Arbitrarily deformed Kerr-Newman black hole in an external gravitational field

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An exact axisymmetric solution of the Einstein-Maxwell equations possessing two infinite sets of arbitrary real parameters and able to describe a deformed Kerr-Newman black hole in an external gravitational field is presented in a concise analytic form. The validity of Smarr's mass formula is demonstrated for a Kerr-Newman black hole surrounded by an external static gravitational field. [S0556-2821(98)05406-X]

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I. INTRODUCTION

Distorted black holes have been studied by different authors. The distortion of a Schwarzschild black hole by internal deformations was first considered by Erez and Rosen [1], and in [2] it was shown that the quadrupole deformation in the Erez-Rosen solution makes the horizon completely singular. However, the inner mass-multipole moments may also cause the horizon to be singular only in a countable number of points [3], but the appearance of singularities on the horizon or in its vicinity is inevitable for the asymptotically flat static vacuum Weyl solutions in view of Israel's theorem [4]. The most concise description of a Schwarzschild black hole possessing an arbitrary set of inner mass-multipole moments provides the solution [5], and its stationary generalizations representing the deformed Kerr [6] and Kerr-Newman [7] black holes were obtained by Manko and Novikov [8] (generalizations of the Kerr and Kerr-Newman metrics involving the asymptotically flat part of the Erez-Rosen solution as a background metric were constructed by Quevedo and Mashhoon [9,10]).

A black hole can also be distorted by an external gravitational field, and until recently this type of distortion was analyzed with the aid of the outer mass-multipole moments represented by the asymptotically nonflat part of the Erez-Rosen solution. Doroshkevich, Zel'dovich, and Novikov [2] were the first ones to consider the Schwarzschild black hole in an external (quadrupole) gravitational field. They constructed a whole metric corresponding to that spacetime and showed that the Schwarzschild horizon in the external quadrupole field remains regular. In [11] Chandrasekhar obtained the equilibrium condition for a black hole in a static external gravitational field, and the global properties of spacetimes representing static distorted black holes obtained a detailed analysis in a paper by Geroch and Hartle [12]. It can be remarked that in view of Chandrasekhar's result, e.g., the metric representing a Schwarzschild black hole in an external dipole field [13] and its stationary generalizations [14,15] are not satisfactory from the physical viewpoint since in all of them the upper or lower parts of the symmetry axis are not regular (it is precisely for guaranteeing the regularity of the axis that the external field in [2] was chosen to have a quadrupole character). A Kerr black hole surrounded by an external static field was first studied by Tomimatsu [16] who applied the inverse scattering method [17] to the static potential considered by Chandrasekhar [11], the latter potential being a particular case of the Erez-Rosen solution [1]. An important fact established by Tomimatsu is that the well-known Smarr's mass formula [18] for a black hole also holds in the presence of an external gravitational field. It might be mentioned that Tomimatsu's stationary vacuum solution, as well as the static one analyzed by Chandrasekhar, is constructed up to one metric function defined implicitly via first-order differential equations. The whole metric describing the Kerr black hole in an external field has been obtained in a concise explicit form in [19] thanks to the representation of the outer mass-multipole moments in a way different to that of Erez and Rosen.

In [19] some possible straightforward extensions of the results obtained have been outlined, in particular to the case of a charged Kerr black hole in an external gravitational field. The present paper aims at yet a more general objective: We shall give concise expressions for all the metrical fields which will represent a Kerr-Newman black hole possessing two arbitrary sets of mass-multipole moments, the set of inner multipoles describing the deformations of the black hole due to internal perturbations and the set of outer multipoles standing for an arbitrary static and axisymmetric external gravitational field. In Sec. II we shall consider a static metric representing a deformed Schwarzschild particle in an external field, and in Sec. III we shall give its charged stationary generalization describing a Kerr-Newman black hole distorted by internal deformations and by external gravitational field. In Sec. IV we prove the validity of Smarr's mass formula in the case when a black hole is surrounded by an external gravitational field. Here, as an interesting auxiliary result, we give a general expression for the magnetic potential A_3 involving an arbitrary static Weyl potential ψ. Some concluding remarks are contained in Sec. V.

