Symplectic current for the field perturbations in dilaton-axion gravity coupled with Abelian fields

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Using the self-adjointness of the perturbation equations in the most general four-dimensional dilaton-axion gravity coupled with Abelian fields, a covariantly conserved current associated with the coupled field perturbations is obtained. By particular choices of the background quantities, the low-energy limit of the string theory is covered and the part of the current corresponding to the purely Einstein-Maxwell perturbations is fully consistent with that obtained previously by Burnett and Wald (by a different approach). The possible extensions of the present results are discussed. [S0556-2821(98)04206-4]

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I. INTRODUCTION

The more recent unification theories such as supergravity or superstrings predict the existence of long-range scalar partners to the usual tensor gravity of ordinary Einstein theory. The main feature of these scalar fields is that they appear nonminimally coupled to the gravity and matter fields. Although the presence of these scalar fields changes radically the structure of the solutions, their implications and the understanding of general aspects of the ordinary gravity, some fundamental problems even persist. For example, the usual concept of energy and momentum as conserved quantities in special-relativistic theory does not exist when gravity is involved; it is not clear how to extend this concept to the setting of general relativity and, of course, to the more general formulations mentioned above. However, in perturbation theory of black holes in the framework of the ordinary Einstein-Maxwell (EM) theory, some conservation laws are known; in fact, it has been demonstrated that the existence of a conserved current for the coupled gravitational and electromagnetic perturbations, is a very general feature of the EM theory [1].

The aim of this paper is to demonstrate that in the more general framework of the Einstein-Abelian fields interacting with the scalar partners, namely, dilaton and axion fields, a conserved quantity for coupled field perturbations can be obtained, generalizing in this manner the results reached in Ref. [1]. In addition of giving the explicit expression for this conserved quantity (known as the symplectic current in the literature), the main novelty of this work is the method of derivating this expression, which has been used for the first time in Ref. [2].

Our starting point is to consider the following generalized bosonic action on a curved space-time manifold:

$$S=\int \sqrt{-g}d^4x\Bigg\{R-2(\partial_\mu\phi)\partial^\mu\phi-\frac{1}{2}\zeta(\phi)(\partial_\mu\eta)\partial^\mu\eta+\xi(\phi)F_{\mu\nu}F^{\mu\nu}+\omega(\eta)F_{\mu\nu}\widetilde{F}^{\mu\nu}+V(\phi,\eta)\Bigg\}, \tag{1}$$

where R is the (four dimensional) scalar curvature, $F_{\mu\nu}=2\nabla_{[\mu}A_{\nu]}$ is the Abelian gauge field, $\widetilde{F}^{\mu\nu}=(1/\sqrt{-g})\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$ corresponds to the dual of $F_{\mu\nu}$. On the other hand, ϕ represents the dilaton (scalar) field and η the axion (pseudoscalar) field; the arbitrary functions $\zeta(\phi)$, $\xi(\phi)$, and $\omega(\eta)$ are collectively known as the coupling functions (chosen arbitrary for generality). The presence of these functions makes that the dilaton and axion appear nonminimally coupled to the gravity and matter fields. $V(\phi,\eta)$ represents the dilaton-axion potential, which is a (well-behaved) function of the dilaton and the axion alone, and contains no derivatives of these fields. S covers a large family of nontrivial actions for four-dimensional gravity appearing in the modern literature; a specific choice of the coupling functions and the dilaton-axion potential corresponds to a particular gravity theory. For example, for the special choice $\zeta=e^{4\phi}$, $\xi=e^{-2\phi}$, and $\omega=\eta$, the action (1) reduces to the usual low-energy effective action for the heterotic string theory. V may be a Liouville-type dilaton potential, $\Lambda e^{b\phi}$, i.e., a cosmological constant term with dilaton coupling (see [3] and references therein). The factors –2 and $-\frac{1}{2}$ appearing in the action (1) are introduced for future convenience.

Let us derive now the field equations from the action (1). Variation of S with respect to the gauge field A_μ gives the modified Maxwell equations

$$\nabla_\mu(\omega\widetilde{F}^{\mu\nu}+\xi F^{\mu\nu})=0, \tag{2}$$

together with the Bianchi identities

$$\nabla_\mu\widetilde{F}^{\mu\nu}=0. \tag{3}$$

Variation with respect to the dilaton field gives the following equation with scalar and vector sources:

$$\nabla_\mu\nabla^\mu\phi+\frac{1}{4}\left[\frac{d\xi}{d\phi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\frac{d\zeta}{d\phi}(\partial_\mu\eta)\partial^\mu\eta+\frac{\partial V}{\partial\phi}\right]=0, \tag{4}$$

similarly the axion field satisfies the equation with scalar and pseudoscalar (invariant made out of vector fields) sources:

$$\nabla_\mu(\zeta\partial^\mu\eta)+\frac{d\omega}{d\eta}F_{\mu\nu}\widetilde{F}^{\mu\nu}+\frac{\partial V}{\partial\eta}=0. \tag{5}$$

