Magnetohydrodynamics in the extreme relativistic limit

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(Received 9 July 1997; published 26 February 1998)

We present two new formulations of magnetohydrodynamics (MHD), in the limit where the inertia of the charge carriers can be neglected. The first employs Lagrangian coordinates and generalizes Newcomb's formalism to allow for a variable time slicing. It contains an extremely simple prescription for generalizing the action of a relativistic Nambu-Goto string to four dimensions. It is also related by a duality transformation to the action presented by Achterberg. This transformation causes the perturbed and unperturbed Lagrangian coordinates to exchange roles as dynamical fields and background spacetime. Our second formulation introduces massless electrically charged fermions as the current carrying modes, and considers long wavelength perturbations with $\omega^2, k_\perp^2 \ll eB$. Because the Fermi zero mode can be bosonized separately on each magnetic flux line, the current density may be written in terms of a four-dimensional axion field that acts as a Lagrange multiplier to enforce the MHD condition. The fundamental modes of the magnetofluid in this limit comprise two oppositely directed Alfvén modes and the fast mode, all of which propagate at the speed of light. We calculate the nonlinear interaction between two Alfvén modes, and show that in both formulations it satisfies the same simple expression. This provides the first exact treatment of the effects of compressibility on nonlinear interactions between MHD waves. We then summarize the interactions between Alfvén modes, between Alfvén modes and fast modes, and between fast modes in terms of a simplified Lagrangian. The three-mode interaction between fast modes is a magnetohydrodynamic analogue of the QED process of photon splitting, but occurs in background magnetic fields of arbitrary strength. The scaling behavior of an Alfvén wave cascade in a box is derived, paying close attention to boundary conditions. This result also applies to nonrelativistic MHD media and differs from those obtained by previous authors in the nonrelativistic regime. Finally, we briefly outline the physical processes which determine the inner scale of such a cascade in neutron star magnetospheres, black hole accretion disks, and γ-ray burst sources. At low charge density, the waves at the inner scale may become charge starved; whereas Compton drag is the dominant dissipative mechanism at large optical depth to electron scattering. A turbulent cascade leads to effective dissipation even in optically thick media, and in particular can significantly raise the entropy-baryon ratio in the relativistic outflows that power cosmological γ-ray bursts. [S0556-2821(98)03806-5]

PACS number(s): 95.30.Qd, 11.10.Lm, 11.27.+d, 52.35.Bj

I. INTRODUCTION

Relativistic formulations of magnetohydrodynamics (MHD) generally accommodate the effects of both matter and field inertia. We focus here on the extreme relativistic limit, in which matter inertia can be neglected. This is an excellent approximation in the magnetosphere of a neutron star, for example, even when the density of charge carriers is more than sufficient to enforce the MHD condition

$$\mathbf{E}\cdot\mathbf{B}=0. \tag{1.1}$$

The equation of motion in the extreme relativistic limit states that the net Lorentz force on the charge carriers vanishes,[1]

$$j^\mu F_{\mu\nu}=\partial_\rho F^{\rho\mu}F_{\mu\nu}=0; \tag{1.2}$$

i.e., the conducting medium is force-free. The normal modes of the magnetofluid then simplify dramatically.[2] In the case of a uniform background magnetic field $\mathbf{B}_0=B_0\hat{\mathbf{z}}$, they are the two Alfvén modes A^+ and A^- (with dispersion relation $\omega=\pm k_z$ and polarization transverse to the background magnetic field) and the fast mode F (with dispersion relation $\omega^2=k^2$ and single polarization state $\delta\mathbf{E}\cdot\mathbf{B}_0=0$). In each case, the magnitude of the group velocity equals the speed of light.

In marked contrast with the behavior of vacuum electromagnetic waves, these three modes undergo nonlinear interactions even at the classical level. The lowest order perturbative interactions turn out to be

$$A^+ + A^- \leftrightarrow A^+ + A^-, \tag{1.3}$$

$$A^+ + A^- \leftrightarrow F,$$

[1]Throughout this paper we use Heaviside-Lorentz units with $\hbar=c=1$.

[2]For a derivation of these normal modes in the nonrelativistic case, see, for example [1].

and

$$F+F\leftrightarrow F. \tag{1.4}$$

These interactions are nonvanishing for the fully four-dimensional MHD modes, namely, those with nonvanishing momentum $k_\perp$ perpendicular to the background magnetic field: the current-carrying A modes and the F mode. Analogous interactions between Alfvén waves in the nonrelativistic regime have been studied by a number of authors [2–6]. This work focused on incompressible fluids, and has generated a continuing debate on the role of nonperturbative effects.

In this paper we develop two equivalent variational formulations of extreme relativistic MHD, calculate the form of the interactions (1.3) and (1.4), and derive the scaling relations for relativistic Alfvén turbulence. Since there is no such thing as an incompressible relativistic fluid, our work provides the first exact treatment of the effects of compressibility on nonlinear MHD mode interactions.

The limitations of the fluid description of relativistic MHD deserve to be emphasized at the beginning. The existence of current-carrying modes with group velocity equal to the speed of light requires the existence of *massless* electric charge carriers.[3] We show that, in turn, this allows a greatly simplified description of the dynamics in terms of the electromagnetic field coupled to a pseudoscalar (axionlike) field. The neglect of the inertia of the charge carriers implies a restriction to *long wavelength* modes, with perpendicular momentum $k_\perp$ too small to excite individual fermions from the lowest energy state (with longitudinal momentum $|p|=E$) into higher Landau states.

The advection of magnetic field lines by a perfectly conducting fluid provides an equivalent description of the dynamics. To achieve a Lagrangian description of this physical model, one must choose between treating the perturbed positions x^μ of the fluid particles as field variables in a space defined by the initial coordinates x_0^μ, or vice versa. The first route leads to a simple covariant action

$$S=+\frac{1}{4}\int d^4x\widetilde{F}^{\mu\nu}\widetilde{F}_{\mu\nu}, \tag{1.5}$$

in which the dual $\widetilde{F}_{\mu\nu}$ of the field strength tensor is expressed as a function of the x^μ. This is a direct four-dimensional generalization of the standard Lagrangian formulation of nonrelativistic MHD [7]. An equivalent Lagrangian description of the perturbed magnetofluid in which the *unperturbed* coordinates appear as dynamical variables was worked out several years ago by Achterberg [8]. This second description involves the electric variable $F^{\mu\nu}$, and is therefore related by a duality transformation. The net effect of this transformation is to cause the initial and final coordinates x_0^μ and x^μ to exchange roles as field variable and background spacetime. The perturbed Lagrangian coordinates are the dynamical fields in the ''magnetic'' formalism described in this paper, and because of this the action (1.5) turns out to be a direct four-dimensional analogue of the action of a relativistic string.