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II. STATIC SEED SOLUTION

The static axisymmetric vacuum problem reduces to solving the equations

$$[(x^2-1)\psi_x]_{,x}+[(1-y^2)\psi_y]_{,y}=0 \tag{1}$$

and

$$\gamma_{,x}=\frac{1-y^2}{x^2-y^2}[x(x^2-1)\psi_{,x}^2-x(1-y^2)\psi_{,y}^2 -2y(x^2-1)\psi_{,x}\psi_{,y}],$$

$$\gamma_{,x}=\frac{x^2-1}{x^2-y^2}[y(x^2-1)\psi_{,x}^2-y(1-y^2)\psi_{,y}^2 +2x(1-y^2)\psi_{,x}\psi_{,y}], \tag{2}$$

Eq. (1) being the integrability condition of the system (2). Here $\psi(x,y)$ and $\gamma(x,y)$ are the metric functions in Weyl's line element

$$ds^2=m^2e^{-2\psi}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right) +(x^2-1)(1-y^2)d\varphi\right]-e^{2\psi}dt^2 \tag{3}$$

(throughout the paper units are used such that $c=G=1$), and x and y are the prolate spheroidal coordinates first introduced by Erez and Rosen [1] and related, e.g., to Weyl cylindrical coordinates (ρ,z) by the formulas

$$x=\frac{1}{2m}(r_++r_-),\quad y=\frac{1}{2m}(r_+-r_-),$$

$$r_\pm:=\sqrt{\rho^2+(z\pm m)^2},$$

$$\rho=m\sqrt{(x^2-1)(1-y^2)},\quad z=mxy,\quad m=\text{const.} \tag{4}$$

Were one wished to use the Erez-Rosen solution to describe a deformed Schwarzschild particle in an external field, one should have to choose ψ in the form

$$\psi=\frac{1}{2}\ln\frac{x-1}{x+1}+\sum_{n=1}^{\infty}[p_nP_n(x)+q_nQ_n(x)]P_n(y), \tag{5}$$

where the first term on the right-hand side of Eq. (5) is the Schwarzschild solution, $P_n()$ are the Legendre polynomials, and $Q_n()$ are Legendre functions of the second kind. Then the real constants q_n would describe the deformations of the source, while the real parameters p_n would describe the external static gravitational field. We mention that the case $p_n=0$, $q_n\neq 0$ was considered by Quevedo [20] who obtained the corresponding metric function γ by integrating the system (2), the resulting γ being defined by very cumbersome expressions. The case $q_n=0$, $p_n\neq 0$ was considered by Chandrasekhar [11] and it was used by Tomimatsu [16] as a static seed solution for the construction of his stationary metric. A general expression for γ in this case has not yet been obtained, and the only two particular solutions for which γ has been found in the explicit form are (i) $p_2\neq 0$, $p_n=0$, $n\neq 2$ (Doroshkevich, Zel'dovich, and Novikov [2]) which represents a Schwarzschild black hole in an external quadrupole field; (ii) $p_1\neq 0$, $p_n=0, n>1$ (Kerns and Wild [13]) for which the resulting γ does not satisfy the elementary flatness condition on the symmetry axis. In [15] case (ii) was superposed with a set of q_n terms, but again the respective function γ was not given there explicitly, the solution itself possessing the same inherent defect as the one considered by Kerns and Wild. Apparently, the metric function γ corresponding to the general form of ψ in Eq. (5) must be by far more complicated than its particular case involving only the constants q_n, and to our knowledge, no attempt has been made to obtain a general expression for γ.

On the other hand, the potential (5) admits an equivalent representation, namely,

$$\psi=\frac{1}{2}\ln\frac{x-1}{x+1}+\sum_{n=1}^{\infty}\left(\frac{a_n}{r^{n+1}}+b_nr^n\right)P_n\left(\frac{xy}{r}\right),$$

$$r:=\sqrt{x^2+y^2-1}, \tag{6}$$

with the constants a_n describing the deformations of the source and constants b_n defining the external field. When $b_n=0$, the potential (6) reduces to the solution considered by Manko [5] for which the corresponding metric coefficient γ was found in a concise analytic form, and if $a_n=0$, the potential (6) goes over to the solution recently used by Bretón *et al.* for a description of the Kerr black hole in an external gravitational field [19] for which the function γ is also defined by a very concise formula.