Finally, the Einstein equations resulting from the variation of the action with respect to $g_{\mu\nu}$ read

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu}, \tag{6}$$

with the matter energy-momentum tensor containing scalar and vector sources:

$$\begin{aligned}T_{\mu\nu}=&2(\partial_\mu\phi)\partial_\nu\phi+\frac{1}{2}\zeta(\partial_\mu\eta)\partial_\nu\eta\\&-g_{\mu\nu}\left[(\partial^\alpha\phi)\partial_\alpha\phi+\frac{1}{4}\zeta(\partial^\alpha\eta)\partial_\alpha\eta\right]\\&-2\xi\left(F_{\mu\lambda}F_\nu{}^\lambda-\frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}\right)+\frac{1}{2}g_{\mu\nu}V.\end{aligned} \tag{7}$$

In Sec. II we will focus in the perturbed versions of the equations of motion (2)–(7) around a general curved background. Section III presents the general relationship between the adjoint character of the operators governing the field perturbations and the existence of the corresponding symplectic current; moreover, the purely EM field perturbations are compared with those given in Ref. [1]. In Sec. IV we finish with some concluding remarks and future extensions of the present work.

II. PERTURBED FIELD EQUATIONS

In this section and throughout the superscript B denotes the corresponding first-order perturbations. In particular, the metric, gauge potential, dilaton and axion perturbations are represented by $h_{\mu\nu}$, b_μ, $\phi^{\rm B}$, and $\eta^{\rm B}$ respectively.

In addition, one easily finds that

$$(g^{\mu\nu})^{\rm B}=-h^{\mu\nu},$$

$$g^{\rm B}=gg_{\mu\nu}h^{\mu\nu},$$

$$F^{\rm B}_{\mu\nu}=\partial_\mu b_\nu-\partial_\nu b_\mu,$$

$$(\widetilde{F}^{\mu\nu})^{\rm B}=\frac{2}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha b_\beta-\frac{1}{2}\widetilde{F}^{\mu\nu}g_{\alpha\beta}h^{\alpha\beta},$$

$$\xi^{\rm B}=\frac{d\xi}{d\phi}\phi^{\rm B},\quad\left[\left(\frac{d\xi}{d\phi}\right)^{\rm B}=\frac{d^2\xi}{d\phi^2}\phi^{\rm B}\right],$$

$$\zeta^{\rm B}=\frac{d\zeta}{d\phi}\phi^{\rm B},$$

$$\omega^{\rm B}=\frac{d\omega}{d\eta}\eta^{\rm B},$$

$$V^{\rm B}=\frac{\partial V}{\partial\phi}\phi^{\rm B}+\frac{\partial V}{\partial\eta}\eta^{\rm B},$$

$$\left(\frac{\partial V}{\partial\phi}\right)^{\rm B}=\frac{\partial^2 V}{\partial^2\phi}\phi^{\rm B}+\frac{\partial^2 V}{\partial\eta\partial\phi}\eta^{\rm B},$$

$$\left(\frac{\partial V}{\partial\eta}\right)^{\rm B}=\frac{\partial^2 V}{\partial\phi\partial\eta}\phi^{\rm B}+\frac{\partial^2 V}{\partial^2\eta}\eta^{\rm B},$$

$$(\Gamma^\lambda_{\mu\nu})^{\rm B}=\frac{1}{2}g^{\lambda\rho}[\nabla_\mu h_{\nu\rho}+\nabla_\nu h_{\mu\rho}-\nabla_\rho h_{\mu\nu}], \tag{8}$$

in these expressions (and in what follows), the covariant derivative ∇_μ is with respect to the background metric $g_{\mu\nu}$, which raises and lowers the indices, for example, $h^{\mu\nu}=g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta}$, that will be used implicitly below.

In order to obtain the first of the equations governing the perturbations, it is suitable to rewrite Eq. (2) in the form

$$\widetilde{F}^\mu{}_\nu\partial_\mu\omega+g^{\mu\alpha}\nabla_\alpha(\xi F_{\mu\nu})=0,$$

and now taking linear perturbations around a general background spacetime, using Eqs. (8) and after a few rearrangements, the linearization of the modified Maxwell equations takes the form

