Renormalization group approach to the Einstein equation in cosmology

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The renormalization group method has been adapted to the analysis of the long-time behavior of the nonlinear partial differential equation and has demonstrated its power in the study of critical phenomena of gravitational collapse. In the present work we apply the renormalization group to the Einstein equation in cosmology and carry out a detailed analysis of renormalization group flow in the vicinity of the scale invariant fixed point in the spherically symmetric and inhomogeneous dust filled universe model. [S0556-2821(98)02506-5]

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I. INTRODUCTION

Recently the renormalization group (RG) idea was applied to study the long-time asymptotics of nonlinear partial differential equations [1,2]. The RG transformation there is the integration of the equation up to a finite time followed by a rescaling of the dependent and independent variables. The RG transformation together with the original differential equation gives a RG equation. Using the RG transformation, the problem at infinite time is reduced to the problem at finite time. A fixed point of the RG transformation corresponds to a scale-invariant solution of the differential equation. We can obtain the long-time behavior of the equation by studying the flow around fixed points.

As an application of this RG method to the system of gravity, Koike, Hara, and Adachi [3] analyzed the Einstein equation to understand the problem of the critical behavior of the black hole mass in gravitational collapse found by numerical study [4]. A pedagogical exposition of the RG method in the deterministic system is given by Tasaki [5] in a simple but very illustrative example of the motion of a point particle in the Newtonian gravity.

Here we apply the RG to the Einstein equations in the cosmological situation. For simplicity, we shall consider only two cases. One is a homogeneous and isotropic universe filled with a perfect fluid and the other is a spherically symmetric universe filled with dust. We shall study the flow near the fixed points of the RG equations, which have self-similarity.

The astronomical observations indicate that the present universe has a hierarchical structure such as galaxies, clusters of galaxies, and superclusters, and that the two-point correlation function of the galaxies and of the clusters of galaxies can be expressed roughly by a single power law [6]. The scale-invariant Harrison-Zel'dovichi spectrum for the primordial density perturbation has been successful in the study of the structure formation of the universe [7]. These suggest that the present universe has some self-similarity and that the scale-invariant solution plays an important role in cosmology.

In this paper we apply the renormalization group method to the Einstein equations in the cosmological context. In Sec. II, we illustrate the application of the RG method to the heat equation with a nonlinear term. In Sec. III, we apply the RG method to the Einstein equations. Section IV is devoted to a summary and discussion.

II. RENORMALIZATION GROUP TRANSFORMATION: HEAT EQUATION WITH A NONLINEAR TERM

In this section, we review the RG method for nonlinear partial differential equations [1]. First we consider the heat equation with a nonlinear term as a simple example:

$$\frac{\partial u(x,t)}{\partial t}=\frac{1}{2}[u''(x,t)+\lambda u^2(x,t)], \tag{2.1}$$

where the prime denotes the spatial derivative and λ is a coupling constant. Equation (2.1) has scale invariance under the following scale transformation:

$$x\to Lx,$$
$$t\to L^2 t,$$
$$u(x,t)\to L^2 u(Lx,L^2 t), \tag{2.2}$$

where L is a parameter of the scale transformation and is taken to be larger than 1. Namely, if $u(x,t)$ is a solution of Eq. (2.1), the scaled function

$$\overset{(L)}{u}(x,t)=L^2 u(Lx,L^2 t) \tag{2.3}$$

is also a solution of Eq. (2.1). We can thus obtain a one-parameter family of solutions, provided that $u(x,t)$ is a solution.

Here we define the RG transformation $\mathcal{R}_L$ of a function of x by

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$$\mathcal{R}_L u(x,1) = \overset{(L)}{u}(x,1). \tag{2.4}$$

In short, the $\mathcal{R}_L$ is a map from one set of initial data to another. It is convenient to take the initial time to be $t=1$. The RG transformation $\mathcal{R}_L$ has a semigroup property:

$$\mathcal{R}_{L^n} = \mathcal{R}_{L^{n-1}} \circ \mathcal{R}_L. \tag{2.5}$$

Letting $t=1$ and then $L^2=t$ in Eqs. (2.3) and (2.4), we can express an arbitrary solution $u(x,t)$ as an initial data $u(\cdot,1)$ transformed by RG transformations:

$$u(x,t) = t^{-1} \overset{(t^{1/2})}{u}(xt^{-1/2},1). \tag{2.6}$$

The large L means late time. Repeating the RG transformation (2.4), we can see the long-time behavior of the solution $u(x,t)$ in Eq. (2.1).

Denoting $L=e^{\tau}$, we have from Eq. (2.3) that

$$\frac{d\overset{(L)}{u}}{d\tau} = L\frac{d\overset{(L)}{u}}{dL} = 2\overset{(L)}{u} + x\overset{(L)}{u}{}' + 2\frac{\partial \overset{(L)}{u}}{\partial t}. \tag{2.7}$$

Using the original partial differential equation (2.1) we have

$$\frac{d\overset{(L)}{u}}{d\tau} = 2\overset{(L)}{u} + \lambda \overset{(L)}{u}{}^2 + x\overset{(L)}{u}{}' + \overset{(L)}{u}{}''. \tag{2.8}$$

This is the equation satisfied by the scaled function $\overset{(L)}{u}$, which we call the RG equation. We note that Eq. (2.8) has no explicit scale L dependence because of the scale invariance of the original equation (2.1).

We investigate the fixed point of the RG equation (2.8). The fixed point u^* is defined by

$$\mathcal{R}_L u^* = u^* \tag{2.9}$$

for any $L>1$. This condition means that the field profile is unchanged after time evolution followed by a suitable rescaling. In general this condition is equivalent to

$$\frac{d\overset{(L)}{u}{}^*}{d\tau} = 0. \tag{2.10}$$

From Eq. (2.8), u^* satisfies the following equation:

$$2u^* + \lambda u^{*2} + xu^{*\prime} + u^{*\prime\prime} = 0. \tag{2.11}$$

In the homogeneous case, we can easily obtain the fixed points

$$u^* = 0 \quad \text{and} \quad -\frac{2}{\lambda}. \tag{2.12}$$

To investigate the character of the fixed point, Eq. (2.12), we consider a linear perturbation around the fixed point, Eq. (2.12). The perturbed quantity $\overset{(L)}{\delta u}$ is defined by

$$\overset{(L)}{u} = u^* + \overset{(L)}{\delta u}, \tag{2.13}$$

where $\overset{(L)}{\delta u}$ is assumed small. Substituting Eq. (2.13) into Eq. (2.8) and neglecting the second-order term $\overset{(L)}{\delta u}{}^2$, we obtain the linearized equation for $\overset{(L)}{\delta u}$:

$$\frac{d\overset{(L)}{\delta u}}{d\tau} = \begin{cases} 2\overset{(L)}{\delta u} + x\overset{(L)}{\delta u}{}' + \overset{(L)}{\delta u}{}'' & (u^*=0), \\ -2\overset{(L)}{\delta u} + x\overset{(L)}{\delta u}{}' + \overset{(L)}{\delta u}{}'' & \left(u^* = -\dfrac{2}{\lambda}\right). \end{cases} \tag{2.14}$$

We require the boundary condition

$$\overset{(L)}{\delta u} \to 0 \quad (|x| \to \infty). \tag{2.15}$$

We are going to find the normal modes with the ansatz

$$\overset{(L)}{\delta u} = f(x) e^{-x^2/2 + \omega\tau}, \tag{2.16}$$

where $f(x)$ is a function to be determined below and ω is a constant. From Eqs. (2.14) and (2.16), we have

$$\begin{aligned} &f'' - xf' - (\omega-1)f = 0 \quad (u^*=0), \\ &f'' - xf' - (\omega+3)f = 0 \quad \left(u^* = -\frac{2}{\lambda}\right). \end{aligned} \tag{2.17}$$

The regularity at $x=0$ and the boundary condition at $|x|=\infty$ imply

$$f(x) = H_n(x), \tag{2.18}$$

$$\omega = \begin{cases} 1-n & (u^*=0), \\ -3-n & \left(u^* = -\dfrac{2}{\lambda}\right), \end{cases} \tag{2.19}$$

where $H_n(x)$ is the Hermite polynomial and $n=0,1,2,\ldots$.