It is remarkable that the general case with nonzero a_n and b_n can be treated almost as conveniently as the particular cases mentioned above. Note that if the sum in Eq. (6) started from $n=0$, this sum would be just the representation of the general Weyl solution considered by Hoenselaers [21] in which the Schwarzschild black hole solution is contained as an infinite series with the following choice of parameters a_n $(b_n=0)$:

$$a_{2k}=-\frac{1}{2k+1},\quad a_{2k+1}=0,\quad k\geqslant 0, \tag{7}$$

so that after cutting off the sum the solution would lose the black hole limit. On the other hand, since in the potential (6) the Schwarzschild solution is introduced explicitly as the simplest physical case, one is able to restrict one's consideration by any desired number of multipole moments in accordance with a concrete problem. Of course, the apparent advantage of having the spherically symmetric solution as the leading term in Eq. (6) causes some additional technical difficulties for the integration of Eqs. (2) compared to the representation used by Hoenselaers since now Schwarzschild-multipole cross terms appear; however, fortunately the input of these terms into the expression for γ can be written in an elegant way. The result of the integration of the system (2) for the potential (6) is the following:

$$\gamma=\frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}+\sum_{n=1}^{\infty}\Bigg(a_n\sum_{k=0}^{n}\{[(-1)^{n-k}(x+y)+x-y]$$
$$\times r^{-k-1}P_k-1+(-1)^{n+1}\}$$
$$+b_n\sum_{l=0}^{n-1}[(-1)^{n-l+1}(x+y)-x+y]r^lP_l\Bigg)$$
$$+\sum_{n,p=1}^{\infty}\Bigg\{\frac{(n+1)(p+1)a_na_p}{(n+p+2)r^{n+p+2}}(P_{n+1}P_{p+1}-P_nP_p)$$
$$+\frac{npb_nb_pr^{n+p}}{n+p}(P_nP_p-P_{n-1}P_{p-1})+\gamma_{n,p}\Bigg\}, \tag{8}$$

where

$$\gamma_{n,p}=\begin{cases}\dfrac{2n(p+1)a_pb_n}{(p-n+1)r^{p-n+1}}(P_nP_p-P_{n-1}P_{p+1}), & p\neq n-1,\\ n^2a_{n-1}b_n(P_n\pi_{n-1}-P_{n-1}\pi_n), & p=n-1.\end{cases} \tag{9}$$

Hoenselaers' functions π_n which are defined as derivatives of the Legendre polynomials P_n with respect to the index n are completely determined by the first two functions π_0 and π_1 via the recursion formula [21]

$$(n+1)\pi_{n+1}-(2n+1)\not{c}\,\pi_n+n\pi_{n-1}$$
$$=-P_{n+1}+2\not{c}P_n-P_{n-1},$$

$$\pi_0=\frac{1}{2}(1+\not{c}),\quad \pi_1=P_1\pi_0+\not{c}-1,\quad \not{c}:=xy/r. \tag{10}$$

The regularity of the static metric (6),(8) on the symmetry axis implies the vanishing of γ when $y=\pm 1$. This leads us to the following two conditions:

$$\sum_{k=1}^{\infty}b_{2k-1}=0,\quad \sum_{n=2}^{\infty}na_{n-1}b_n=0, \tag{11}$$

which should be satisfied to guarantee the regularity of the symmetry axis outside the Schwarzschild black hole; Eqs. (11) are also the equilibrium condition for a deformed black hole in the external field. Note that the regularity axis condition will be automatically fulfilled if the potential (6) is symmetric with respect to the equatorial plane ($y=0$) which means that all the parameters a_{2k+1} and b_{2k+1}, $k=0,1,\ldots$, with odd indices are zeros.

The interior mass multipoles described by a_n cause the Schwarzschild horizon to be singular at the equator [5], whereas the horizon remains completely regular if all $a_n=0$ and, besides, Chandrasekhar's equilibrium condition [11] holds, i.e.,

$$\sum_{k=1}^{\infty}b_{2k-1}=0, \tag{12}$$

which is a particular case of the conditions (11).

