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Decoherent histories approach to the arrival time problem

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What is the probability of a particle entering a given region of space at any time between t_1 and t_2? Standard quantum theory assigns probabilities to alternatives at a fixed moment of time and is not immediately suited to questions of this type. We use the decoherent histories approach to quantum theory to compute the probability of a nonrelativistic particle crossing $x=0$ during an interval of time. For a system consisting of a single nonrelativistic particle, histories coarse grained according to whether or not they pass through spacetime regions are generally not decoherent, except for very special initial states, and thus probabilities cannot be assigned. Decoherence may, however, be achieved by coupling the particle to an environment consisting of a set of harmonic oscillators in a thermal bath. Probabilities for spacetime coarse grainings are thus calculated by considering restricted density operator propagators of the quantum Brownian motion model. We also show how to achieve decoherence by replicating the system N times and then projecting onto the number density of particles that cross during a given time interval, and this gives an alternative expression for the crossing probability. The latter approach shows that the relative frequency for histories is approximately decoherent for sufficiently large N, a result related to the Finkelstein-Graham-Hartle theorem. [S0556-2821(98)01404-0]

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I. INTRODUCTION

In nonrelativistic quantum mechanics, the probability of finding a particle between points x and $x+dx$ at a fixed time t is given by

$$p(x,t)dx=|\Psi(x,t)|^2dx, \tag{1.1}$$

where $\Psi(x,t)$ is the wave function of the particle. More generally, the variety of questions one might ask about a particle at a fixed moment of time may be represented by a projection operator P_α, and the probability of a particular alternative is given by

$$p(\alpha)=\mathrm{Tr}(P_\alpha\rho), \tag{1.2}$$

where ρ is the density operator of the system.

Equations (1.1) and (1.2) refer to questions about the properties of the particle at a fixed moment of time. However, it is of interest to ask questions about the particle that do not refer to a particular moment of time. One could ask, for example, for the probability that the particle entered the region between x and $x+dx$ at *any* moment of time between t_1 and t_2. That is, for the probability of finding the particle in a region of *spacetime*. What predictions does quantum mechanics make for questions of this type?

This question is clearly a physically relevant one since time is measured by physical devices which are generally limited in their precision. It is therefore never possible to say that a physical event occurs at a precise value of time, only that it occurs in some range of times. Furthermore, there has been considerable recent experimental and theoretical interest in the question of tunneling times [1,2]. That is, the question, given that a particle has tunneled through a barrier region, how much time did it spend inside the barrier?

The question of time in nonrelativistic quantum mechanics is also closely related to the so-called ''problem of time'' in quantum gravity. In quantum cosmology, the wave function of the universe satisfies not a Schrödinger equation, but the Wheeler-DeWitt equation

$$\mathcal{H}\Psi[h_{ij},\phi]=0 \tag{1.3}$$

The wave function Ψ depends on the three-metric h_{ij} and the matter field configurations ϕ on a closed spacelike three-surface [3–5]. There is no time label. ''Time'' is somehow already present amongst the dynamical variables h_{ij},ϕ. Although a comprehensive scheme for interpreting the wave function is yet to be put forward, one possible view is that the interpretation will involve treating all the dynamical variables h_{ij},ϕ on an equal footing, rather than trying to single out one particular combination of them to act as time. For this reason, it is of interest to see if one can carry out a similar exercise in nonrelativistic quantum mechanics. That is, to see what the predictions quantum mechanics makes about *spacetime* regions, rather than regions of space at fixed moments of time.

Spacetime questions tend to be rather nontrivial. As stressed by Hartle, who has carried out a number of investigations in this area [6–8], time plays a ''peculiar and central role'' in nonrelativistic quantum mechanics. It is not represented by a self-adjoint operator and there is no obstruction to assuming that it may be measured with arbitrary precision. It enters the Schrödinger equation as an external parameter. As such, it is perhaps best thought of as a label referring to a classical, external measuring device, rather than as a fundamental quantum observable. Yet time is measured by physical systems, and all physical systems are believed to be subject to the laws of quantum theory.

Given these features, means more elaborate that those usually employed are required to define quantum-mechanical probabilities that do not refer to a specific moment of time, and the issue has a long history [9]. One may find in the literature a variety of attempts to define questions of time in a quantum-mechanical way. These include attempts to define time operators [10–12], the use of internal physical clocks [6,7], and path integral approaches [8,13–15]. The literature on tunneling times is a particularly rich source of ideas on this topic [1]. Many of these attempts also tie in with the time-energy uncertainty relations [16,17].

The approach we shall use in this paper involves the decoherent histories approach to quantum theory [18–21]. This is an approach to quantum theory suitable for genuinely closed systems. It was developed in part for quantum cosmology, but it has been very fruitful in enhancing understanding of non-relativistic quantum systems, especially the emergence of classical behavior.

For our purposes, the particular attraction of this approach is that it assigns probabilities directly to the possible histories of a system, rather than to events at a single moment of time. It is therefore very suited to the question of spacetime probabilities considered here. This is because the question of whether a particle did or did not enter a given region at *any* time between t_1 and t_2 clearly cannot be reduced to a question about the state of the particle at a fixed moment of time, but depends on the entire history of the system during that time interval.

The decoherent histories approach, for spacetime questions, turns out to be most clearly formulated in terms of path integrals over paths in configuration space [14,8,4]. The desired spacetime amplitudes are obtained by summing $e^{iS[x(t)]}$, where $S[x(t)]$ is the action, over paths $x(t)$ passing through the spacetime region in question, and consistent with the initial state. The probabilities are obtained by squaring the amplitudes in the usual way. The decoherent histories approach is not inextricably tied to path integrals, however. Operator approaches to the same questions are also available, but are often more cumbersome.

The decoherent histories approach brings a new element into the game which, it is clear from the literature, has so far only been partially appreciated. This new feature is decoherence—the destruction of interference between histories.

When computed according to the path integral scheme outlined above, the probability of entering a spacetime region added to the probability of not entering that region is not equal to 1, in general. This is because of interference. The question of whether a particle enters a spacetime region, when carefully broken down, is actually a quite complicated combination of questions about the positions of the particle at a sequence of times. It is therefore, in essence, a complicated combination of double slit situations. Not surprisingly, there is therefore interference and probabilities cannot be assigned.

This feature has been exhibited very clearly by the extensive work of Yamada and Takagi [14]. They considered a number of spacetime coarse grainings for a free, nonrelativistic particle. They found that probabilities could be assigned, in the decoherent histories approach, only for very special initial states, and the probabilities were then rather uninteresting, e.g., probability zero for entering the region, and 1 for not entering it.

There is an important lesson here. For a free, nonrelativistic particle, probabilities for whether or not the particle enters a spacetime region *cannot be assigned in general*, due to the presence of interference. Physically, this may at first appear unreasonable, because one could imagine situating a measuring device in the spatial region in question, and then asking whether it registers the presence of a particle during a given time interval. The point, of course, is that introducing a measuring device modifies the physical situation. A measuring device typically has a large number of internal degrees of freedom, and, from the point of view of the decoherent histories approach, these provide an ''environment'' which produces the decoherence necessary for the assignment of probabilities. (This is in keeping with the general point made by Landauer in the context of tunneling times—that time in quantum mechanics only makes sense if the mechanism by which it is measured is fully specified [2].)

Generally, therefore, we might expect that by making suitable modifications to the basic physical situation, decoherence may be achieved and probabilities may be assigned to spacetime coarse grainings. In this paper, we will consider two simple modifications which lead to decoherence for spacetime coarse grainings of a point particle.

The first modification consists of coupling the point particle to a bath of harmonic oscillators in a thermal state (the quantum Brownian motion model [22,23]). Interference is destroyed as a result of the interaction with the bath, and probabilities can be assigned for essentially arbitrary initial states of the point particle. This modification is a model of continuous position measurements.

The second modification is to replicate the system N times, where N is large, and then ask for the probability that some fraction of the particles $f=n/N$ enters the spacetime region. Coarse graining f over a small range, together with large N statistics then ensures decoherence, and probabilities may then be assigned to f. The reason we expect decoherence here is that we are effectively projecting onto number density, which is expected to be decoherent because it is typically a slowly varying quantity [24]. This modification is less obviously tied to a particular type of measurement, but it can be shown that decoherent sets of histories correspond, in a certain sense, to *some* kind of measurement (not necessarily a physically realizable one) [19,25].

