Perturbations in dilaton-axion gravity coupled with electromagnetic fields

R. Cartas-Fuentevilla
Instituto de Física, Universidad Autónoma de Puebla, Apartado postal J-48 72570 Puebla, Puebla, México
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Explicit solutions of the perturbed Einstein-Maxwell-dilaton-axion theory are obtained by means of complex scalar potentials in the case when the background geometry is the spacetime corresponding to plane waves in the regions prior to the collision. These expressions are derived using Wald's method of adjoint operators and a decoupled system of equations; no gauge-fixing condition on the perturbed tetrad is imposed. Our results cover the low energy limit of string theory for some fixed values of the dilaton coupling constants. The existence of purely incoming perturbations is discussed. [S0556-2821(98)02304-2]

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I. INTRODUCTION

Dilaton-axion gravity coupled with matter fields has been extensively studied in the last years. This gravity theory arises in a natural way in $N=4$ supergravity and in the low-energy limit of heterotic string theory after the compactification of six of the ten dimensions of the string theory. When the dilaton and axion fields are incorporated, the properties of the solutions appearing in ordinary Einstein gravity can be drastically modified, for example, changes in the global causal structure of the solutions, new implications on the black-hole thermodynamics, solutions suggesting that Penrose's conjecture on cosmic censorship may be violated [1–3], etc.

In particular, when the matter field is the electromagnetic field, the four-dimensional field equations for the Einstein-Maxwell (EM) theory interacting with the dilaton and axion fields are

$$\nabla_\mu(\eta\tilde{F}^{\mu\nu}+\xi F^{\mu\nu})=0, \quad \nabla_{[\mu}F_{\nu\lambda]}=0 \quad (\text{Maxwell}), \tag{1}$$

$$\nabla_\mu\nabla^\mu\phi+\frac{1}{2}a\xi F^2-\frac{1}{2}b\zeta(\partial_\mu\eta)\partial^\mu\eta=0,$$
$$F^2\equiv F_{\mu\nu}F^{\mu\nu} \quad (\text{dilaton}), \tag{2}$$

$$\nabla_\mu(\zeta\partial^\mu\eta)-F_{\mu\nu}\tilde{F}^{\mu\nu}=0,$$
$$\tilde{F}^{\mu\nu}\equiv\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad (\text{axion}), \tag{3}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu} \quad (\text{Einstein}), \tag{4}$$

with the matter energy-momentum tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu}=2(\partial_\mu\phi)\partial_\nu\phi+\frac{1}{2}\zeta(\partial_\mu\eta)\partial_\nu\eta-g_{\mu\nu}\bigg[(\partial^\alpha\phi)\partial_\alpha\phi+\frac{1}{4}\zeta(\partial^\alpha\eta)\partial_\alpha\eta\bigg]+2\xi\bigg(F_{\mu\lambda}F_\nu{}^\lambda-\frac{1}{4}g_{\mu\nu}F^2\bigg), \tag{5}$$

where ϕ and η denote the dilaton and axion scalar fields, respectively, and we define $\xi\equiv e^{-2a\phi}$ and $\zeta\equiv e^{4b\phi}$; the dimensionless parameters a and b correspond to the coupling constants which govern the coupling of the dilaton to the Maxwell field ($F_{\mu\nu}$) and to the axion field, respectively. In addition, R is the scalar curvature, $\tilde{F}^{\mu\nu}$ is the dual of $F^{\mu\nu}$, $g=\det(g_{\mu\nu})$, and $\mu,\nu=0,1,2,3$. Special theories are contained in the field equations (1)–(5), for example for $a=1$ and $b=1$; these equations are obtained from the bosonic sector of the four-dimensional effective action obtained after the compactification in the string theory.

Although there exists an extensive body of literature on the exact solutions of the field equations (1)–(5) and their implications, no great progress has been made in the study of their perturbations, which would provide us with a way of understanding the structure of such solutions. With this idea in mind, in this work we will focus our attention in the perturbed versions of Eqs. (1)–(5) around a general background solution. Therefore, in Sec. II we introduce the notation and the full equations for the metric, electromagnetic field, dilaton, and axion field perturbations.

Various procedures have been applied in the study of systems of perturbation equations. A method usually employed in this study, for example in the linearized EM theory [4–6], consists in trying to solve the set of equations for the complete perturbations directly. This procedure has some disadvantages—for example, it involves a lot of differential equations to solve, in addition to the fact that in the present case, the situation becomes more complicated due to the presence of the dilaton and axion fields. Fortunately, there are other procedures which permit massive reductions of the number of differential equations to solve and computations, besides providing us with expressions for the perturbations in terms of derivatives of complex scalar potentials, which automatically gives the correct relative normalization between all the components of the perturbations. One of these procedures is the Wald method of adjoint operators, which applies when we can obtain a decoupled set of equations from the original set of equations for the perturbations, provided the self-adjointness of this system of equations has been established [7]. Therefore, in Sec. III the self-adjoint character of the linearized Einstein-Maxwell-dilaton-axion (EMDA) theory is discussed.

Among all exact solutions of Eqs. (1)–(5), the plane wave solutions are especially important, since these geometries correspond to exact solutions of the string theory at all orders of the string tension parameter [8] and in higher dimensions lead to exact extreme black hole solutions when the dimensional reduction is performed [9]. The perturbations of plane wave geometries have been studied by Chandrasekhar and Xanthopoulos in the framework of the EM theory and their conclusions showed the absence of purely incoming perturbations [5,6]; they believed that this fact would be connected with some ‘‘no hair’’ theorem to gravitational waves waiting to be discovered [6]. However, recently it has been demonstrated in the same framework of the EM theory that the existence of purely incoming perturbations is a property of the most general spacetime representing plane waves bound to collision [10,11]. With these preliminary ideas, in Sec. IV we shall study the perturbations of the spacetime corresponding to gravitational plane waves coupled to the electromagnetic waves, dilaton, and axion waves in the regions previous to the collision in the scheme of the EMDA theory. Using the Newman-Penrose formalism, we find a decoupled system of equations for the perturbations without imposing any gauge condition on the perturbed tetrad. In this manner, the complete perturbations for the metric, electromagnetic potential, dilaton, and axion fields are expressed in terms of complex scalar potentials. These expressions allow us in Sec. V to demonstrate the existence of purely incoming perturbations even in the more general framework of the EMDA theory. The Newman-Penrose formulation of the field equations (1)–(5) is summarized in the Appendix, which is useful in Sec. IV and for future reference.

II. LINEARIZATION OF THE EMDA THEORY

In this section and throughout the superscript ‘‘B’’ denotes the corresponding first-order perturbations. In particular, the metric, electromagnetic potential, dilaton, and axion perturbations are represented by $h_{\mu\nu}$, b_μ, $\phi^{\rm B}$, and $\eta^{\rm B}$, respectively.

In addition, one easily can demonstrate that

$$
\begin{aligned}
(g^{\mu\nu})^{\rm B} &= -h^{\mu\nu},\\
g^{\rm B} &= g g_{\mu\nu} h^{\mu\nu},\\
F^{\rm B}_{\mu\nu} &= \partial_\mu b_\nu - \partial_\nu b_\mu,\\
(\widetilde{F}^{\mu\nu})^{\rm B} &= \frac{2}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha b_\beta - \frac{1}{2}\widetilde{F}^{\mu\nu} g_{\alpha\beta} h^{\alpha\beta},\\
\xi^{\rm B} &\equiv (e^{-2a\phi})^{\rm B} = -2a\xi\phi^{\rm B},\\
\zeta^{\rm B} &\equiv (e^{4b\phi})^{\rm B} = 4b\zeta\phi^{\rm B},\\
(\Gamma^{\lambda}_{\mu\nu})^{\rm B} &= \frac{1}{2} g^{\lambda\rho}[\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\mu\rho} - \nabla_\rho h_{\mu\nu}],\\
R^{\rm B} &= g^{\mu\nu} R^{\rm B}_{\mu\nu} - R_{\mu\nu} h^{\mu\nu},
\end{aligned}
\tag{6}
$$

where the covariant derivative ∇_μ is with respect to the background metric $g_{\mu\nu}$, which raises and lowers the indices, for example $h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta}$, that will be used below. The explicit form of $R^{\rm B}_{\mu\nu}$ in terms of $h_{\mu\nu}$ is not required, since its properties are well known in other references [7].