$$\begin{aligned}&4\widetilde{F}^\mu{}_\nu\partial_\mu\left(\frac{d\omega}{d\eta}\eta^{\rm B}\right)+4\nabla^\mu\left(\frac{d\xi}{d\phi}F_{\mu\nu}\phi^{\rm B}\right)+4\left\{g_\nu{}^\mu\nabla^\rho(\xi\nabla_\rho)-\nabla^\mu(\xi\nabla_\nu)+\frac{2}{\sqrt{-g}}g_{\alpha\nu}\epsilon^{\rho\alpha\lambda\mu}(\partial_\rho\omega)\partial_\lambda\right\}b_\mu\\&+4\left\{(\partial_\rho\omega)\left[-\frac{1}{2}g^{\mu\alpha}\widetilde{F}^\rho{}_\nu+g^\alpha{}_\nu\widetilde{F}^{\rho\mu}\right]-[\nabla^\alpha(\xi F^\mu{}_\nu)]-\xi\left[F^\alpha{}_\nu\nabla^\mu+g^\mu{}_\nu F^{\rho\alpha}\nabla_\rho-\frac{1}{2}g^{\mu\alpha}F^\rho{}_\nu\nabla_\rho\right]\right\}h_{\mu\alpha}=0,\end{aligned} \tag{9}$$

where we have multiplied by a factor 4 for future convenience [4].

Similarly, the dilaton equation (4) can be rewritten as

$$g^{\mu\alpha}[\partial_\alpha\partial_\mu\phi-\Gamma^\lambda_{\alpha\mu}\partial_\lambda\phi]-\frac{1}{8}\frac{d\zeta}{d\phi}g^{\mu\alpha}(\partial_\mu\eta)\partial_\alpha\eta+\frac{1}{4}\left[\frac{d\xi}{d\phi}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}+\frac{\partial V}{\partial\phi}\right]=0,$$

whose perturbed version is (after multiplying by a factor −4)

$$\left\{\frac{d\zeta}{d\phi}(\partial^{\mu}\eta)\partial_{\mu}-\frac{\partial^{2}V}{\partial\phi\partial\eta}\right\}\eta^{\mathrm{B}}-4\left\{\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu}\frac{d^{2}\xi}{d^{2}\phi}-\frac{1}{2}\frac{d^{2}\zeta}{d^{2}\phi}(\partial^{\mu}\eta)(\partial_{\mu}\eta)\right]+\nabla^{\mu}\nabla_{\mu}\right\}\phi^{\mathrm{B}}-4\frac{d\xi}{d\phi}F^{\mu\nu}\partial_{\mu}b_{\nu}$$
$$-4\left\{-(\nabla^{\alpha}\nabla^{\mu}\phi)+\frac{1}{2}\frac{d\xi}{d\phi}F^{\lambda\mu}F^{\alpha}{}_{\lambda}+\frac{1}{8}\frac{d\zeta}{d\phi}(\partial^{\mu}\eta)(\partial^{\alpha}\eta)-(\nabla^{\alpha}\phi)\nabla^{\mu}+\frac{1}{2}g^{\mu\alpha}(\nabla^{\rho}\phi)\nabla_{\rho}\right\}h_{\mu\alpha}=0. \tag{10}$$

Similarly, the linearized axion equation can be obtained from Eq. (5),

$$-\left\{\zeta\nabla_{\mu}\partial^{\mu}+(\partial^{\mu}\zeta)\partial_{\mu}+\frac{d^{2}\omega}{d\eta^{2}}F_{\mu\nu}\widetilde{F}^{\mu\nu}+\frac{\partial^{2}V}{\partial^{2}\eta}\right\}\eta^{\mathrm{B}}-\left\{\left[\nabla_{\mu}\left(\frac{d\zeta}{d\phi}\partial^{\mu}\eta\right)\right]+\frac{d\zeta}{d\phi}(\partial^{\alpha}\eta)\partial_{\alpha}+\frac{\partial^{2}V}{\partial\eta\partial\phi}\right\}\phi^{\mathrm{B}}-4\frac{d\omega}{d\eta}\widetilde{F}^{\mu\nu}\partial_{\mu}b_{\nu}$$
$$+\left\{\zeta\left[(\nabla^{\alpha}\partial^{\mu}\eta)+(\partial^{\alpha}\eta)\nabla^{\mu}-\frac{1}{2}g^{\mu\alpha}(\partial^{\rho}\eta)\nabla_{\rho}\right]+(\partial^{\mu}\zeta)(\partial^{\alpha}\eta)+\frac{1}{2}\frac{d\omega}{d\eta}F_{\rho\gamma}\widetilde{F}^{\rho\gamma}g^{\mu\alpha}\right\}h_{\mu\alpha}=0, \tag{11}$$

where we have multiplied by a factor −1.