From Eq. (2.19), $u^* = -2/\lambda$ is an attractor because all ω's are negative. On the other hand, $u^*=0$ has only one relevant mode $(n=0)$.

We can discuss the long-time behavior of a solution of the nonlinear diffusion equation (2.1), if $u(x,1)$ is sufficiently close to the self-similar profile u^*. From Eq. (2.12), we obtain the two self-similar profiles. Suppose the initial spatial profile of $\overset{(L)}{\delta u}$ is expressed as a superposition of the normal modes $H_n e^{-x^2/2}$. As we have seen from Eq. (2.19), if $u(x,1)$ is sufficiently close to the fixed point $u^* = -2/\lambda$, the solution approaches $-2t^{-1}/\lambda$ in the course of time because all modes of perturbation are irrelevant. On the other hand, there is only one growing mode $(n=0)$ of perturbation around the fixed point $u^*=0$. As time goes on, the behavior of the solution near this fixed point is dominated by the relevant mode $n=0$. This relevant mode corresponds to a Gaussian distribution $t^{-1/2}e^{-x^2/(2t)}$.

From this instructive example, we see that if the perturbation around the fixed point has a finite number of relevant modes or no relevant modes, we have some prediction power for the long-time behavior of the nonlinear partial differential equation.

III. RENORMALIZATION GROUP FOR THE EINSTEIN EQUATIONS

In this section, we apply the RG method, which is explained in the previous section, to the Einstein equations.

Take a synchronous reference frame where the line element is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= -dt^2 + \gamma_{ij}(x^k,t)dx^i dx^j. \tag{3.1}$$

Throughout this paper Latin letters will denote spatial indices and Greek letters spacetime indices. The matter is taken to be a perfect fluid characterized by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{3.2}$$

where p, ρ, and u_μ are pressure, energy density, and four-velocity, respectively. We assume that the equation of state of the fluid is

$$p = (\Gamma - 1)\rho, \tag{3.3}$$

where Γ is a constant. The Einstein equations are

$$\dot{\gamma}_{ij} = 2K_{ij}, \tag{3.4}$$

$$\dot{K}_{ij} = -{}^3R_{ij} - KK_{ij} + 2K_i^l K_{lj} + \frac{\kappa\rho}{2}[2\Gamma u_i u_j + (2-\Gamma)\gamma_{ij}], \tag{3.5}$$

$$\kappa\rho = \frac{{}^3R + K^2 - K_l^m K_m^l}{2(1 + \Gamma u_l u^l)}, \tag{3.6}$$

$$\kappa\Gamma\rho u_i = -\frac{1}{\sqrt{1 + u_l u^l}}(K^j_{i;j} - K_{,i}), \tag{3.7}$$

where K_{ij} is the extrinsic curvature, ${}^3R_{ij}$ is the Ricci tensor associated with γ_{ij}, and $\kappa \equiv 8\pi G$. An overdot denotes the derivative with respect to t and a semicolon denotes the covariant derivative with respect to γ_{ij}.

Hereafter we consider the RG transformation for the dynamical variables γ_{ij} and K_{ij}. In the following subsections, we investigate the two cases; the one is a homogeneous and isotropic universe, and the other is a spherically symmetric inhomogeneous dust universe.

A. Homogeneous and isotropic case

We consider the homogeneous and isotropic universe as a simple case. This case is rather trivial because the field equation becomes an ordinary differential equation. Nonetheless, this gives a nice warming up model to familiarize us with the RG approach to the universe. In this case, the spatial metric is written by

$$\gamma_{ij}(x^k,t) = \frac{a^2(t)}{\left(1 + \frac{k(t)r^2}{4}\right)^2}\delta_{ij}, \tag{3.8}$$

where $r^2 \equiv \delta_{ij}x^i x^j$, and $a(t)$ and $k(t)$ are the functions of time t to be studied. Substituting Eq. (3.8) into Eqs. (3.4)–(3.7), we get

$$\dot{a} = aH, \tag{3.9}$$

$$\dot{H} = -3H^2 - \frac{2k}{a^2} + \frac{2-\Gamma}{2}\kappa\rho, \tag{3.10}$$

$$\dot{k} = 0, \tag{3.11}$$

$$\kappa\rho = 3\left[H^2 + \frac{k}{a^2}\right], \tag{3.12}$$

where H is the Hubble parameter. From the conservation law of the energy density, we have

$$\kappa\rho = Ma^{-3\Gamma}, \tag{3.13}$$

where M is an arbitrary constant.

First, we consider the following scale transformation:

$$t \to Lt,$$

$$a(t) \to \overset{(L)}{a}(t) \equiv L^{-2/(3\Gamma)}a(Lt), \tag{3.14}$$

$$k(t) \to \overset{(L)}{k}(t) \equiv L^{2(3\Gamma-2)/(3\Gamma)}k(Lt), \tag{3.15}$$

where L is a parameter of the scale transformation and larger than 1. From Eqs. (3.10) and (3.13), under this scale transformation the variables H and ρ are scaled in the following way:

$$H(t) \to \overset{(L)}{H}(t) = LH(Lt), \tag{3.16}$$

$$\rho(t) \to \overset{(L)}{\rho}(t) = L^2\rho(Lt). \tag{3.17}$$

The equations of motion (3.9)–(3.12) are invariant under the scale transformation, Eqs. (3.14)–(3.17), and equivalently the scaled variables satisfy the original equation.

Second, we define the RG transformation $\mathcal{R}_L$:

$$\mathcal{R}_L a(1) = \overset{(L)}{a}(1), \quad \mathcal{R}_L H(1) = \overset{(L)}{H}(1),$$
$$\mathcal{R}_L k(1) = \overset{(L)}{k}(1). \tag{3.18}$$

Letting $t = L$, we have the formulas

$$a(t) = t^{2/(3\Gamma)}\overset{(t)}{a}(1), \tag{3.19}$$

$$H(t) = t^{-1}\overset{(t)}{H}(1), \tag{3.20}$$

$$k(t) = t^{-2(3\Gamma-2)/(3\Gamma)}\overset{(t)}{k}(1), \tag{3.21}$$

which we shall use later to see the long-time behavior of a, H, and k.

Third, we derive the RG equation. Letting $L = e^\tau$, the infinitesimal transformation of $\overset{(L)}{a}$, $\overset{(L)}{H}$, and $\overset{(L)}{k}$ with respect to τ is

$$\frac{d\overset{(L)}{a}}{d\tau} = -\frac{2}{3\Gamma}\overset{(L)}{a} + \frac{\partial\overset{(L)}{a}}{\partial t},$$

$$\frac{d\overset{(L)}{H}}{d\tau}=\overset{(L)}{H}+\frac{\partial\overset{(L)}{H}}{\partial t},$$

$$\frac{d\overset{(L)}{k}}{d\tau}=\frac{2(3\Gamma-2)}{3\Gamma}\overset{(L)}{k}+\frac{\partial\overset{(L)}{k}}{\partial t}. \tag{3.22}$$

Using the equations of motion (3.9), (3.10), and (3.11), Eqs. (3.22) can be rewritten as

$$\frac{d\overset{(L)}{a}}{d\tau}=-\frac{2}{3\Gamma}\overset{(L)}{a}+\overset{(L)}{a}\,\overset{(L)}{H},$$

$$\frac{d\overset{(L)}{H}}{d\tau}=\overset{(L)}{H}+\left[-\overset{(L)}{H}{}^2+\frac{2-3\Gamma}{6}\kappa\overset{(L)}{\rho}\right],$$

$$\frac{d\overset{(L)}{k}}{d\tau}=\frac{2(3\Gamma-2)}{3\Gamma}\overset{(L)}{k}. \tag{3.23}$$

These equations (3.23) are the RG equations.