In order to linearize the Maxwell equation (1), it is suitable to write this equation in the form

$$
\widetilde{F}^{\mu}{}_{\nu}\partial_\mu \eta + g^{\mu\alpha}\nabla_\alpha(\xi F_{\mu\nu}) = 0,
$$

and now considering linear perturbations around a general background solution, using Eqs. (6) and grouping suitably, the linearization of the preceding equation takes the form

$$
\begin{aligned}
&-4\widetilde{F}^{\mu}{}_{\nu}\partial_\mu \eta^{\rm B} + 8a\nabla_\alpha \xi F^{\alpha}{}_{\nu}\phi^{\rm B} - 4\Big\{ g_\nu{}^{\mu}\nabla^\rho(\xi\nabla_\rho) - \nabla^\mu(\xi\nabla_\nu)\\
&\quad + \frac{2}{\sqrt{-g}} g_{\alpha\nu}\epsilon^{\rho\alpha\lambda\mu}(\partial_\rho\eta)\partial_\lambda\Big\} b_\mu + 4\Big\{(\partial_\rho\eta)\Big[\frac{1}{2}g^{\mu\alpha}\widetilde{F}^{\rho}{}_{\nu}\\
&\quad - g^{\alpha}{}_{\nu}\widetilde{F}^{\rho\mu}\Big] + [\nabla^\alpha(\xi F^{\mu}{}_{\nu})] + \xi\Big[F^{\alpha}{}_{\nu}\nabla^\mu + g^{\mu}{}_{\nu}F^{\rho\alpha}\nabla_\rho\\
&\quad - \frac{1}{2}g^{\mu\alpha}F^{\rho}{}_{\nu}\nabla_\rho\Big]\Big\} h_{\mu\alpha} = 0,
\end{aligned}
\tag{7}
$$

where we have multiplied by a factor of –4 for future convenience [7].

Similarly, we can write the dilaton equation (2) as

$$
g^{\mu\alpha}[\partial_\alpha\partial_\mu\phi - \Gamma^{\lambda}_{\alpha\mu}\partial_\lambda\phi] - \frac{1}{2}b\zeta g^{\mu\alpha}(\partial_\mu\eta)\partial_\alpha\eta + \frac{1}{2}a\xi F^2 = 0,
$$

now taking linear perturbations and multiplying by a factor of 4, the linearized dilaton equation is

$$
\begin{aligned}
&4b\zeta(\partial^\mu\eta)\partial_\mu\eta^{\rm B} + 4[a^2\xi F^2 + 2b^2\zeta(\partial^\mu\eta)(\partial_\mu\eta) - \nabla^\mu\nabla_\mu]\phi^{\rm B}\\
&\quad - 8a\xi F^{\mu\nu}\nabla_\mu b_\nu + 4\Big\{(\nabla^\alpha\nabla^\mu\phi) + a\xi F^{\mu\lambda}F^{\alpha}{}_{\lambda}\\
&\quad - \frac{1}{2}b\zeta(\partial^\mu\eta)(\partial^\alpha\eta) + (\nabla^\alpha\phi)\nabla^\mu - \frac{1}{2}g^{\mu\alpha}(\nabla^\rho\phi)\nabla_\rho\Big\}\\
&\quad \times h_{\mu\alpha} = 0.
\end{aligned}
\tag{8}
$$

By the same procedure we can obtain the linearized axion equation from Eq. (3) (multiplying by a factor of –1):

$$
\begin{aligned}
&-\nabla^\mu\zeta\partial_\mu\eta^{\rm B} - 4b[F_{\mu\nu}\widetilde{F}^{\mu\nu} + \zeta(\partial^\mu\eta)\partial_\mu]\phi^{\rm B} + 4\widetilde{F}^{\mu\nu}\nabla_\mu b_\nu\\
&\quad - \Big\{\frac{1}{2}g^{\mu\nu}[\zeta(\partial^\alpha\eta)\nabla_\alpha + F_{\alpha\gamma}\widetilde{F}^{\alpha\gamma}] - \zeta(\partial^\nu\eta)\nabla^\mu\\
&\quad - [\nabla^\mu(\zeta\partial^\nu\eta)]\Big\} h_{\mu\nu} = 0.
\end{aligned}
\tag{9}
$$

In order to linearize the Einstein equations (4), we first derive the perturbed energy-momentum tensor of matter given in Eq. (5). It is not difficult to find that the linearized first and second terms are given by

$$2[(\partial_\mu\phi)\partial_\nu\phi]^{\rm B}+\frac{1}{2}[\zeta(\partial_\mu\eta)\partial_\nu\eta]^{\rm B}$$
$$=2[2(\partial_{(\mu}\phi)\partial_{\nu)}+b\zeta(\partial_\mu\eta)(\partial_\nu\eta)]\phi^{\rm B}+\zeta(\partial_{(\mu}\phi)\partial_{\nu)}\eta^{\rm B}. \tag{10}$$

The third term can be suitably written as

$$-g_{\mu\nu}\left[(\partial^\alpha\phi)\partial_\alpha\phi+\frac{1}{4}\zeta(\partial^\alpha\eta)\partial_\alpha\eta\right]=-g_{\mu\nu}g^{\lambda\alpha}\left[(\partial_\lambda\phi)\partial_\alpha\phi+\frac{1}{4}\zeta(\partial_\lambda\eta)\partial_\alpha\eta\right];$$

then,

$$-\left\{g_{\mu\nu}\left[(\partial^\alpha\phi)\partial_\alpha\phi+\frac{1}{4}\zeta(\partial^\alpha\eta)\partial_\alpha\eta\right]\right\}^{\rm B}$$
$$=\left\{g_{\mu\nu}\left[(\partial^\lambda\phi)(\partial^\alpha\phi)+\frac{1}{4}\zeta(\partial^\lambda\eta)(\partial^\alpha\eta)\right]-g_\nu{}^\lambda g_\mu{}^\alpha\left[(\partial^\rho\phi)(\partial_\rho\phi)+\frac{1}{4}\zeta(\partial^\rho\eta)(\partial_\rho\eta)\right]\right\}h_{\lambda\alpha}$$
$$-g_{\mu\nu}[2(\partial^\alpha\phi)\partial_\alpha+b\zeta(\partial^\alpha\eta)(\partial_\alpha\eta)]\phi^{\rm B}-\frac{1}{2}g_{\mu\nu}\zeta(\partial^\alpha\eta)\partial_\alpha\eta^{\rm B}. \tag{11}$$

The linearized third term can be expressed as

$$(\xi T^{\rm M}_{\mu\nu})^{\rm B}=T^{\rm M}_{\mu\nu}\xi^{\rm B}+\xi(T^{\rm M}_{\mu\nu})^{\rm B}, \tag{12}$$

where

$$T^{\rm M}_{\mu\nu}=2\left[F_{\mu\lambda}F_\nu{}^\lambda-\frac{1}{4}g_{\mu\nu}F^2\right]$$

is the usual energy-momentum tensor of the electromagnetic field. Using the formulas (6), the expression (12) can be written as follows:

$$(\xi T^{\rm M}_{\mu\nu})^{\rm B}=-2a\xi T^{\rm M}_{\mu\nu}\phi^{\rm B}-2\xi\left[F_\mu{}^\alpha F_\nu{}^\gamma+\frac{1}{4}F^2g_\mu{}^\alpha g_\nu{}^\gamma+\frac{1}{2}g_{\mu\nu}F^{\lambda\gamma}F^\alpha{}_\lambda\right]h_{\alpha\gamma}$$
$$-2\xi\left[2F^\gamma{}_{(\mu}F^{\rm B}_{\nu)\gamma}+\frac{1}{2}g_{\mu\nu}F^{\alpha\gamma}F^{\rm B}_{\alpha\gamma}\right]. \tag{13}$$

The metric perturbations coming from Eq. (13) are the same that appear in the case when the only matter field present is the electromagnetic field [7], except for the phase factor ξ; in this manner, the overall factor acting on $h_{\alpha\gamma}$ continues to be a function.