Similarly, writing the perturbations of the left-hand side of Einstein equation (6) as $R^{\mathrm{B}}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^{\mathrm{B}}-\frac{1}{2}Rh_{\mu\nu}$, the perturbed version of this equation can be written in the form

$$-\left\{\zeta(\partial_{(\mu}\eta)\partial_{\alpha)}-\frac{1}{2}\zeta g_{\mu\alpha}(\partial^{\rho}\eta)\partial_{\rho}+\frac{1}{2}g_{\mu\alpha}\frac{\partial V}{\partial\eta}\right\}\eta^{\mathrm{B}}-\left\{4(\partial_{(\mu}\phi)\partial_{\alpha)}-2g_{\mu\alpha}(\partial^{\rho}\phi)\partial_{\rho}+\frac{1}{2}\frac{d\zeta}{d\phi}\left[(\partial_{\mu}\eta)(\partial_{\alpha}\eta)-\frac{1}{2}g_{\mu\alpha}(\partial^{\rho}\eta)(\partial_{\rho}\eta)\right]\right.$$
$$\left.+\frac{1}{2}g_{\mu\alpha}\frac{\partial V}{\partial\phi}-\frac{d\xi}{d\phi}T^{M}_{\mu\alpha}\right\}\phi^{\mathrm{B}}-2\xi\{g_{\mu\alpha}F^{\lambda\gamma}\partial_{\lambda}+2[F^{\gamma}{}_{(\mu}\partial_{\alpha)}+g^{\gamma}{}_{(\mu}F_{\alpha)}{}^{\lambda}\partial_{\lambda}]\}b_{\gamma}+[(\mathcal{E}_{V}+\mathcal{E}_{S})h_{\rho\gamma}]_{\alpha\mu}=0, \tag{12}$$

with

$$T^{M}_{\mu\nu}=2\left[F_{\mu\lambda}F_{\nu}{}^{\lambda}-\frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}\right],$$

where the metric perturbations coming from perturbed tensor of matter (plus that coming from the term $-\frac{1}{2}Rh_{\mu\nu}$) have been represented by the operator $\mathcal{E}_S$, which only is a function (and contains no differential operators). The explicit form of this function is not important, as will be seen below where we will discuss the concept of the adjoint of an operator. $\mathcal{E}_V$ (coming from the perturbed terms $R^{\mathrm{B}}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^{\mathrm{B}}$) is the operator describing gravitational perturbations of vacuum space-times, i.e., it is that given in Eq. (3) of Ref. [4].

The complete set of coupled equations for the field perturbations (9)–(12) can be expressed in compact form in the following matrix form:

$$\begin{bmatrix}\mathcal{E}_{A} & \mathcal{E}_{AD} & \mathcal{E}_{AE} & \mathcal{E}_{AG}\\ \mathcal{E}_{DA} & \mathcal{E}_{D} & \mathcal{E}_{DE} & \mathcal{E}_{DG}\\ \mathcal{E}_{EA} & \mathcal{E}_{ED} & \mathcal{E}_{E} & \mathcal{E}_{EG}\\ \mathcal{E}_{GA} & \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_{G}\end{bmatrix}\begin{bmatrix}\eta^{\mathrm{B}}\\ \phi^{\mathrm{B}}\\ (b_{\mu})\\ (h_{\mu\nu})\end{bmatrix}=0, \tag{13}$$

where the first, second, third and fourth rows correspond to the equations (11), (10), (9) and (12), respectively, and from these equations, the explicit forms of the $\mathcal{E}$'s (linear partial differential operators involving the background fields) can be read.

Note that the presence of matter fields nonminimally coupled to gravity makes that all field perturbations appear in each of the perturbation equations (9)–(12). As will be seen in the next section, in spite of the very complicated appearance of these equations, they possess the self-adjoint character, which will allow us to establish the existence of a symplectic current for the coupled field perturbations.

III. SELF-ADJOINTNESS AND THE SYMPLECTIC CURRENT

In accordance with Wald's definition [4], if $\mathcal{E}$ corresponds to a linear partial differential operator [such as the matrix operator of Eqs. (13)] which maps m-index tensor fields into n-index tensor fields, then, the adjoint operator of $\mathcal{E}$, denoted by $\mathcal{E}^{\dagger}$, is that linear partial differential operator mapping n-index tensor fields into m-index tensor fields such that

$$f^{\rho\sigma\cdots}[\mathcal{E}(g_{\mu\nu\ldots})]_{\rho\sigma\ldots}-[\mathcal{E}^{\dagger}(f^{\rho\sigma\cdots})]^{\mu\nu\cdots}g_{\mu\nu\ldots}=\nabla_{\mu}J^{\mu}, \tag{14}$$

where J^{μ} is some vector field, and similarly for any other operator. From this definition, if $\mathcal{A}$ and $\mathcal{B}$ are any two linear operators, one obtains that

$$(\mathcal{A}\mathcal{B})^{\dagger}=\mathcal{B}^{\dagger}\mathcal{A}^{\dagger},\quad (\mathcal{A}+\mathcal{B})^{\dagger}=\mathcal{A}^{\dagger}+\mathcal{B}^{\dagger}. \tag{15}$$