Here we investigate the fixed point of the RG equations. The fixed point (a^*,H^*,k^*) is defined by

$$\mathcal{R}_L a^*=a^*,\quad \mathcal{R}_L H^*=H^*,\quad \mathcal{R}_L k^*=k^*. \tag{3.24}$$

The above conditions can be rewritten as

$$\frac{d\overset{(L)}{a}{}^*}{d\tau}=0,\quad \frac{d\overset{(L)}{H}{}^*}{d\tau}=0,\quad \frac{d\overset{(L)}{k}{}^*}{d\tau}=0. \tag{3.25}$$

From Eqs. (3.13) and (3.23), the fixed point is

$$a^*=\left(\frac{3M\Gamma^2}{4}\right)^{1/(3\Gamma)},\quad H^*=\frac{2}{3\Gamma},\quad k^*=0,\quad \kappa\rho^*=3H^{*2}. \tag{3.26}$$

This fixed point corresponds to a flat Friedmann universe.

Note that if Γ is taken to be 2/3, there is another fixed point where k^* is nonzero. For the nonzero k^* case, the term of the spatial curvature can be absorbed into the term of the energy density of matter because the dependence of the scale factor on each term is the same. Thus the $k^*=0$ case includes the nonzero k^* case. Hereafter we concentrate on only the $k^*=0$ case.

In order to study the flow in the RG around the fixed point, we consider the perturbation around the fixed point. The perturbed quantities $\overset{(L)}{\delta a}$, $\overset{(L)}{\delta H}$, and $\overset{(L)}{\delta k}$ are defined by

$$\overset{(L)}{a}=a^*+\overset{(L)}{\delta a},\quad \overset{(L)}{H}=H^*+\overset{(L)}{\delta H},\quad \overset{(L)}{k}=k^*+\overset{(L)}{\delta k}, \tag{3.27}$$

where $\overset{(L)}{\delta a}$, $\overset{(L)}{\delta H}$, and $\overset{(L)}{\delta k}$ are assumed to be small qualities. From Eqs. (3.23), the perturbed quantities satisfy the linearized equations

$$\frac{d\overset{(L)}{\delta a}}{d\tau}=a^*\overset{(L)}{\delta H}, \tag{3.28}$$

$$\frac{d\overset{(L)}{\delta H}}{d\tau}=-\overset{(L)}{\delta H}-\frac{3\Gamma}{2a^*}\overset{(L)}{\delta k}, \tag{3.29}$$

$$\frac{d\overset{(L)}{\delta k}}{d\tau}=\frac{2(3\Gamma-2)}{3\Gamma}\overset{(L)}{\delta k}, \tag{3.30}$$

where we neglect the second-order term δa^2 and δH^2 and use the linearized equation of Eq. (3.12):

$$\kappa\overset{(L)}{\delta\rho}=\frac{4}{\Gamma}\overset{(L)}{\delta H}+\frac{3}{a^{*2}}\overset{(L)}{\delta k}. \tag{3.31}$$

Substituting Eq. (3.28) into Eq. (3.29), $\overset{(L)}{\delta a}$ satisfies

$$\frac{d^2\overset{(L)}{\delta a}}{d\tau^2}+\frac{d\overset{(L)}{\delta a}}{d\tau}+\frac{3\Gamma}{2a^*}\overset{(L)}{\delta k}=0. \tag{3.32}$$

We solve Eq. (3.32):

$$\overset{(L)}{\delta a}=f_1e^{-\tau}+f_2e^{[2(3\Gamma-2)/(3\Gamma)]\tau}, \tag{3.33}$$

where f_1 and f_2 are arbitrary constants. From the solution (3.33), we can see the flow in the RG around the fixed point. If $3\Gamma-2<0$, this fixed point is an attractor. On the other hand, if $3\Gamma-2>0$, there is a single relevant mode. Note that in the case $3\Gamma-2>0$, the matter we consider satisfies the strong energy condition.

From the flow in the RG around the fixed point, we can see the long-time behavior of the homogeneous and isotropic universe. If $3\Gamma-2<0$ and setting the initial profile in the vicinity of the fixed point (a^*,H^*,k^*), the spacetime will approach the flat Friedmann universe $a(t)=a^*t^{2/(3\Gamma)}$. On the other hand, if $3\Gamma-2>0$, the spacetime will deviate from the flat Friedmann universe because there is a relevant mode $\delta a(t)=f_2t^{2(3\Gamma-2)/(3\Gamma)}$.

In the context of the usual cosmological perturbation around a flat Friedmann universe, the f_1 mode corresponds to the decaying mode and the f_2 mode corresponds to the growing mode, which implies the gravitational instability, because the matter should satisfy the strong energy condition.

B. Spherically symmetric case

We consider the spherically symmetric *inhomogeneous* case. In this case, the spatial metric is written by

$$\gamma_{ij}(x^k,t)dx^idx^j=A^2(r,t)dr^2+B^2(r,t)(d\theta^2+\sin^2\theta d\phi^2). \tag{3.34}$$

Namely, $\gamma_{rr}=A^2(r,t), \gamma_{\theta\theta}=B^2(r,t)$, and $\gamma_{\phi\phi}=B^2\sin^2\theta$ while the other components of the spatial metric vanish. As a simple case, we investigate the universe filled with dust, i.e., $\Gamma=1$, and we can set $u^i=0$ in Eqs. (3.4)–(3.7). The Einstein equations are

$$\dot{\gamma}_{ij}=2K_{ij}, \tag{3.35}$$

$$\dot{K}_{ij}=-{}^3R_{ij}+2K_{il}K^l_j-KK_{ij}+\frac{1}{2}\kappa\rho\gamma_{ij}, \tag{3.36}$$

$$\kappa\rho=\frac{1}{2}[{}^{3}R+K^{2}-K^{m}_{l}K^{l}_{m}], \tag{3.37}$$

$$K^{j}_{i;j}-K_{,i}=0. \tag{3.38}$$

Here we consider the following scale transformation:

$$r\rightarrow Lr,$$

$$t\rightarrow L^{\alpha}t,$$

$$\gamma_{rr}(r,t)\rightarrow \overset{(L)}{\gamma}_{rr}(r,t)\equiv L^{2-2\alpha}\gamma_{rr}(Lr,L^{\alpha}t), \tag{3.39}$$

$$\gamma_{\theta\theta}(r,t)\rightarrow \overset{(L)}{\gamma}_{\theta\theta}(r,t)\equiv L^{-2\alpha}\gamma_{\theta\theta}(Lr,L^{\alpha}t), \tag{3.40}$$

$$K_{rr}(r,t)\rightarrow \overset{(L)}{K}_{rr}(r,t)\equiv L^{2-\alpha}K_{rr}(Lr,L^{\alpha}t), \tag{3.41}$$

$$K_{\theta\theta}(r,t)\rightarrow \overset{(L)}{K}_{\theta\theta}(r,t)\equiv L^{-\alpha}K_{\theta\theta}(Lr,L^{\alpha}t), \tag{3.42}$$

$$\rho(r,t)\rightarrow \overset{(L)}{\rho}(r,t)\equiv L^{2\alpha}\rho(Lr,L^{\alpha}t), \tag{3.43}$$

where α is an arbitrary constant because the coordinate transformation $r\rightarrow r^{\beta}$ yields a substitution of α/β for α in Eqs. (3.39)–(3.43) and above Eq. (3.39). Without loss of generality, we take α to be positive. Because of the scale invariance of the Einstein equations Eqs. (3.35) – (3.38), the scaled variables $\overset{(L)}{\gamma}_{rr}$, $\overset{(L)}{\gamma}_{\theta\theta}$, $\overset{(L)}{K}_{rr}$, $\overset{(L)}{K}_{\theta\theta}$, and $\overset{(L)}{\rho}$ also satisfy Eqs. (3.35)–(3.38).