In Sec. II, we briefly review the decoherent histories approach. In Sec. III, we briefly review the work of Hartle and of Yamada and Takagi on spacetime coarse grainings. In Sec. IV we sketch our results on spacetime coarse grainings for quantum Brownian motion models. In Sec. V, we describe the large N case. We summarize and conclude in Sec. VI.

II. DECOHERENT HISTORIES APPROACH TO QUANTUM THEORY

We give here a very brief summary of the decoherent histories approach to quantum theory. Far more extensive descriptions can be found in many other places [4,18–21,26–29].

In quantum mechanics, propositions about the attributes of a system at a fixed moment of time are represented by sets of projections operators. The projection operators P_α effect a partition of the possible alternatives α a system may exhibit at each moment of time. They are exhaustive and exclusive:

$$\sum_{\alpha} P_\alpha = 1, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha. \tag{2.1}$$

A projector is said to be *fine grained* if it is of the form $|\alpha\rangle\langle\alpha|$, where $\{|\alpha\rangle\}$ are a complete set of states. Otherwise it is *coarse grained*. A quantum-mechanical history (strictly, a *homogeneous* history [28]) is characterized by a string of time-dependent projections $P^1_{\alpha_1}(t_1),\ldots,P^n_{\alpha_n}(t_n)$ together with an initial state ρ. The time-dependent projections are related to the time-independent ones by

$$P^k_{\alpha_k}(t_k) = e^{iH(t_k - t_0)} P^k_{\alpha_k} e^{-iH(t_k - t_0)}, \tag{2.2}$$

where H is the Hamiltonian. The candidate probability for these homogeneous histories is

$$p(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathrm{Tr}[P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha_1}(t_1)\cdots P^n_{\alpha_n}(t_n)]. \tag{2.3}$$

It is straightforward to show that Eq. (2.3) is both nonnegative and normalized to unity when summed over $\alpha_1,\cdots\alpha_n$. However, Eq. (2.3) does not satisfy all the axioms of probability theory, and for that reason it is referred to as a candidate probability. It does not satisfy the requirement of additivity on disjoint regions of sample space. More precisely, for each set of histories, one may construct coarser-grained histories by grouping the histories together. This may be achieved, for example, by summing over the projections at each moment of time:

$$\bar{P}_{\bar{\alpha}} = \sum_{\alpha\in\bar{\alpha}} P_\alpha \tag{2.4}$$

(although this is not the most general type of coarse graining—see below). The additivity requirement is then that the probabilities for each coarser-grained history should be the sum of the probabilities of the finer-grained histories of which it is comprised. Quantum-mechanical interference generally prevents this requirement from being satisfied. Histories of closed quantum systems cannot in general be assigned probabilities.

There are, however, certain types of histories for which interference is negligible, and the candidate probabilities for histories do satisfy the sum rules. These histories may be found using the decoherence functional

$$D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}[P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha_1'}(t_1)\cdots P^n_{\alpha_n'}(t_n)]. \tag{2.5}$$

Here $\underline{\alpha}$ denotes the string $\alpha_1,\alpha_2,\ldots,\alpha_n$. Intuitively, the decoherence functional measures the amount of interference between pairs of histories. It may be shown that the additivity requirement is satisfied for all coarse grainings if and only if

$$\mathrm{Re}\, D(\underline{\alpha},\underline{\alpha}') = 0 \tag{2.6}$$

for all distinct pairs of histories $\underline{\alpha},\underline{\alpha}'$ [20]. Such sets of histories are said to be *consistent*, or *weakly decoherent*. The consistency condition (2.6) is typically satisfied only for coarse-grained histories, and this then often leads to satisfaction of the stronger condition of *decoherence*

$$D(\underline{\alpha},\underline{\alpha}') = 0 \tag{2.7}$$

for $\underline{\alpha}\neq\underline{\alpha}'$. The condition of decoherence is associated with the existence of so-called generalized records, corresponding to the idea that information about the variables followed is stored in the variables ignored in the coarse graining procedure [19,25].

For histories characterized by projections onto ranges of position at different times, the decoherence functional may be represented by a path integral:

$$D(\alpha,\alpha') = \int_\alpha \mathcal{D}x \int_{\alpha'} \mathcal{D}y\, \exp\left(\frac{i}{\hbar} S[x] - \frac{i}{\hbar} S[y]\right)\rho(x_0,y_0). \tag{2.8}$$

The integral is over paths $x(t),y(t)$, starting at x_0,y_0, and both ending at the same final point x_f, where x_f, x_0, and y_0 are all integrated over, and weighted by the initial state $\rho(x_0,y_0)$. The paths are also constrained to pass through spatial gates at a sequence of times corresponding to the projection operators.

However, the path integral representation of the decoherence functional also points the way towards asking types of questions that are not represented by homogeneous histories [8]. Consider for example the following question. Suppose a particle starts at $t=0$ in a state with nonzero support only in $x>0$. What is the probability that the particle will either cross or never cross $x=0$ during the time interval $[0,\tau]$? In the path integral of the form (2.8) it is clear how to proceed. One sums over paths that, respectively, either always cross or never cross $x=0$ during the time interval.

How does this look in operator language? The operator form of the decoherence functional is

$$D(\alpha,\alpha') = \mathrm{Tr}(C_\alpha \rho C^\dagger_{\alpha'}), \tag{2.9}$$

where

$$C_\alpha = P_{\alpha_n}(t_n)\cdots P_{\alpha_1}(t_1), \tag{2.10}$$

The histories that never cross $x=0$ are represented by taking the projectors in C_α to be onto the positive x axis, and then taking the limit $n\to\infty$ and $t_k - t_{k-1}\to 0$. The histories that always cross $x=0$ are then represented by the object

$$\bar{C}_\alpha = 1 - C_\alpha. \tag{2.11}$$

This is called an *inhomogeneous* history, because it cannot be represented as a single string of projectors. It can however, be represented as a *sum* of strings of projectors [8,28].

The proper framework in which these operations, in particular Eq. (2.11), are understood, is the so-called generalized quantum theory of Hartle [8] and Isham *et al.* [28]. It is called ‘‘generalized’’ because it admits inhomogeneous histories as viable objects, while standard quantum theory concerns itself entirely with homogeneous histories. We will make essential use of inhomogeneous histories in what follows.

In practice, for point particle systems, decoherence is readily achieved by coupling to an environment. Here, we will use the much studied case of the quantum Brownian motion model, in which the particle is linearly coupled through position to a bath of harmonic oscillators in a thermal state at temperature T and characterized by a dissipation coefficient γ. The details of this model may be found elsewhere [22,23,26,27].

We consider histories characterized only by the position of the particle and the environmental coordinates are traced out. The path integral representation of the decoherence functional then has the form

$$D(\alpha,\alpha')=\int_\alpha \mathcal{D}x\int_{\alpha'}\mathcal{D}y\ \exp\left(\frac{i}{\hbar}S[x]-\frac{i}{\hbar}S[y]+\frac{i}{\hbar}W[x,y]\right)\rho(x_0,y_0), \tag{2.12}$$

where $W[x,y]$ is the Feynman-Vernon influence functional phase, and is given by

$$W[x,y]=-m\gamma\int dt(x-y)(\dot{x}+\dot{y})+i\,\frac{2m\gamma kT}{\hbar}\times\int dt(x-y)^2. \tag{2.13}$$

The first term induces dissipation in the effective classical equations of motion. The second term is responsible for thermal fluctuations. It is also responsible for suppressing contributions from paths $x(t)$ and $y(t)$ that differ widely, and produces decoherence of configuration space histories.

The corresponding classical theory is no longer the mechanics of a single point particle, but a point particle coupled to a heat bath. The classical correspondence is now to a stochastic process which may be described by either a Langevin equation, or by a Fokker-Planck equation for a phase space probability distribution $w(p,x,t)$:

$$\frac{\partial w}{\partial t}=-\frac{p}{m}\frac{\partial w}{\partial x}+2\gamma\frac{\partial(pw)}{\partial p}+D\frac{\partial^2 w}{\partial p^2}, \tag{2.14}$$

where $w\geqslant 0$ and

$$\int dp\int dx w(p,x,t)=1.$$

When the mass is sufficiently large, this equation describes near-deterministic evolution with small thermal fluctuations about it.