In the nonrelativistic, incompressible case, it has been shown that colliding A modes are subject to nonlocal kinematic interactions in addition to the more familiar resonant dynamical interactions [5,6]. In the relativistic case, we show that the additional freedom to choose a time slicing leads us to a gauge in which these kinematic interactions drop out, so that the coordinate transformation $x^\mu(x_0^\nu)$ is incompressible in a four-dimensional sense.

The structure of this paper is as follows. The Lagrangian formulation of relativistic MHD is presented in Sec. II, and the axionic formulation in Sec. III. The nonlinear collision between two Alfvén wave packets is calculated in Sec. IV, and the equivalence of the two formalisms is demonstrated. In Sec. V we consider wave interactions including the relativistic fast mode, and summarize the lowest order interactions in a simplified Lagrangian. We then derive the scaling relations for relativistic Alfven turbulence in Sec. VI, emphasizing how the cascade properties depend sensitively on boundary conditions. This work also has applications to nonrelativistic, incompressible MHD turbulence. Finally, we discuss astrophysical applications of our results in Sec. VII, including wave damping in neutron star magnetospheres and cosmological γ-ray burst sources.

II. LAGRANGIAN PERTURBATIONS OF A RELATIVISTIC MHD FLUID

As is well known in the nonrelativistic case [9,7] one obtains a simple solution to the constraint equations

$$\nabla\cdot\mathbf{B}=0;\quad \nabla\times\mathbf{E}+\partial\mathbf{B}/\partial t=0 \tag{2.1}$$

by labeling fluid particles with Lagrangian coordinates $\mathbf{x}_0$,

$$\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}(\mathbf{x}_0,t). \tag{2.2}$$

In other words, $\mathbf{x}_0$ is the initial position of a fluid particle and $\boldsymbol{\xi}$ its displacement. The solution is

$$B_i=\frac{1}{J}\frac{\partial x_i}{\partial x_{0j}}B_{0j}, \tag{2.3}$$

or, equivalently,

$$\mathbf{B}=J^{-1}[\mathbf{B}_0+(\mathbf{B}_0\cdot\boldsymbol{\nabla}_0)\boldsymbol{\xi}]. \tag{2.4}$$

Here, J is the Jacobian of the transformation from $\mathbf{x}_0$ to $\mathbf{x}$: i.e.,

$$J=\det(\delta_{ij}+\partial\xi_i/\partial x_{0j}). \tag{2.5}$$

The vanishing of $\mathbf{E}$ in the fluid rest frame implies

$$\mathbf{E}=-\frac{d\mathbf{x}}{dt}\times\mathbf{B}. \tag{2.6}$$

A simple variational principle for nonrelativistic MHD, based on these Lagrangian variables, was written down long ago by Newcomb [7]. Starting from the action

[3]In practice, the dynamics is very nearly equivalent if the charge carriers are massive, but their space density is much larger than the minimum value J/e needed to support a current density J.

$$S=\int dtd^3x\left[\frac{1}{2}\rho\left(\frac{d\mathbf{x}}{dt}\right)^2-\frac{1}{2}B^2\right], \tag{2.7}$$

and substituting $d^3x=Jd^3x_0$, $\rho=\rho_0/J$, and Eq. (2.3), one obtains

$$S=\int dtd^3x_0\left[\frac{\rho_0}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2-\frac{1}{2J}\left(B_{0j}\frac{\partial x_i}{\partial x_{0j}}\right)^2\right] \tag{2.8}$$

when the background fluid is static. Varying Eq. (2.8) with respect to x_i yields the usual MHD equation. We focus in this paper on wave propagation and interactions in a static medium with a uniform magnetic field

$$\mathbf{B}_0=B_0\hat{\mathbf{z}}. \tag{2.9}$$

The Jacobian J plays a crucial role in this formalism. Without it, the action depends only on derivatives with respect to two coordinates t and z_0. All the terms in the MHD equation involving derivatives with respect to the transverse coordinates arise from the variation of J.

This action has a simple covariant generalization when the inertia of the conducting fluid can be neglected,

$$S'=-\int d^4x\,\frac{1}{4}F^{\mu\nu}F_{\mu\nu}=\int d^4x\,\frac{1}{2}(E^2-B^2). \tag{2.10}$$

To make sense of this expression, we must introduce a background time coordinate τ (e.g., the time coordinate far from a localized region of perturbed magnetofluid), and allow for variable time slicings of the fluid. Then the dual of the field strength tensor takes the simple form

$$\widetilde{F}^{\mu\nu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}=\frac{\widetilde{F}_0^{\alpha\beta}}{J_4}\frac{\partial x^\mu}{\partial x_0^\alpha}\frac{\partial x^\nu}{\partial x_0^\beta}, \tag{2.11}$$

where $J_4=\det(\partial x^\mu/\partial x_{0\nu})$ is the Jacobian of the corresponding coordinate transformation. This is the four-dimensional generalization of Eq. (2.3) and (2.6). In parallel with the nonrelativistic case, the equation of motion (1.2) follows by varying S' with respect to x^μ, as is demonstrated in Appendix A.