We derive the RG equation. Letting $L=e^{\tau}$, the infinitesimal transformation of $\overset{(L)}{\gamma}_{ij}$ and $\overset{(L)}{K}_{ij}$ with respect to τ is

$$\frac{d\overset{(L)}{\gamma}_{rr}}{d\tau}=2(1-\alpha)\overset{(L)}{\gamma}_{rr}+r\partial_{r}\overset{(L)}{\gamma}_{rr}+2\alpha\overset{(L)}{K}_{rr}, \tag{3.44}$$

$$\frac{d\overset{(L)}{\gamma}_{\theta\theta}}{d\tau}=-2\alpha\overset{(L)}{\gamma}_{\theta\theta}+r\partial_{r}\overset{(L)}{\gamma}_{\theta\theta}+2\alpha\overset{(L)}{K}_{\theta\theta}, \tag{3.45}$$

$$\frac{d\overset{(L)}{K}_{rr}}{d\tau}=(2-\alpha)\overset{(L)}{K}_{rr}+r\partial_{r}\overset{(L)}{K}_{rr}+\alpha\left[\frac{1}{4}{}^{3}\overset{(L)}{R}\overset{(L)}{\gamma}_{rr}-{}^{3}\overset{(L)}{R}_{rr}+2\overset{(L)}{K}_{rl}\overset{(L)}{K}{}^{l}_{r}-\overset{(L)}{K}\,\overset{(L)}{K}_{rr}+\frac{1}{4}(\overset{(L)}{K}{}^{2}\overset{(L)}{\gamma}_{rr}-\overset{(L)}{K}{}^{m}_{l}\overset{(L)}{K}{}^{l}_{m}\overset{(L)}{\gamma}_{rr})\right], \tag{3.46}$$

$$\frac{d\overset{(L)}{K}_{\theta\theta}}{d\tau}=-\alpha\overset{(L)}{K}_{\theta\theta}+r\partial_{r}\overset{(L)}{K}_{\theta\theta}+\alpha\left[\frac{1}{4}{}^{3}\overset{(L)}{R}\overset{(L)}{\gamma}_{\theta\theta}-{}^{3}\overset{(L)}{R}_{\theta\theta}+2\overset{(L)}{K}_{\theta l}\overset{(L)}{K}{}^{l}_{\theta}-\overset{(L)}{K}\,\overset{(L)}{K}_{\theta\theta}+\frac{1}{4}(\overset{(L)}{K}{}^{2}\overset{(L)}{\gamma}_{\theta\theta}-\overset{(L)}{K}{}^{m}_{l}\overset{(L)}{K}{}^{l}_{m}\overset{(L)}{\gamma}_{\theta\theta})\right], \tag{3.47}$$

where ${}^{3}\overset{(L)}{R}_{ij}$ is the Ricci tensor associated with $\overset{(L)}{\gamma}_{ij}$. In the derivation of Eqs. (3.44)–(3.47), the equations of motion (3.35) and (3.36) are used. These equations (3.44)–(3.47) are the RG equations.

In terms of $\overset{(L)}{A}$ and $\overset{(L)}{B}$ ($\overset{(L)}{A}{}^{2}=\overset{(L)}{\gamma}_{rr}$ and $\overset{(L)}{B}{}^{2}=\overset{(L)}{\gamma}_{\theta\theta}$), the RG equations (3.44)–(3.47) read

$$\frac{d^{2}\overset{(L)}{A}}{d\tau^{2}}+\left[2(\alpha-1)+\frac{1}{\overset{(L)}{B}}\frac{d\overset{(L)}{B}}{d\tau}-\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}\right]\frac{d\overset{(L)}{A}}{d\tau}-2r\frac{d\overset{(L)}{A}{}'}{d\tau}=-r^{2}\overset{(L)}{A}{}''+(2\alpha-3)r\overset{(L)}{A}{}'-\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}\overset{(L)}{A}+\frac{1}{2}\left(\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}\right)^{2}\overset{(L)}{A}-\frac{r^{2}\overset{(L)}{A}{}'\overset{(L)}{B}{}'}{\overset{(L)}{B}}-\frac{\alpha^{2}-4\alpha+2}{2}\overset{(L)}{A}+\frac{\alpha^{2}(\overset{(L)}{A}{}^{3}-2\overset{(L)}{A}{}'\overset{(L)}{B}\overset{(L)}{B}{}'-\overset{(L)}{A}\overset{(L)}{B}{}'^{2}+2\overset{(L)}{A}\overset{(L)}{B}\overset{(L)}{B}{}'')}{2\overset{(L)}{A}{}^{2}\overset{(L)}{B}{}^{2}}+\frac{1}{\overset{(L)}{B}{}^{2}}\left[\frac{\overset{(L)}{A}}{2}\frac{d\overset{(L)}{B}}{d\tau}+\overset{(L)}{A}\overset{(L)}{B}-r\overset{(L)}{A}\overset{(L)}{B}{}'+r\overset{(L)}{A}{}'\overset{(L)}{B}\right]\frac{d\overset{(L)}{B}}{d\tau}, \tag{3.48}$$

$$\frac{d^{2}\overset{(L)}{B}}{d\tau^{2}}+\left[2\alpha+\frac{1}{2\overset{(L)}{B}}\frac{d\overset{(L)}{B}}{d\tau}-\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}\right]\frac{d\overset{(L)}{B}}{d\tau}-2r\frac{d\overset{(L)}{B}{}'}{d\tau}=-r^{2}\overset{(L)}{B}{}''+(2\alpha-1)r\overset{(L)}{B}{}'-\frac{r^{2}\overset{(L)}{B}{}'^{2}}{2\overset{(L)}{B}}+\frac{\alpha^{2}(\overset{(L)}{B}{}'^{2}-\overset{(L)}{A}{}^{2})}{2\overset{(L)}{A}{}^{2}\overset{(L)}{B}}-\frac{\alpha^{2}}{2}\overset{(L)}{B}, \tag{3.49}$$

where the prime denotes the derivative with respect to r.

From Eq. (3.37), the scaled variable $\overset{(L)}{\rho}$ is

$$\kappa\overset{(L)}{\rho}=\frac{2}{\alpha^2\overset{(L)}{A}}\left[\alpha+\frac{1}{\overset{(L)}{B}}\frac{d\overset{(L)}{B}}{d\tau}-\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}\right]\frac{d\overset{(L)}{A}}{d\tau}+\frac{1}{\alpha^2\overset{(L)}{B}}\left[2(2\alpha-1)-2\frac{r\overset{(L)}{A}{}'}{\overset{(L)}{A}}-2\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}+\frac{1}{\overset{(L)}{B}}\frac{d\overset{(L)}{B}}{d\tau}\right]\frac{d\overset{(L)}{B}}{d\tau}+\frac{3\alpha-2}{\alpha}-\frac{2}{\alpha}\frac{r\overset{(L)}{A}{}'}{\overset{(L)}{A}}-\frac{2(2\alpha-1)}{\alpha^2}\frac{r\overset{(L)}{B}{}'}{\overset{(L)}{B}}$$
$$+\frac{2}{\alpha^2}\frac{r^2\overset{(L)}{A}{}'\overset{(L)}{B}{}'}{\overset{(L)}{A}\overset{(L)}{B}}+\frac{1}{\alpha^2}\frac{r^2\overset{(L)}{B}{}'^2}{\overset{(L)}{B}{}^2}+\frac{1}{\overset{(L)}{A}{}^3\overset{(L)}{B}{}^2}[\overset{(L)}{A}{}^3+2\overset{(L)}{A}{}'\overset{(L)}{B}\overset{(L)}{B}{}'-\overset{(L)}{A}\overset{(L)}{B}{}'^2-2\overset{(L)}{A}\overset{(L)}{B}\overset{(L)}{B}{}'']. \tag{3.50}$$