Finally, from Eqs. (4), (10)–(13) the linearized Einstein equations are given by

$$\zeta\left[\frac{1}{2}g_{\mu\nu}(\partial^\alpha\eta)\partial_\alpha-(\partial_{(\mu}\eta)\partial_{\nu)}\right]\eta^{\rm B}$$
$$+2\left\{a\xi T^{\rm M}_{\mu\nu}+\frac{1}{2}b\zeta g_{\mu\nu}(\partial^\alpha\eta)(\partial_\alpha\eta)+g_{\mu\nu}(\partial^\alpha\phi)\partial_\alpha-2(\partial_{(\mu}\phi)\partial_{\nu)}-b\zeta(\partial_\mu\eta)(\partial_\nu\eta)\right\}\phi^{\rm B}$$
$$-2\xi[2g^\alpha{}_{(\mu}F_{\nu)}{}^\gamma-2g^\gamma{}_{(\mu}F_{\nu)}{}^\alpha-g_{\mu\nu}F^{\alpha\gamma}]\nabla_\alpha b_\gamma$$
$$+\left\{\mathcal{E}_{\rm G}'+g_\mu{}^\alpha g_\nu{}^\lambda\left[(\partial^\rho\phi)(\partial_\rho\phi)+\frac{1}{4}\zeta(\partial^\rho\eta)(\partial_\rho\eta)\right]-g_{\mu\nu}\left[(\partial^\lambda\phi)(\partial^\alpha\phi)+\frac{1}{4}\zeta(\partial^\lambda\eta)(\partial^\alpha\eta)\right]\right\}h_{\alpha\lambda}=0, \tag{14}$$

where the operators acting on the metric perturbations $h_{\mu\nu}$ coming from the linearization of the first member of Eq. (4) $R^{\rm B}_{\mu\nu}-\frac{1}{2}(g_{\mu\nu}R)^{\rm B}$, and those coming from Eq. (13) have been represented by the operator $\mathcal{E}_{\rm G}'$, whose explicit form is not important, because it is essentially the same are appearing in the framework of the EM theory [see the paragraph after Eqs. (6) and (13)] and is well known [7].

The complete set of perturbed EMDA equations (7), (8), (9), and (14) can be expressed in the following matrix form for future convenience:

$$\begin{bmatrix}\mathcal{E}_A & \mathcal{E}_{AD} & \mathcal{E}_{AE} & \mathcal{E}_{AG}\\ \mathcal{E}_{DA} & \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG}\\ \mathcal{E}_{EA} & \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG}\\ \mathcal{E}_{GA} & \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G\end{bmatrix}\begin{bmatrix}\eta^{\rm B}\\ \phi^{\rm B}\\ (b_\mu)\\ (h_{\mu\nu})\end{bmatrix}=0, \tag{15}$$

where the $\mathcal{E}$'s are linear partial differential operators involving the background fields, whose explicit forms can be read from Eqs. (7), (8), (9), and (14), which correspond to the third, second, first, and fourth rows, respectively.

III. SELF-ADJOINTNESS OF THE PERTURBED EMDA THEORY

In order to find expressions for the complete solutions of systems of linear partial differential equations in terms of scalar potentials, Wald introduced a method which makes use of the concept of the adjoint of a linear operator [7]. If $\mathcal{E}$ corresponds to a linear partial differential operator which maps m-index tensor fields into n-index tensor fields then, the adjoint operator of $\mathcal{E}$, denoted by $\mathcal{E}^\dagger$, is the linear partial differential operator mapping n-index tensor fields into m-index tensor fields such that

$$t^{\rho\sigma\cdots}[\mathcal{E}(f_{\mu\nu\ldots})]_{\rho\sigma\ldots}=[\mathcal{E}^\dagger(t^{\rho\sigma\cdots})]^{\mu\nu\cdots}f_{\mu\nu\ldots}+\nabla_\mu v^\mu, \tag{16}$$

where v^μ is some vector field, and similarly for any other operator. For example, in the Newman-Penrose formalism we have that

$$D^\dagger = -(D+\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho}), \quad \Delta^\dagger = -(\Delta-\gamma-\bar{\gamma}+\mu+\bar{\mu}),$$

$$\delta^\dagger = -(\delta+\beta-\bar{\alpha}-\tau+\bar{\pi}), \quad \bar{\delta}^\dagger = -(\bar{\delta}+\bar{\beta}-\alpha-\bar{\tau}+\pi), \tag{17}$$

which will be useful below. In the case of a function f,

$$f^\dagger = f. \tag{18}$$

Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are any two linear operators, then from the definition (16) one obtains that

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger, \quad (\mathcal{A}+\mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger. \tag{19}$$

For more details of Wald's method, see for example [7].

With the aim of demonstrating the self-adjointness of the matrix operator governing the perturbations in Eqs. (15), we will determine the adjoint operator of each of the operators appearing in that equation.

The first diagonal element $\mathcal{E}_A$ can be read from Eq. (9); it is that acting on η^{B},

$$\mathcal{E}_A = -\nabla^\mu \zeta \partial_\mu,$$

which maps scalar fields into themselves. Using the identity

$$\nabla^\mu(\psi_2\zeta\partial_\mu\psi_1) \equiv \psi_2\nabla^\mu\zeta\partial_\mu\psi_1 - \psi_1\nabla^\mu\zeta\partial_\mu\psi_2 + \nabla_\mu(\zeta\psi_1\partial^\mu\psi_2),$$

where ψ_1 and ψ_2 are any two scalar fields, it is straightforward to show that

$$\psi_2\mathcal{E}_A\psi_1 = \psi_1\mathcal{E}_A\psi_2 + \nabla_\mu\zeta(\psi_1\partial^\mu\psi_2 - \psi_2\partial^\mu\psi_1).$$

This expression has the form (16) and allows us to identify that

$$\mathcal{E}_A{}^\dagger = \mathcal{E}_A. \tag{20}$$

Similarly, using the definition (16), the properties (18) and (19), and assuming that the background fields satisfy Eqs. (1)–(5), one can demonstrate that

$$\mathcal{E}_D^\dagger = \mathcal{E}_D, \quad \mathcal{E}_E^\dagger = \mathcal{E}_E, \quad \mathcal{E}_G^\dagger = \mathcal{E}_G,$$

$$\mathcal{E}_{DE}^\dagger = \mathcal{E}_{ED}, \quad \mathcal{E}_{DG}^\dagger = \mathcal{E}_{GD}, \quad \mathcal{E}_{GE}^\dagger = \mathcal{E}_{EG},$$

$$\mathcal{E}_{AD}^\dagger = \mathcal{E}_{DA}, \quad \mathcal{E}_{AG}^\dagger = \mathcal{E}_{GA}, \quad \mathcal{E}_{AE}^\dagger = \mathcal{E}_{EA}, \tag{21}$$

then, from Eqs. (20)–(21) we have found that the adjoint of the matrix operator appearing in Eq. (15) is given by [12]

$$\begin{bmatrix} \mathcal{E}_A & \mathcal{E}_{AD} & \mathcal{E}_{AE} & \mathcal{E}_{AG} \\ \mathcal{E}_{DA} & \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG} \\ \mathcal{E}_{EA} & \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG} \\ \mathcal{E}_{GA} & \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G \end{bmatrix}^\dagger = \begin{bmatrix} \mathcal{E}_A & \mathcal{E}_{AD} & \mathcal{E}_{AE} & \mathcal{E}_{AG} \\ \mathcal{E}_{DA} & \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG} \\ \mathcal{E}_{EA} & \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG} \\ \mathcal{E}_{GA} & \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G \end{bmatrix}, \tag{22}$$

which means that this matrix operator is self-adjoint. It is important to point out that, on a curved background, the operators corresponding to the Weyl neutrino equation and the linearized Yang-Mills equations are also self-adjoint. On the other hand, the operator corresponding to the usual free massless field equations of spin greater than one is not self-adjoint on a curved spacetime.