In the case of a function f [for example, the operator $\mathcal{E}_S$ appearing in Eq. (12)],

$$f^{\dagger}=f. \tag{16}$$

From Eq. (14) we can easily see that this definition automatically guarantees that, if the fields f and g are two independent solutions of the linear system $\mathcal{E}(f)=0=\mathcal{E}(g)$ and the operator $\mathcal{E}$ is self-adjoint (i.e. $\mathcal{E}^{\dagger}=\mathcal{E}$, or antiself-adjoint, $\mathcal{E}^{\dagger}=-\mathcal{E}$), then J^{μ} is a covariantly conserved quantity (which depends on the fields f and g) [2]. This fact means that there exists an intrinsic connection between self-adjointness and the existence of a conserved quantity.

In this manner, the goal of this paper is to demonstrate that the operator governing the field perturbations in Eq. (13) is self-adjoint. For this purpose, let us consider the first diagonal element $\mathcal{E}_A$ which can be read from Eq. (11), it is that acting on $\eta^{\rm B}$, and mapping scalar fields into themselves. Using the identity

$$\nabla^\mu(\psi_2\zeta\partial_\mu\psi_1)\equiv\psi_2\nabla^\mu\zeta\partial_\mu\psi_1-\psi_1\nabla^\mu\zeta\partial_\mu\psi_2+\nabla_\mu(\zeta\psi_1\partial^\mu\psi_2),$$

where ψ_1 and ψ_2 are any two scalar fields, it is straightforward to show that

$$\psi_2\mathcal{E}_A\psi_1=\psi_1\mathcal{E}_A\psi_2+\nabla_\mu\zeta(\psi_1\partial^\mu\psi_2-\psi_2\partial^\mu\psi_1),$$

this expression has the form (14) and allows us to identify that

$$\mathcal{E}_A{}^\dagger=\mathcal{E}_A. \tag{17}$$

Similarly, using the definition (14), the properties (15) and (16) and assuming that the background fields satisfy Eqs. (2)–(7), one can demonstrate that

$$\psi_2\mathcal{E}_{DA}\psi_1=\psi_1\mathcal{E}_{AD}\psi_2+\nabla_\mu\left[\frac{d\zeta}{d\phi}\psi_1\psi_2\partial^\mu\eta\right]\quad(\mathcal{E}_{DA}^\dagger=\mathcal{E}_{AD}),$$

$$\psi_1(\mathcal{E}_{AE}B_\nu)=B^\nu(\mathcal{E}_{EA}\psi_1)_\nu+\nabla_\mu\left[4\frac{d\omega}{d\eta}\tilde{F}^{\nu\mu}B_\nu\psi_1\right]\quad(\mathcal{E}_{AE}^\dagger=\mathcal{E}_{EA}),$$

$$A^{\mu\alpha}(\mathcal{E}_{GA}\psi_1)_{\mu\alpha}=\psi_1(\mathcal{E}_{AG}A_{\mu\alpha})+\nabla_\mu\zeta\psi_1\left[\frac{1}{2}A_\alpha{}^\alpha\partial^\mu\eta-A^{\mu\nu}\partial_\nu\eta\right]\quad(\mathcal{E}_{AG}^\dagger=\mathcal{E}_{GA}),$$

$$\psi_2\mathcal{E}_D\psi_1=\psi_1\mathcal{E}_D\psi_2+\nabla_\mu 4(\psi_1\partial^\mu\psi_2-\psi_2\partial^\mu\psi_1)\quad(\mathcal{E}_D^\dagger=\mathcal{E}_D),$$

$$\psi_1(\mathcal{E}_{DE}B_\nu)=B^\nu(\mathcal{E}_{ED}\psi_1)_\nu+\nabla_\mu\left(4\frac{d\xi}{d\phi}F^{\nu\mu}B_\nu\psi_1\right)\quad(\mathcal{E}_{DE}^\dagger=\mathcal{E}_{ED}),$$

$$\psi_1(\mathcal{E}_{DG}A_{\mu\alpha})=A^{\mu\alpha}(\mathcal{E}_{GD}\psi_1)_{\mu\alpha}+\nabla_\mu 2\psi_1[2A^\mu{}_\alpha\partial^\alpha\phi-A_\alpha{}^\alpha\partial^\mu\phi]\quad(\mathcal{E}_{DG}^\dagger=\mathcal{E}_{GD}),$$

$$A^\nu(\mathcal{E}_E B_\mu)_\nu=B^\nu(\mathcal{E}_E A_\mu)_\nu+\nabla_\mu 8\left\{\xi[A_\rho\nabla^{[\mu}B^{\rho]}+B_\rho\nabla^{[\rho}A^{\mu]}]+\frac{1}{\sqrt{-g}}\epsilon^{\rho\alpha\mu\lambda}A_\alpha B_\lambda\partial_\rho\eta\right\}\quad(\mathcal{E}_E^\dagger=\mathcal{E}_E),$$