To conclude this section, let us write out a particular solution which represents a Schwarzschild black hole possessing an arbitrary mass-quadrupole moment which is surrounded by a quadrupole external field, both quadrupole moments, the inner and outer, being the first physically nontrivial ones in the respective multipole sets (we denote $a_2=a$ and $b_2=b$):

$$\psi_{qq}=\frac{1}{2}\ln\frac{x-1}{x+1}+\frac{1}{2}\left(\frac{a}{r^5}+b\right)(3x^2y^2-r^2),$$

$$\gamma_{qq}=\frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}-(x^2-1)(1-y^2)\Bigg\{\frac{3}{8}a^2[r^2(r^2-14x^2y^2)$$
$$+25x^4y^4]r^{-12}-3ab(r^2+x^2y^2)r^{-5}$$
$$+\frac{1}{4}b^2(9x^2y^2-r^2)\Bigg\}-x(1-y^2)[a(5x^2-3)r^{-5}+2b]$$
$$+2a(x^5r^{-5}-1). \tag{13}$$

As far as we know, solution (13) is the first static solution which might be interesting from the astrophysical point of view as describing a black hole distorted by both internal and external gravitational fields.

III. DEFORMED KERR-NEWMAN BLACK HOLE IN AN EXTERNAL FIELD

Now we are turning to the construction of an exact solution of the Einstein-Maxwell equations describing a deformed Kerr-Newman black hole in an external static axisymmetric gravitational field. The stationary axially symmetric electrovac problem reduces to solving the Ernst equations [22]

$$(\mathrm{Re}\mathcal{E}+\Phi\bar{\Phi})\Delta\mathcal{E}=(\nabla\mathcal{E}+2\bar{\Phi}\nabla\Phi)\nabla\mathcal{E},$$

$$(\mathrm{Re}\mathcal{E}+\Phi\bar{\Phi})\Delta\Phi=(\nabla\mathcal{E}+2\bar{\Phi}\nabla\Phi)\nabla\Phi, \tag{14}$$

where the complex potentials $\mathcal{E}$ and Φ are defined via the relations (an overbar denotes complex conjugation)

$$\mathcal{E}=f-\Phi\bar{\Phi}+i\Omega,\quad \Phi=A_4+iA_3',$$

$$A_{3,x}=\omega A_{4,x}+m(y^2-1)f^{-1}A_{3,y}',$$

$$A_{3,y}=\omega A_{4,y}+m(x^2-1)f^{-1}A_{3,x}', \tag{15}$$

$$\omega_{,x}=m(y^2-1)f^{-2}[\Omega_{,y}+2\mathrm{Im}(\bar{\Phi}\Phi_y)],$$

$$\omega_{,y}=m(x^2-1)f^{-2}[\Omega_{,x}+2\mathrm{Im}(\bar{\Phi}\Phi_x)],$$

f and ω being the metric coefficients in the Papapetrou stationary axisymmetric line element [23]

$$ds^2=m^2f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)\right.$$
$$\left.+(x^2-1)(1-y^2)d\varphi\right]-f(dt-\omega d\varphi)^2, \qquad (16)$$

and A_4 and A_3 being, respectively, the electric and magnetic components of the electromagnetic four-potential.

To generalize the metric considered in the previous section to the stationary electrovac case we shall use the general formulas representing a nonlinear superposition of the Kerr-Newman solution with an arbitrary static vacuum Weyl field obtained in [8]. In order to make our further consideration most illustrative, we choose the simplest way of carrying out such a superposition, and below we write out the respective formulas for the Ernst complex potentials and corresponding metric functions in the form most convenient for concrete applications:

