partial L}{\partial(\partial_{\alpha}\overline{\psi^l})}(\overline{J^{\mu\nu}})^l{}_m\overline{\psi^m}\\&-\frac{\partial L}{\partial(\partial_{\mu}\psi^l)}(J^{\alpha\nu})^l{}_m\psi^m-\frac{\partial L}{\partial(\partial_{\mu}\overline{\psi^l})}(\overline{J^{\alpha\nu}})^l{}_m\overline{\psi^m}\\&-\frac{\partial L}{\partial(\partial_{\nu}\psi^l)}(J^{\alpha\mu})^l{}_m\psi^m-\frac{\partial L}{\partial(\partial_{\nu}\overline{\psi^l})}(\overline{J^{\alpha\mu}})^l{}_m\overline{\psi^m}\Bigg]\end{aligned} \tag{4}$$

should be used, where $\theta^{\mu\nu}$ is the canonical energy momentum tensor, $J^{\mu\nu}$ is the generator of Lorentz transformations for ψ and $\overline{J^{\mu\nu}}$ is the generator of Lorentz transformations for $\bar{\psi}$. A short calculation gives

$$T_{\mu\nu}=\frac{\imath}{4}[\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi+\bar{\psi}\gamma^{\nu}\overleftrightarrow{\partial}^{\mu}\psi]. \tag{5}$$

Thus

$$T_{00}=\frac{\imath}{2}[\psi^{\dagger}\dot{\psi}-\dot{\psi}^{\dagger}\psi]. \tag{6}$$

Now define

$$\langle\rho\rangle=\langle :T_{00}:\rangle \tag{7}$$

and consider the Dirac field in a box of volume V. The field operator can be written in terms of creation and annihilation operators as

$$\psi(x)=\sum_k \sum_{\alpha=1,2} [b_\alpha(k)u^\alpha(k)e^{ik\cdot x}+d_\alpha^\dagger(k)v^\alpha(k)e^{-ik\cdot x}] \tag{8}$$

where

$$u^\alpha(k)=\begin{pmatrix} \sqrt{\frac{\omega+m}{2\omega V}}\phi^\alpha \\ \frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{2\omega(\omega+m)V}}\phi^\alpha \end{pmatrix}, \tag{9}$$

$$v^\alpha(k)=\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{2\omega(\omega+m)V}}\phi^\alpha \\ \sqrt{\frac{\omega+m}{2\omega V}}\phi^\alpha \end{pmatrix}, \tag{10}$$

$\phi^{1\dagger}=(1,0)$, and $\phi^{2\dagger}=(0,1)$. The creation and annihilation operators satisfy

$$\{b_\alpha(k),b_{\alpha'}^\dagger(k')\}=\delta_{\alpha,\alpha'}\delta_{k,k'} \tag{11}$$

and

$$\{d_\alpha(k),d_{\alpha'}^\dagger(k')\}=\delta_{\alpha,\alpha'}\delta_{k,k'} \tag{12}$$

with all other anticommutators vanishing. Substituting Eq. (8) into Eq. (7) gives

$$\langle\rho\rangle=\frac{1}{2}\sum_{k,k'}\sum_{\alpha,\alpha'}\{(\omega_k+\omega_{k'})[\langle b_\alpha^\dagger(k)b_{\alpha'}(k')\rangle u^{\dagger\alpha}(k)u^{\alpha'}(k')e^{-i(k-k')\cdot x}+\langle d_{\alpha'}^\dagger(k')d_\alpha(k)\rangle v^{\dagger\alpha}(k)v^{\alpha'}(k')e^{i(k-k')\cdot x}]\}$$
$$+(\omega_{k'}-\omega_k)[\langle d_\alpha(k)b_{\alpha'}(k')\rangle v^{\dagger\alpha}(k)u^{\alpha'}(k')e^{i(k+k')\cdot x}-\langle b_\alpha^\dagger(k)d_{\alpha'}^\dagger(k')\rangle u^{\dagger\alpha}(k)v^{\alpha'}(k')e^{-i(k+k')\cdot x}]\} \tag{13}$$

where I have used $:d_\alpha(k)d_{\alpha'}(k'):=-d_{\alpha'}^\dagger(k')d_\alpha(k)$.