From Eq. (3.38), the momentum constraint reads

$$\overset{(L)}{B}{}'\frac{d\overset{(L)}{A}}{d\tau}-\overset{(L)}{A}\frac{d\overset{(L)}{B}{}'}{d\tau}-r\overset{(L)}{A}{}'\overset{(L)}{B}{}'+r\overset{(L)}{A}\overset{(L)}{B}{}''=0. \tag{3.51}$$

Letting $t=L^{\alpha}$, the original variables $A(r,t)$ and $B(r,t)$ are expressed by the scaled variables $\overset{(L)}{A}$ and $\overset{(L)}{B}$:

$$A(r,t)=t^{(\alpha-1)/\alpha}\overset{(t^{1/\alpha})}{A}(rt^{-1/\alpha},1), \tag{3.52}$$

$$B(r,t)=t\overset{(t^{1/\alpha})}{B}(rt^{-1/\alpha},1). \tag{3.53}$$

Here we investigate the fixed point of the RG equations (3.48) and (3.49) defined by

$$\frac{dA^*}{d\tau}=0,\quad \frac{dB^*}{d\tau}=0. \tag{3.54}$$

At the fixed point,

$$A(r,t)=t^{(\alpha-1)/\alpha}\overset{(t^{1/\alpha})}{A^*}(rt^{-1/\alpha},1)=t^{(\alpha-1)/\alpha}\times(\text{ function of } rt^{-1/\alpha} \text{ only }) \tag{3.55}$$

and

$$B(r,t)=t\overset{(t^{1/\alpha})}{B^*}(rt^{-1/\alpha},1)=t\times(\text{ function of } rt^{-1/\alpha} \text{ only }) \tag{3.56}$$

is a self-similar solution.

In the spherically symmetric spacetime filled with dust, the general solution of the Einstein equations is the Tolman-Bondi solution (A1) and (A2) in the Appendix. Therefore we can obtain the fixed point from the Tolman-Bondi solution with self-similarity rather than solving Eqs. (3.54) directly. The precise form of these are as follows.

For $c=0$,

$$A^*(r,1)=\frac{\alpha r^{\alpha/3-1}(1-3pr^{\alpha})}{3(1-pr^{\alpha})^{1/3}}, \tag{3.57}$$

$$B^*(r,1)=r^{\alpha/3}(1-pr^{\alpha})^{2/3}, \tag{3.58}$$

$$\kappa\rho^*(r,1)=\frac{4}{3(1-pr^{\alpha})(1-3pr^{\alpha})}. \tag{3.59}$$

For $c>0$,

$$A^*(r,1)=\frac{\alpha}{(1+c)^{1/2}r}\left[\frac{2}{9c}(\cosh\eta-1)r^{\alpha}-c^{1/2}\frac{\sinh\eta}{\cosh\eta-1}\right], \tag{3.60}$$

$$B^*(r,1)=\frac{2}{9c}r^{\alpha}(\cosh\eta-1), \tag{3.61}$$

$$\sinh\eta-\eta=\frac{9c^{3/2}}{2}(r^{-\alpha}-p), \tag{3.62}$$

$$\kappa\rho^*(r,1)=\frac{9c^2}{r^{\alpha}(\cosh\eta-1)^2\left[\frac{2r^{\alpha}}{9c}(\cosh\eta-1)-c^{1/2}\frac{\sinh\eta}{\cosh\eta-1}\right]}. \tag{3.63}$$

For $c<0$,

$$A^*(r,1)=\frac{\alpha}{(1-|c|)^{1/2}r}\left[\frac{2}{9|c|}(1-\cos\eta)r^{\alpha}-|c|^{1/2}\frac{\sin\eta}{1-\cos\eta}\right], \tag{3.64}$$

$$B^*(r,1)=\frac{2}{9|c|}r^{\alpha}(1-\cos\eta), \tag{3.65}$$

$$\eta-\sin\eta=\frac{9|c|^{3/2}}{2}(r^{-\alpha}-p), \tag{3.66}$$

$$\kappa\rho^*(r,1)=\frac{9c^2}{r^{\alpha}(1-\cos\eta)^2\left[\frac{2r^{\alpha}}{9|c|}(1-\cos\eta)-|c|^{1/2}\frac{\sin\eta}{1-\cos\eta}\right]}, \tag{3.67}$$

where c and p are constants.

The constant c can be interpreted as the total energy of the universe in the analogy of the Newtonian mechanics. By the signature of the constant c, these fixed points are classified into the following three. The universe with $c=0$ is similar to the flat Friedmann universe. The universes with a positive c or a negative c are similar to the open and closed Friedmann universes, respectively. Especially when

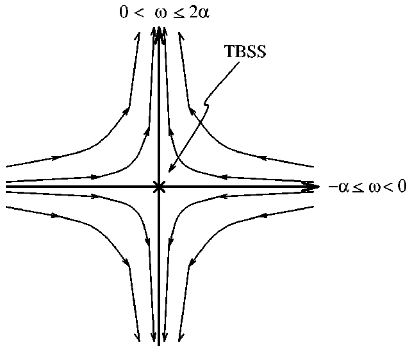

FIG. 1. TBSS represents the Tolman-Bondi solution with self-similarity in the case of $c=0$ and $p<0$. The axes correspond to the modes of linear perturbation, Eq. (3.76).

$c=p=0$, the above fixed point coincides with the flat Friedmann universe and the spacetime becomes homogeneous.

In the context of the RG, we can treat the time evolution of the field variables as the map from one set of initial data to another. If the initial data is taken to be the above fixed point A^*, B^*, and ρ^*, the spacetime will evolve into the Tolman-Bondi solution with self-similarity. In the cases of $c=0$ and $c>0$, the fixed point is not regular if $p>0$. For $c<0$, the fixed point has singularities irrespective of the signature of p because the spacetime is similar to the closed Friedmann universe and will recollapse. Since we should set a regular initial data in the physical situation, we investigate only the case of $c=0$ ($p<0$) and $c>0$ ($p\leqslant 0$) where the fixed point is everywhere regular. Note we exclude the case of $c=p=0$ because we have already studied it in the previous subsection.

To study the behavior of the flow in the RG around the fixed point, we consider the linear perturbation around the fixed point. The linear perturbation around the self-similar Tolman-Bondi solution has also been discussed by Tomita [8] in a different context from ours [9]. For simplicity, we concentrate on the spherical modes of linear perturbation. The perturbed quantities $\overset{(L)}{\delta A}$ and $\overset{(L)}{\delta B}$ are defined by

$$\overset{(L)}{\gamma}_{rr}=A^{*2}+2A^*\overset{(L)}{\delta A},$$

$$\overset{(L)}{\gamma}_{\theta\theta}=B^{*2}+2B^*\overset{(L)}{\delta B}. \tag{3.68}$$

We assume the spatial metric variables for $\overset{(L)}{\delta A}$ and $\overset{(L)}{\delta B}$ in the following form:

$$\overset{(L)}{\delta A}=a(r)e^{\omega\tau},$$

$$\overset{(L)}{\delta B}=b(r)e^{\omega\tau}. \tag{3.69}$$

The perturbed quantities $a(r)$ and $\overset{(L)}{\delta\rho}$ are expressed by $b(r)$;

$$\frac{a}{A^*}=\frac{b'}{B^{*\prime}}-\frac{c_1}{2(1+c)}r^{\omega}, \tag{3.70}$$

$$\frac{\overset{(L)}{\delta\rho}}{\rho^*}=e^{\omega\tau}\left[\frac{9(\omega+\alpha)}{4\alpha}c_2r^{\omega}-2\frac{b}{B^*}-\frac{b'}{B^{*\prime}}\right], \tag{3.71}$$

where c_1 and c_2 are arbitrary constants (see the Appendix).