In the case of a uniform background magnetic field (2.9)

$$\widetilde{F}^{\mu\nu}=\frac{B_0}{J_4}\left(\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial z_0}-\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\mu}{\partial z_0}\right), \tag{2.12}$$

and the magnetic and electric fields can be written separately as

$$\mathbf{B}=\frac{B_0}{J_4}\left(\frac{\partial t}{\partial\tau}\frac{\partial\mathbf{x}}{\partial z_0}-\frac{\partial t}{\partial z_0}\frac{\partial\mathbf{x}}{\partial\tau}\right); \tag{2.13}$$

$$\mathbf{E}=-\frac{B_0}{J_4}\left(\frac{\partial\mathbf{x}}{\partial\tau}\times\frac{\partial\mathbf{x}}{\partial z_0}\right). \tag{2.14}$$

Making these substitutions in Eq. (2.10) yields

$$S'=\frac{B_0^2}{2}\int d^4x_0\,\frac{1}{J_4}\left[\left(\frac{\partial\mathbf{x}}{\partial\tau}\times\frac{\partial\mathbf{x}}{\partial z_0}\right)^2-\left(\frac{\partial t}{\partial\tau}\frac{\partial\mathbf{x}}{\partial z_0}-\frac{\partial t}{\partial z_0}\frac{\partial\mathbf{x}}{\partial\tau}\right)^2\right]. \tag{2.15}$$

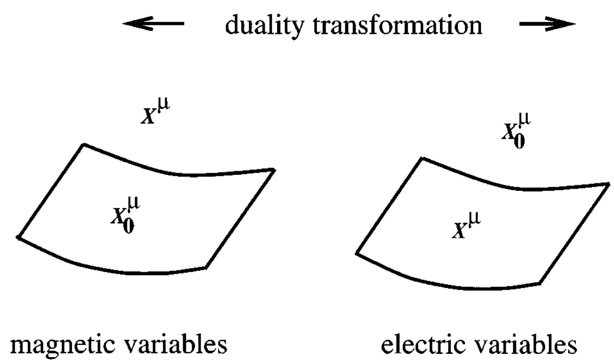

FIG. 1. In the action employing magnetic Lagrangian variables (2.11), the perturbed coordinates x^μ of the magnetofluid play the role of dynamical fields in a background spacetime labeled by the unperturbed coordinates x_0^μ. The roles of x^μ and x_0^μ are reversed by a duality transformation into the electric Lagrangian variables (2.16).

The nonlinear interactions between fully four-dimensional MHD waves are encoded in the factor of of J_4^{-1}. Without it, the dynamics is essentially two-dimensional and equivalent to that of noninteracting waves on relativistic Nambu-Goto strings. One immediate benefit of introducing the new time coordinate t is that by an appropriate choice of time slicings one can set $J_4=1$ everywhere in the equations of motion, after introducing appropriate longitudinal components of the displacement function ξ^μ. We demonstrate this for the collision between two Alfvén wave packets in Sec. III, and for the collision between two F modes in Sec. IV.

An equivalent Lagrangian description of the perturbed magnetofluid involves the dual variable $F^{\mu\nu}$ [8],

$$F^{\mu\nu}=F_0^{\alpha\beta}\frac{\partial x_{0\alpha}}{\partial x_\mu}\frac{\partial x_{0\beta}}{\partial x_\nu}. \tag{2.16}$$

This expression arises from the fact that the Lie derivative of the field strength two-form $F=\frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$ vanishes along the fluid four-velocity, if the MHD condition (1.1) is satisfied. The corresponding electric current density is

$$J_0^\alpha=-F_0^{\alpha\beta}\Box x_{0\beta}, \tag{2.17}$$

in the unperturbed coordinate system. This vanishes unless $k^2\neq 0$, which is possible only for A modes that are sheared perpendicular to $\mathbf{B}_0$, namely, those with $k_\perp\neq 0$. The transformations (2.11) and (2.16) both preserve the MHD condition. In addition, the Lorentz scalar E^2-B^2 is everywhere negative if the unperturbed field is magnetic; as follows from the inequality $(\partial x^\mu/\partial\tau\cdot\partial x^\mu/\partial z_0)^2-(\partial x^\mu/\partial\tau)^2(\partial x^\mu/\partial z_0)^2<0$.

The important thing to note here is that, in the actions built out of Eqs. (2.11) and (2.16), the initial and final coordinates x_0^μ and x^μ play opposite roles as field variable and background spacetime (Fig. 1). The perturbed Lagrangian coordinates are the dynamical fields in the "magnetic" formula (2.11), and so the corresponding action (2.15) provides a more direct four-dimensional analog of the action of a relativistic string.

The connection between (2.15) and the classical string action is obvious when the latter is written in the following form [10]:

$$S_{\rm Schild}=T\int d\tau d\sigma\left[\left(\frac{\partial \mathbf{x}}{\partial\tau}\times\frac{\partial \mathbf{x}}{\partial\sigma}\right)^2-\left(\frac{\partial t}{\partial\tau}\frac{\partial \mathbf{x}}{\partial\sigma}-\frac{\partial t}{\partial\sigma}\frac{\partial \mathbf{x}}{\partial\tau}\right)^2\right] \tag{2.18}$$

(in flat space). Here, T is the string's tension and σ is the coordinate along the string. The dynamics of a thin magnetic flux tube is essentially two dimensional, with the excitations of the tube being limited to transverse modes that are equivalent (in the extreme relativistic limit) to Nambu-Goto waves. The stress energy tensor averaged over the transverse coordinates of the tube (or string) satisfies $\langle T_{tt}\rangle=-\langle T_{zz}\rangle$ in both cases.[4] By contrast, the fully four-dimensional MHD modes (the F mode and the current-carrying A modes) undergo nonlinear interactions that Nambu-Goto waves do not. In this sense, four-dimensional relativistic MHD provides a highly nontrivial generalization of Nambu-Goto dynamics. These wave interactions are the main subject of this paper.

The dynamics of a perfect fluid composed of strings has been studied previously by Stachel [12], although the precise correspondence between the two actions was not written down by him. The main result of the preceding discussion is that this correspondence is remarkably simple:

$$J_2\leftrightarrow J_4,\qquad \int d^2x_0\leftrightarrow\int d^4x_0. \tag{2.19}$$

The dynamics of a perfect fluid composed of k-dimensional objects in D-dimensional spacetime was investigated in [13], by employing energy flux and ''particle'' flux as the basic variables. In the specific case $k=1$, $D=4$, it was shown that the dynamics is given by equations that are formally equivalent to the force free equation (1.2) and the constraint equations (2.1). However, this approach is less useful for studying interactions between waves, and will not be adopted here. Finally, the dynamics of a relativistic superfluid has been studied recently in [14]. The basic variable in this formalism is a Kalb-Ramond field that couples to the vorticity.[5]

One new complication that arises in the relativistic regime is that the displacement current and the electrostatic force both play important dynamical roles. As a result, the equation of motion is fourth order in ξ, as compared with second order in the nonrelativistic case. This is seen most easily by writing Eq. (1.2) in component form:

$$(\nabla\cdot\mathbf{E})\mathbf{E}+(\nabla\times\mathbf{B})\times\mathbf{B}-\partial_t\mathbf{E}\times\mathbf{B}=0, \tag{2.20}$$

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}-\partial_t\mathbf{E})=0. \tag{2.21}$$

The first of these equations has only two independent components, given the MHD condition $\mathbf{E}\cdot\mathbf{B}=0$, and the second is a straightforward consequence of the first. From this, one expects that the propagating MHD modes have only two physical degrees of freedom (the components of $\mathbf{E}$ and $\mathbf{B}$ perpendicular to $\mathbf{B}_0$ for the A modes, and the components perpendicular to $\mathbf{k}$ for the F mode).