III. SPACETIME COARSE GRAININGS

We are generally interested in spacetime coarse grainings which consist of asking for the probability that a particle does or does not enter a certain region of space during a certain time interval. However, the essentials of this question boil down to the following simpler question: what is the probability that the particle will either cross or not cross $x=0$ at any time in the time interval $[0,t]$? We will concentrate on this question.

In this section we briefly review the result of Yamada and Takagi [14], Hartle [8,6,4], and Micanek and Hartle [30]. We will compute the decoherence functional using the path integral expression (2.8), which may be written

$$D(\alpha,\alpha')=\int dx_f\Psi_t^\alpha(x_f)[\Psi_t^{\alpha'}(x_f)]^*, \tag{3.1}$$

where $\Psi_t^\alpha(x_f)$ denotes the amplitude obtained by summing over paths ending at x_f at time t, consistent with the restriction α and consistent with the given initial state, so we have

$$\Psi_t^\alpha(x_f)=\int_\alpha \mathcal{D}x(t)\exp\left(\frac{i}{\hbar}S[x]\right)\Psi_0(x_0). \tag{3.2}$$

Suppose the system starts out in the initial state $\Psi_0(x)$ at $t=0$. The amplitude for the particle to start in this initial state, and end up at x at time t, but without ever crossing $x=0$, is

$$\Psi_t^r(x)=\int_{-\infty}^{\infty}dx_0 g_r(x,t|x_0,0)\Psi_0(x_0), \tag{3.3}$$

where g_r is the restricted Green function, i.e., the sum over paths that never crosses $x=0$. For the free particle considered here (and also for any system with a potential symmetric about $x=0$), g_r may be constructed by the method of images:

$$g_r(x,t|x_0,0)=[\theta(x)\theta(x_0)+\theta(-x)\theta(-x_0)][g(x,t|x_0,0)-g(x,t|-x_0,0)], \tag{3.4}$$

where $g(x,t|x_0,0)$ is the unrestricted propagator.

The amplitude to cross $x=0$ is

$$\Psi_t^c(x)=\int_{-\infty}^{\infty}dx_0 g_c(x,t|x_0,0)\Psi_0(x_0), \tag{3.5}$$

where $g_c(x,t|x_0,0)$ is the crossing propagator, i.e., the sum over paths which always cross $x=0$. This breaks up into two parts. If x and x_0 are on opposite sides of $x=0$, it is clearly just the usual propagator $g(x,t|x_0,0)$. If x and x_0 are on the same side of $x=0$, it is given by $g(-x,t|x_{0,0})$. This may be seen by reflecting the segment of the path after last crossing about $x=0$ [31]. (Alternatively, this is just the usual propagator minus the restricted one.) Hence,

$$g_c(x,t|x_0,0)=[\theta(x)\theta(-x_0)+\theta(-x)\theta(x_0)]g(x,t|x_0,0)+[\theta(x)\theta(x_0)+\theta(-x)\theta(-x_0)]\times g(-x,t|x_0,0). \tag{3.6}$$

The crossing propagator may also be expressed in terms of the so-called path decomposition expansion, a form which is sometimes useful [31–35].

Inserting these expressions in the decoherence function, Yamada and Takagi found that the consistency condition may be satisfied exactly by states which are antisymmetric about $x=0$. The probability of crossing $x=0$ is then 0 and the probability of not crossing is 1. What is happening in this case is that the probability flux across $x=0$, which clearly has nonzero components going both to the left and the right, averages to zero.

Less trivial probabilities are obtained in the case where one asks for the probability that the particle remains always in $x>0$ or not, with an initial state with support along the entire x axis [4]. The probabilities become trivial again, however, in the interesting case of an initial state with support only in $x>0$.

Yamada and Takagi have also considered the case of the probability of finding the particle in a spacetime region [14]. That is, the probability that the particle enters, or does not enter, the spatial interval Δ, at any time during the time interval $[0,t]$. Again the consistency condition is satisfied only for very special initial states and the probabilities are then rather trivial.

In an attempt to assign probabilities for arbitrary initial states, Micanek and Hartle considered the above results in the limit that the time interval $[0,t]$ becomes very small [30]. Such an assignment must clearly be possible in the limit $t \to 0$. They found that both the off-diagonal terms of the decoherence functional D and the crossing probability p are of order $\epsilon=(\hbar t/m)^{1/2}$ for small t, and the probability $\bar{p}$ for not crossing is of order 1. Hence $p+\bar{p}\approx 1$. They therefore argued that probabilities can be assigned if t is sufficiently small.

On the other hand, we have the exact relation

$$p+\bar{p}+2\,\mathrm{Re}\,D=1. \tag{3.7}$$

Re D represents the degree of fuzziness in the definition of the probabilities. Since it is of the same order as $\bar{p}$, one may wonder whether it is then valid to claim approximate consistency. Another condition that may be relevant is the condition

$$|D|^2 \ll p\bar{p} \tag{3.8}$$

which was suggested in Ref. [29] as a measure of approximate decoherence, and is clearly satisfied in this case. Ultimately, the question of which mathematical conditions best characterize approximate decoherence or approximate consistency can only be settled by examining the means by which the predicted probabilities could be tested experimentally, and this has not yet been considered.

For a system consisting of a single point particle, therefore, crossing probabilities can be assigned to histories only in a limited class of circumstances. In the following sections, we will see how probabilities may be assigned in a wider variety of situations.

IV. DECOHERENCE OF SPACETIME COARSE-GRAINED HISTORIES IN THE QUANTUM BROWNIAN MOTION MODEL

In order to achieve decoherence for a wide class of initial states, and hence assign probabilities to quantum-mechanical histories for spacetime regions, it is necessary to modify the point particle system in some way. In this section, we discuss a modification consisting of coupling the particle to a bath of harmonic oscillators in a thermal state. We are therefore considering the quantum Brownian motion model, a model that has been discussed very extensively in the decoherence literature [22].

This explicit modification of the single particle system means that the corresponding classical problem (to which the quantum results should reduce under certain circumstances) is in fact a stochastic process described by either a Langevin equation or by a Fokker-Planck equation. It is therefore appropriate to first study the arrival problem in the corresponding classical stochastic process (see for example, Refs. [36–40], and references therein).

A. The arrival time problem in classical Brownian motion

Classical Brownian motion may be described by the Fokker-Planck equation (2.14) for the phase space probability distribution $w(p,x,t)$. For simplicity we will work in the limit of negligible dissipation, hence the equation is

$$\frac{\partial w}{\partial t}=-\frac{p}{m}\frac{\partial w}{\partial x}+D\frac{\partial^2 w}{\partial p^2}, \tag{4.1}$$

where $D=2m\gamma kT$. The Fokker-Planck equation is to be solved subject to the initial condition

$$w(p,x,0)=w_0(p,x). \tag{4.2}$$

Consider now the arrival time problem in classical Brownian motion. The question is this. Suppose the initial state is localized in the region $x>0$. What is the probability that, under evolution according to the Fokker-Planck equation (4.1), the particle either crosses or does not cross $x=0$ during the time interval $[0,t]$?

A useful way to formulate spacetime questions of this type is in terms of the Fokker-Planck propagator $K(p,x,t|p_0,x_0,0)$. The solution to Eq. (4.1) with the initial condition (4.2) may be written in terms of K as

$$w(p,x,t)=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dx K(p,x,t|p_0,x_0,0)w_0(p,x). \tag{4.3}$$

The Fokker-Planck propagator satisfies the Fokker-Planck equation (4.1) with respect to its final arguments, and satisfies delta function initial conditions:

$$K(p,x,0|p_0,x_0,0)=\delta(p-p_0)\,\delta(x-x_0). \tag{4.4}$$

For the free particle without dissipation, it is given explicitly by

$$K(p,x,t|p_0,x_0,0)=N\,\exp\left[-\alpha(p-p_0)^2-\beta\left(x-x_0-\frac{p_0 t}{m}\right)^2 +\epsilon(p-p_0)\left(x-x_0-\frac{p_0 t}{m}\right)\right], \tag{4.5}$$

where N, α, β, and ϵ are given by

$$\alpha=\frac{1}{Dt},\quad \beta=\frac{3m^2}{Dt^3},\quad \epsilon=\frac{3m}{Dt^2},\quad N=\left(\frac{3m^2}{4\pi D^2t^4}\right)^{1/2}. \tag{4.6}$$

(with $D=2m\gamma kT$). An important property it satisfies is the composition law

$$K(p,x,t|p_0,x_0,0)=\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dx_1K(p,x,t|p_1,x_1,t_1)\times K(p_1,x_1,t_1|p_0,x_0,0), \tag{4.7}$$

where $t>t_1>0$.

