Colliding axisymmetric *pp* waves

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An exact solution is found describing the collision of a class of axisymmetric pp waves. They are impulsive in character and their coordinate singularities become point curvature singularities at the boundaries of the interaction region. The solution is conformally flat. Concrete examples are given involving an ultrarelativistic black hole against a burst of pure radiation or two colliding beamlike waves. [S0556-2821(98)01306-X]

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I. INTRODUCTION

The problem of colliding plane waves in general relativity has been thoroughly investigated by now [1,2]. Even more interesting and realistic is the collision of the more general class of pp waves of finite extent and energy. One particular example, the collision of ultrarelativistic black holes, has been studied by approximate methods [3–5]. The main reason for the lack of exact solutions is that pp waves are written easily in Brinkmann coordinates, but the analogue of the Rosen transformation has not been known. Recently, the diagonalization of axisymmetric pp waves was achieved [6]. They are described by the line element in cylindrical coordinates

$$ds^2 = 2du dv - e^{-U}(e^{V}dr^2 + e^{-V}d\varphi^2), \tag{1}$$

where $u=(1/\sqrt{2})(t-z)$, $v=(1/\sqrt{2})(t+z)$ and U, V depend on u, r for a left-moving wave and on v, r for a right-moving wave.

The standard description of a head-on collision of two waves divides the u, v space into four regions [1]. Regions II ($u>0$, $v<0$) and III ($u<0$, $v>0$) are occupied by the approaching waves with line element (1). Region I ($u<0$, $v<0$) represents the flat spacetime between the waves. Region IV ($u>0$, $v>0$) describes their collision and interaction. We suppose that in the interaction region the line element preserves its axial symmetry and is described by functions $g_{uv}=e^{-M}$, U and V which depend on u, v, r. In the present paper we shall find all solutions with $M=0$ in a manner similar to the classification of diagonal plane waves with $M=0$ [7].

In Sec. II the general solution with $M=0$ is found in the interaction region. It is extended to a global solution in Sec. III and its parameters are linked to the characteristics of the approaching waves. The structure of the solution is elucidated further in Sec. IV by studying its invariants. In Sec. V two examples are given. Section VI contains some conclusions.

II. SOLUTION WITH $M=0$ IN THE INTERACTION REGION

The vacuum Einstein equations in the interaction region simplify when Eq. (1) is rewritten as

$$ds^2 = 2du dv - Q^2 dr^2 - P^2 d\varphi^2. \tag{2}$$

Then they read

$$PQ_{uu} + QP_{uu} = 0, \tag{3}$$

$$PQ_{vv} + QP_{vv} = 0, \tag{4}$$

$$Q_{uv} = 0, \tag{5}$$

$$P_{uv} = 0, \tag{6}$$

$$Q_v P_r = Q P_{vr}, \tag{7}$$

$$Q_u P_r = Q P_{ur}, \tag{8}$$

$$Q^2(P_v Q_u + P_u Q_v) - QP_{rr} + P_r Q_r = 0. \tag{9}$$

When $P_r = Q_r = 0$, Eqs. (7) and (8) become trivial and the others reduce to the equations for plane waves [7].

Equations (7) and (8) are easily integrated and give $Q = e^{h(r)}P_r$ with an arbitrary $h(r)$. However, when $u=v=0$ we are in region I with Minkowskian background and $P(0,0,r)=r$, $Q(0,0,r)=1$. This condition fixes $h(r)$ to zero and the result is

$$Q = P_r \tag{10}$$

coinciding with the condition for a single pp wave [6]. In fact, the final conclusion drawn below does not depend on $h(r)$. Now Eqs. (3) and (4) become

$$(P_{uu}P)_r = 0, \tag{11}$$

$$(P_{vv}P)_r = 0. \tag{12}$$

Equations (5) and (6) show that the u and v dependence separate:

$$P = f(u,r) + g(v,r). \tag{13}$$

The remaining equation (9) may be written in two equivalent forms:

$$(e^{-U})_{uv} = 0, \tag{14}$$

$$(P_u P_v)_r = 0. \tag{15}$$

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The first form is well-known from the study of colliding plane waves.

We want to prove that the solution of Eqs. (11)–(15) is

$$P=b_0(r)-b_1(r)u-b_2(r)v \tag{16}$$

for some functions $b_i(r)$. If $P_{uu}=P_{vv}=0$, Eq. (16) immediately follows. Suppose that $P_{vv}\neq 0$. Then Eqs. (11), (12) and (13)–(15) give