It is remarkable that, in direct analogy with nonrelativistic MHD, Eq. (2.20) admits *exact* nonlinear solutions. For the A mode,

$$\mathbf{B}=\mathbf{B}_0+\mathbf{b}(z\mp t,x,y) \tag{2.22}$$

and

$$\mathbf{E}=\pm\mathbf{b}\times\hat{\mathbf{z}} \tag{2.23}$$

(with $\mathbf{b}\cdot\hat{\mathbf{z}}=\nabla\cdot\mathbf{b}=0$). This describes a wave propagating in a single direction along the background magnetic field. On the other hand, two waves traveling in opposite directions cannot be superposed and still remain an exact nonlinear solution. A modes with $\mathbf{k}$ parallel to $\mathbf{B}_0$ may be viewed as vacuum electromagnetic waves. However, the shear between neighboring field lines in a wave with $\mathbf{k}\times\mathbf{B}_0\neq 0$, or in a collection of such waves, is supported by a current density along $\mathbf{B}_0$,

$$\mathbf{j}=\left(\frac{\partial b_y}{\partial x}-\frac{\partial b_x}{\partial y}\right)\hat{\mathbf{z}}. \tag{2.24}$$

The fast magnetosonic modes are essentially vacuum electromagnetic waves, with zero charge and current densities. Hence they also constitute exact nonlinear solutions. For plane waves these are given by

$$\mathbf{B}=\mathbf{B}_0+\mathbf{b}(\hat{\mathbf{k}}\cdot\mathbf{x}-t) \tag{2.25}$$

and

$$\mathbf{E}=-\hat{\mathbf{k}}\times\mathbf{b}, \tag{2.26}$$

where $\hat{\mathbf{k}}$ is a constant unit vector, $\nabla\cdot\mathbf{b}=0$, and $\mathbf{b}\cdot(\hat{\mathbf{k}}\times\mathbf{B}_0)=0$. In contrast with true vacuum electromagnetic waves, the plasma response (i.e., $\mathbf{E}\cdot\mathbf{B}=0$) prevents superposed fast modes with arbitrary $\hat{\mathbf{k}}$ from being an exact solution.

Given the existence of the exact nonlinear solutions, we expect that A modes propagating in the same direction along $\mathbf{B}_0$ will be noninteracting. A modes propagating in different directions will interact with each other and also with F modes. F modes will undergo nonlinear interactions, unless they are traveling in the same direction.

III. AXIONIC FORMULATION OF RELATIVISTIC MHD

Until now we have assumed the presence of electromagnetic fields and charged matter which enforces the MHD condition $\mathbf{E}\cdot\mathbf{B}=0$, but made no further attempt to prescribe the charged fields. These fields must have modes which propagate at the speed of light in both directions along the background magnetic field. Massless, electrically charged fermions of charge e in a background magnetic field $B_0\hat{\mathbf{z}}$ have one zero mode with dispersion relation $\omega^2=k_z^2$ and spin $\sigma_z=eB_0/|eB_0|$ (e.g., [15]). Thus, we require at least two species of fermions with opposite helicities $\chi=\sigma_z k_z/|k_z|$, since particles and antiparticles have identical values of $\chi/e=\mathrm{sgn}(k_z)/|e|$.

Because these fermi zero modes are effectively two di-

[4]Indeed, a number of years ago, when considering the dynamics of thin magnetic flux tubes, Nielsen and Olesen [11] wrote down a related expression for $\tilde{F}_{\mu\nu}$, but without the factor of $(J_4)^{-1}$.

[5]This field is physically distinct from the pseudoscalar field introduced below, which is excited only in the presence of an electromagnetic current—that is, only if the magnetofluid supports waves that are sheared perpendicular to the background magnetic field.

mensional, we can apply the technique of bosonization [16,17]. In two dimensions, one can write the electric current j^a $(a=t,z)$ as[6]

$$j^a=(2\pi e)\varepsilon^{ab}\partial_b\theta. \tag{3.1}$$

When $\delta\mathbf{B}=0$, this procedure can be applied directly to the fermi fields on each quantum of magnetic flux; the four-dimensional current density $j^\mu$is obtained by multiplying j^a by the transverse density of states $|eB_0|/2\pi$, and setting $j^x=j^y=0$. More generally, when the magnetic field experiences a long wavelength perturbation $k^2\ll eB$, one can find a local Lorentz frame in which $\mathbf{E}=0$ and repeat this procedure. The net result is that j^μ can be written in a simple covariant form:

$$j^\mu=-\frac{e^2}{2}\,\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\theta F_{\rho\sigma}, \tag{3.2}$$

Since the electromagnetic field was force-free in the original (local) Lorentz frame, it is not surprising to find that this expression exactly satisfies Eq. (1.2),

$$j^\mu F_{\mu\nu}=e^2\partial_\nu\theta(\mathbf{E}\cdot\mathbf{B})=0, \tag{3.3}$$

as long as the MHD condition is satisfied.

Expression (3.2) is formally equivalent to the current density induced by a spatially variable axion field in a background electromagnetic field, and indeed follows from the action

$$S''=\int d^4x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{e^2}{4}\,\theta\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \tag{3.4}$$

as long as

$$\int_{-\infty}^{\infty}dz\,\partial_z\theta=\int_{-\infty}^{\infty}dt\,\partial_t\theta=0. \tag{3.5}$$

These conditions are equivalent to

$$\int_{-\infty}^{\infty}dz\,\partial_z(\delta B_{x,y})=\int_{-\infty}^{\infty}dt\,\partial_t(\delta B_{x,y})=0, \tag{3.6}$$

which corresponds to a vanishing net transverse displacement of the background magnetic field lines across an Alfvén wave packet. An additional surface term is present otherwise, representing a net charge on the Alfvén wave packet coupled to a background gauge field. The effect of this term on wave interactions is examined in Secs. IV and V.