For our purposes, the utility of the Fokker-Planck propagator is that it may be used to assign probabilities to individual paths in phase space. Divide the time interval $[0,t]$ into subintervals, $t_0=0$, $t_1,t_2,\ldots,t_{n-1}$, $t_n=t$. Then in the limit that the subintervals go to zero, and $n\to\infty$ but with t held constant, the quantity

$$\prod_{k=1}^{n} K(p_k,x_k,t_k|p_{k-1},x_{k-1},t_{k-1}) \tag{4.8}$$

is proportional to the probability for a path in phase space. The probability for various types of coarse grained paths (including spacetime coarse grainings) can therefore be calculated by summing over this basic object.

We are interested in the probability $w_r(p_n,x_n,t)$ that the particle follows a path which remains always in the region $x>0$ during the time interval $[0,t]$ and ends at the point $x_n>0$ with momentum p_n. The desired total probabilities for crossing or not crossing can then be constructed from this object. w_r is clearly given by

$$w_r(p_n,x_n,t)=\int_0^{\infty}dx_{n-1}\cdots\int_0^{\infty}dx_1\int_0^{\infty}dx_0\int_{-\infty}^{\infty}dp_{n-1}\cdots\times\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_0\times\prod_{k=1}^{n}K(p_k,x_k,t_k|p_{k-1},x_{k-1},t_{k-1})w_0(p_0,x_0) \tag{4.9}$$

in the continuum limit.

Now it is actually more useful to derive a differential equation and boundary conditions for $w_r(p,x,t)$, rather than attempt to evaluate the above multiple integral. First of all, it is clear from the properties of the propagator that $w_r(p,x,t)$ satisfies the Fokker-Planck equation (4.1) and the initial condition (4.2). However, we also expect some sort of condition at $x=0$. From the explicit expression for the propagator (4.5), (4.6), we see that in the continuum limit, the propagator between p_{n-1}, x_{n-1} and the final point p_n, x_n becomes proportional to the delta function

$$\delta(x_n-x_{n-1}-p_nt/m). \tag{4.10}$$

Since $x_{n-1}\geqslant 0$, when $x_n=0$ this δ function will give zero when $p_n>0$, but could be nonzero when $p_n<0$. Hence we deduce that the boundary condition on $w_r(p,x,t)$ is

$$w_r(p,0,t)=0 \quad \text{if } p>0. \tag{4.11}$$

This is the absorbing boundary condition usually given for the arrival time problem [39,41] (although this argument for it does not seem to have appeared elsewhere).

It is now convenient to introduce a restricted propagator $K_r(p,x,t|p_0,x_0,0)$, which propagates $w_r(p,x,t)$. That is, K_r satisfies the delta function initial conditions (4.4) and the same boundary conditions as w_r, Eq. (4.11). Since the original Fokker-Planck equation is not invariant under $x\to -x$, we cannot expect that a simple method of images (of the type used in Sec. III), will readily yield the restricted propagator K_r. K_r has recently been found [38], using a modified method of images technique due to Carslaw [42], and we briefly summarize those results.

Consider first the usual Fokker-Planck propagator (4.5). Introducing the coordinates

$$X=\frac{p}{m}-\frac{3x}{2t},\quad Y=\frac{\sqrt{3}x}{2t}, \tag{4.12}$$

$$X_0=-\frac{p_0}{2m}-\frac{3x_0}{2t},\quad Y_0=\frac{\sqrt{3}}{2}\left(\frac{p_0}{m}+\frac{x_0}{t}\right), \tag{4.13}$$

the propagator (4.5) becomes

$$K=\frac{\sqrt{3}}{2\pi\tilde{t}^2}\exp\left(-\frac{(X-X_0)^2}{\tilde{t}}-\frac{(Y-Y_0)^2}{\tilde{t}}\right). \tag{4.14}$$

Here, $\tilde{t}=Dt/m^2$. Now go to polar coordinates:

$$X=r\cos\theta,\quad Y=r\sin\theta, \tag{4.15}$$

$$X_0=r'\cos\theta',\quad Y_0=r'\sin\theta'. \tag{4.16}$$

Then from Eq. (4.14), it is possible to construct a so-called multiform Green function [42]

$$g(r,\theta,r',\theta')=\frac{\sqrt{3}}{2\pi^{3/2}\tilde{t}^2}\times\exp\left(-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{\tilde{t}}\right)\times\int_{-\infty}^{a}d\lambda\, e^{-\lambda^2}, \tag{4.17}$$

where

$$a=2\left(\frac{rr'}{\tilde{t}}\right)^{1/2}\cos\left(\frac{\theta-\theta'}{2}\right). \tag{4.18}$$

As with the original Fokker-Planck propagator, this object is a solution to the Fokker-Planck equation with δ function initial conditions, but differs in that it has the property that it is defined on a two-sheeted Riemann surface and has period 4π. The desired restricted propagator K_r is then given by

$$K_r(p,x,t|p_0,x_0,0)=g(r,\theta,r',\theta')-g(r,\theta,r',-\theta'). \tag{4.19}$$

The point $x=0$ for $p>0$ is $\theta=0$ in the new coordinates, and the above object indeed vanishes at $\theta=0$. Furthermore, the second term in the above goes to zero at $t=0$, while the first one goes to a δ function as required.

The probability of not crossing the surface during the time interval $[0,t]$ is then given by

$$p_r=\int_{-\infty}^{\infty}dp\int_0^{\infty}dx\int_{-\infty}^{\infty}dp_0 \times\int_0^{\infty}dx_0K_r(p,x,t|p_0,x_0,0)w_0(p_0,x_0). \tag{4.20}$$

The probability of crossing must then be $p_c=1-p_r$, which can also be written

$$p_c=\int_{-\infty}^{0}dp\int_{-\infty}^{\infty}dp_0\int_0^{\infty}dx_0\,\frac{p}{m} \times K_r(p,x=0,t|p_0,x_0,0)w_0(p_0,x_0). \tag{4.21}$$

This completes the discussion of the classical stochastic problem.

B. The arrival time problem in quantum Brownian motion

We now consider the analagous problem in the quantum case. We therefore attempt to repeat the analysis of Sec. III, but using instead of Eq. (3.1), the decoherence function

$$D(\alpha,\alpha')=\int_\alpha \mathcal{D}x\int_{\alpha'}\mathcal{D}y\,\exp\left(\frac{i}{\hbar}\mathcal{S}[x]-\frac{i}{\hbar}\mathcal{S}[y]+\frac{i}{\hbar}W[x,y]\right)\rho_0(x_0,y_0). \tag{4.22}$$

Here, $W[x,y]$ is the influence functional phase (2.13), but with the dissipation term neglected. The sum is over all paths $x(t)$, $y(t)$ which are consistent with the coarse graining α, α'. Hartle has discussed how this case might be carried out, and we follow his discussion [8].

Let the initial density operator have support only on the positive axis, and we ask for the probability that the particle either crosses or never crosses $x=0$ during the time interval $[0,t]$. The history label α takes two values, which we denote $\alpha=c$ and $\alpha=r$. The decoherence functional is conveniently rewritten

$$D(\alpha,\alpha')=\mathrm{Tr}(\rho_{\alpha\alpha'}), \tag{4.23}$$

where

$$\langle x_f|\rho_{\alpha\alpha'}|y_f\rangle\equiv\rho_{\alpha\alpha'}(x_f,y_f) =\int_\alpha \mathcal{D}x\int_{\alpha'}\mathcal{D}y\,\exp\left(\frac{i}{\hbar}\mathcal{S}[x]-\frac{i}{\hbar}\mathcal{S}[y]+\frac{i}{\hbar}W[x,y]\right)\rho_0(x_0,y_0), \tag{4.24}$$

where here the sum is over paths consistent with the coarse graining, but they end at fixed final points x_f, y_f. This object actually obeys a master equation:

$$i\hbar\,\frac{\partial\rho}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2\rho}{\partial x^2}-\frac{\partial^2\rho}{\partial y^2}\right)-\frac{i}{\hbar}D(x-y)^2\rho. \tag{4.25}$$

This is the usual master equation for the evolution of the density operator of quantum Brownian motion [22].