Now consider a state vector of the form

$$|\psi\rangle=\frac{1}{\sqrt{1+\lambda^2}}[|k_z,1\rangle+\lambda|k_x,2\rangle] \tag{14}$$

where $|k,\alpha\rangle=b_\alpha^\dagger(k)|0\rangle$ and λ is real. Since the state $|\psi\rangle$ contains only electrons all expectation values in Eq. (13) containing $d_\alpha(k)$ or $d_\alpha^\dagger(k)$ vanish. Substituting Eqs. (14) and (9) into Eq. (13) gives

$$\langle\rho\rangle=\frac{1}{(1+\lambda^2)V}[\omega_{k_z}+\lambda\beta+\lambda^2\omega_{k_x}] \tag{15}$$

where

$$\beta=\frac{k_xk_z(\omega_{k_x}+\omega_{k_z})\cos\theta}{2\sqrt{\omega_{k_x}\omega_{k_z}(\omega_{k_x}+m)(\omega_{k_z}+m)}} \tag{16}$$

and $\theta=(k_x-k_z)x$. Note that $\langle\rho\rangle=\omega_{k_z}/V$ for $\lambda=0$ and $\langle\rho\rangle=\omega_{k_x}/V$ as $\lambda\to\infty$, as expected. It is easy to see that $\langle\rho\rangle$ will be negative if

$$\beta^2>4\omega_{k_x}\omega_{k_z} \tag{17}$$

and if

$$-\frac{\beta}{2}-\sqrt{\left(\frac{\beta}{2}\right)^2-\omega_{k_x}\omega_{k_z}}<\omega_{k_x}\lambda<-\frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2-\omega_{k_x}\omega_{k_z}}. \tag{18}$$

Consider the ultrarelativistic limit, $k_x,k_z\gg m$. In this limit

$$\beta=\frac{1}{2}(\omega_{k_x}+\omega_{k_z})\cos\theta. \tag{19}$$

Equation (17) becomes

$$\cos^2\theta>\frac{16\omega_{k_x}\omega_{k_z}}{(\omega_{k_x}+\omega_{k_z})^2}. \tag{20}$$

For a solution to exist it is necessary that $16\omega_{k_x}\omega_{k_z}\leqslant(\omega_{k_x}+\omega_{k_z})^2$. This will be satisfied if $\omega_{k_x}\geqslant(7+\sqrt{48})\omega_{k_z}$ or if $\omega_{k_z}\geqslant(7+\sqrt{48})\omega_{k_x}$. Thus it is possible to produce negative energy densities for state vectors of the form (14) if λ is chosen to satisfy Eq. (18).

It is now easy to show that the energy density is not bounded from below. To simplify the discussion take $x^\mu=0$, so that $\cos\theta=1$. In the ultrarelativistic limit with $\omega_{k_x}\gg\omega_{k_z}$,

$$\beta=\frac{1}{2}\omega_{k_x} \tag{21}$$

and $-1/2\leqslant\lambda\leqslant0$. The energy density is given by

$$\langle\rho\rangle\simeq-\frac{\lambda\omega_{k_x}}{(1+\lambda^2)V}\left(\lambda+\frac{1}{2}\right). \tag{22}$$

Thus in the limit $\omega_{k_x}/V\to\infty$ the energy density at the space-time point $x^\mu=0$ goes to $-\infty$, for $-1/2<\lambda<0$.

Now consider, within the above limits, the energy density on the spacetime. For general x^{μ},

$$\langle\rho\rangle=\frac{\lambda\omega_{k_x}}{(1+\lambda^2)V}\left[\lambda+\frac{1}{2}\cos(\omega_{k_x}(t-x))\right]. \tag{23}$$

Thus $\langle\rho\rangle$ is a cosine wave propagating at the speed of light superimposed on a positive background. The energy density at a fixed spatial point will be negative for a time Δt, which satisfies

$$-V\langle\rho\rangle_{min}\Delta t=\frac{|\lambda|(1+2\lambda)}{1+\lambda^2}\cos^{-1}(2|\lambda|), \tag{24}$$

where $\langle\rho\rangle_{min}$ is the minimum value of $\langle\rho\rangle$ for fixed ω_{k_x} and λ. Since the wave propagates at the speed of light the extent of the negative energy density will satisfy the same expression as above with Δt replaced by Δx. For large values of $|\langle\rho\rangle|V$ (and λ not too close to $-1/2$ or 0) the energy density will undergo rapid oscillations from positive to negative values. Note that the time average of the energy density is positive.