The θ parameter in the action S'' has a simple interpretation as a Lagrange multiplier which enforces the MHD condition:

$$\frac{\partial L}{\partial\theta}=0\Rightarrow\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}=-8\mathbf{E}\cdot\mathbf{B}=0. \tag{3.7}$$

[6]We choose a normalization different from the conventional one by a factor $2\pi^{3/2}$.

A kinetic term for θ is absent because we are neglecting the energy

$$\frac{eB}{2\pi}\int^{p_F}\frac{dp_z}{2\pi}\,|p_z|=\frac{eB}{2\pi}\,\frac{p_F^2}{4\pi} \tag{3.8}$$

of the fluctuating fermi fields compared with the energy

$$\frac{1}{2}\,[(\delta\mathbf{B})^2+(\delta\mathbf{E})^2]=(\delta\mathbf{B})^2 \tag{3.9}$$

of the fluctuating gauge field. The z component p_F of the Fermi momentum of the excited Dirac sea is related to the perpendicular wave number $k_\perp$ of the Alfvén excitation through

$$|j|=|k_\perp\delta B|=\frac{e^2B}{2\pi}\,\frac{|p_F|}{2\pi}. \tag{3.10}$$

Hence the neglect of the kinetic energy of the Fermi fields is justified when the wave is weakly sheared,

$$k_\perp^2\ll\frac{e^2}{2\pi^2}\,eB, \tag{3.11}$$

that is, when the perpendicular momentum of the wave is too small to excite individual fermions into higher Landau levels.

There is a physical distinction between this model for an MHD fluid, and the picture advanced in Sec. II, that deserves to be emphasized. Consider, for example, an A mode with vanishing space charge ($\mathbf{k}\|\mathbf{B}_0$) exerts no net force on the Fermi zero modes propagating in the same direction as the wave, and does not perturb them. Modes of energy $|\mathcal{E}|\lesssim(eB_0)^{1/2}$ propagating against the wave receive transverse momentum $p_\perp/|\mathcal{E}|=-2ie(\omega|\mathcal{E}|/B_0)\delta B/B_0$ in the direction of $\delta\mathbf{E}$. Even for a large amplitude wave, this vanishes as $\sim e^2\omega/(eB)^{1/2}$ in the low frequency limit that we are exploring. In this case, the MHD wave can be thought of as a vacuum electromagnetic wave superposed on a static background magnetic field and a static Dirac sea.

IV. EQUIVALENCE OF THE TWO FORMULATIONS: COLLISION OF TWO ALFVÉN WAVE PACKETS

We have presented two distinct formulations of magnetohydrodynamics in the extreme relativistic limit. The basic variable in the first formulation is the displacement ξ^μ of each fluid particle from its equilibrium position (and time). The second formulation involves the electromagnetic field tensor $F^{\mu\nu}$ as well as an axionlike field θ that acts as a Lagrange multiplier to enforce the MHD condition. Let us now demonstrate the equivalence of these two formulations by calculating the collision between two Alfvén wave packets $+$ and $-$ propagating in opposite directions along $\mathbf{B}_0$.

We emphasize that the interaction that we now calculate is *dynamic*, and will be related to third-order terms in the MHD Lagrangian. Colliding Alfvén waves also undergo kinematic interactions that involve a mixing between the transverse positions of the field lines in Lagrangian and Eulerian coordinates, as has been demonstrated in [6] for a nonrelativistic magnetofluid. The perturbative expansion of these additional kinematic interactions breaks down when the transverse displacement of each field line is free to undergo a random walk along $\mathbf{B}_0$. While analogous interactions are also present in the relativistic regime, we are able to choose a temporal gauge in which they vanish, so that only the true dynamical interactions survive. An additional difference between a relativistic magnetofluid and a nonrelativistic (incompressible) magnetofluid is that in the relativistic regime, the lowest order dynamical interaction is third order in ξ^μ (as compared with fourth order in the nonrelativistic regime [3]).

The displacement vectors $\xi_+^\mu(z-t,x,y)$ and $\xi_-^\mu(z+t,x,y)$ (or, more precisely, their derivatives with respect to $z_+\equiv z+t$ and $z_-\equiv z-t$) provide a covariant generalization of the Elsasser variables

$$\mathbf{u}=\frac{\mathbf{b}}{B_0}+\frac{\mathbf{v}}{V_A};\quad \mathbf{w}=\frac{\mathbf{b}}{B_0}-\frac{\mathbf{v}}{V_A}\quad (V_A\equiv B_0/\sqrt{4\pi\rho}) \tag{4.1}$$

commonly used in treatments of nonrelativistic MHD (e.g., [18]). We further choose a gauge in which $\xi_\pm^t=\xi_\pm^z=0$ and $J_4=1$ when the wave packets are well separated. The corresponding time-dependent magnetic and electric fields are

$$\delta\mathbf{B}_\pm=B_0\boldsymbol{\xi}'_\pm;\quad \delta\mathbf{E}_\pm=\pm B_0(\boldsymbol{\xi}'_\pm\times\hat{\mathbf{z}}), \tag{4.2}$$

where

$$\boldsymbol{\xi}'_\pm=\frac{1}{2}(\partial_z\mp\partial_t)\boldsymbol{\xi}_\pm\equiv\partial_\mp\boldsymbol{\xi}_\pm . \tag{4.3}$$

When the waves overlap, longitudinal displacements are excited, as well as a new transverse wave $\{\delta\mathbf{B}_I,\delta\mathbf{E}_I,\xi_I\}(z,t,x,y)$ whose time evolution we now calculate.

Substituting $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}_++\delta\mathbf{B}_-+\delta\mathbf{B}_I$, $\mathbf{E}=\delta\mathbf{E}_++\delta\mathbf{E}_-+\delta\mathbf{E}_I$ into the induction equation and the equation of motion (2.20), one finds

$$\boldsymbol{\nabla}\times\delta\mathbf{E}_I+\partial_t\delta\mathbf{B}_I=0, \tag{4.4}$$

and

$$B_0[(\boldsymbol{\nabla}\times\delta\mathbf{B}_I)\times\hat{\mathbf{z}}-\partial_t\delta\mathbf{E}_I\times\hat{\mathbf{z}}]$$
$$=2\boldsymbol{\nabla}_\perp(\delta\mathbf{B}_+\cdot\delta\mathbf{B}_-)-2(\delta\mathbf{B}_+\cdot\boldsymbol{\nabla})\delta\mathbf{B}_--2(\delta\mathbf{B}_-\cdot\boldsymbol{\nabla})\delta\mathbf{B}_+ . \tag{4.5}$$

These two equations govern the collision of two Alfvén wave packets to second order in $\delta B/B$.