$$A^{\mu\alpha}(\mathcal{E}_{GE}B_\nu)_{\mu\alpha}=B^\nu(\mathcal{E}_{EG}A_{\mu\alpha})_\nu+\nabla_\mu 2\xi\{4B_\nu F_\alpha{}^{[\nu}A^{\mu]\alpha}-A_\alpha{}^\alpha B_\nu F^{\mu\nu}\}\quad(\mathcal{E}_{GE}^\dagger=\mathcal{E}_{EG}), \tag{18}$$

where A_μ and B_μ are any two vector fields and $A_{\mu\nu}$ any 2-index (symmetric) tensor field. Finally, since $\mathcal{E}_G=\mathcal{E}_V+\mathcal{E}_S$, taking the expression for $\mathcal{E}_V$ given in Ref. [4] [see the paragraph after Eq. (12)] and the property (16), one finds that

$$A^{\mu\nu}(\mathcal{E}_G B_{\alpha\rho})_{\mu\nu}=B^{\alpha\rho}(\mathcal{E}_G A_{\mu\nu})_{\alpha\rho}+\nabla_\mu S^{\mu\alpha\beta\lambda\rho\gamma}\times(A_{\alpha\beta}\nabla_\lambda B_{\rho\gamma}-B_{\alpha\beta}\nabla_\lambda A_{\rho\gamma})\quad(\mathcal{E}_G^\dagger=\mathcal{E}_G), \tag{19}$$

where $B_{\mu\nu}$ is another 2-index (symmetric) tensor field and [1]

$$S^{\mu\alpha\beta\lambda\rho\gamma}=g^{\mu(\rho}g^{\gamma)(\alpha}g^{\beta)\lambda}-\frac{1}{2}g^{\mu\lambda}g^{\alpha(\rho}g^{\gamma)\beta}-\frac{1}{2}g^{\mu(\alpha}g^{\beta)\lambda}g^{\rho\gamma}-\frac{1}{2}g^{\alpha\beta}g^{\mu(\rho}g^{\gamma)\lambda}+\frac{1}{2}g^{\alpha\beta}g^{\mu\lambda}g^{\rho\gamma}. \tag{20}$$

Then, from Eqs. (17)–(19) we have found that the adjoint of the matrix operator appearing in Eq. (13) is given by [5]

$$\begin{bmatrix}\mathcal{E}_A & \mathcal{E}_{AD} & \mathcal{E}_{AE} & \mathcal{E}_{AG}\\ \mathcal{E}_{DA} & \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG}\\ \mathcal{E}_{EA} & \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG}\\ \mathcal{E}_{GA} & \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G\end{bmatrix}^\dagger=\begin{bmatrix}\mathcal{E}_A & \mathcal{E}_{AD} & \mathcal{E}_{AE} & \mathcal{E}_{AG}\\ \mathcal{E}_{DA} & \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG}\\ \mathcal{E}_{EA} & \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG}\\ \mathcal{E}_{GA} & \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G\end{bmatrix}, \tag{21}$$

which means that this operator is self-adjoint. It is worth to point out that any operator appearing in physics is not necessarily self-adjoint. For example, the operator corresponding to the usual free massless field equations of spin greater that one is not self-adjoint on a curved spacetime; although, those corresponding to the Weyl neutrino equation and the linearized Yang-Mills are self-adjoint on a curved background.

With the purpose of obtaining the symplectic current for the coupled field perturbations, let S_1 and S_2 be any two solutions of the system (13) given by

$$(S_1)=\begin{pmatrix}\eta_1^{\rm B}\\ \phi_1^{\rm B}\\ (b_\mu)\\ (h_{\mu\nu})\end{pmatrix},\quad (S_2)=\begin{pmatrix}\eta_2^{\rm B}\\ \phi_2^{\rm B}\\ (B_\mu)\\ (H_{\mu\nu})\end{pmatrix}, \tag{22}$$

and from definition (14)

$$S_2(\mathcal{E}S_1)-S_1(\mathcal{E}^\dagger S_2)=\nabla_\mu J^\mu, \tag{23}$$

where $\mathcal{E}$ is the matrix operator of Eq. (13). In this manner, from Eqs. (17)–(23) we can easily find the explicit form for the symplectic current,

$$J^\mu=J_{\rm A}{}^\mu+J_{\rm AD}{}^\mu+J_{\rm AE}{}^\mu+J_{\rm AG}{}^\mu+J_{\rm D}{}^\mu+J_{\rm DE}{}^\mu+J_{\rm DG}{}^\mu+J_{\rm E}{}^\mu+J_{\rm EG}{}^\mu+J_{\rm G}{}^\mu, \tag{24}$$