The objects $\rho_{\alpha\alpha'}$ are then found by solving this equation subject to matching the initial state ρ_0, and also to the following boundary conditions (which follow from the path integral representation):

$$\rho_{rr}(x,y)=0 \quad \text{for } x\leqslant 0 \quad \text{and } y\leqslant 0, \tag{4.26}$$

$$\rho_{rc}(x,y)=0 \quad \text{for } x\leqslant 0, \tag{4.27}$$

$$\rho_{cr}(x,y)=0 \quad \text{for } y\leqslant 0. \tag{4.28}$$

Given ρ_{rr}, ρ_{rc}, ρ_{cr}, the quantity ρ_{cc} may be calculated from the relation

$$\rho_{rr}+\rho_{rc}+\rho_{cr}+\rho_{cc}=\rho. \tag{4.29}$$

In the unitary case, this problem was solved very easily using the method of images. From the results of Sec. III, for example, it can be seen that in the unitary case

$$\rho_{rr}(x,y)=\theta(x)\theta(y)[\rho(x,y)-\rho(-x,y)-\rho(x,-y)+\rho(-x,-y)], \tag{4.30}$$

where $\rho(x,y)$ is the unrestricted solution to the master equation matching the prescribed initial condition.

The problem in the nonunitary case treated here, however, is that the master equation is *not* invariant under $x\to -x$ (or under $y\to -y$), hence $\rho(-x,y)$ and $\rho(x,-y)$ are *not* solutions to the master equation: The method of images is therefore not applicable in this case (contrary to the claim in Ref. [8]). As far as an analytic approach goes, this represent a very serious technical problem. Restricted propagation problems are very hard to solve analytically in the absence of the method of images.

We will pursue an approximate analytic solution to the problem. First, we will make use of the well-known fact that evolution according to the master equation (4.25) forces every initial density operator to become approximately diagonal in position on a very short time scale [43–46]. (More generally, the density operator approaches a form which is approximately diagonal in a set of phase space localized states [47].) Therefore, all density operators will satisfy the condition $\rho(x,0)=0=\rho(0,y)$ approximately, except perhaps when x and y are close to zero. So for ρ_{rc} and ρ_{cr} the only solution satisfying the boundary conditions (4.26), (4.27), in this approximation, is

$$\rho_{rc}(x,y)\approx 0, \quad \rho_{cr}(x,y)\approx 0. \tag{4.31}$$

This is actually not surprising, since these terms represent the interference between histories, and the mechanism that makes the density operator diagonal is also known to strongly suppress interference between histories. For ρ_{rr} we still need to satisfy the boundary condition (4.25) close to $x=y$.

To proceed further, we make use of the Wigner representation of the density operator [48]:

$$W(p,x)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}d\xi e^{-(i/\hbar)p\xi}\rho\left(x+\frac{\xi}{2},x-\frac{\xi}{2}\right), \tag{4.32}$$

$$\rho(x,y)=\int_{-\infty}^{\infty}dp e^{(i/\hbar)p(x-y)}W\left(p,\frac{x+y}{2}\right). \tag{4.33}$$

The Wigner representation is very useful in studies of the master equation, since it is similar to a classical phase space distribution function. Indeed, for quantum Brownian motion model with a free particle, the Wigner function obeys the same Fokker-Planck equation (4.1) as the analagous classical phase space distribution function. What makes it fail to be a classical phase space distribution is that it can take negative values. However, it can be shown that the Wigner function becomes positive after a short time (typically the decoherence time), and numerous authors have discussed its use as an approximate classical phase space distribution, under these conditions.

These properties suggest that we can get an approximate solution to the quantum problem by taking the solution to the classical stochastic problem $w_r(p,x,t)$, and regarding it as the Wigner function of the density operator. The desired density operator is then obtained from the Wigner transform (4.33). The main issue is to demonstrate the connection between the quantum and classical boundary conditions (4.26) and (4.11).

The quantity $\rho_{rr}(x_f,y_f)$ is given by the path integral expression

$$\rho_{rr}(x_f,y_f)=\int_r\mathcal{D}x\int_r\mathcal{D}y\ \exp\left[\frac{im}{2\hbar}\int dt(\dot{x}^2-\dot{y}^2)-\frac{2m\gamma kT}{\hbar^2}\int dt(x-y)^2\right]\rho_0(x_0,y_0), \tag{4.34}$$

where the subscript r denotes the fact that the path integral is over paths $x(t)$, $y(t)$ that lie in $x>0$, $y>0$. This path integral is the exact solution to the master equation (4.25) and the boundary conditions (4.26). Now introduce $X=\frac{1}{2}(x+y)$, $\xi=x-y$. Then the path integral becomes

$$\rho_{rr}(x_f,y_f)=\int_r\mathcal{D}X\int_r\mathcal{D}\xi\ \exp\left(-\frac{im}{\hbar}\int dt\xi\ddot{X}+\frac{im}{\hbar}\xi_f\dot{X}_f-\frac{im}{\hbar}\xi_0\dot{X}_0-\frac{2m\gamma kT}{\hbar^2}\int dt\xi^2\right)\times\rho_0\left(X_0+\frac{1}{2}\xi_0,X_0-\frac{1}{2}\xi_0\right), \tag{4.35}$$

where an integration by parts has been performed in the exponent.

The ranges of integration of X and ξ are now unfortunately not so simple. The region $x>0$, $y>0$ translates into $X>0$, $-X<\xi<X$. However, we may exploit the fact that the constant $2m\gamma kT/\hbar^2$ is typically very large[1] (it is this that gives decoherence) so the integral over ξ is strongly concentrated around $\xi=0$. This means that the range of ξ may be extended to $(-\infty,\infty)$ and the Gaussian integral over ξ may be carried out. Furthermore, the integral over ξ_0 effectively performs a Wigner transform of the initial state, and we obtain

$$\rho_{rr}(x_f,y_f)=\int_r\mathcal{D}X\ \exp\left(\frac{im}{\hbar}\xi_f\dot{X}_f-\frac{m}{8\gamma kT}\int dt\ddot{X}^2\right)W_0(m\dot{X}_0,X_0), \tag{4.36}$$

where in the functional integral over X, X_f is fixed. (A similar trick was used in Ref. [19].) Denoting the Wigner transform of ρ_{rr} by W_{rr}, this equation is readily rewritten as

$$W_{rr}(m\dot{X}_f,X_f)=\int_r\mathcal{D}X\ \exp\left(-\frac{m}{8\gamma kT}\int dt\ddot{X}^2\right)\times W_0(m\dot{X}_0,X_0), \tag{4.37}$$

where the functional integral over $X(t)$ is over paths which lie in $X>0$, and match X_f and $\dot{X}_f$ at the final time.

Now the point is that the path integral (4.37) is in fact exactly the same as the continuum limit of the expression (4.9) for classical Brownian motion (with, of course, the classical phase space distribution function replaced by the Wigner function). To prove this assertion, consider first the case of unrestricted propagation. Denote the path integral occuring in Eq. (4.37) by $\tilde{K}$, so

$$\tilde{K}(\dot{X}_f,X_0,\tau|\dot{X}_0,X_0,0)=\int\mathcal{D}X\ \exp\left(-\frac{m}{8\gamma kT}\int_0^{\tau}dt\ddot{X}^2\right), \tag{4.38}$$

where the integral is over all paths $X(t)$ satisfying $X(0)=X_0$, $\dot{X}(0)=\dot{X}_0$, $X(\tau)=X_f$, $\dot{X}(\tau)=\dot{X}_f$. This path integral is readily evaluated. The integral is dominated by paths satisfying $d^4X/dt^4=0$ and the above boundary conditions. These paths may be written

[1]On dimensional grounds, ''large'' here means much greater than $\tau^{-1}\sigma^{-2}$, for some time scale τ and length scale σ. τ may be taken to be the time scale of the entire history (the range of the t integration in the action). Another choice might be the localization time $(\hbar/\gamma kT)^{1/2}$ [47]. The only possible choice for σ is the width of the initial state. More detailed calculations are required to confirm these order of magnitude estimates, but experience suggests that for macroscopic values of the various parameters (order 1 in cgs units), the constant in question is indeed very large.

$$X(t)=X_0+t\dot{X}_0+\frac{(X_f-X_0-\tau\dot{X}_0)}{\tau^2}\,t^2+\left(\frac{(\dot{X}_f-\dot{X}_0)}{\tau^2}-2\,\frac{(X_f-X_0-\tau\dot{X}_0)}{\tau^3}\right)t^2(t-\tau). \tag{4.39}$$

Inserting this in the exponent and evaluating, it is readily shown that $\tilde{K}$ is in fact exactly the same as the Fokker-Planck propagator (4.5), with $p=m\dot{X}_f$, $x=X_f$, $p_0=m\dot{X}_0$, $x_0=X_0$. Therefore Eq. (4.38) is a path integral representation of the Fokker-Planck propagator, in the unrestricted case [49].