The only technical subtlety here is that expression (2.11) for $\tilde{F}_{\mu\nu}$ involves derivatives with respect to the unperturbed coordinates τ and z_0, whereas Maxwell's equations involve derivatives with respect to the perturbed coordinates x^μ. These derivatives are equivalent for the noninteracting wave packets, but one generically finds that $t\neq\tau$ and $z\neq z_0$ when two MHD modes overlap. Nonetheless, it is possible to choose a temporal gauge in which $J_4=1$ and the derivatives remain identical to a high order in ξ,

$$\frac{\partial\xi^\mu}{\partial\tau}=\frac{\partial\xi^\mu}{\partial t}+O(\xi_\pm^4);\quad \frac{\partial\xi^\mu}{\partial z_0}=\frac{\partial\xi^\mu}{\partial z}+O(\xi_\pm^4), \tag{4.6}$$

as we sketch in Appendix B. In this gauge, the distorted magnetofluid in the interaction region is incompressible in a four-dimensional sense.

Even though expression (2.11) directly solves the induction equation, the gauge choice $J_4=1$ leads to constraints on ξ. These are found by substituting $\xi^\mu=\xi_+^\mu+\xi_-^\mu+\xi_I^\mu$ into expressions (2.13) and (2.14) for $\mathbf{B}$ and $\mathbf{E}$ and then into the induction equation. Keeping all terms to second order in $\xi_\pm$ (note that ξ_I is itself second order), this gives

$$(\xi'_-\cdot\partial)\xi'_+-(\xi'_+\cdot\partial)\xi'_-+(\partial\cdot\xi'_-)\xi'_+-(\partial\cdot\xi'_+)\xi'_-$$
$$=-\frac{1}{2}\partial_\tau(\partial\cdot\xi_I)\hat{\mathbf{z}}. \tag{4.7}$$

Since the transverse components of $\xi_\pm$ are linearly independent, this requires

$$\partial\cdot\xi_+=\partial\cdot\xi_-=\partial\cdot\xi_I=0 \tag{4.8}$$

together with

$$(\xi'_-\cdot\partial)\xi_+=(\xi'_+\cdot\partial)\xi_-=0. \tag{4.9}$$

Here, the derivative ∂ is with respect to the *perturbed* coordinates $\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}$, whereas J_4 involves derivatives of ξ^μ with respect to the *unperturbed* coordinates $\mathbf{x}_0$. Nonetheless it is easy to show, as we do in Appendix B, that conditions (4.8) and (4.9) are equivalent to the conditions

$$\partial_0\cdot\xi=0;\quad (\xi\cdot\partial_0)\xi=0, \tag{4.10}$$

where $(\partial_0)_\mu\equiv\partial/\partial x_0^\mu$. Equations (4.10) in turn guarantee that $J_4=1$.

Although $\partial\cdot\xi_\pm=\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\pm=0$ for the noninteracting Alfvén wave packets, longitudinal components are required to enforce equations (4.9) when two A waves overlap. The unique choice which also satisfies Eq. (4.8) is

$$(\xi'_\pm)_\parallel=-\frac{k_\mp\cdot\xi_\pm}{(k_\mp)_\parallel\cdot(k_\pm)_\parallel}(k_\pm)_\parallel=\frac{(\mathbf{k}_\mp)_\perp\cdot(\boldsymbol{\xi}'_\pm)_\perp}{2\omega_+\omega_-}(k_\pm)_\parallel, \tag{4.11}$$

in a Fourier representation $\xi^\mu=\xi_0^\mu e^{-ik^\mu x_\mu}$ with $k^\mu=k_\perp^\mu+k_\parallel^\mu$ and $k_\parallel^\mu=(\omega,0,0,k_z)$. Equation (4.11) determines the fluctuating components of t and z in the interaction region.

The equation of motion now takes a much simpler form. Expressing $\mathbf{B}$ and $\mathbf{E}$ in terms of $\xi^\mu=\xi_+^\mu+\xi_-^\mu+\xi_I^\mu$, and making use of the constraints (4.8) and (4.9), we obtain

$$(\partial_t^2-\partial_z^2)\boldsymbol{\xi}_I-\boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_I)=2\boldsymbol{\nabla}_\perp(\xi'_+\cdot\xi'_-). \tag{4.12}$$

This demonstrates that the physical components of ξ_I are purely transverse. Without loss of generality, we may express $\boldsymbol{\xi}_I$ as the sum of irrotational and solenoidal vector fields, i.e.,

$$\boldsymbol{\xi}_I=\boldsymbol{\nabla}_\perp\times(\psi\hat{\mathbf{z}})+\boldsymbol{\nabla}_\perp\chi\equiv\boldsymbol{\xi}_I(A)+\boldsymbol{\xi}_I(F). \tag{4.13}$$

Here ψ and χ are scalar fields, which we have respectively identified with the A and F modes produced in the interaction. Equation (4.12) now becomes

$$(\partial_t^2-\partial_z^2)\boldsymbol{\xi}_I(A)+(\partial_t^2-\nabla^2)\boldsymbol{\xi}_I(F)=2\boldsymbol{\nabla}_\perp(\xi'_+\cdot\xi'_-). \tag{4.14}$$

Operating $\nabla_\perp x$ on both sides of Eq. (4.14) shows that only the F mode is sourced, $A^+ + A^- \leftrightarrow F$. The main advantage of this formalism is that it allows us to separate the gauge-dependent longitudinal components of ξ that are excited during the interaction, from the physical transverse wave ξ_I that survives asymptotically.