As for the spherical modes of the perturbation, we can easily obtain the solutions.

For $c=0$,

$$b(r)=r^{\omega}\left[\frac{9}{20}c_1r^{-\alpha/3}(1-pr^{\alpha})^{4/3}+\frac{3}{4}c_2r^{\alpha/3}(1-pr^{\alpha})^{2/3}\right.$$
$$\left.-\frac{2}{3}c_3r^{4\alpha/3}(1-pr^{\alpha})^{-1/3}\right], \tag{3.72}$$

where c_3 is another arbitrary constant. The density contrast is

$$\frac{\overset{(L)}{\delta\rho}}{\rho^*}=e^{\omega\tau}r^{\omega}\left\{-\frac{9}{20\alpha}c_1r^{-2\alpha/3}(1-pr^{\alpha})^{2/3}(1-3pr^{\alpha})^{-1}\right.$$
$$\times[3\omega+\alpha-3(\omega+3\alpha)pr^{\alpha}]$$
$$-\frac{9\omega}{2\alpha}c_2pr^{\alpha}(1-3pr^{\alpha})^{-1}$$
$$+\frac{2}{\alpha}c_3r^{\alpha}(1-pr^{\alpha})^{-1}(1-3pr^{\alpha})^{-1}$$
$$\left.\times[\omega+2\alpha-(\omega+3\alpha)pr^{\alpha}]\right\}. \tag{3.73}$$

In the expression for the linear perturbation, Eq. (3.73), there are three terms corresponding to c_1, c_2, and c_3 so that there should be a gauge mode hidden in Eq. (3.73) because the number of physical modes has to be 2. Actually there remains a gauge freedom corresponding to the coordinate transformation of r, Eq. (A8). The gauge mode is given by

$$\frac{\overset{(L)}{\delta\rho}_g}{\rho^*}=fr^{\omega+1}e^{\omega\tau}\frac{\rho^{*\prime}}{\rho^*}$$
$$=e^{\omega\tau}r^{\omega}f[2\alpha pr^{\alpha}(2-3pr^{\alpha})(1-pr^{\alpha})^{-1}$$
$$\times(1-3pr^{\alpha})^{-1}], \tag{3.74}$$

with f being an arbitrary constant. Because we should fix the freedom of gauge, we choose f so that $(\overset{(L)}{\delta\rho}+\overset{(L)}{\delta\rho}_g)/\rho^*$ behave as nicely as possible at $r=\infty$ because we are interested in the perturbation modes which are finite at $r=\infty$.

We use the following condition as a convenient gauge condition.

$$f=-\frac{9(\omega+3\alpha)}{40\alpha^2}p^{2/3}c_1+\frac{3\omega}{4\alpha^2}c_2-\frac{\omega+3\alpha}{3\alpha^2p}c_3. \tag{3.75}$$

We fix the gauge mode by the above condition and obtain the physical perturbation $\overset{(L)}{\delta\tilde{\rho}}=\overset{(L)}{\delta\rho}+\overset{(L)}{\delta\rho}_g$ as follows:

$$\frac{\overset{(L)}{\delta\rho}}{\rho^*}=e^{\omega\tau}r^{\omega}\{\Delta_1(1-3pr^{\alpha})^{-1}[p^{-2/3}r^{-2\alpha/3}(1-pr^{\alpha})^{2/3}\times[3\omega+\alpha-3(\omega+3\alpha)pr^{\alpha}]+(\omega+3\alpha)pr^{\alpha}(2-3pr^{\alpha})(1-pr^{\alpha})^{-1}]+\Delta_2 pr^{\alpha}(1-pr^{\alpha})^{-1}(1-3pr^{\alpha})^{-1}\},\tag{3.76}$$

where

$$\Delta_1=-\frac{9p^{2/3}}{20\alpha}c_1,\tag{3.77}$$

$$\Delta_2=-\frac{3\omega}{2\alpha}\left(c_2-\frac{4}{9p}c_3\right).\tag{3.78}$$

Since we consider only the case of $p<0$, the coordinate r can be taken from 0 to ∞. We demand the regularity condition that $\overset{(L)}{\delta\rho}/\rho^*$ should be finite at the boundary, $r=0$ and $r=\infty$. This condition implies that Δ_1 modes with $2\alpha/3\leqslant\omega\leqslant\alpha$ and Δ_2 modes with $-\alpha\leqslant\omega\leqslant\alpha$ are allowed. The mode with $\omega=0$ corresponds to a change of p in the self-similar solution, Eqs. (3.59), and ρ^* remains constant independent of τ in the direction. Although this fixed point is not a repeller, it has many relevant modes, Δ_1 with $0<\omega\leqslant\alpha$ and Δ_2 with $0<\omega\leqslant\alpha$. Note that a suitable linear combination of the Δ_1 and Δ_2 modes will have an asymptotic behavior $\approx r^{\omega-2\alpha}$ at $r=\infty$. For such modes, $2\alpha/3\leqslant\omega\leqslant 2\alpha$ is allowed. These modes which satisfy the regularity condition are not discrete but continuous. We suppose that this special feature arises because our matter is assumed to be dust. To summarize, the possible value of ω ranges from $-\alpha$ to 2α. The flow of the RG in the vicinity of the fixed point is shown in Fig. 1.

In the case of $c=p=0$ where the fixed point corresponds to the flat Friedmann universe, the c_2 modes in the density contrast (3.73) are gauge modes. The regularity condition implies that the c_1 mode with $\omega=2\alpha/3$ and the c_3 mode with $\omega=-\alpha$ are allowed. This result corresponds to the homogeneous and isotropic case in the previous section. Compared with the usual cosmological perturbation in the synchronous comoving reference frame, this result appears to be strange because the only spherical modes of the linear perturbation allowed are constant in space. However, the time coordinate used in this RG method is different from the usual cosmic time coordinate, and the solution allowed by the regularity condition in each case does not coincide in general. Moreover, in the homogeneous universe, there is no nontrivial characteristic profile of field variables. If the fixed point is a homogeneous universe, the RG method may have no advantage since the RG approach respects the self-similar profile. But if the fixed point is an inhomogeneous universe, we believe that the RG method may be useful.

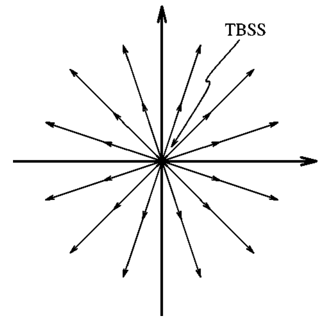

FIG. 2. TBSS represents the Tolman-Bondi solution with self-similarity in the case of $c>0$ and $p<0$. The axes correspond to the modes of linear perturbation, Eq. (3.83).

Compared with the $c=0(p<0)$ case, the value of ω allowed in the case $(p=0)$ is the lower limit in the case $(p<0)$. The effect of the nonlinearity of gravity makes the growth rate of the density contrast large.