In the restricted case, the restricted path integral (4.37) may be written as a composition of propagators over a large number of successive small time intervals, with $X(t)$ integrated over a positive range on each time slice. However, in the limit that the small time intervals go to zero, the *unrestricted* propagator may be used to describe the propagation between neighboring slices. In this way we see that the restricted path integral (4.37) for the Wigner function (4.37) coincides with our previous expression (4.9) for the classical phase space distribution.

What we have shown may therefore be summarized as follows. We have assumed that the parameters of the model are such that the factor $2m\gamma kT/\hbar^2$ is very large. This ensures decoherence of histories and/or density matrices. It also allows us to approximately evaluate the integral over ξ in the path integral (4.36), leading to the expression (4.37). It then follows that, in this approximation, the probabilities for not crossing and for crossing $x=0$ are given by the expressions (4.20), (4.21), with the classical phase space distribution function w_0 replaced by the initial Wigner function W_0 in the quantum case. This is the main result of this section.

C. Properties of the solution

The properties of the expressions (4.20), (4.21) are not readily seen because the form (4.17)–(4.19) of the restricted propagator is not particularly transparent. However, some simple properties of our results may be seen by examining the path integral form (4.37) or (4.38).

It is of interest to consider the motion of a wave packet. That is, we take an initial state consisting of a wave packet concentrated at some $x>0$, and moving towards the origin. We are interested in the probability of whether it will cross $x=0$ or not during some time interval, under the evolution by the path integral (4.37) or (4.38).

The integrand in Eq. (4.37) is peaked about the unique path for which $\ddot{X}=0$ with the prescribed values of X_0 and $\dot{X}_0$. This is of course the classical path with the prescribed initial data. From Eq. (4.37) [or the unrestricted propagator (4.5)], the spatial width $(\Delta X)^2$ of the peak is of order $\gamma kT/(m\tau^3)$, where τ is the total time interval. If the classical path does not cross $x=0$ and approaches $x=0$ no closer than a distance ΔX during the time interval, then it will lie well within the integration range $X>0$, and the propagation is essentially the same as unrestricted propagation, since the dominant contribution to the integral comes from the region $X>0$. It is then easy to see, from the normalization of the Wigner function, that the probability of not crossing is approximately 1, the intuitively expected result.

If the classical path crosses $x=0$ during the time interval, it will lie outside the integration range of $X(t)$ for time slices after the time at which it crossed. If it crosses sufficiently early that an entire wave packet of width ΔX may enter $x<0$ before time τ, then the functional integration will sample only the exponentially small tail of the integrand, so W_{rr} will be very small. The probability of not crossing will therefore be close to zero, again the intuitively expected result.

These results are, as stated, intuitively expected, but it is of interest to contrast them with the unitary case described in Sec. III, which has a slightly surprising feature. Consider again, therefore, a wave packet that starts at $x_0>0$ moving towards the origin. The amplitude for not crossing is given by the restricted amplitude (3.4) and the restricted propagator (3.5). However, in the case where the centre of the wave packet reaches the origin during the time interval, it is easily seen from the propagator (3.5) that after hitting the origin there is a piece of the wave packet which is reflected back into $x>0$ (this is the image wave packet that has come from $x<0$). This means that we have the counterintuitive result that the probability for remaining in $x>0$ is not in fact close to zero [8,50].

Although counterintuitive, it is not particularly disturbing, since with this initial state, the histories for crossing and not crossing do not satisfy the consistency condition (2.6), so we should not expect them to agree with our physical intuition. Furthermore, as we have just shown, intuitively sensible results are obtained when the particle is coupled to an environment to produce decoherence. In particularly, there is no reflection of wave packets off the origin.

The nonunitary case also gives sensible results in the case of an initial state consisting of a superposition of wave packets. For example, let the initial state be of the form

$$|\psi\rangle=\alpha|\psi_1\rangle+\beta|\psi_2\rangle, \tag{4.40}$$

where $|\psi_1\rangle$ is a wave packet concentrated at some point in $x>0$ heading towards the origin, and $|\psi_2\rangle$ is also concentrated in $x>0$ but is heading away from the origin. The Wigner function of this state has the form

$$W(p,x)=|\alpha|^2W_1(p,x)+|\beta|^2W_2(p,x)+\text{interference terms}, \tag{4.41}$$

where W_1, W_2 are the Wigner functions of $|\psi_1\rangle,|\psi_2\rangle$. On inserting this in Eq. (4.38), we find the following. First, the interference terms are strongly suppressed [this is a well-known property of evolution according to Eq. (4.1)]. Secondly, using the above results on a single wave packet, it is easy to see that the probabilities for crossing and not crossing are $|\alpha|^2$ and $|\beta|^2$, respectively, again the expected results.

In the above simple examples, the crossing probabilities are independent of the details of the environment, at least approximately. It is clear that more generally, the crossing probabilities will in fact depend on the features of the environment (e.g., its temperature). One might find this slightly unsettling, at least in comparison to quantum-mechanical probabilities at a fixed moment of time, which depend only on the state at that time and not on the details of where the property in question might be measured. This possible dependence on the decoherence mechanism, however, is in keeping with the point made by Landauer and mentioned in the Introduction—that to specify time in quantum mechanics one has to specify the physical mechanism by which it is measured. Furthermore, one can then expect that the results obtained might depend to some degree on the choice of mechanism.

V. HISTORIES OF CROSSING DENSITIES

We now consider a very different type of modification of the original situation of Sec. III, which leads to decoherence, and hence to the assignment of probabilities for histories which cross or do not cross $x=0$. We consider a system of N noninteracting free particles, and consider histories of imprecisely specified values of number density. That is, we ask for the probability that between $n-\Delta n$ and $n+\Delta n$ particles cross $x=0$ during the time interval $[0,t]$, for $0\leqslant n\leqslant N$, and Δn typically much smaller than n. As we shall see, such histories are generally decoherent, essentially as a result of large N statistics. This modification was inspired by the results of Ref. [24] on hydrodynamic histories, in which a similar feature was observed.

We first summarize the one-particle case. Let C be the class operator for histories crossing $x=0$ during the time interval $[0,t]$, and $\overline{C}$ the class operator for not crossing, so $C+\overline{C}=1$. The (candidate) probabilities for crossing and not crossing are

$$p=\mathrm{Tr}(C\rho C^{\dagger}), \quad \overline{p}=\mathrm{Tr}(\overline{C}\rho\overline{C}^{\dagger}), \tag{5.1}$$

respectively, and the off-diagonal term of the decoherence functional is

$$D=\mathrm{Tr}(C\rho\overline{C}^{\dagger}). \tag{5.2}$$

These quantities satisfy the relation

$$p+\overline{p}+2\ \mathrm{Re}\ D=1. \tag{5.3}$$

Consider the two particle case. There are three class operators, corresponding to zero, one or two particles crossing $x=0$ during the time interval $[0,t]$. These are given by, respectively,

$$C_0=\overline{C}\otimes\overline{C}, \tag{5.4}$$

$$C_1=\overline{C}\otimes C+C\otimes\overline{C}, \tag{5.5}$$

$$C_2=C\otimes C, \tag{5.6}$$

and clearly $C_0+C_1+C_2=1$. The expressions for the case of three or more particles rapidly become complicated, but we are saved by a useful trick, used in Ref. [24] (and similar to a trick used in studies of random walks [51]). In the N particle case, the class operator corresponding to n particles crossing is given by

$$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\lambda\, e^{-i\lambda n}(\overline{C}+e^{i\lambda}C)\otimes(\overline{C}+e^{i\lambda}C)\otimes\cdots, \tag{5.7}$$

where there are N terms in the tensor product. How this expression works is that in tensor product terms, the coefficient of $e^{-i\lambda n}$ consists of all possible combinations of terms consisting of n C's and $(N-n)$ $\overline{C}$'s. Eventually we will be interested in a coarse graining over n, which consists of binning n into ranges of width $2\Delta n$, labeled by $\overline{n}$,

$$C_{\overline{n}}=\sum_{n\in\overline{n}}C_n. \tag{5.8}$$

We will not carry this out explicitly, since the result of doing this is intuitively clear. Explicit coarse grainings of this type in a related problem were carried out in Ref. [24].