Now let us repeat this calculation in the axionic formulation of MHD. To begin, it is useful to rederive the sheared A modes and evaluate θ explicitly. We look for a single fourier mode $\delta\mathbf{B}$, $\delta\mathbf{E}\propto e^{-ik_\mu x^\mu}$. The gradient of θ can be decomposed as

$$\partial^\mu\theta=iK^\mu(k)+ik^\mu\delta\theta_0 e^{-ik_\mu x^\mu}, \tag{4.15}$$

where the first (spatially constant) term depends only on (some components of) k^μ. The absence of a constant current or charge density implies $K^0=K^z=0$. Then one has

$$\frac{j^0}{e^2}=\boldsymbol{\nabla}\theta\cdot\mathbf{B}=i\mathbf{K}\cdot\delta\mathbf{B}+ik_z\delta\theta B_0, \tag{4.16}$$

and

$$\frac{\mathbf{j}}{e^2}=-(\dot{\theta}\mathbf{B}+\boldsymbol{\nabla}\theta\times\mathbf{E})=-i\mathbf{K}_\perp\times\mathbf{E}+i\omega\delta\theta\mathbf{B}_0, \tag{4.17}$$

where we define $\mathbf{K}_\perp=\mathbf{K}-K_z\hat{\mathbf{z}}$ and make use of the constraints $\omega\delta\mathbf{B}-\mathbf{k}\times\delta\mathbf{B}=0$ and $\mathbf{k}\cdot\delta\mathbf{B}=0$.

This shows that j^μ is precisely first order in $\delta\mathbf{B}$. Furthermore, the induction equation and the MHD condition together guarantee that $\mathbf{E}\cdot\mathbf{B}_0=0$, and so $\mathbf{j}$ is parallel to $\mathbf{B}_0$. (We could have guessed this at the beginning, since the current carrying Fermi zero modes propagate parallel to the unperturbed magnetic field; but the result is more general.) Finally, dividing $\mathbf{k}$ into components perpendicular and parallel to $\delta\mathbf{E}$, and making use of the Maxwell equation

$$\mathbf{j}=i(\mathbf{k}_{\perp,\delta E}+\mathbf{k}_{\parallel,\delta E})\times\delta\mathbf{B}+i\omega\delta\mathbf{E}=i\mathbf{k}_{\parallel,\delta E}\times\delta\mathbf{B}, \tag{4.18}$$

one sees that the Poynting flux $\frac{1}{2}\delta\mathbf{E}\times\delta\mathbf{B}$ lies parallel to $\mathbf{B}_0$. To obtain the sheared A modes, it is sufficient to set $\delta\theta=0$ and take

$$\mathbf{K}_\pm=(\mathbf{K}_\pm)_\perp=\pm(\hat{\mathbf{z}}\times\mathbf{k}_\pm). \tag{4.19}$$

Now consider the interactions between two A modes $\{\delta\mathbf{B}_+,\mathbf{K}_+\}$ and $\{\delta\mathbf{B}_-,\mathbf{K}_-\}$. One observes new interaction terms in the expression for the charge density

$$\frac{\Delta j^0}{e^2}=i\mathbf{K}_+\cdot\delta\mathbf{B}_-+i\mathbf{K}_-\cdot\delta\mathbf{B}_+. \tag{4.20}$$

In order to cancel these terms (and the corresponding terms in j^z) it is necessary to introduce a fluctuating longitudinal component of $\partial^\mu\theta$,

$$\partial_t\theta_l=\frac{i}{B_0}\hat{\mathbf{z}}\cdot[\mathbf{k}_\perp^+\times\delta\mathbf{B}_-+\mathbf{k}_\perp^-\times\delta\mathbf{B}_+];$$

$$\partial_z\theta_l=\frac{i}{B_0}\hat{\mathbf{z}}\cdot[\mathbf{k}_\perp^+\times\delta\mathbf{B}_--\mathbf{k}_\perp^-\times\delta\mathbf{B}_+]. \tag{4.21}$$

A new longitudinal electric-field enforces the MHD condition

$$\delta\mathbf{E}_l=\frac{2}{B_0}[\delta\mathbf{B}_+\times\delta\mathbf{B}_-]. \tag{4.22}$$

Substituting $K^\mu=K_+^\mu+K_-^\mu$, $\partial^\mu\theta=\partial^\mu\theta_l$ and $\mathbf{E}=\delta\mathbf{E}_+ +\delta\mathbf{E}_-+\delta\mathbf{E}_l$ into Eq. (4.17), and then taking $\hat{\mathbf{z}}\times$, leads to the following time evolution equation for the transverse field components:

$$\partial_z(\delta\mathbf{B}_I)_\perp-\boldsymbol{\nabla}_\perp\delta B_{Iz}+\hat{\mathbf{z}}\times\partial_t\delta\mathbf{E}_I=\frac{2}{B_0}[\boldsymbol{\nabla}_\perp(\delta\mathbf{B}_+\cdot\delta\mathbf{B}_-)$$
$$-(\delta\mathbf{B}_-\cdot\boldsymbol{\nabla})\delta\mathbf{B}_+-(\delta\mathbf{B}_+\cdot\boldsymbol{\nabla})\delta\mathbf{B}_-]. \tag{4.23}$$

This is equivalent to Eq. (4.5).

V. OTHER WAVE INTERACTIONS

There is, in addition to the three-mode interaction between two A modes analyzed in the last section, a three-mode interaction between F modes,

$$F(1)+F(2)\leftrightarrow F(3). \tag{5.1}$$

This is easier to calculate, since none of the participating modes supports an electric current. The incident waves are

$$\delta\mathbf{B}_{1,2}=B_0\cos\,\theta_{1,2}\boldsymbol{\xi}'_{1,2};$$

$$\delta\mathbf{E}_{1,2}=\frac{1}{\cos\,\theta_{1,2}}\delta\mathbf{B}_{1,2}\times\hat{\mathbf{z}}=B_0(\boldsymbol{\xi}'_{1,2}\times\hat{\mathbf{z}}), \tag{5.2}$$

where θ is the angle between the wave propagation direction and the background magnetic field, and $\boldsymbol{\xi}=\boldsymbol{\xi}(\hat{\mathbf{k}}\cdot\mathbf{x}-t)$. The MHD condition is satisfied in the zone where the waves overlap only if a third, interaction component to the electric field,

$$\delta\mathbf{E}_I=-\frac{\delta\mathbf{B}_1\times\delta\mathbf{B}_2}{B_0\cos\,\theta_1\cos\,\theta_2}(\cos\,\theta_1-\cos\,\theta_2), \tag{5.3}$$

is present. The equation of motion (2.20) for the new fast mode then becomes

$$B_0[(\nabla\times\delta\mathbf{B}_3)\times\hat{\mathbf{z}}-\partial_t\delta\mathbf{E}_3\times\hat{\mathbf{z}}]$$
$$=\frac{\cos\,\theta_2-\cos\,\theta_1}{\cos\,\theta_1\cos\,\theta_2}\partial_t(\delta\mathbf{B}_1\times\delta\mathbf{B}_2)\times\hat{\mathbf{z}}. \tag{5.4}$$