$$\begin{aligned}
&\mathcal{E}=e^{2\psi}A_-/A_+, \quad \Phi=\beta B/A_+,\\
&f=(1-\beta^2)^2e^{2\psi}A/D, \quad e^{2\gamma}=c_1e^{2\gamma'}A/(x^2-1),\\
&\omega=(2me^{-2\psi}CA^{-1}+c_2)/(1-\beta^2)^2,\\
&A_\pm:=(1-\beta^2e^{\pm2\psi})[x(1+ab)+iy(b-a)]\pm(1\\
&\quad+\beta^2e^{\pm2\psi})(1-ia)(1-ib),\\
&A:=(x^2-1)(1+ab)^2-(1-y^2)(b-a)^2,\\
&B:=(1-e^{2\psi})[x(1+ab)+iy(b-a)]\\
&\quad+(1+e^{2\psi})(1-ia)(1-ib),\\
&C:=(x^2-1)(1+ab)[(b-a)(\beta^4e^{4\psi}+1)+y(a+b)\\
&\quad\times(\beta^4e^{4\psi}-1)]+(1-y^2)(b-a)[(1+ab)(\beta^4e^{4\psi}+1)\\
&\quad-x(1-ab)(\beta^4e^{4\psi}-1)],\\
&D:=[x(1+ab)(1-\beta^2e^{2\psi})+(1-ab)(1+\beta^2e^{2\psi})]^2\\
&\quad+[y(a-b)(1-\beta^2e^{2\psi})+(a+b)(1+\beta^2e^{2\psi})]^2.
\end{aligned} \qquad (17)$$

In the above formulas the potential ψ is any solution of Laplace's equation (1), γ' is a function γ of the static solution $\psi'=\frac{1}{2}\ln[(x-1)/(x+1)]+\psi$, the functions a and b corresponding to a given ψ should be found from the first-order differential equations

$$\begin{aligned}
&(x-y)a_{,x}=2a[(xy-1)\psi_{,x}+(1-y^2)\psi_{,y}],\\
&(x-y)a_{,y}=2a[-(x^2-1)\psi_{,x}+(xy-1)\psi_{,y}],\\
&(x+y)b_{,x}=-2b[(xy+1)\psi_{,x}+(1-y^2)\psi_{,y}],\\
&(x+y)b_{,y}=-2b[-(x^2-1)\psi_{,x}+(xy+1)\psi_{,y}],
\end{aligned} \qquad (18)$$

and from Eqs. (18) it follows that both a and b are defined up to an arbitrary constant factor. The integration constants c_1 and c_2 can assume arbitrary real values, and these should be chosen in such a way that the elementary flatness condition of the symmetry axis is preserved. In formulas (17) we assume the charge parameter β to be a real constant which means the absence of a magnetic monopole moment. We mention, however, that the magnetic charge can be introduced trivially in the above superposition relations by considering β as a complex constant and by formally substituting $|\beta|^2\equiv\beta\bar{\beta}$ instead of β^2.

To have the metric (6),(8) as a static vacuum limit for our stationary electrovac solution, we choose ψ in the form

$$\psi=\sum_{n=1}^{\infty}\left(\frac{a_n}{r^{n+1}}+b_nr^n\right)P_n\left(\frac{xy}{r}\right). \qquad (19)$$

This choice of ψ means that the function γ' in Eqs. (17) is simply the function γ defined by Eq. (8).

The next step in the construction of the solution is to find expressions of the corresponding functions a and b by integrating Eqs. (18); the result of the integration is

$$\begin{aligned}
a=&-\alpha\exp\left\{2\sum_{n=1}^{\infty}\left(-a_n\left[(x-y)\sum_{k=0}^{n}r^{-k-1}P_k-1\right]\right.\right.\\
&\left.\left.+b_n(x-y)\sum_{l=0}^{n-1}r^lP_l\right)\right\},\\
b=&\alpha\exp\left\{2\sum_{n=1}^{\infty}\left(a_n\left[(x+y)\sum_{k=0}^{n}(-1)^{n-k+1}r^{-k-1}P_k\right.\right.\right.\\
&\left.\left.\left.+(-1)^n\right]+b_n(x+y)\sum_{l=0}^{n-1}(-1)^{n-l}r^lP_l\right)\right\},
\end{aligned} \qquad (20)$$

where the integration constants are chosen in such a way that the solution is asymptotically flat in the absence of an external field ($b_n=0$).