The self-adjointness shown in Eq. (22) for linearized EMDA theory (for any coupling constants a and b) is not sufficient to find solutions of this set of equations in terms of scalar potentials in some particular cases, since one requires also that the corresponding decoupled system of equations be found. In the next section, such a decoupled system from Eqs. (15) is obtained when the background solution is the spacetime corresponding to plane waves bound to a collision, which can also be expressed in matrix form [see Eq. (47)].

IV. INCOMING WAVES AND THEIR PERTURBATIONS

A. Background solution to be perturbed

The spacetime corresponding to the colliding plane waves in the regions prior to the collision (which contains one of the approaching waves) can be specified by [13]

$$ds^2 = 2e^{-M}du\,dv - e^{-U}[e^{-V}(dx^2)^2 + e^V(dx^1)^2], \tag{23}$$

where $u = x^0 + x^3$ and $v = x^0 - x^3$, and with the metric components, electromagnetic, dilaton, and axion fields depending only on v:

$$U(v), \quad V(v), \quad M(v), \quad A_\mu(v), \quad \phi(v), \quad \eta(v). \tag{24}$$

The diagonal line element (23) can be described by the null tetrad

$$D = \frac{\sqrt{2}}{N}\partial_u, \quad \Delta = \frac{\sqrt{2}}{N}\partial_v,$$

$$\delta = \frac{1}{\sqrt{2H}}(\chi^{-1/2}\partial_{x^1} + i\chi^{1/2}\partial_{x^2}),$$

$$\bar{\delta} = \frac{1}{\sqrt{2H}}(\chi^{-1/2}\partial_{x^1} - i\chi^{1/2}\partial_{x^2}), \tag{25}$$

where we have defined, for simplicity,

$$N^2 \equiv 2e^{-M}, \quad H \equiv e^{-U}, \quad \chi \equiv e^V.$$

In addition, the only nonvanishing spin coefficients are

$$\gamma(v) = -\frac{1}{\sqrt{2}N}\frac{d}{dv}\ln N, \quad \mu(v) = -\frac{1}{\sqrt{2}N}\frac{d}{dv}\ln H,$$

$$\lambda(v) = \frac{1}{\sqrt{2}N}\frac{d}{dv}\ln\chi, \tag{26}$$

the only nonvanishing component of the spinor Weyl is

$$\Psi_4 = -\frac{1}{2}\left[\frac{d^2V}{dv^2} - \frac{dV}{dv}\left(\frac{dU}{dv} - \frac{dM}{dv}\right)\right]. \tag{27}$$

The incoming regions are filled with Petrov type-N gravitational fields, radiation fields, and null electromagnetic fields;

therefore, if we take the tetrad vector l^μ along the principal null direction of the background electromagnetic field, then we have that

$$\varphi_0 = 0 = \varphi_1, \tag{28}$$

$\varphi_2(v)$ being the only nonvanishing component. Using Eq. (28), from Eqs. (A9) the Einstein field equations reduce to

$$\Phi_{22} = -(\Delta\phi)^2 - \frac{1}{4}\zeta(\Delta\eta)^2 + 2\xi\varphi_2\overline{\varphi_2}, \tag{29}$$

since $\Delta\phi$ and $\Delta\eta$ are the only nonvanishing derivatives of dilaton and axion fields.

B. Decoupled equations and master equations

From the Maxwell equations (A1) and (A3) and from Eqs. (24), (26), and (28) we obtain

$$\bar{\delta}\varphi_0^{\rm B} - D\varphi_1^{\rm B} - \varphi_2\kappa^{\rm B} = 2\pi\xi^{-1}l^\mu j_\mu, \tag{30}$$

$$[\Delta - 2\gamma + \mu - a(\Delta\phi) - i\xi^{-1}(\Delta\eta)]\varphi_0^{\rm B} - \delta\varphi_1^{\rm B} - \varphi_2\sigma^{\rm B} + a\overline{\varphi_2}\,(D\phi)^{\rm B} - \overline{\varphi_2}\xi^{-1}(\Delta\eta)^{\rm B} = 2\pi\xi^{-1}m^\mu j_\mu, \tag{31}$$

where j_μ represents a source for the electromagnetic perturbations (see Ref. [10] and references cited therein). Note that the components of the perturbed tetrad do not appear, since they are acting on φ_0 and φ_1 and these background quantities vanish according to Eq. (28).

Moreover, from the Ricci identities and Eqs. (24) and (26) one finds that

$$D\sigma^{\rm B} - \delta\kappa^{\rm B} = \Psi_0^{\rm B}, \quad D\overline{\rho}^{\rm B} - \delta\overline{\kappa}^{\rm B} = 0. \tag{32}$$

Applying δ to Eq. (30) and D to Eq. (31), subtracting and considering the left-hand side of Eqs. (32) (and that $[D,\delta]=0$), we obtain that

$$\mathcal{O}_E\varphi_0^{\rm B} - \varphi_2\Psi_0^{\rm B} + a\overline{\varphi_2}D(D\phi)^{\rm B} - \overline{\varphi_2}\xi^{-1}D(D\eta)^{\rm B} = 2\pi\mathcal{S}_E(j_\mu), \tag{33}$$

with

$$\mathcal{O}_E = D[\Delta - 2\gamma + \mu - a(\Delta\phi) - i\xi^{-1}(\Delta\eta)] - \delta\bar{\delta},$$

$$\mathcal{S}_E(j_\mu) = \xi^{-1}[D(m^\mu j_\mu) - \delta(l^\mu j_\mu)]. \tag{34}$$

On the other hand, from the Bianchi identities and by considering that the only nonvanishing spinor Weyl component is given in Eq. (27) we obtain that

$$\bar{\delta}\Psi_0^{\rm B} - D\Psi_1^{\rm B} = 4\pi[\delta(l^\mu l^\nu T_{\mu\nu}) - D(l^\mu m^\nu T_{\mu\nu})], \tag{35}$$

$$(\Delta - 4\gamma + \mu)\Psi_0^{\rm B} - \delta\Psi_1^{\rm B} + 2\xi\overline{\varphi_2}D\varphi_0^{\rm B} = 4\pi[\delta(l^\mu m^\nu T_{\mu\nu}) - \overline{\lambda}l^\mu l^\nu T_{\mu\nu} - D(m^\mu m^\nu T_{\mu\nu})], \tag{36}$$

where we also have included an additional source for the gravitational perturbations, $T_{\mu\nu}$ [10]. Applying the same procedure used in Eqs. (30) and (31) to eliminate $\varphi_1^{\rm B}$, we can cancel the terms with $\Psi_1^{\rm B}$ of Eqs. (35) and (36) and we get

$$\mathcal{O}_G\Psi_0^{\rm B} + 2\xi\overline{\varphi_2}D^2\varphi_0^{\rm B} = 4\pi\mathcal{S}_G(T_{\mu\nu}), \tag{37}$$

where

$$\mathcal{S}_G(T_{\mu\nu}) = D[\delta(l^\mu m^\nu T_{\mu\nu}) - D(m^\mu m^\nu T_{\mu\nu}) - \overline{\lambda}l^\mu l^\nu T_{\mu\nu}] + \delta[D(l^\mu m^\nu T_{\mu\nu}) - \delta(l^\mu l^\nu T_{\mu\nu})],$$

$$\mathcal{O}_G = D(\Delta - 4\gamma + \mu) - \delta\bar{\delta}. \tag{38}$$

In order to complete the system of Eqs. (33) and (37) and to avoid the appearence of undesirable perturbed quantities [14], before considering the perturbations, we apply D to Eq. (A5) (dilaton equation) and we obtain

$$D(\Delta + \mu - \gamma - \overline{\gamma})D\phi - \overline{\rho}D\Delta\phi - (\Delta\phi)D\overline{\rho} + D(\overline{\tau}\delta\phi) + D(-\delta + \overline{\alpha} - \beta + \tau)\bar{\delta}\phi + \frac{1}{4}aD(\xi F^2) - \frac{1}{2}bD\{\zeta[(D\eta)\Delta\eta - (\delta\eta)\bar{\delta}\eta]\} = 0. \tag{39}$$