$$\left(\frac{P_{uu}}{P_{vv}}\right)_r=0, \tag{17}$$

$$(P_{uu}P_{vv})_r=0, \tag{18}$$

which are equivalent to

$$P_{uur}P_{vv}=0, \tag{19}$$

$$P_{uu}P_{vvr}=0. \tag{20}$$

There are two possibilities: $P_{uur}=P_{vvr}=0$ $(P_{uu}\neq 0)$, or $P_{uu}=0$. The first possibility combined with Eq. (13) means that $P_r=0$. Hence Eqs. (3)–(9) reduce to the plane wave case, as was already mentioned, which is discussed in [7]. The second possibility means that $f=c_1(r)u+c_2(r)$. Putting this result into Eqs. (13)–(15) we get $c_1(r)g_v=g_1(v)$. Equations (11) and (12) become

$$g_{1v}\left(u+\frac{c_2+g}{c_1}\right)_r=0. \tag{21}$$

If $g_{1v}=0$ it easily follows that P takes the form of Eq. (16). If $g_{1v}\neq 0$, $g=c_1g_2(v)-c_2$ and Eqs. (17) and (18) give $(c_1^2g_{2v})_r=0$. Again, P is of the form of Eq. (16).

Suppose, at last, that $P_{vv}=0$ but $P_{uu}\neq 0$. Since the equations are symmetric with respect to u, v the same argument leads to the same conclusion. Inserting Eq. (16) into Eq. (15) we obtain the constraint

$$b_1(r)b_2(r)=a, \tag{22}$$

where $a\neq 0$ is some constant.

III. GLOBAL SOLUTION WITH $M=0$

It is obtained by taking into account the Minkowski boundary condition and extending the solution from region IV to regions II and III with the help of the Penrose ansatz:

$$P=r-b_1(r)u\,\theta(u)-b_2(r)v\,\theta(v), \tag{23}$$

$$Q=1-b_{1r}u\,\theta(u)-b_{2r}v\,\theta(v). \tag{24}$$

Going to region II or region III we see that the approaching waves are impulsive [6]:

$$b_i(r)=H_i(r)_r, \tag{25}$$

$$ds^2=2du_idw_i+2H_i(P)\,\delta(u_i)du_i^2-dP^2-P^2d\varphi^2, \tag{26}$$

where $u_1=u$, $u_2=v$. They are induced by some impulse with energy density

$$\rho_i(r,u_i)=\frac{1}{2r}(rH_i(r)_r)_r\delta(u_i) \tag{27}$$

due to a beam of pure radiation [8], light [9], or a point particle moving with the speed of light [10]. Then Eq. (22) becomes

$$H_2(r)_r=\frac{a}{H_1(r)_r}. \tag{28}$$

It is clear that Eq. (28) prevents the study of two equal colliding waves, e.g., two ultrarelativistic black holes. This is a consequence of the simplifying assumption $M=0$. Positive energy-density induces positive and increasing H_i, hence $b_i>0$, $b_{ir}>0$, $a>0$. Applying the constraint (22) to Eqs. (23) and (24) and changing notation to $b_1\equiv b$ yields

$$P=r-bu\,\theta(u)-\frac{a}{b}v\,\theta(v), \tag{29}$$

$$Q=1-b_ru\,\theta(u)+\frac{ab_r}{b^2}v\,\theta(v). \tag{30}$$

The change of the relative sign in Q is reminiscent of the similar change in the Babala solution [11], which is one of the three diagonal vacuum plane waves with $M=0$ [7]. Thus Eqs. (29) and (30) may be considered as a one-function analogue of the Babala solution, although they do not reduce to it when b is constant. Equations (29) and (30) also give

$$e^{-U}=r-(rb_r+b-bb_ru)u\,\theta(u)+\frac{a}{b^2}\left(rb_r-b-\frac{ab_r}{b}v\right)v\,\theta(v). \tag{31}$$

The presence of null matter at the boundaries is signalled in the coordinates (1) by the discontinuities in U_u, which break the O'Brien-Synge boundary conditions [1]. The terms linear in u and v disappear from Eq. (31) when $rb_r\pm b=0$ and these conditions are satisfied simultaneously only by a trivial b.

IV. STRUCTURE OF THE INVARIANTS

More information about the solution may be learned from its invariants. The only nontrivial Ricci scalars are

$$\Phi_{22}=\frac{1}{2}R_{uu}=-\frac{1}{2}e^U(P_{uu}P)_r=re^U\rho_1(r,u), \tag{32}$$

$$\Phi_{00}=\frac{1}{2}R_{vv}=-\frac{1}{2}e^U(P_{vv}P)_r=re^U\rho_2(r,v), \tag{33}$$

from which one can deduce the energy-momentum tensor:

$$T_{\mu\nu}=2re^U(\rho_1l_\mu l_\nu+\rho_2n_\mu n_\nu), \tag{34}$$

where l_μ, n_μ are the first two vectors of the usual NP tetrad for Eq. (2) [1]. There are two planes of null dust with variable energy densities. In regions II, III they coincide with Eq. (27), but along the boundaries of IV they become dependent on the other null coordinate because $re^U\rho_i \neq \rho_i$. The factor e^U in Eqs. (32)–(34) is well known in plane wave solutions with thin shells of null matter [11,12]. Equation (29) shows that in regions II and III the single pp waves have coordinate singularities $P=0$. Equations (32) and (33) tell that they turn into curvature singularities on the boundaries of the interaction region at points $v=0$, $u=r/b$ and $u=0$, $v=rb/a$.