For $c>0$,

$$b(r)=r^{\omega}\left\{-\frac{1}{9c^2}c_1\left[2(\cosh\eta-1)-\frac{3\sinh\eta(\sinh\eta-\eta)}{\cosh\eta-1}\right]+\frac{1}{2c}c_2\left[\cosh\eta-1-\frac{\sinh\eta(\sinh\eta-\eta)}{\cosh\eta-1}\right]-\frac{c^{1/2}\sinh\eta}{\cosh\eta-1}c_3\right\}.\tag{3.79}$$

The density contrast is

$$\frac{\overset{(L)}{\delta\rho}}{\rho^*}=\frac{e^{\omega\tau}r^{\omega}}{\alpha\left[\frac{2r^{\alpha}}{9c}(\cosh\eta-1)-\frac{c^{1/2}\sinh\eta}{\cosh\eta-1}\right]}\left\{\frac{1}{9c^2}c_1\left[(\omega+3\alpha)r^{\alpha}\left(2(\cosh\eta-1)-\frac{3\sinh\eta(\sinh\eta-\eta)}{\cosh\eta-1}\right)+\frac{27c^{3/2}\alpha}{2}\left(-\frac{\sinh\eta}{\cosh\eta-1}+\frac{(2\cosh\eta+1)(\sinh\eta-\eta)}{(\cosh\eta-1)^2}\right)\right]-\frac{1}{2c}c_2\left[r^{\alpha}\left(2\alpha(\cosh\eta-1)-(\omega+3\alpha)\frac{\sinh\eta(\sinh\eta-\eta)}{\cosh\eta-1}\right)+\frac{9c^{2/3}}{2}\left(\frac{(\omega-\alpha)\sinh\eta}{\cosh\eta-1}+\frac{\alpha(2\cosh\eta+1)(\sinh\eta-\eta)}{(\cosh\eta-1)^2}\right)\right]+c^{1/2}c_3\left((\omega+3\alpha)r^{\alpha}\frac{\sinh\eta}{\cosh\eta-1}-\frac{9\alpha c^{2/3}(2\cosh\eta+1)}{2(\cosh\eta-1)^2}\right)\right\}.\tag{3.80}$$

The gauge mode is given by

$$\frac{\delta \overset{(L)}{\rho}_g}{\rho^*}=fr^{\omega+1}e^{\omega\tau}\frac{\rho^{*\prime}}{\rho^*}$$

$$=-\frac{\alpha f e^{\omega\tau}r^{\omega}}{\left[\frac{2r^{\alpha}}{9c}(\cosh\eta-1)-\frac{c^{1/2}\sinh\eta}{\cosh\eta-1}\right]}\left[\frac{4}{9c}r^{\alpha}(\cosh\eta-1)-\frac{4c^{1/2}\sinh\eta}{\cosh\eta-1}+\frac{9c^2r^{-\alpha}(2\cosh\eta+1)}{2(\cosh\eta-1)^2}\right]. \tag{3.81}$$

Similarly to the case of $c=0$, we demand the regularity condition at $r=0$ and $r=\infty$.

(i) Case 1 ($p<0$). We use the following convenient gauge condition:

$$f=\frac{1}{4\alpha^2}\left\{\frac{\omega+3\alpha}{c}c_1\left[2-3\frac{\sinh\eta_0(\sinh\eta_0-\eta_0)}{(\cosh\eta_0-1)^2}\right]-\frac{9}{2}c_2\left[2\alpha-(\omega+3\alpha)\frac{\sinh\eta_0(\sinh\eta_0-\eta_0)}{(\cosh\eta_0-1)^2}\right]+\frac{9c^{2/3}(\omega+3\alpha)\sinh\eta_0}{(\cosh\eta_0-1)^2}c_3\right\}, \tag{3.82}$$

where $\eta=\eta_0$ corresponds to $r\to\infty$ and η_0 is thus given by $\sinh\eta_0-\eta_0=-9pc^{3/2}/2$.

By using the above condition, Eq. (3.82), we obtain the physical perturbation $\delta\overset{(L)}{\tilde{\rho}}=\delta\overset{(L)}{\rho}+\delta\overset{(L)}{\rho}_g$ as follows:

$$\frac{\delta\overset{(L)}{\tilde{\rho}}}{\rho^*}=\frac{e^{\omega\tau}r^{\omega}}{\alpha\left[\frac{2r^{\alpha}}{9c}(\cosh\eta-1)-\frac{c^{1/2}\sinh\eta}{\cosh\eta-1}\right]}\left(\Delta_1^+\left\{3(\omega+3\alpha)r^{\alpha}(\cosh\eta-1)\left(\frac{\sinh\eta_0(\sinh\eta_0-\eta_0)}{(\cosh\eta_0-1)^2}-\frac{\sinh\eta(\sinh\eta-\eta)}{(\cosh\eta-1)^2}\right)\right.\right.$$
$$\left.+\frac{9c^{3/2}}{2}\left[\left(4\omega+9\alpha-6(\omega+3\alpha)\frac{\sinh\eta_0(\sinh\eta_0-\eta_0)}{(\cosh\eta_0-1)^2}\right)\frac{\sinh\eta}{\cosh\eta-1}+3\alpha\frac{(2\cosh\eta+1)(\sinh\eta-\eta)}{(\cosh\eta-1)^2}\right]\right\}$$
$$+\Delta_2^+\left[(\omega+3\alpha)r^{\alpha}(\cosh\eta-1)\left(\frac{\sinh\eta}{(\cosh\eta-1)^2}-\frac{\sinh\eta_0}{(\cosh\eta_0-1)^2}\right)-\frac{9c^{3/2}\alpha(2\cosh\eta+1)}{2(\cosh\eta-1)^2}\right.$$
$$\left.\left.+\frac{9(\omega+3\alpha)c^{2/3}\sinh\eta_0}{4(\cosh\eta_0-1)^2}\left(\frac{4\sinh\eta}{\cosh\eta-1}\,\frac{9c^{3/2}(2\cosh\eta+1)}{2(\cosh\eta-1)^2r^{\alpha}}\right)\right]\right), \tag{3.83}$$

where

$$\Delta_1^+=\frac{1}{9c^2}c_1, \tag{3.84}$$

$$\Delta_2^+=-\frac{9pc^{1/2}}{4}\left(c_2-\frac{4}{9p}c_3\right), \tag{3.85}$$

and $\sinh\eta-\eta=9c^{3/2}(r^{-\alpha}-p)/2$.

The regularity condition at $r=0$ and $r=\infty$ implies that Δ_1^+ modes with $\omega=\alpha$ and Δ_2^+ modes with $0\leqslant\omega\leqslant\alpha$ are allowed. In this case, this fixed point is a repeller up to the zero mode because all other modes which satisfy the regularity condition have a positive ω. Note that a suitable linear combination of the Δ_1^+ and Δ_2^+ modes will have an asymptotic behavior $\approx r^{\omega-2\alpha}$. Therefore the possible value of ω ranges from 0 to 2α. The flow of the RG in the vicinity of the fixed point is shown in Fig. 2.

(ii) Case 2 ($p=0$). The gauge condition which we use is

$$f=-\frac{9}{4\alpha}c_2. \tag{3.86}$$

By using the above condition, Eq. (3.86), we obtain the physical perturbation

$$\frac{\delta\overset{(L)}{\tilde{\rho}}}{\rho^*}=\frac{e^{\omega\tau}r^{\omega}}{\alpha\left[\frac{2r^{\alpha}}{9c}(\cosh\eta-1)-\frac{c^{1/2}\sinh\eta}{\cosh\eta-1}\right]}$$
$$\times\left\{\Delta_1^+\left[(\omega+3\alpha)r^{\alpha}\left(2(\cosh\eta-1)-\frac{3\sinh\eta(\sinh\eta-\eta)}{\cosh\eta-1}\right)+\frac{27c^{3/2}\alpha}{2}\right.\right.$$
$$\left.\times\left(-\frac{\sinh\eta}{\cosh\eta-1}+\frac{(2\cosh\eta+1)(\sinh\eta-\eta)}{(\cosh\eta-1)^2}\right)\right]$$
$$\left.+\Delta_2^+\left(\frac{(\omega+3\alpha)r^{\alpha}\sinh\eta}{\cosh\eta-1}-\frac{9\alpha c^{2/3}(2\cosh\eta+1)}{2(\cosh\eta-1)^2}\right)\right\}, \tag{3.87}$$

where

$$\Delta_1^+ = \frac{1}{9c^2} c_1, \tag{3.88}$$

$$\Delta_2^+ = c^{1/2} c_3. \tag{3.89}$$

The regularity condition at $r=0$ and $r=\infty$ implies that no modes are allowed.