Using the commutation relations the fifth term can be expressed as

$$\begin{aligned} D(-\delta + \overline{\alpha} - \beta + \tau)\bar{\delta}\phi &= -(\delta - \overline{\alpha} - \beta + \overline{\pi})(\bar{\delta} - \alpha - \overline{\beta} + \pi)D\phi \\ &\quad + (\delta - \overline{\alpha} - \beta + \overline{\pi})\kappa\Delta\phi - (\delta - \overline{\alpha} - \beta + \overline{\pi}) \\ &\quad \times[(\rho + \overline{\varepsilon} - \varepsilon)\bar{\delta}\phi + \overline{\sigma}\delta\phi] \\ &\quad + \kappa\Delta\bar{\delta}\phi - (\overline{\rho} + \overline{\varepsilon} - \varepsilon)\delta\bar{\delta}\phi \\ &\quad - \sigma\bar{\delta}^2\phi + D[(\overline{\alpha} - \beta + \tau)\bar{\delta}\phi], \end{aligned} \tag{40}$$

and now using Eqs. (24) and (26) it is very easy to demonstrate that

$$[D(-\delta + \overline{\alpha} - \beta + \tau)\bar{\delta}\phi]^{\rm B} = -\delta\bar{\delta}(D\phi)^{\rm B} + \Delta\phi\delta\overline{\kappa}^{\rm B}. \tag{41}$$

Furthermore, the linearization of nonvanishing remaining terms of Eq. (39) is given by

$$[D(\Delta + \mu - \gamma - \overline{\gamma})D\phi]^{\rm B} = D(\Delta + \mu - \gamma - \overline{\gamma})(D\phi)^{\rm B},$$

$$-[(\Delta\phi)D\overline{\rho}]^{\rm B} = -(\Delta\phi)D\overline{\rho}^{\rm B},$$

$$[D\{\zeta[(D\eta)\Delta\eta - (\delta\eta)\bar{\delta}\eta]\}]^{\rm B} = \zeta(\Delta\eta)D(D\eta)^{\rm B},$$

$$[D(\xi F^2)]^{\rm B} = 4\xi[\varphi_2 D\varphi_0^{\rm B} + \overline{\varphi_2}D\overline{\varphi_0}^{\rm B}], \tag{42}$$

then, from Eqs. (41) and (42) and the right-hand side of Eqs. (32), the perturbed version of Eq. (39) takes the form

$$\mathcal{O}_D(D\phi)^{\mathrm{B}}-\frac{1}{2}b\zeta(\Delta\eta)D(D\eta)^{\mathrm{B}}+a\xi[\varphi_2 D\varphi_0^{\mathrm{B}}+\overline{\varphi_2}D\overline{\varphi_0}^{\mathrm{B}}]$$
$$=2\pi D\phi_s+4\pi(\Delta\phi)l^\mu l^\nu T_{\mu\nu}, \tag{43}$$

where

$$\mathcal{O}_D=D(\Delta+\mu-\gamma-\overline{\gamma})-\delta\overline{\delta}, \tag{44}$$

and ϕ_s represents a source for the dilaton field perturbations.

The procedure employed in order to obtain the preceding equation from the dilaton equation can be straightforwardly applied on Eq. (A7) (axion equation), and to obtain

$$\mathcal{O}_A(D\eta)^{\mathrm{B}}+2b\zeta(\Delta\eta)D(D\phi)^{\mathrm{B}}+4i[\varphi_2 D\varphi_0^{\mathrm{B}}-\overline{\varphi_2}D\overline{\varphi_0}^{\mathrm{B}}]$$
$$=8\pi D\eta_s+4\pi\zeta(\Delta\eta)l^\mu l^\nu T_{\mu\nu}, \tag{45}$$

where

$$\mathcal{O}_A=\zeta[D(\Delta+\mu-2\gamma+2b\Delta\phi)-\delta\overline{\delta}], \tag{46}$$

and η_s represents a source for the axion field perturbations.

The set of four equations (33), (37), (43), and (45) involves actually five unknowns $(D\phi)^{\mathrm{B}}$, $(D\eta)^{\mathrm{B}}$, φ_0^{B}, Ψ_0^{B}, and $\overline{\varphi_0}^{\mathrm{B}}$. In order to rectify this situation, we must consider the complex conjugates of Eqs. (33) and (37) to obtain two additional equations, which complete our linear system for six unknowns (the five ones mentioned above) plus $\overline{\Psi_0}^{\mathrm{B}}$ [$(D\phi)^{\mathrm{B}}$ and $(D\eta)^{\mathrm{B}}$ are real quantities]. Note that the complex conjugates of Eqs. (43) and (45) are themselves. This system of six equations can be expressed in the following matrix form:

$$\begin{pmatrix} \mathcal{O}_G & 2\xi\overline{\varphi_2}D^2 & 0 & 0 & 0 & 0 \\ -\varphi_2 & \mathcal{O}_E & 0 & 0 & a\overline{\varphi_2}D & -i\overline{\varphi_2\xi}^{-1}D \\ 0 & 0 & \overline{\mathcal{O}_G} & 2\xi\varphi_2 D^2 & 0 & 0 \\ 0 & 0 & -\overline{\varphi_2} & \overline{\mathcal{O}_E} & a\varphi_2 D & i\varphi_2\xi^{-1}D \\ 0 & a\xi\varphi_2 D & 0 & a\xi\overline{\varphi_2}D & \mathcal{O}_D & -\frac{1}{2}b\zeta(\Delta\eta)D \\ 0 & 4i\varphi_2 D & 0 & -4i\overline{\varphi_2}D & 2b\zeta(\Delta\eta)D & \mathcal{O}_A \end{pmatrix} \begin{pmatrix} \Psi_0^{\mathrm{B}} \\ \varphi_0^{\mathrm{B}} \\ \overline{\Psi_0}^{\mathrm{B}} \\ \overline{\varphi_0}^{\mathrm{B}} \\ (D\phi)^{\mathrm{B}} \\ (D\eta)^{\mathrm{B}} \end{pmatrix} = 4\pi \begin{pmatrix} \mathcal{S}_G T_{\mu\nu} \\ \frac{1}{2}\mathcal{S}_E j_\mu \\ \overline{\mathcal{S}_G} T_{\mu\nu} \\ \frac{1}{2}\overline{\mathcal{S}_E} j_\mu \\ \frac{1}{2}D\phi_s+(\Delta\phi)l^\mu l^\nu T_{\mu\nu} \\ 2D\eta_s+\zeta(\Delta\eta)l^\mu l^\nu T_{\mu\nu} \end{pmatrix}, \tag{47}$$

where the operators $\overline{\mathcal{S}_E}$ and $\overline{\mathcal{S}_G}$ correspond to the complex conjugates of those of Eqs. (34) and (38). The right-hand side corresponds to

$$\mathcal{S}\begin{bmatrix} (T_{\mu\nu}) \\ (j_\mu) \\ \phi_s \\ \eta_s \end{bmatrix},$$

where $\mathcal{S}$ is given by the following 6×4 matrix:

$$\mathcal{S}=\begin{pmatrix} \mathcal{S}_G & 0 & 0 & 0 \\ 0 & \frac{1}{2}\mathcal{S}_E & 0 & 0 \\ \overline{\mathcal{S}_G} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\overline{\mathcal{S}_E} & 0 & 0 \\ (\Delta\phi)l^\mu l^\nu & 0 & \frac{1}{2}D & 0 \\ \zeta(\Delta\eta)l^\mu l^\nu & 0 & 0 & 2D \end{pmatrix}. \tag{48}$$