Now we can find the explicit form of the real parameters c_1 and c_2 in Eqs. (17) by demanding the regularity of the functions γ and ω on the symmetry axis outside the horizon; assuming that the conditions (11) hold, the resulting expressions for c_1 and c_2 are

$$c_1=\frac{1}{(1-\alpha^2)^2}, \quad c_2=-\frac{4m\alpha(1+\beta^4\sigma^2)}{(1-\alpha^2)\sigma},$$
$$\sigma:=\exp\left\{2\sum_{k=1}^{\infty}b_{2k}\right\}. \qquad (21)$$

Formulas (17),(19)–(21) together with the expression for γ' defined by Eqs. (8),(9) fully describe a deformed Kerr-Newman black hole in an external static gravitational field. The main limiting cases of this electrovac solution are the following.

(a) When $a_n=0$, $b_n\neq0$, we have the case of a Kerr-Newman black hole distorted only by an external gravitational field.

(b) In the case of $b_n=0$, $a_n\neq0$, one arrives at the asymptotically flat solution constructed in Ref. [8] which represents the exterior field of a charged stationary rotating mass possessing the whole set of the inner mass-multipole moments.

(c) The deformed Kerr [6] or Reissner-Nordström [24] black holes in the external field correspond to the case $\beta=0$ or $\alpha=0$, respectively. When both parameters α and β are equal to zero, we come to the static vacuum solution considered in Sec. II.

(d) The well-known Kerr-Newman solution [7] for a charged rotating black hole is contained in our formulas as the simplest stationary electrovac case $a_n=b_n=0$.

It is worth mentioning that the general superposition formulas from Ref. [8] also include a different possibility to construct a generalization of the Kerr-Newman metric due to the presence of an arbitrary Newman-Unti-Tamburino (NUT) parameter [25], and this requires additional knowledge of the function $\hat{p}$ satisfying the first-order differential equations (see Ref. [8] for more details)

$$\hat{p}_{,x}=2(y^2-1)\psi_{,y}, \quad \hat{p}_{,y}=2(x^2-1)\psi_{,x}. \tag{22}$$

To make the results of this section to be fully applicable to other situations, below we give the form of this function $\hat{p}$ when ψ is defined by Eq. (19):

$$\hat{p}=-2\sum_{n=1}^{\infty}\left(\frac{a_n}{r^n}+\frac{nb_n}{n+1}r^{n+1}\right)\left(\frac{xy}{r}P_n-P_{n-1}\right). \tag{23}$$

At the same time, since for our purposes we only need the simplest solution above considered, we shall not analyze here other possibilities of carrying out the nonlinear superposition of the Kerr-Newman and Weyl solutions.

An important characteristic feature of our stationary electrovac metric is the existence of an event horizon defined by the hypersurface $x=1$ [26]. The horizon inevitably contains singular points if any of the constants a_n describing the deformations of a massive source are not equal to zero [8]. On the other hand, when all a_n are zeros, the horizon is regular for any nonzero parameters b_n, provided Chandrasekhar's condition (12) holds. However, in this latter case some naked ring singularities may appear in the region exterior to the horizon, and similar to the solution representing a Kerr black hole in the external gravitational field [19] a sufficient condition of the regularity of our solution outside the horizon (save, of course, the points at infinity) can be formulated as ($|\alpha|<1$, $|\beta|<1$)

$$a_n=0, \quad b_{2k-1}=0, \quad b_{2k}\leqslant 0, \quad n,k=1,2,\dots. \tag{24}$$

As a nontrivial, significant application of the metric presented in this section one could consider the verification of the general Smarr's mass formula for black holes [18], and in the next section we shall demonstrate the validity of this formula in the presence of an external gravitational field.

IV. SMARR'S MASS FORMULA AND THE MAGNETIC POTENTIAL

Smarr's mass formula [18] discovered for the case of an isolated Kerr-Newman black hole is an elegant relation between several quantities characterizing a black hole, and it reads

$$M=\frac{1}{4\pi}\kappa S+2\Omega^H J+\Phi^H Q, \tag{25}$$

where M, J, and Q are, respectively, the total mass, total angular momentum, and total charge of a black hole, while κ, S, Ω^H, and Φ^H defined on the horizon are, respectively, the surface gravity, area of the horizon, its angular velocity, and electric potential. The general analysis of Smarr's formula and its generalization to the case of nonzero electric current and matter contributions were given by Carter [26], and Tomimatsu verified its validity for different axisymmetric problems [16,27]. Thus, in [16] Tomimatsu showed that Smarr's formula holds for a Kerr black hole in the presence of an external static gravitational field, and in his paper [27], devoted to the analysis of equilibrium states in a binary system of charged rotating black holes, he demonstrated that Eq. (25) is not valid for each black hole because of some specific features of the charging transformation [22,28] he used. An important ''technical'' contribution of Tomimatsu is the adjustment of the Ernst formalism [22] to the calculation of the quantities entering Eq. (25) that simplifies considerably the verification of Smarr's formula. In what follows we shall use the results of the Ref. [27] to show that relation (25) does hold for a Kerr-Newman black hole in an external gravitational field (we shall put all a_n equal to zero to avoid singularities on the horizon).