The decoherence functional for histories of precisely specified values of n is

$$D(n,n')=\mathrm{Tr}(C_n\rho\otimes\rho\otimes\cdots\otimes\rho C_{n'}^{\dagger}), \tag{5.9}$$

where we have assumed a factored initial state for the N particle system. Inserting the above expression for C_n, this may be written

$$D(n,n')=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}d\lambda\int_{-\pi}^{\pi}d\lambda'\, e^{-i\lambda n+i\lambda' n'}(e^{i(\lambda-\lambda')}p+e^{i\lambda}D+e^{-i\lambda'}D^*+\overline{p})^N. \tag{5.10}$$

Using the binomial expansion to expand the integrand, the integral over λ may be carried out, with the result,

$$D(n,n')=\frac{1}{2\pi}\binom{N}{n}\int_{-\pi}^{\pi}d\lambda'\, e^{i\lambda' n'}(\overline{p}+e^{-i\lambda'}D^*)^{N-n}\times(D+e^{-i\lambda'}p)^n. \tag{5.11}$$

Further use of the binomial theorem permits the remaining integral to be done, with the result

$$D(n,n')=\binom{N}{n}\overline{p}^{N-n-n'}(D^*)^{n'}D^n\sum_{k=0}^{n}\binom{N-n}{n'-k}\binom{n}{k}\left(\frac{p\overline{p}}{|D|^2}\right)^k. \tag{5.12}$$

for $n\leqslant n'$. For $n\geqslant n'$, on the other hand, one obtains

$$D(n,n')=\binom{N}{n}\overline{p}^{N-n}p^{n'}D^{n-n'}\sum_{k=0}^{N-n}\binom{N-n}{k}\binom{n}{n'-k}\times\left(\frac{|D|^2}{p\overline{p}}\right)^k. \tag{5.13}$$

It is useful in Eq. (5.11) to rewrite the integral as a complex contour integral. Let $z=e^{-i\lambda'}$. Then we obtain

$$D(n,n')=\frac{1}{2\pi i}\binom{N}{n}\int\frac{dz}{z^{n'+1}}(\overline{p}+D^*z)^{N-n}(D+pz)^n, \tag{5.14}$$

where the integral is along any closed contour about the origin. Now performing the rescaling $z\rightarrow(\overline{p}/D^*)z$, this becomes

$$D(n,n')=\frac{1}{2\pi i}\binom{N}{n}\bar{p}^{N-n-n'}(D^*)^{n'}D^n\int dz\,z^{-n'-1}\times(1+z)^{N-n}(1+\alpha z)^n, \tag{5.15}$$

where $\alpha=p\bar{p}/|D|^2$.

The discrete sums in Eqs. (5.12) and (5.13) can be evaluated in terms of a hypergeometric function F. For example, Eq. (5.12) yields

$$D(n,n')=\binom{N}{n}\binom{N-n}{n'}\bar{p}^{N-n-n'}(D^*)^{n'}\times D^nF(-n,-n';N-n-n'+1,\alpha). \tag{5.16}$$

However, the hypergeometric function is of the degenerate type (and can be written as a finite hypergeometric series) for which asymptotic forms are not easily found, although this exact expression may be of use for computer plots. We will instead therefore consider asymptotic forms of the expressions (5.12), (5.13), and (5.15).

Consider first the case of very large α. This is the case in which there is some degree of decoherence of the one particle system, but perhaps not sufficient to assign probabilities defined to satisfactory precision. We shall see that this is exponentially enhanced in the N particle case.

Taking N, n, n' to be of the same order (although not necessarily large), for $\alpha\gg N^2$, the discrete sum (5.12) is dominated by the $k=n$ term, and we find

$$D(n,n')=\frac{N!}{n!(N-n')!(n'-n)!}\bar{p}^{N-n-n'}(D^*)^{n'}D^n\alpha^n\times\left[1+O\left(\frac{N^2}{\alpha}\right)\right]. \tag{5.17}$$

A reasonable measure of approximate decoherence is the size of the decoherence functional in comparison to its diagonal terms. Here, this is given by

$$\epsilon\equiv\frac{|D(n,n')|^2}{D(n,n)D(n',n')}\approx\frac{1}{\alpha^{n'-n}}\frac{n'!(N-n)!}{n!(N-n')![(n'-n)!]^2}. \tag{5.18}$$

Since $\alpha\gg N^2$, the dominant term is the term depending on α. For $n'-n$ reasonably large (recall that this is the case $n'>n$), the degree of decoherence of the N particle case is exponentially enhanced compared to the one particle case.

Of course, n' and n may differ by a small number, such as 1 or 2, in which case the degree of decoherence would then not be very good. The point is, however, that we are envisaging the further coarse graining (5.8). As can be seen from similar calculations in Ref. [24], this would have the effect of replacing n and n' by coarse grained variables $\bar{n}$ and $\bar{n}'$. These can differ by no less than the coarse graining parameter $2\Delta n$, which is taken to be large. The degree of decoherence is therefore of order $\alpha^{-2\Delta n}$, which will be very small.

Given decoherence for the case of large α, we may now assign probabilities. These are given by

$$p(n)=\binom{N}{n}p^n\bar{p}^{N-n}\left(1+\frac{(N-n)n}{\alpha}+\cdots\right). \tag{5.19}$$

For large N, n, this becomes, to leading order,

$$p(n)\sim\exp\left[-\frac{N}{2n(N-n)}\left(n-\frac{pN}{(p+\bar{p})}\right)^2\right]. \tag{5.20}$$

Inserting the most probable value of n in the width, this becomes

$$p(n)\sim\exp\left[-\frac{N(p+\bar{p})^2}{2p\bar{p}}\left(\frac{n}{N}-\frac{p}{(p+\bar{p})}\right)^2\right]. \tag{5.21}$$

Note that we cannot take $p+\bar{p}=1$ since these are not consistent probabilities.

This is a gratifying result. It shows that the relative frequency with which the particles cross is strongly peaked about the value $p/(p+\bar{p})$. Also notice that

$$\left(\frac{n}{N}-\frac{p}{(p+\bar{p})}\right)^2=\left(\frac{N-n}{N}-\frac{\bar{p}}{(p+\bar{p})}\right)^2 \tag{5.22}$$

which is consistent with the notion that the relative frequency of not crossing is $\bar{p}/(p+\bar{p})$. These results are tantamount to taking the probabilities for crossing and not crossing in the single particle case to be not p and $\bar{p}$, but $p/(p+\bar{p})$ and $\bar{p}/(p+\bar{p})$ (which clearly add to 1, as required). Again we should be considering coarse grained values of n but it is clear that this will effect only the width of the peak and not the configurations about which the distribution is peaked.

Another case which is amenable to straightforward analysis is the case $\alpha=1$. This might not be exactly reachable in practice, but it represents the extreme case in which the decoherence of the one particle case is as bad as it can possibly get. From either Eq. (5.12) or Eq. (5.15), we find

$$D(n,n')=\binom{N}{n}\binom{N}{n'}\bar{p}^{N-n-n'}(D^*)^{n'}D^n. \tag{5.23}$$

It is straightforward to show that, in this case,

$$|D(n,n')|^2=D(n,n)D(n',n'), \tag{5.24}$$

hence the decoherence in the N particle case is just as bad as the one particle case.

Now we consider the somewhat harder and more general case of $\alpha>1$ but not arbitrarily large. Here we resort to some more sophisticated techniques to expand the contour integral (5.15) in the limit of large N,n,n'.