$\mathcal{O}$ being the matrix operator appearing on the left-hand side and using Eqs. (17), (34), (38), (44), and (46), we find that

$$\mathcal{O}^\dagger = \begin{pmatrix} \mathcal{O}_G^\dagger & -\varphi_2 & 0 & 0 & 0 & 0 \\ 2\xi\overline{\varphi_2}D^2 & \mathcal{O}_E^\dagger & 0 & 0 & -a\xi\varphi_2 D & -4i\varphi_2 D \\ 0 & 0 & \overline{\mathcal{O}_G^\dagger} & -\overline{\varphi_2} & 0 & 0 \\ 0 & 0 & 2\xi\varphi_2 D^2 & \overline{\mathcal{O}_E^\dagger} & -a\xi\overline{\varphi_2}D & 4i\overline{\varphi_2}D \\ 0 & -a\overline{\varphi_2}D & 0 & -a\varphi_2 D & \mathcal{O}_D^\dagger & -2b\zeta(\Delta\eta)D \\ 0 & i\overline{\varphi_2}\xi^{-1}D & 0 & -i\varphi_2\xi^{-1}D & \frac{1}{2}b\zeta(\Delta\eta)D & \mathcal{O}_A^\dagger \end{pmatrix}, \tag{49}$$

where

$$\mathcal{O}_G^\dagger = (\Delta + 2\gamma + \mu)D - \overline{\delta}\delta,$$

$$\mathcal{O}_E^\dagger = [\Delta + \mu + a(\Delta\phi) + i\xi^{-1}(\Delta\eta)]D - \overline{\delta}\delta,$$

$$\mathcal{O}_D^\dagger = (\Delta + \mu)D - \overline{\delta}\delta,$$

$$\mathcal{O}_A^\dagger = \zeta[D(\Delta + \mu + 2b\Delta\phi) - \overline{\delta}\delta], \tag{50}$$

and also

$$\mathcal{S}^\dagger = \begin{pmatrix} \mathcal{S}_G^\dagger & 0 & \overline{\mathcal{S}_G^\dagger} & 0 & (\Delta\phi)l^\mu l^\nu & \zeta(\Delta\eta)l^\mu l^\nu \\ 0 & \frac{1}{2}\mathcal{S}_E^\dagger & 0 & \frac{1}{2}\overline{\mathcal{S}_E^\dagger} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}D & 0 \\ 0 & 0 & 0 & 0 & 0 & -2D \end{pmatrix}, \tag{51}$$

where

$$\mathcal{S}_G^\dagger = -l^\mu l^\nu[\delta^2 - \overline{\lambda}D] - m^\mu m^\nu D^2 + 2l^{(\mu}m^{\nu)}\delta D,$$

$$\mathcal{S}_E^\dagger = \xi^{-1}[l^\mu\delta - m^\mu D]. \tag{52}$$

In this manner, if the matrix potential (ψ) satisfies $\mathcal{O}^\dagger(\psi) = 0$ with

$$(\psi) = \begin{pmatrix} \psi_G \\ \psi_E \\ \overline{\psi_G} \\ \overline{\psi_E} \\ \psi_D \\ \psi_A \end{pmatrix}, \tag{53}$$

(ψ_G, ψ_E, ψ_D, and ψ_A have types $\{-4,0\}$,$\{-2,0\}$,$\{-1,-1\}$, and $\{-1,-1\}$, respectively, in the sense of the Geroch-Held-Penrose formalism), it means, using Eq. (46), that

$$\mathcal{O}_G^\dagger\psi_G - \varphi_2\psi_E = 0,$$

$$2\xi\overline{\varphi_2}D^2\psi_G + \mathcal{O}_E^\dagger\psi_E - a\xi\varphi_2 D\psi_D - 4i\varphi_2 D\psi_A = 0,$$

$$-a\overline{\varphi_2}D\psi_E - a\varphi_2 D\overline{\psi_E} + \mathcal{O}_D^\dagger\psi_D - 2b\zeta(\Delta\eta)D\psi_A = 0,$$

$$\overline{\mathcal{O}_G^\dagger}\,\overline{\psi_G} - \overline{\varphi_2}\,\overline{\psi_E} = 0,$$

$$2\xi\varphi_2 D^2\overline{\psi_G} + \overline{\mathcal{O}_E^\dagger}\,\overline{\psi_E} - a\xi\overline{\varphi_2}D\psi_D + 4i\overline{\varphi_2}D\psi_A = 0,$$

$$i\overline{\varphi_2}\xi^{-1}D\psi_E - i\varphi_2\xi^{-1}D\overline{\psi_E} + \frac{1}{2}b\zeta(\Delta\eta)D\psi_D + \mathcal{O}_A^\dagger\psi_A = 0. \tag{54}$$

Then the metric, vector potential, axion, and dilaton field perturbations are given by [7]

$$\begin{pmatrix} h_{\mu\nu} \\ b_\mu \\ \phi^{\mathrm{B}} \\ \eta^{\mathrm{B}} \end{pmatrix} = \mathcal{S}^\dagger(\psi) = \begin{pmatrix} \mathcal{S}_G^\dagger\psi_G + \overline{\mathcal{S}_G^\dagger}\,\overline{\psi_G} + l_\mu l_\nu[(\Delta\phi)\psi_D + \zeta(\Delta\eta)\psi_A] \\ \frac{1}{2}\mathcal{S}_E^\dagger\psi_E + \frac{1}{2}\overline{\mathcal{S}_E^\dagger}\,\overline{\psi_E} \\ -\frac{1}{2}D\psi_D \\ -2D\psi_A \end{pmatrix}, \tag{55}$$

where the last equality follows from Eqs. (51) and (53). Using Eqs. (52), we have finally that the *real* perturbations are

$$h_{\mu\nu} = -2\{l_\mu l_\nu[\delta^2 - \overline{\lambda}D] + m_\mu m_\nu D^2 - 2l_{(\mu}m_{\nu)}\delta D\psi_G + l_\mu l_\nu[(\Delta\phi)\psi_D + \zeta(\Delta\eta)\psi_A] + \text{c.c.},$$

$$b_\mu = \frac{1}{2}\xi^{-1}(l_\mu\delta - m_\mu D)\psi_E + \text{c.c.},$$

$$\phi^{\mathrm{B}} = -\frac{1}{2}D\psi_D,$$

$$\eta^{\mathrm{B}} = -2D\psi_A. \tag{56}$$

In this manner, the perturbations given in Eq. (56) are defined completely algebraically by means of the six scalar potentials ψ_G, $\overline{\psi_G}$, ψ_E, $\overline{\psi_E}$, ψ_D, and ψ_A which satisfy the six coupled equations (54), called *the master equations* [11]. Before closing this section, let us recall that these expressions for the perturbations have been obtained without fixing any gauge condition on the perturbed null tetrad. They are independent on the six degrees of perturbed tetrad gauge freedom, contrary to other approaches which make use of this gauge freedom in order to simplify the equations for the perturbations [5,6]. On the other hand, explicit forms of the background quantities have not been required, only the dependence on the coordinate v of these quantities [see Eq. (24)], which is a more general property of the incoming waves.

C. Gauge-independent field perturbations

We can derive the expressions for the components of the electromagnetic field perturbations using the second of Eqs. (56), the formula $F^{\mathrm{B}}_{\mu\nu}=\partial_\mu b_\nu-\partial_\nu b_\mu$, and the following definitions:

$$\overline{\varphi_0}^{\mathrm{B}}\equiv l^\mu \overline{m}^\nu F^{\mathrm{B}}_{\mu\nu}=\frac{1}{2}\xi^{-1}D^2\psi_{\mathrm{E}},$$

$$\overline{\varphi_1}^{\mathrm{B}}\equiv\frac{1}{2}(l^\mu n^\nu+m^\mu\overline{m}^\nu)F^{\mathrm{B}}_{\mu\nu}=\frac{1}{2}\xi^{-1}\delta D\psi_{\mathrm{E}},$$

$$\overline{\varphi_2}^{\mathrm{B}}\equiv m^\mu n^\nu F^{\mathrm{B}}_{\mu\nu}=\frac{1}{2}\xi^{-1}\{(\delta^2-\overline{\lambda}D)\psi_{\mathrm{E}} -[(\Delta+\mu+2a(\Delta\phi))D-\delta\overline{\delta}]\overline{\psi_E}\}. \qquad (57)$$