However, for achieving this goal, besides the formulas obtained in the previous section, we still need, as will be seen later on, an expression for the magnetic potential A_3 which can be found by integrating the respective first-order differential equations in Eq. (15). The integration can be facilitated by introducing Yamazaki's ansatz [29] and finally leads to the expression

$$A_3=\omega A_4-\frac{\beta}{(1-\beta^2)^2}(2me^{-2\psi}FA^{-1}+c_3), \tag{26}$$

where

$$\begin{aligned}
&A_4=\mathrm{Re}\Phi=\beta E/D,\\
&E:=(1-e^{2\psi})(1-\beta^2e^{2\psi})[x^2(1+ab)^2+y^2(a-b)^2]\\
&\quad+2(1-\beta^2e^{4\psi})[x(1-a^2b^2)+y(a^2-b^2)]\\
&\quad+(1+e^{2\psi})(1+\beta^2e^{2\psi})(1+a^2)(1+b^2),\\
&F:=(x^2-1)(1+ab)[(b-a)(\beta^2e^{4\psi}+1)+y(a+b)\\
&\quad\times(\beta^2e^{4\psi}-1)]+(1-y^2)(b-a)[(1+ab)(\beta^2e^{4\psi}+1)\\
&\quad-x(1-ab)(\beta^2e^{4\psi}-1)],\\
&c_3:=-\frac{4m\alpha(1+\beta^2\sigma^2)}{(1-\alpha^2)\sigma}
\end{aligned} \tag{27}$$

(the above choice of the integration constant c_3 guarantees the regularity of the potential A_3 on the symmetry axis outside the horizon).

Now we can write down Tomimatsu's formulas, slightly changing them for our particular coordinate system (note that in our notations Ω, A_4, A_3, A_3', and $\widetilde{A}_3$ are Tomimatsu's φ, $-A_t$, A_ϕ, A_ϕ', and $\overline{A_\phi}$, respectively); taking into account that all the potentials and functions which are coming below should be evaluated for $x=1$, we have

$$M=-\frac{1}{4}\omega[\Omega(y=1)-\Omega(y=-1)],$$

$$S=4\pi m(-\omega^2 e^{2\gamma})^{1/2},\quad \kappa=(-\omega^2 e^{2\gamma})^{-1/2},\quad \Omega^H=\omega^{-1},$$

$$J=\frac{1}{4}\omega\left\{-2m-\frac{1}{2}\omega[\Omega(y=1)-\Omega(y=-1)]\right.$$
$$\left.+\widetilde{A}_3[A_3'(y=1)-A_3'(y=-1)]\right\}, \tag{28}$$

$$\Phi^H=-\Omega^H\widetilde{A}_3=-\Omega^H(A_3-\omega A_4),$$

$$Q=\frac{1}{2}\omega[A_3'(y=1)-A_3'(y=-1)],$$

Ω and A_3' being the imaginary parts of the Ernst complex potentials $\mathcal{E}$ and Φ, respectively (Ω should not be confused with Ω^H).