From the linear perturbation analysis around the fixed point, we see that the long-time behavior of the spherically symmetric dust universe is separated into two types. One is the case that the fixed point is a repeller. In this case, the Tolman-Bondi solution with self-similarity does not play an important role in an expanding universe because this fixed point is unstable and the spacetime will diverge from this fixed point. In the other case, the fixed point has both relevant and irrelevant modes. Although this fixed point is not a repeller, it has continuously many relevant modes. Thus it is not as straightforward as in the case of gravitational collapse [3] to extract the long-time behavior of the universe, because it is sensitive to the initial condition and therefore we cannot uniquely predict the outcome. In the final section, we briefly discuss how to treat the fixed point which has many relevant modes of the perturbation.

IV. SUMMARY AND DISCUSSION

We considered the spherically symmetric but inhomogeneous universe filled with dust, where the Einstein equations have scale invariance, Eqs. (3.39)–(3.43), and applied the renormalization group (RG) method to study its long-time asymptotics. The fixed point of the RG transformation is a self-similar solution with scale invariance of the Einstein equations. In order to study the flow of the RG around this fixed point, the linear perturbation analysis is used. We impose the perturbation on the regularity at the boundary where the radial coordinate r equals zero or infinity. This boundary means that the area radius equals zero or infinity in the case of $c=0$; on the other hand, in the case of $c>0$, it equals a finite or infinity. The fixed point is the Tolman-Bondi solution with self-similarity, which includes the flat Friedmann universe. The behavior of the fixed point is separated into two types. Both types have many relevant modes of the perturbation. The fixed points of the RG flow are self-similar solutions of the Einstein equations, which are worth studying in their own right and have been studied by many people [10,11]. Our approach has an obvious advantage that the fixed point is in general not always spherical symmetric like the Tolman-Bondi solution of the dust filled universe. More importantly, we can systematically treat the *dynamics* of the universe near the self-similar solution, adopting the scale as a ''time'' of the evolution. The RG flow near the fixed points is a new aspect in the study of the self-similarity in the universe [12].

The Tolman-Bondi solution with self-similarity is unstable against almost all spherical modes of linear perturbation. The spacetime will deviate from this fixed point. It is necessary to study the nonspherical mode of perturbation to say something more definite. In the cosmological problem, only the statistical quantities are meaningful if we think of a comparison with observations. There are some works in the RG approach [13–15] on the universe which has a hierarchical structure. We may contemplate further development of the RG approach to the cosmology formulated in the present work by introducing some kind of volume or statistical average for observables such as the energy density and the Hubble constant of the universe. A statistical concept is needed not only for comparison with observations but also for us to proceed further in the analysis of the RG equation because we have continuously many relevant (growing) modes around the fixed points. That is, the long-time behavior of the universe is sensitive to the initial configuration over which we have no *a priori* control and we have to consider the statistical likelihood of the initial values.

We remark that the introduction of a volume average in a finite region of the universe potentially introduces the scale invariance violation by hand because the exact scale invariance holds only for an infinite space. Note that in quantum field theories and statistical physics of the second-order phase transition the scale invariance violations are hidden in the form of a cutoff of the spectrum of physical modes. We shall elaborate our present observation in our future work.

The self-similar solution given by Eqs. (3.52) and (3.53) through the fixed point of the RG equation is essentially a function of $t^{-1/\alpha}r = t^{1-1/\alpha}r/t$, which is roughly the fraction of the physical distance to the horizon scale of the Friedmann universe. Also note that in the case of a nonlinear diffusion equation, Eq. (2.6) implies that the self-similar solution is a function of the ratio of the distance x to the diffusion length $\sqrt{t}$. In the both cases, the self-similar solution is a function of the distance in units of a physically relevant time-dependent scale. We believe this is a general phenomenon and the physical background of the RG equation which governs how dynamical variables deviate from the self-similar solution.

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APPENDIX: TOLMAN-BONDI SOLUTION WITH SELF-SIMILARITY

In the spherically symmetric universe filled with dust, the most general solution of the Einstein equations is the Tolman-Bondi solution [16]:

$$A(r,t) = \frac{B'(r,t)}{\sqrt{1+C_1(r)}}, \tag{A1}$$

$$B(r,t)=\begin{cases}\left(\frac{9C_2(r)}{4}\right)^{1/3}[t-C_3(r)]^{2/3} & \text{for } C_1(r)=0,\\ \frac{C_2(r)}{2C_1(r)}(\cosh\eta-1) \quad \left(t-C_3(r)=\frac{C_2(r)}{2C_1^{3/2}(r)}(\sinh\eta-\eta)\right) & \text{for } C_1(r)>0,\\ \frac{C_2(r)}{2|C_1(r)|}(1-\cos\eta) \quad \left(t-C_3(r)=\frac{C_2(r)}{2|C_1^{3/2}(r)|}(\eta-\sin\eta)\right) & \text{for } C_1(r)<0,\end{cases} \tag{A2}$$

$$\kappa\rho(r,t)=\frac{C_2'(r)}{B^2B'}, \tag{A3}$$

where $C_1(r)$, $C_2(r)$, and $C_3(r)$ are arbitrary functions of r and a prime denotes the derivative with respect to r. By taking $C_1(r)=c$, $C_2(r)=4r^\alpha/9$, and $C_3(r)=pr^\alpha$, we obtain the Tolman-Bondi solution with self-similarity, Eqs. (3.57)–(3.67).

As for the calculation of linear perturbation, since we concentrate on the spherical modes of perturbation around a self-similar solutions, it is enough to consider the linear perturbation of the arbitrary functions $C_1(r)$, $C_2(r)$, and $C_3(r)$. The perturbed quantities $\delta C_1(r)$, $\delta C_2(r)$, and $\delta C_3(r)$ can be expressed by a superposition of modes with different ω and taken in the following form:

$$\delta C_1(r)=c_1r^\omega, \tag{A4}$$

$$\delta C_2(r)=c_2r^{\omega+\alpha}, \tag{A5}$$

$$\delta C_3(r)=c_3r^{\omega+\alpha}. \tag{A6}$$

By a coordinate transformation of r,

$$r\rightarrow r+F(r), \tag{A7}$$

where $F(r)$ is an arbitrary function of r, we obtain the gauge mode of linear perturbation. This function $F(r)$ also can be expressed by the superposition of modes with different ω in the form

$$F(r)=fr^{\omega+1}. \tag{A8}$$

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[9] The RG equations also can be seen as the Einstein equations in the new coordinate system (τ,ξ,θ,ϕ), where $\tau=(\ln t)/\alpha$, $\xi=r/t^{1/\alpha}$, $\overset{(L)}{A}(\xi,\tau)=t^{(1-\alpha)/\alpha}A(r,t)$, and $\overset{(L)}{B}(\xi,\tau)=t^{-1}B(r,t)$. One might be puzzled by observing that in this new coordinate system, the lapse function can be zero at $\xi^2\overset{(L)}{A}=\alpha^2$. However, there is no physical meaning of this point because the Einstein equations are found to be regular at this point.

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