Similarly, the components of the Weyl spinor perturbations can be obtained from the first of Eqs. (56) making use of the formula

$$\Psi^{\mathrm{B}}_{ACDE}=\frac{1}{2}\nabla^{R'}{}_{(A}\nabla^{S'}{}_{C}h_{DE)R'S'}+\frac{1}{2}h_{(AC}{}^{R'S'}\Phi_{DE)R'S'}.$$

Then, we find that

$$\overline{\Psi}^{\mathrm{B}}_0=-D^4\psi_{\mathrm{G}},$$

$$\overline{\Psi}^{\mathrm{B}}_1=-\delta D^3\psi_{\mathrm{G}},$$

$$\overline{\Psi}^{\mathrm{B}}_2=-(\delta^2-\overline{\lambda}D)D^2\psi_{\mathrm{G}} +\frac{1}{6}D^2[(\Delta\phi)\psi_D+\zeta(\Delta\eta)\psi_A],$$

$$\overline{\Psi}^{\mathrm{B}}_3=-(\delta^2-3\overline{\lambda}D)\delta D\psi_{\mathrm{G}} +\frac{1}{4}\delta D[(\Delta\phi)\psi_D+\zeta(\Delta\eta)\psi_A],$$

$$\overline{\Psi}^{\mathrm{B}}_4=-(\delta^2-3\overline{\lambda}D)(\delta^2-\overline{\lambda}D)\psi_{\mathrm{G}}-\delta^2(\overline{\delta}^2-\lambda D)\overline{\psi_{\mathrm{G}}} +\left[\frac{1}{2}\Phi_{22}-(\Delta+2\gamma+\mu)(\Delta+\mu)\right]D^2\overline{\psi_{\mathrm{G}}} +(\overline{\delta}^2-\overline{\lambda}D)[(\Delta\phi)\psi_D+\zeta(\Delta\eta)\psi_A] +[2(\Delta+2\gamma+2\mu)\delta+\overline{\lambda\delta}]\overline{\delta}D\overline{\psi_{\mathrm{G}}}, \qquad (58)$$

where Φ_{22} is given by Eq. (29). Since the field perturbations given by Eqs. (57) and (58) are completely defined by the quantities given in Eqs. (56), they have the same gauge independence [see the paragraph after Eq. (56)], which allows us, for example, to define appropriately fluxes of energy [15], and to study the matching conditions between the different regions occurring in the collision of plane waves [16].

On the other hand, from the definition $\tau=-n^\nu l^\mu\nabla_\nu m_\mu$, using the fact that the only nonvanishing spin coefficients are those given in Eq. (26) and the formula for $h_{\mu\nu}$ given in Eq. (56) we can find that

$$\tau^{\mathrm{B}}=-\overline{\delta}D^2\overline{\psi_{\mathrm{G}}}-(\Delta-2\gamma)l^\mu m^{\mathrm{B}}_\mu, \qquad (59)$$

where $l^\mu m^{\mathrm{B}}_\mu$ corresponds to one of six degrees of freedom of the perturbed tetrad [6,11].

V. EXISTENCE OF PURELY INCOMING PERTURBATIONS

The purely incoming perturbations correspond to the u-independent perturbations. We will demonstrate that the existence of purely incoming perturbations found in Refs. [10,11] in the scheme of the EM theory (contrary to the findings of Chandrasekhar and Xanthopoulos in Refs. [5,6]) is a property that persists in the more general framework of the EMDA theory and, for this purpose it is convenient to define the complex variable

$$z\equiv\frac{1}{\sqrt{2}}[\chi^{1/2}x^1+i\chi^{-1/2}x^2], \qquad (60)$$

and its complex conjugate to replace the real coordinates x^1 and x^2. With this definition, the relevant components of the null tetrad (25) can be rewritten as $\overline{\delta}=1/\sqrt{H}\partial_z$ and $\delta=1/\sqrt{H}\partial_{\overline{z}}$. A direct way to obtain u-independient field perturbations is to assume that the potentials ψ_E, ψ_G, ψ_D, and ψ_A do not depend on u, then the master equations (54) reduce to

$$\frac{1}{H}\partial_z\partial_{\overline{z}}\psi_G+\varphi_2\psi_E=0,\quad \partial_z\partial_{\overline{z}}\psi_E=0,$$
$$\partial_z\partial_{\overline{z}}\psi_D=0,\quad \partial_z\partial_{\overline{z}}\psi_A=0, \qquad (61)$$

whose solutions are

$$\psi_E=-\frac{1}{\varphi_2}H\partial_z F(v,z),$$

$$\psi_G=H^2[\overline{z}F(v,z)+G(v,z)],$$

$$\psi_D=J_D(v,z),$$

$$\psi_A=J_A(v,z), \qquad (62)$$

where $F(v,z)$, $G(v,z)$, $J_D(v,z)$, and $J_A(v,z)$ are arbitrary functions and the factors $1/\varphi_2 H$ and H are introduced for convenience. The only nonvanishing field perturbations can be obtained from Eqs. (56)–(58) and (62):

$$\overline{\varphi_2}^{\mathrm{B}} = -\frac{1}{2\varphi_2}\xi^{-1}\partial_z^3 F(v,z),$$

$$\overline{\Psi}_4^{\mathrm{B}} = -\partial_z^4\{\bar{z}F(v,z)+G(v,z)\} + \frac{1}{H}\partial_z^2[(\Delta\phi)J_D(v,z) + \zeta(\Delta\eta)J_A(v,z)], \tag{63}$$

and from Eq. (59), choosing $l^\mu m_\mu^{\mathrm{B}}$ equal to zero,

$$\tau^{\mathrm{B}} = 0, \tag{64}$$

and since F and G are arbitrary functions, Eqs. (63) imply the existence of nontrivial (type-N) incoming perturbations. Equation (64) implies that in this particular case, the resulting spacetime $g+h$ corresponds to plane waves. The expressions (63) and (64) have been obtained under the assumption that the scalar potentials do not depend on u, but it is not the only way to obtain u-independent perturbations. In fact, the more general solution for this kind of perturbation can be obtained from Eqs. (56)–(58) assuming u-independent functions on the left-hand sides of these equations, provided that the potentials appearing on the right-hand side satisfy the master equations (54); but in this general case, the solution is not necessarily type-N, nor corresponds, in general, to a plane wave geometry. However, the particular solution given in Eqs. (63) and (64) is sufficient to demonstrate the existence of a nontrivial one.

On the other hand, in the framework of the EM theory (where only electromagnetic and gravitational perturbations are present), if the electromagnetic field vanishes (i.e., $\partial_z^3 F = 0$) in Eq. (63), the remaining nonvanishing purely gravitational perturbations not only would correspond to a solution of the linearized EM equations but they also would correspond to an *exact* solution of the EM equations [17]. In the present case (since the dilaton and axion field perturbations vanish), would the nonvanishing purely gravitational perturbation (63) also correspond to an exact solution of the EMDA theory? The establishment of a similar result for the EMDA theory to the one given in Ref. [17] for the EM theory would allow us to answer this open question. In addition, it may allow us to consider gravitational perturbations without perturbing the matter fields.

VI. CONCLUDING REMARKS

Our approach based on the self-adjointness and decoupled systems is very general and it can be applied, in principle, to the whole of solutions of the EMDA theory, as long as the corresponding decoupled system of equations is found. In addition, one may avoid the imposition of any gauge condition on the perturbed tetrad, which is a great advantage. Some interesting perturbation cases to study are the solution corresponding to black holes which carry both electric and magnetic charge, and no static solutions for the extremal electrically charged black holes in EMD theory, which suggest the possible violation of the cosmic censorship [3]. On the other hand, the finding of the perturbations in the interaction regions for colliding plane waves will allow us to study the stability of singularities emerging in these regions and the junction conditions with those found in this work in the incoming regions. Futhermore, recently it has been pointed out that the self-adjoint character of a set of perturbation equations is connected with the existence of a conserved current for such perturbations [18]; works along these lines are in progress and a more detailed demonstration of the self-adjointness of the perturbed EMDA theory will be given.