The integral (5.15) may be written

$$D(n,n')=\binom{N}{n}\bar{p}^{N-n-n'}(D^*)^{n'}D^nJ, \tag{5.25}$$

where

$$J=\frac{1}{2\pi i}\int dz z^{-n'-1}[f(z)]^N \tag{5.26}$$

and

$$f(z)=(1+z)^{1-n/N}(1+\alpha z)^{n/N}. \tag{5.27}$$

This integral, for large N, has the asymptotic form

$$J\sim \frac{(f(\rho))^N}{[2\pi N\kappa_2(\rho)]^{1/2}\rho^{n'}}\left[1+O\left(\frac{1}{N}\right)\right]. \tag{5.28}$$

Here, ρ is the unique positive solution to the equation

$$N\rho\,\frac{f'(\rho)}{f(\rho)}=n', \tag{5.29}$$

which, in this case, reads

$$\alpha(N-n')\rho^2+[N-n-n'+\alpha(n-n')]\rho-n'=0 \tag{5.30}$$

and

$$\kappa_2(\rho)=\rho\,\frac{f'(\rho)}{f(\rho)}+\rho^2\left[\frac{f''(\rho)}{f(\rho)}-\left(\frac{f'(\rho)}{f(\rho)}\right)^2\right], \tag{5.31}$$

The origin of this formula is as follows [52,53]. The integration contour in Eq. (5.26) is any closed contour about the origin. Let $z=\rho e^{i\theta}$, where ρ is arbitrary. The idea is to take a circular contour whose radius is chosen in such a way that the dominant contribution to the integral for large N comes from the immediate neighborhood of $\theta=0$. In terms of ρ and θ the integral becomes

$$J=\frac{1}{2\pi\rho^{n'}}\int_{-\pi}^{\pi}d\theta e^{-in'\theta}[f(\rho e^{i\theta})]^N. \tag{5.32}$$

Now expand the integrand about $\theta=0$. We have

$$[f(\rho e^{i\theta})]^N=\exp[N\ln f(\rho e^{i\theta})]=[f(\rho)]^N\exp\left(iN\theta_\rho\,\frac{f'(\rho)}{f(\rho)}-\frac{1}{2}N\theta^2\kappa_2(\rho)+O(N\theta^3)\right), \tag{5.33}$$

where κ_2 is given by Eq. (5.32). Now clearly if ρ, which is so far arbitrary, is chosen to satisfy Eq. (5.29), the linear term in the exponent in the whole integrand vanishes. For large N the integral over θ is then a Gaussian strongly concentrated around $\theta=0$, and may be done with the desired result (5.28).

(Note that it was not necessary to use this more elaborate asymptotic expansion technique in Ref. [24]. There, the integral analogous to Eq. (5.10) has the property that the modulus of the integrand is less than 1 and equal to 1 when the λ parameters are zero, so it was possible to evaluate for large N by expanding about zero. Here, the norm of the integrand (5.10) does not have this property.)

The decoherence functional is therefore given by Eq. (5.25) with, to leading order,

$$J=J_{nn'}=(1+\rho_{nn'})^{N-n}(1+\alpha\rho_{nn'})^n\rho_{nn'}^{-n'} \tag{5.34}$$

and

$$\rho_{nn'}=\frac{-N+n+n'-\alpha(n-n')+\{[N-n-n'+\alpha(n-n')]^2+4\alpha n'(N-n')\}^{1/2}}{2\alpha(N-n')}. \tag{5.35}$$

The candidate probabilities for the histories are

$$p(n)=\binom{N}{n}\overline{p}^{N-2n}|D|^{2n}(1+\rho_n)^{N-n}(1+\alpha\rho_n)^n\rho_n^{-n}, \tag{5.36}$$

where

$$\rho_n=\frac{-N+2n+[(N-2n)^2+4\alpha n(N-n)]^{1/2}}{2\alpha(N-n)}. \tag{5.37}$$

The probabilities may be assigned when the degree of decoherence

$$\epsilon=\frac{|D(n,n')|^2}{D(n,n)D(n',n')}=\frac{n'!(N-n')!}{n!(N-n)!}\,\frac{|J_{nn'}|^2}{J_{nn}J_{n'n'}} \tag{5.38}$$

is small. Equations (5.34)–(5.37) give the degree of decoherence and the expressions for the probabilities for all values of α when N,n,n' are large. Since these are not very transparent, it is useful to examine them in more detail for special cases.

Above we extracted the leading order for very large α (essentially $\alpha\gg N^2$). We may now improve on this by expanding Eqs. (5.34)–(5.37) for the case $\alpha\gg 1$, if we also assume that $|n'-n|$ is about the same order of magnitude as N,n,n'. A straightforward but tedious calculation shows that the degree of decoherence is

$$\epsilon\sim\alpha^{-|n-n'|}. \tag{5.39}$$

to leading order, which will be very small. Furthermore, the probabilities are given by Eq. (5.21). Hence the result obtained for the case $\alpha\gg N^2$ above also hold for $\alpha\gg 1$.

Another case easily handled is the case $\alpha=1+\delta$, where $0<\delta\ll 1$. Recall that for $\alpha=1$ there is no decoherence [Eq. (5.24)], so it is interesting to see how large α needs to be before decoherence is achieved. Again a straightforward calculation shows that, to leading order, the degree of decoherence is

$$\epsilon \sim \exp\left(-\frac{(n-n')^2}{N}\,\delta\right). \tag{5.40}$$

Assuming again that $n-n'$ and N are of about the same order, approximate decoherence is achieved if $\delta \gg 1/|n-n'|$. Hence α does not have to be very much greater than 1 in order to achieve approximate decoherence. The probabilities in this case are, to leading order,

$$p(n) \sim \binom{N}{n}^2 \bar{p}^{N-n} p^n \exp\left(\delta \frac{n^2}{N}\right) \tag{5.41}$$

For large N, n, and recalling that $\delta \ll 1$, this has the asymptotic form

$$p(n) \sim \exp\left[-\frac{N}{n(N-n)}\left(n - \frac{p^{1/2}N}{(p^{1/2}+\bar{p}^{1/2})}\right)^2\right]. \tag{5.42}$$

[This is easily seen by noting that Eq. (5.41) is the square of the leading order term in Eq. (5.19) with p, $\bar{p}$ replaced by $p^{1/2}$, $\bar{p}^{1/2}$.] This case therefore corresponds to regarding the expressions $p^{1/2}/(p^{1/2}+\bar{p}^{1/2})$ and $\bar{p}^{1/2}/(p^{1/2}+\bar{p}^{1/2})$ as the probabilities for crossing and not crossing in the one particle case.

Finally, we note that all of the analysis of this section does not in fact specifically concern the crossing time problem. It would apply to any situation in which the original system consists of a coarse graining into just two histories, the system is replicated N times, and projections onto the relative frequency $f=n/N$, suitably coarse grained, are considered. This is not unrelated to the Finkelstein-Graham-Hartle theorem [54], which shows that the conventional probabilistic interpretation of quantum theory can arise from consideration of the eigenstates of relative frequency operator of the entire closed system. Here, we have shown that the relative frequency for histories is typically decoherent for large N (in this connection, see also Ref. [55]).

VI. SUMMARY AND CONCLUSIONS

For the closed system consisting of a single point particle in nonrelativistic quantum mechanics, probabilities generally cannot be assigned to histories partitioned according to whether or not they cross $x=0$ during a fixed time interval. We have shown in this paper, however, that by making modifications to this basic physical situation, decoherence may be achieved and probabilities assigned for arbitrary initial states.

The first modification we considered was to couple the particle to a thermal environment. This corresponds to continuous imprecise measurements of the particle's position. The desired probabilities are given by Eqs. (4.20), (4.21), where w_0 is taken to be the initial Wigner function.

The second modification consisted of replicating the system N times, and then considering the number density of particles crossing $x=0$ in the limit of large N. This less obviously corresponds to a particular type of measurement, but on general grounds, since there is decoherence (rather than just consistency), there is a correspondence with some kind of measurement (although not necessarily a physically realizable one). The probabilities in a regime of interest are given by Eq. (5.21). In each case, when decoherence is achieved, the resultant probabilities depend, at least to some degree, on the mechanism producing decoherence, and this is to be expected.

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