QUANTUM INEQUALITIES

In this section I will show that the energy density satisfies the quantum inequality (1) for the state given in Eq. (14). To show this I will take the limit $m\to 0$ for the Dirac field. Substituting Eq. (23) into

$$\hat{\rho}=\frac{t_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle\rho\rangle}{t^2+t_0^2}dt \tag{25}$$

and taking $\vec{x}=0$ gives

$$\hat{\rho}=\frac{\lambda\omega_{k_x}}{(1+\lambda^2)V}[\lambda+\tfrac{1}{2}e^{-\omega_{k_x}t_0}]. \tag{26}$$

Thus $\hat{\rho}$ will be negative if

$$\omega_{k_x}t_0<-\ln(2|\lambda|). \tag{27}$$

Now consider the quantum inequality (1) with $A=1$. If periodic boundary conditions are imposed,

$$\omega=\frac{2\pi}{L}\sqrt{n_x^2+n_y^2+n_z^2} \tag{28}$$

where n_x, n_y, and n_z are integers. Next note that

$$\frac{1}{\sqrt{3}}(|n_x|+|n_y|+|n_z|)\leqslant\sqrt{n_x^2+n_y^2+n_z^2}\leqslant|n_x|+|n_y|+|n_z|. \tag{29}$$

To see this define $f(n_x,n_y,n_z)$ by

$$f(n_x,n_y,n_z)=\sqrt{n_x^2+n_y^2+n_z^2}-\alpha(|n_x|+|n_y|+|n_z|) \tag{30}$$

where α is a positive constant. Now let

$$|n_x|=r\cos\phi\sin\theta$$
$$|n_y|=r\sin\phi\sin\theta \tag{31}$$
$$|n_z|=r\cos\theta,$$

where $0\leqslant(\theta,\phi)\leqslant\pi/2$. Thus f can be written as

$$f(r,\theta,\phi)=r(1-\alpha g(\theta,\phi)) \tag{32}$$

where

$$g(\theta,\phi)=\sqrt{1+2\sin^2(\phi+\pi/4)}\sin(\theta+\psi) \tag{33}$$

and

$$\cot(\psi)=\sqrt{2}\sin(\phi+\pi/4). \tag{34}$$

Given that $0\leqslant(\theta,\phi)\leqslant\pi/2$, it is easy to see that $1\leqslant g(\theta,\phi)\leqslant\sqrt{3}$. The function $g(\theta,\phi)$ has its maximum value of $\sqrt{3}$ when $\phi=\pi/4$ and $\theta=\pi/2-\cot^{-1}(\sqrt{2})$. The minimum value of $g(\theta,\phi)$ occurs when $\theta=0$. In this case $g(0,\phi)=1$ for all ϕ. The function $g(\theta,\phi)$ also equals 1 at $\theta=\pi/2$, $\phi=0,\pi/2$. Now if $\alpha=1$ in Eq. (30) then $f=r(1-g)\leqslant 0$. This gives the inequality $\sqrt{n_x^2+n_y^2+n_z^2}\leqslant|n_x|+|n_y|+|n_z|$. On the other hand, if $\alpha=1/\sqrt{3}$ then $f=r(1-g/\sqrt{3})\geqslant 0$. This gives the inequality $3^{-1/2}[|n_x|+|n_y|+|n_z|]\leqslant\sqrt{n_x^2+n_y^2+n_z^2}$. Thus Eq. (29) is proved.

To show that Eq. (1) is satisfied I will show that an even more restrictive inequality is satisfied. From Eq. (29)

$$\omega_k^-\leqslant\omega_k\leqslant\omega_k^+ \tag{35}$$

where

$$\omega_k^-=\frac{2\pi}{\sqrt{3}L}[|n_x|+|n_y|+|n_z|] \tag{36}$$

and

$$\omega_k^+=\frac{2\pi}{L}[|n_x|+|n_y|+|n_z|]. \tag{37}$$

Since

$$\omega_k^-e^{-2\omega_k^+t_0}\leqslant\omega_k e^{-2\omega_k t_0} \tag{38}$$

the quantum inequality (1) will be satisfied if

$$\hat{\rho}\geqslant-\frac{1}{2V}\sum_k\omega_k^-e^{-2\omega_k^+t_0} \tag{39}$$

is satisfied. Substituting in the expressions for ω_k^+ and ω_k^- gives

$$\hat{\rho}\geqslant-\frac{2\sqrt{3}\pi}{LV}\left[\sum_{n=0}^{\infty}ne^{-\alpha n}\right]\left[2\sum_{k=0}^{\infty}e^{-\alpha k}-1\right]^2, \tag{40}$$