The only nontrivial Weyl scalars are

$$\Psi_4=\frac{P_{uu}}{P}+\Phi_{22}=-\frac{b^2\delta(u)}{rb-av\,\theta(v)}+\Phi_{22}, \tag{35}$$

$$\Psi_0=\frac{P_{vv}}{P}+\Phi_{00}=-\frac{a\,\delta(v)}{b[r-bu\,\theta(u)]}+\Phi_{00}. \tag{36}$$

From Eqs. (27), (32), (33), (35), and (36) it is clear that the interaction region is conformally flat. The Weyl scalars confirm that the approaching waves are impulsive. They, like the Ricci scalars, become singular when P vanishes. These point curvature singularities are generic and cannot be avoided by a careful choice of b. In regions II and III Eqs. (35) and (36) coincide with the expressions derived in [6].

V. SOME EXAMPLES

The energy densities may be given as functions of b by Eqs. (22), (25), and (27):

$$\rho_1=\frac{(rb)_r}{2r}\,\delta(u), \tag{37}$$

$$\rho_2=\frac{a}{2r}\left(\frac{r}{b}\right)_r\delta(v). \tag{38}$$

The second density is positive when r/b is an increasing function. One possible solution includes an ultrarelativistic black hole approaching from region II and is given by

$$b=\frac{4\mu}{r}, \tag{39}$$

$$H_1=4\mu\ln r, \tag{40}$$

$$H_2=\frac{ar^2}{8\mu}, \tag{41}$$

$$\rho_1=\frac{\mu}{2}\,\delta(r)\,\delta(u), \tag{42}$$

$$\rho_2=\frac{a}{4\mu}\,\delta(v), \tag{43}$$

$$P=r-\frac{4\mu}{r}u\,\theta(u)-\frac{ar}{4\mu}v\,\theta(v), \tag{44}$$

$$Q=1+\frac{4\mu}{r^2}u\,\theta(u)-\frac{a}{4\mu}v\,\theta(v), \tag{45}$$

$$\Phi_{00}=\frac{ar^4\,\delta(v)}{4\mu[r^4-16\mu^2u^2\theta(u)]}, \tag{46}$$

$$\Phi_{22}=\frac{8\mu^3\,\delta(r)\,\delta(u)}{[4\mu-av\,\theta(v)]^2}, \tag{47}$$

$$\Psi_0=-\frac{ar^2u\,\theta(u)\,\delta(v)}{r^4-16\mu^2u^2\theta(u)}, \tag{48}$$

$$\Psi_4=\frac{8\mu^2\,\delta(u)}{4\mu-av\,\theta(v)}\left(\frac{\mu\,\delta(r)}{4\mu-av\,\theta(v)}-\frac{2}{r^2}\right), \tag{49}$$

where μ is the momentum of the null point particle. The wave arriving from region III is induced by a pure radiation burst of constant density across the wavefront. In fact, this is a plane wave [6]. Speaking loosely, this example describes the collision of an almost pure exterior solution with the simplest interior solution.

There is another solution with positive ρ_i which are finite on the axis $r=0$ and decrease when $r\rightarrow\infty$. It is given by

$$b=\frac{1}{r}\ln(c+r^2), \tag{50}$$

$$\rho_1=\frac{1}{c+r^2}\,\delta(u), \tag{51}$$

$$\rho_2=\frac{a}{\ln(c+r^2)}\left[1-\frac{r^2}{(c+r^2)\ln(c+r^2)}\right]\delta(v), \tag{52}$$

with $c>e$. One can say that the waves are beam like, i.e., they have finite transverse extent, but their energy diverges. It seems impossible to arrange for finite energy of both waves when $M=0$. We omit the lengthy expressions for the metric and its invariants because the solution possesses the general features established above.

VI. CONCLUSION

The assumption $M=0$ in the case of colliding axisymmetric pp waves is almost as restrictive as in the case of colliding plane waves, although the freedom in the solution extends to an arbitrary function b instead of arbitrary constants. The solution in the interaction region is still conformally flat and linear in u and v. This indicates the presence of null matter along the boundaries; however, for pp waves the distinction between pure gravitational and matter field components is not so clean cut in view of relations such as Eq. (27). The mechanism by which the coordinate singularities turn into curvature singularities is the same as for plane waves and has its roots in Eq. (14). The constraint (22) does not allow us to study the simplest possible case of two equivalent approaching waves. Obviously, more exact solutions are necessary and nontrivial interactions between pp-waves of finite energy should be possible when the condition $M=0$ is relaxed.

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