where

$$J_{\mathrm{A}}{}^{\mu}=\zeta(\eta_1^{\mathrm{B}}\partial^{\mu}\eta_2^{\mathrm{B}}-\eta_2^{\mathrm{B}}\partial^{\mu}\eta_1^{\mathrm{B}}),$$

$$J_{\mathrm{AD}}{}^{\mu}=\frac{d\zeta}{d\phi}(\partial^{\mu}\eta)[\eta_1^{\mathrm{B}}\phi_2^{\mathrm{B}}-\phi_1^{\mathrm{B}}\eta_2^{\mathrm{B}}],$$

$$J_{\mathrm{AE}}{}^{\mu}=4\frac{d\omega}{d\eta}\tilde{F}^{\mu\rho}[B_{\rho}\eta_1^{\mathrm{B}}-b_{\rho}\eta_2^{\mathrm{B}}],$$

$$J_{\mathrm{AG}}{}^{\mu}=\zeta\left[(\partial_{\rho}\eta)[h^{\mu\rho}\eta_2^{\mathrm{B}}-H^{\mu\rho}\eta_1^{\mathrm{B}}]+\frac{1}{2}(\partial^{\mu}\eta)[H\eta_1^{\mathrm{B}}-h\eta_2^{\mathrm{B}}]\right],$$

$$J_{\mathrm{D}}{}^{\mu}=4[\phi_1^{\mathrm{B}}\partial^{\mu}\phi_2^{\mathrm{B}}-\phi_2^{\mathrm{B}}\partial^{\mu}\phi_1^{\mathrm{B}}],$$

$$J_{\mathrm{DE}}{}^{\mu}=4\frac{d\xi}{d\phi}F^{\mu\rho}[B_{\rho}\phi_1^{\mathrm{B}}-b_{\rho}\phi_2^{\mathrm{B}}],$$

$$J_{\mathrm{DG}}{}^{\mu}=2[2(\partial^{\rho}\phi)[h^{\mu}{}_{\rho}\phi_2^{\mathrm{B}}-H^{\mu}{}_{\rho}\phi_1^{\mathrm{B}}]+(\partial^{\mu}\phi)[H\phi_1^{\mathrm{B}}-h\phi_2^{\mathrm{B}}]],$$

$$J_{\mathrm{E}}{}^{\mu}=8\left\{\xi[B_{\rho}\nabla^{[\mu}b^{\rho]}]-b_{\rho}\nabla^{[\mu}B^{\rho]}\right.$$
$$\left.+\frac{1}{\sqrt{-g}}\epsilon^{\rho\alpha\mu\lambda}(\partial_{\rho}\eta)B_{\alpha}b_{\lambda}\right\},$$

$$J_{\mathrm{EG}}{}^{\mu}=2\xi\{2F_{\rho}{}^{\mu}[h^{\rho\lambda}B_{\lambda}-H^{\rho\lambda}b_{\lambda}]+2F_{\rho\lambda}[H^{\mu\rho}b^{\lambda}-h^{\mu\rho}B^{\lambda}]$$
$$+F^{\mu\rho}(hB_{\rho}-Hb_{\rho})\},$$

$$J_{\mathrm{G}}{}^{\mu}=S^{\mu\alpha\beta\lambda\rho\gamma}(H_{\alpha\beta}\nabla_{\lambda}h_{\rho\gamma}-h_{\alpha\beta}\nabla_{\lambda}H_{\rho\gamma}), \tag{25}$$

where $H=g^{\mu\nu}H_{\mu\nu}$, $h=g^{\mu\nu}h_{\mu\nu}$ and the subscripts just denote the types of field perturbations involved, for example, $J_{\mathrm{DG}}{}^{\mu}$ involves dilaton and gravitational perturbations.

The expression for J^{μ} is not gauge invariant with respect to either gauge transformation of B_{μ} and b_{μ} or transformations $\phi^{\mathrm{B}}\to\phi^{\mathrm{B}}+\mathrm{const}$ and $\eta^{\mathrm{B}}\to\eta^{\mathrm{B}}+\mathrm{const}$. Of course, the last two transformations are not gauge transformations, since the zero modes of the dilaton and of the axion are physically meaningful.

It is important to stress that the existence of the conserved quantity (24) is a direct consequence of the self-adjointness shown in Eq. (21), and in the demonstration of this property only the assumption that the background fields satisfy Eqs. (2)–(7) has been required.