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APPENDIX: THE NEWMAN-PENROSE FORMULATION OF THE EMDA FIELD EQUATIONS

Projecting on the null tetrad l_ν, n_ν, m_ν, and $\overline{m}_\nu$ as usual, the Maxwell field equations (1) take the form

$$\xi\{(\bar{\delta}+\pi-2\alpha)\varphi_0-(D-2\rho)\varphi_1-\kappa\varphi_2-a[\varphi_0\bar{\delta}+\overline{\varphi_0}\delta -(\varphi_1+\overline{\varphi_1})D]\phi\}+i(\overline{\varphi_0}\delta-\varphi_0\bar{\delta}+\varphi_1 D-\overline{\varphi_1}D)\eta=0, \tag{A1}$$

$$\xi\{-(\delta+2\beta-\tau)\varphi_2+(\Delta+2\mu)\varphi_1-\nu\varphi_0 -a[(\varphi_1+\overline{\varphi_1})\Delta-\varphi_2\delta-\overline{\varphi_2\delta}]\phi\} +i[(\overline{\varphi_1}-\varphi_1)\Delta+\varphi_2\delta-\overline{\varphi_2\delta}]\eta=0, \tag{A2}$$

$$\xi\{(\Delta-2\gamma+\mu)\varphi_0-(\delta-2\tau)\varphi_1-\sigma\varphi_2 -a[\varphi_0\Delta+(\overline{\varphi_1}-\varphi_1)\delta-\overline{\varphi_2}D]\phi\} +i[-\varphi_0\Delta+(\overline{\varphi_1}+\varphi_1)\delta-\overline{\varphi_2 D}]\eta=0, \tag{A3}$$

$$\xi\{-(D-\rho+2\varepsilon)\varphi_2+(\bar{\delta}+2\pi)\varphi_1-\lambda\varphi_0 -a[\overline{\varphi_0}\Delta+(\varphi_1-\overline{\varphi_1})\bar{\delta}-\varphi_2 D]\phi\} +i[\overline{\varphi_0}\Delta-(\varphi_1+\overline{\varphi_1})\bar{\delta}+\varphi_2 D]\eta=0. \tag{A4}$$

Writing the operator $\nabla^\mu\nabla_\mu$ in the Newman-Penrose formalism, the dilaton field equation (2) takes the form

$$[(\Delta+\mu-\gamma-\bar{\gamma})D-\bar{\rho}\Delta+\bar{\tau}\delta+(-\delta+\bar{\alpha}-\beta+\tau)\bar{\delta}]\phi -\frac{1}{2}b\zeta[(D\eta)\Delta\eta-(\delta\eta)\bar{\delta}\eta]+\frac{1}{4}a\xi F^2=0, \tag{A5}$$

where

$$F^2=4[\varphi_0\varphi_2+\overline{\varphi_0\varphi_2}-\varphi_1^2-\overline{\varphi_1}^2]. \tag{A6}$$

Similarly, the axion equation (3) is given by

$$\zeta[(\Delta+\mu-\gamma-\bar{\gamma})D-\bar{\rho}\Delta+\bar{\tau}\delta+(-\delta+\bar{\alpha}-\beta+\tau)\bar{\delta}]\eta - \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}+2b\zeta[(D\phi)\Delta\eta+(\Delta\phi)D\eta-(\delta\phi)\bar{\delta}\eta - (\bar{\delta}\phi)\delta\eta]=0, \tag{A7}$$

where

$$F_{\mu\nu}\tilde{F}^{\mu\nu}=-8i(\varphi_0\varphi_2-\overline{\varphi_0\varphi_2}-\varphi_1^2+\overline{\varphi_1}^2). \tag{A8}$$

Moreover, from $\Phi_{\mu\nu}\equiv-\frac{1}{2}(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)$ and from the Ricci tensor given by

$$R_{\mu\nu}=2(\partial_\mu\phi)\partial_\nu\phi+\frac{1}{2}\zeta(\partial_\mu\eta)\partial_\nu\eta + 2\xi\left(F_{\mu\lambda}F_\nu{}^\lambda-\frac{1}{4}g_{\mu\nu}F^2\right),$$

the Ricci scalars can be expressed in the form

$$\Phi_{00}\equiv l^\mu l^\nu\Phi_{\mu\nu}=-(D\phi)^2-\frac{1}{4}\zeta(D\eta)^2+2\xi\varphi_0\overline{\varphi_0},$$

$$\Phi_{11}\equiv l^\mu n^\nu\Phi_{\mu\nu}=-\frac{1}{2}[(D\phi)\Delta\phi+(\delta\phi)\bar{\delta}\phi] - \frac{1}{8}\zeta[(D\eta)\Delta\eta+(\delta\eta)\bar{\delta}\eta]+2\xi\varphi_1\overline{\varphi_1},$$

$$\Phi_{22}\equiv n^\mu n^\nu\Phi_{\mu\nu}=-(\Delta\phi)^2-\frac{1}{4}\zeta(\Delta\eta)^2+2\xi\varphi_2\overline{\varphi_2},$$

$$\Phi_{01}\equiv l^\mu m^\nu\Phi_{\mu\nu}=-(D\phi)\delta\phi-\frac{1}{4}\zeta(D\eta)\delta\eta+2\xi\varphi_0\overline{\varphi_1},$$

$$\Phi_{02}\equiv m^\mu m^\nu\Phi_{\mu\nu}=-(\delta\phi)^2-\frac{1}{4}\zeta(\delta\eta)^2+2\xi\varphi_0\overline{\varphi_2},$$

$$\Phi_{12}\equiv m^\mu n^\nu\Phi_{\mu\nu}=-(\Delta\phi)\delta\phi-\frac{1}{4}\zeta(\Delta\eta)\delta\eta+2\xi\varphi_1\overline{\varphi_2},$$

$$\Lambda\equiv\frac{1}{24}R=\frac{1}{6}[(D\phi)\Delta\phi-(\delta\phi)\bar{\delta}\phi] + \frac{1}{24}\zeta[(D\eta)\Delta\eta-(\delta\eta)\bar{\delta}\eta], \tag{A9}$$

with $\overline{\Phi_{ij}}=\Phi_{ji}$ $(i,j=0,1,2)$.

[1] G. W. Gibbons, Nucl. Phys. **B207**, 337 (1988); G. W. Gibbons and K. Maeda, *ibid.* **B298**, 741 (1988).
[2] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).
[3] J. H. Horne and G. T. Horowitz, Phys. Rev. D **48**, R5457 (1993).
[4] B. C. Xanthopoulos, J. Math. Phys. **32**, 1866 (1991).
[5] S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London **A420**, 93 (1988).
[6] B. C. Xanthopoulos, J. Math. Phys. **30**, 2626 (1989).
[7] R. M. Wald, Phys. Rev. Lett. **41**, 203 (1978).
[8] R. Güven, Phys. Lett. B **191**, 275 (1987).
[9] G. T. Horowitz and A. A. Tseytlin, Phys. Rev. Lett. **73**, 3351 (1994).
[10] G. F. Torres del Castillo, J. Math. Phys. **37**, 4053 (1996).
[11] R. Cartas-Fuentevilla, Phys. Rev. D **56**, 3365 (1997).
[12] B. P. Jeffryes, Class Quantum Grav. **4**, L17 (1987).
[13] M. Gürses and M. Sermutlu, Phys. Rev. D **52**, 809 (1995); A. García and N. Bretón, *ibid.* **53**, 4351 (1996).
[14] A similar situation occurs in the study of perturbations of the Reissner-Nordström solution by S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford, Clarendon, 1983), p. 238.
[15] G. F. Torres del Castillo and R. Cartas-Fuentevilla, Phys. Rev. D **54**, 4886 (1996).
[16] G. F. Torres del Castillo and C. Mendoza-Barrera, Class. Quantum Grav. **13**, 3245 (1996).
[17] G. F. Torres del Castillo, J. Math. Phys. **37**, 4584 (1996).
[18] G. F. Torres del Castillo and J. C. Flores-Urbina, ''Self-adjoint operators and conserved currents,'' report 1997.