where $\alpha=4\pi t_0/L$. The sums can easily be performed, giving

$$\hat{\rho} \geqslant -\frac{2\sqrt{3}\pi}{LV}\frac{e^{-\alpha}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4}. \tag{41}$$

Thus Eq. (1) will be satisfied if

$$\frac{\lambda n_x}{1+\lambda^2}(\lambda+\tfrac{1}{2}e^{-\alpha n_x/2}) \geqslant -\frac{\sqrt{3}e^{-\alpha}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4}. \tag{42}$$

is satisfied. For λ outside the interval $(-\frac{1}{2}e^{-\alpha n_x/2},0)$ the above inequality will obviously be satisfied. Thus consider $-\frac{1}{2}e^{-\alpha n_x/2}<\lambda<0$. Equation (42) will be satisfied if

$$|\lambda| n_x(\lambda+\tfrac{1}{2}e^{-\alpha n_x/2}) \leqslant \frac{\sqrt{3}e^{-\alpha}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4} \tag{43}$$

is satisfied. Now let $\lambda=-(\sigma/2)e^{-\alpha n_x/2}$. The above inequality becomes

$$\frac{1}{4}\sigma n_x e^{-\alpha n_x}(1-\sigma) \leqslant \frac{\sqrt{3}e^{-\alpha}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4}. \tag{44}$$

The left hand side is maximized for $\sigma=1/2$ and for $n_x=1/\alpha$. But n_x is a positive integer. Thus for $\alpha\geqslant 1$ take $n_x=1$. The above inequality will then be satisfied (for $\alpha\geqslant 1$) if

$$\frac{1}{16} \leqslant \frac{\sqrt{3}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4} \tag{45}$$

is satisfied. This is obviously satisfied for all $\alpha\geqslant 1$. For $\alpha<1$ let $n_x=1/\alpha$ (i.e. generalize n_x to a real number). The inequality (44) will be satisfied if

$$\frac{1}{16}e^{-1} \leqslant \frac{\sqrt{3}\alpha e^{-\alpha}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4} \tag{46}$$

is satisfied. In the interval $0<\alpha<1$ the right hand side is a monotonically decreasing function of α with a minimum value of $e^{-1}(1+e^{-1})^2(1-e^{-1})^{-4}$. Thus the above inequality is satisfied. Therefore inequality (42) will be satisfied, which implies that the quantum inequality (1) will be satisfied.

THE KLEIN-GORDON FIELD

In this section I will show that the energy density for a massive scalar field is positive for all states that are arbitrary superpositions of single particle states.

The scalar field operator can be written in terms of creation and annihilation operators as

$$\phi(x)=\imath\sum_k \frac{1}{\sqrt{2V\omega_k}}(a_k e^{\imath k^\mu x_\mu}-a_k^\dagger e^{-\imath k^\mu x_\mu}). \tag{47}$$

The energy-momentum tensor for the scalar field is given by

$$T^{\mu\nu}=\partial^\mu\phi\partial^\nu\phi-\frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi+m^2\phi^2). \tag{48}$$

A short calculation gives

$$\langle\rho\rangle=\frac{1}{2V}\mathrm{Re}\sum_{k,k'}\frac{(\omega_k\omega_{k'}+\vec{k}\cdot\vec{k}'+m^2)}{\sqrt{\omega_k\omega_{k'}}}\langle a_{k'}^\dagger a_k\rangle e^{\imath(k^\mu-k'^\mu)x_\mu} \tag{49}$$

$$+\frac{(\omega_k\omega_{k'}+\vec{k}\cdot\vec{k}'-m^2)}{\sqrt{\omega_k\omega_{k'}}}\langle a_{k'}a_k\rangle e^{\imath(k^\mu+k'^\mu)x_\mu}. \tag{50}$$