In order to compare the purely Einstein-Maxwell current[1] given by $J_{\mathrm{E}}{}^{\mu}+J_{\mathrm{GE}}{}^{\mu}+J_{\mathrm{G}}{}^{\mu}$, with that given in Eq. (3.12) of Ref. [1], it is convenient to note that the linear variations of the electromagnetic field and of the metric appearing in that reference can be rewritten according to our notations as

$$\delta_1 A_{\mu}=b_{\mu},$$

$$\delta_2 A_{\mu}=B_{\mu},$$

$$\delta_1 g_{\mu\nu}=h_{\mu\nu},$$

$$\delta_2 g_{\mu\nu}=H_{\mu\nu},$$

and then

$$\delta_1 F^{\mu\nu}=\nabla^{\mu}b^{\nu}-\nabla^{\nu}b^{\mu}+2F^{\rho[\mu}h^{\nu]}{}_{\rho},$$

$$\delta_2 F^{\mu\nu}=\nabla^{\mu}B^{\nu}-\nabla^{\nu}B^{\mu}+2F^{\rho[\mu}H^{\nu]}{}_{\rho}, \tag{26}$$

in this manner, from Eqs. (25), (26)

$$J_{\mathrm{E}}{}^{\mu}+J_{\mathrm{EG}}{}^{\mu}+J_{\mathrm{G}}{}^{\mu}=S^{\mu\alpha\beta\lambda\rho\gamma}(H_{\alpha\beta}\nabla_{\lambda}h_{\rho\lambda}-h_{\alpha\beta}\nabla_{\lambda}H_{\rho\gamma})$$
$$+4\xi\{B_{\rho}\delta_1F^{\nu\rho}-b_{\rho}\delta_2F^{\mu\rho}\}$$
$$+2\xi F^{\mu\rho}(hB_{\rho}-Hb_{\rho})$$
$$+\frac{8}{\sqrt{-g}}\epsilon^{\rho\alpha\mu\lambda}B_{\alpha}b_{\lambda}\partial_{\rho}\eta, \tag{27}$$

which reduces exactly to the expression (3.12) given in Ref. [1] when $\xi=-k$ (gravitational constant) and $\eta=0$ in the background geometry. Of course, one must be very careful in choosing consistently some background fields since, usually, some restrictions are imposed on the remaining background fields through the field equations. However, we can consistently set

$$\phi=0,$$

$$\eta=0,$$

$$\xi=-k,$$

$$\omega=0,$$

$$V=0,$$

(since, for example, in the low-energy limit of the string theory ξ is of the form $e^{-2\phi}$ and $\omega=\eta$), without imposing any restriction on the remaining electromagnetic field in the background, according to Eqs. (4), (5).

Similarly with $F_{\mu\nu}=0=\eta=V$, the sum $J_{\mathrm{D}}{}^{\mu}+J_{\mathrm{DG}}{}^{\mu}+J_{\mathrm{G}}{}^{\mu}$ should correspond to the symplectic current for field perturbations in the ordinary Einstein-Klein-Gordon theory [2].

In this manner, the current (24) can be considered as the generalization of that of Ref. [1] when the low-energy degrees of freedom most characteristic of string theory, dilaton and axion fields, are incorporated. However, it is worth noting that the present results have been obtained by an approach completely different from that used in Ref. [1].

IV. CONCLUDING REMARKS

Unfortunately, not many solutions of the perturbed equations (9)–(12) (for nontrivial background fields) have been obtained in order to evaluate the symplectic current and to study its physical meaning and implications; however, as shown by Chandrasekhar and Ferrari (see [1] and references cited therein), the current for the Einstein-Maxwell perturbations, in the case of the Reissner-Nordström solution leads to the conservation of energy for the perturbations. On the other

[1] In this point the Abelian gauge field must be considered as the standard electromagnetic field.

hand, in the framework of the Einstein-Maxwell-dilaton-axion theory, when the background geometry is the space-time for colliding plane waves, some explicit solutions have been obtained for the Eqs. (9)–(12) [6,7], and the corresponding symplectic current will be studied in more detail in subsequent works. Furthermore, the self-adjointness of Eqs. (9)–(12) opens a way of finding particular solutions of these equations provided that the corresponding decoupled system of equations is found [4]. In addition, one may avoid the imposition of any gauge condition in the perturbed tetrad [6], which would be a great advantage when the current is evaluated.

Because of the generality of the action (1) considered in this paper, our results can be applied to the study of the perturbations of a great amount of solutions appearing in the literature. Some interesting cases would be the nonstatic solutions for the extremal electrically charged black holes in Einstein-Maxwell-dilaton theory (which suggest the possible violation of the cosmic censorship) [8], solutions corresponding to black holes which carry both electric and magnetic charge [9], etc. On the other hand, in the present paper it has been considered the dilaton-axion gravity coupled only with one Abelian gauge field; however, our results can be easily generalized for coupling to Abelian vector multiplets, which arise in toroidally compactified string theory [10].

Finally, since it has been shown that our approach for obtaining the symplectic current gives the same result as obtained by the Burnett-Wald approach, the possible connection between both methods is an open question.

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