In the case of our solution with all a_n set equal to zero formulas (28) yield

$$M=\frac{m(1+\alpha^2)(1+\beta^2)}{(1-\alpha^2)(1-\beta^2)},$$

$$S=\frac{16\pi m^2(1+\alpha^2)(1+\alpha^2\beta^4\sigma^2)}{(1-\alpha^2)^2(1-\beta^2)^2\sigma},$$

$$\kappa=\frac{(1-\alpha^2)^2(1-\beta^2)^2\sigma}{4m(1+\alpha^2)(1+\alpha^2\beta^4\sigma^2)},$$

$$\Omega^H=-\frac{\alpha(1-\alpha^2)(1-\beta^2)^2\sigma}{2m(1+\alpha^2)(1+\alpha^2\beta^4\sigma^2)}, \tag{29}$$

$$J=-\frac{2m^2\alpha(1+\alpha^2)(1-\beta^4\sigma^2)}{(1-\alpha^2)^2(1-\beta^2)^2\sigma},$$

$$\Phi^H=\frac{\beta(1+\alpha^2\beta^2\sigma^2)}{1+\alpha^2\beta^4\sigma^2},\quad Q=\frac{2m\beta(1+\alpha^2)}{(1-\alpha^2)(1-\beta^2)},$$

and a simple inspection shows that these quantities satisfy identically Eq. (25). Therefore, Smarr's mass formula for a Kerr-Newman black hole is also valid when the black hole is distorted by an external static gravitational field, and we have demonstrated what intuitively might have been expected after Tomimatsu's work on a distorted Kerr black hole [16].

Additional technical difficulties which one has to overcome in the electrovac case to prove formula (25) have their own positive side since now, for instance, having at hand the explicit expression for the potential A_3, we can illustrate the effect of the internal and external mass multipoles on the black hole's magnetic field. In the four diagrams of Fig. 1 the magnetic lines of force are plotted for a nondistorted Kerr-Newman black hole [Fig. 1(i)], for a black hole possessing an additional internal mass-quadrupole moment [Fig. 1(ii)], for a black hole in an external quadrupole gravitational field [Fig. 1(iii)], and for a black hole distorted by both internal and external quadrupole moments [Fig. 1(iv)]. For large absolute values of a_2 and b_2 the magnetic lines of force can have a very exotic aspect.

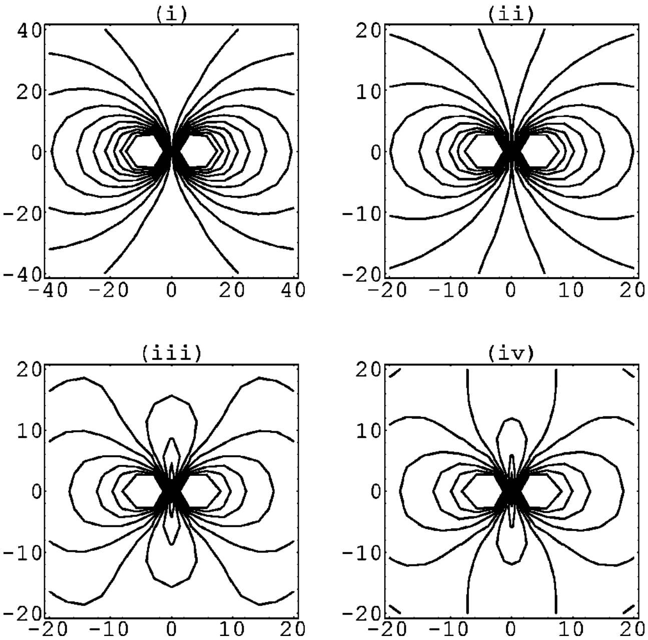

FIG. 1. The magnetic lines of force are plotted for (i) a nondistorted Kerr-Newman black hole, (ii) for a black hole possessing an additional internal mass-quadrupole moment, (iii) for a black hole in an external quadrupole gravitational field, and (iv) for a black hole distorted by both internal and external quadrupole moments.

V. CONCLUSIONS

The family of electrovac spacetimes presented in this paper has several interesting features. First of all, it generalizes the well-known Kerr-Newman black hole solution and involves two infinite sets of arbitrary real parameters which have a clear physical interpretation as describing an exterior static axisymmetric gravitational field and deformations of a massive source. The whole metric is defined by very concise explicit formulas that make it suitable for use in concrete applications. The relevance of our results to astrophysics is evident since they permit one to study the motion of test particles in the vicinities of charged stationary rotating masses, taking into account the external gravitational field of surrounding matter. Last, the electrovac solutions considered in this paper have clearly demonstrated that the general Smarr's mass formula originally derived for an isolated Kerr-Newman black hole is even more important and has more applications than has been previously thought.

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