The state vector can be written as

$$|\psi\rangle=\frac{1}{N}\sum_k \alpha_k|k\rangle \tag{51}$$

where the α_k are arbitrary complex numbers and N is chosen so that $|\psi\rangle$ is normalized. The energy density can now be written as

$$\langle\rho\rangle=\frac{1}{2VN^2}\sum_{k,k'}\frac{(\omega_k\omega_{k'}+\vec{k}\cdot\vec{k}'+m^2)}{\sqrt{\omega_k\omega_{k'}}}\alpha_{k'}^*\alpha_k e^{\imath(k^\mu-k'^\mu)x_\mu}. \tag{52}$$

Now define

$$\beta=\sum_k \sqrt{\omega_k}\alpha_k e^{\imath k^\mu x_\mu}, \quad \lambda=m\sum_k\frac{\alpha_k}{\omega_k}e^{\imath k^\mu x_\mu}, \tag{53}$$

and

$$\vec{\gamma}=\sum_k\frac{\vec{k}}{\sqrt{\omega_k}}\alpha_k e^{\imath k^\mu x_\mu}. \tag{54}$$

Thus for state vectors of the form (51)

$$\langle\rho\rangle=\frac{1}{2VN^2}(|\beta|^2+|\lambda|^2+|\vec{\gamma}|^2), \tag{55}$$

and the energy density is non-negative.

CONCLUSION

In this paper I examined the negative energy densities that can be produced in the Dirac field by state vectors of the form

$$|\psi\rangle=\frac{1}{\sqrt{1+\lambda^2}}(|k_z,1\rangle+\lambda|k_x,2\rangle), \tag{56}$$

where $|k_z,1\rangle$ and $|k_x,2\rangle$ are single particle electron states and λ is real. I showed that if $k_x,k_z\geqslant m$, the energy density at a space-time point x^μ will be negative if λ is chosen so that

$$-\frac{\beta}{2}-\sqrt{\left(\frac{\beta}{2}\right)^2-\omega_{k_x}\omega_{k_z}}<\omega_{k_x}\lambda<-\frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2-\omega_{k_x}\omega_{k_y}} \tag{57}$$

is satisfied, where

$$\beta=\frac{1}{2}(\omega_{k_x}+\omega_{k_z})\cos[(k^\mu-k'^\mu)x_\mu]. \tag{58}$$

Since I am taking λ to be real it is necessary that $\beta \geqslant 4\omega_{k_x}\omega_{k_z}$. This will be satisfied if $\omega_{k_x} \geqslant (7+\sqrt{48})\omega_{k_z}$ or if $\omega_{k_z} \geqslant (7+\sqrt{48})\omega_{k_x}$.

If, in addition to $k_x, k_z \gg m$, one takes $\omega_{k_x} \gg \omega_{k_z}$ then

$$-\frac{1}{2} \leqslant \lambda \leqslant 0, \quad \beta = \frac{1}{2}\omega_{k_x} \tag{59}$$

and

$$\langle \rho \rangle = \frac{\lambda \omega_{k_x}}{(1+\lambda^2)V}\left(\lambda + \frac{1}{2}\right) \tag{60}$$

at the point $x^\mu = 0$. Thus $\langle \rho \rangle \to -\infty$ as $\omega_{k_x}/V \to \infty$, for $-1/2 < \lambda < 0$ and $\langle \rho \rangle$ is not bounded from below.

An observer will see $\langle \rho \rangle$ as a cosine wave propagating at the speed of light superimposed on a positive background. The time average of $\langle \rho \rangle$ is positive and the energy density will be negative for a time interval Δt, which satisfies

$$-V\langle \rho \rangle_{min}\Delta t = \frac{|\lambda|(1+2\lambda)}{1+\lambda^2}\cos^{-1}(2|\lambda|). \tag{61}$$

I also showed that the quantum inequality

$$\hat{\rho} \equiv \frac{t_0}{\pi}\int_{-\infty}^{\infty} \frac{\langle \rho \rangle dt}{t^2+t_0^2} \geqslant -\frac{1}{2V}\sum_k \omega_k e^{-2\omega_k t_0}, \tag{62}$$

which is satisfied by a massless scalar field, is satisfied by the Dirac field for state vectors of the form (56) in the limit $m \to 0$. Finally, I showed that, in contrast to the Dirac field, it is not possible to produce negative energy densities in a scalar field using state vectors that are arbitrary superpositions of single particle states.

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