As with two colliding Alfvén waves, the gauge $J_4=1$ can be imposed by shifting the displacement functions in a manner analogous to Eq. (4.11),

$$\delta\xi_1=-\frac{k_2\cdot\xi_1}{k_1\cdot k_2}k_1;\quad \delta\xi_2=-\frac{k_1\cdot\xi_2}{k_2\cdot k_1}k_2. \tag{5.5}$$

(This gives $k_1 \cdot \delta\xi_1 = k_2 \cdot \delta\xi_2 = 0$ for F modes and charge-free A modes with $k^2=0$.) The equation of motion (5.4) then becomes in Lagrangian coordinates

$$(\partial_t^2 - \partial_z^2)\boldsymbol{\xi}_3 - \boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}_\perp \cdot \boldsymbol{\xi}_3)$$

$$= (\cos\,\theta_2 - \cos\,\theta_1)\partial_t[\boldsymbol{\xi}_2'(\xi_1')^z - \boldsymbol{\xi}_1'(\xi_2')^z], \qquad (5.6)$$

to second order in $\xi_{1,2}$. The right side of Eq. (5.6) vanishes when $\theta_1 = \theta_2$ and the waves propagate at the same angle with respect to $\mathbf{B}_0$ (whether or not $\mathbf{k}_1$ and $\mathbf{k}_2$ are in fact aligned).

This classical process is the magnetohydrodynamic analogue of the QED process of photon splitting [19]. There are, nonetheless, a number of distinctions with photon splitting. First, the spontaneous rate for a F mode to split into two F modes vanishes, as the F mode is an exact, nonlinear solution to the MHD equations. Second, the polarization selection rules are different: whereas all the participating F modes have electric vector perpendicular to $\mathbf{B}_0$, the allowed photon splitting channels in a vacuum are $\perp \leftrightarrow \perp + \parallel$ and $\perp \leftrightarrow \parallel + \parallel$. In a dense plasma, the selection rules are obtained by interchanging $\perp$ and $\parallel$.[7] (Here, $\perp$ and $\parallel$ denote the orientation of $\delta\mathbf{B}$ with respect to $\hat{\mathbf{B}}_0 \times \hat{\mathbf{k}}$.)

When considering all possible interactions between the A modes and F modes, it is useful to simplify the higher order terms in the Lagrangian by imposing the constraint of four-dimensional incompressibility. We follow the Lagrangian formalism developed in Sec. III. Although the general form of the action (2.15) is fairly complicated due to the presence of the Jacobian factor J_4, we have seen that, in practice, it is possible to choose a time-slicing corresponding to $J_4 = 1$. In this gauge,

$$S' = \int d^4x_0[(\partial_\tau\boldsymbol{\xi}\times\hat{\mathbf{z}})^2 - (\partial_{z_0}\boldsymbol{\xi})^2 - 2(\partial_\tau\xi^t\partial_{z_0}\xi^z - \partial_\tau\xi^z\partial_{z_0}\xi^t)$$
$$+ 2(\partial_\tau\boldsymbol{\xi}\times\hat{\mathbf{z}})\cdot(\partial_\tau\boldsymbol{\xi}\times\partial_{z_0}\boldsymbol{\xi}) - 2\partial_\tau\xi^t(\partial_{z_0}\boldsymbol{\xi})^2$$
$$+ 2\partial_{z_0}\xi^t(\partial_\tau\boldsymbol{\xi}\cdot\partial_{z_0}\boldsymbol{\xi}) + (\partial_\tau\boldsymbol{\xi}\times\partial_{z_0}\boldsymbol{\xi})^2$$
$$- (\partial_\tau\xi^t\partial_{z_0}\boldsymbol{\xi} - \partial_{z_0}\xi^t\partial_\tau\boldsymbol{\xi})^2]. \qquad (5.7)$$

Note that the assumption of four-dimensional incompressibility implies that $\xi^{t,z} \neq 0$ in a region where two or more A waves are interacting (Sec. IV). The correct interactions are obtained by substituting into Eq. (5.7) the modified wave displacement functions (4.11). It should be emphasized that the resulting Lagrangian is exact to fourth order in $\boldsymbol{\xi}$, because the gauge choice (4.11) is precisely equivalent to $J_4 = 1$.

The various mode interactions can now easily be read off from the action. The third order terms represent the three-mode couplings (5.1) and

$$A^+ + A^- \leftrightarrow F; \qquad (5.8)$$

and the fourth order terms represent the two four-mode couplings

$$A^+(1) + A^-(1) \leftrightarrow A^+(2) + A^-(2), \qquad (5.9)$$

and

$$A^+(1) + A^-(1) \leftrightarrow F(1) + F(2). \qquad (5.10)$$

The three-mode couplings (5.8) and (5.1) are nonvanishing because the F mode has $\xi^z \neq 0$ (except for the degenerate case $\mathbf{k} \| \mathbf{B}_0$).

The corresponding conservation equations for the longitudinal components of energy-momentum are

$$\omega_+ + \omega_- = \omega_F, \quad k_+^z + k_-^z = \omega_+ - \omega_- = k_F^z \qquad (5.11)$$

for reaction (5.8)

$$\omega_+(1) + \omega_-(1) = \omega_+(2) + \omega_-(2);$$
$$k_+^z(1) + k_-^z(1) = \omega_+(1) - \omega_-(1) = k_+^z(2) + k_-^z(2)$$
$$= \omega_+(2) - \omega_-(2) \qquad (5.12)$$

for reaction (5.9), and similarly for reaction (5.10)

The third order terms in S' require further discussion. We might also consider the following three-mode interaction (cf. [5]):

$$A^+(1) + A^-(1) \leftrightarrow A^+(2) + A^-(1), \qquad (5.13)$$

in which only one of the Alfvén wave packets is distorted during the collision. Unlike the three-mode coupling (5.8) involving the F mode, this reaction is not resonant unless $A^-(1)$ contains a zero-frequency component $\omega_-(1) = 0$ [2–4]. The third order terms in Eq. (5.7) are proportional to

$$\exp\{i[k^+(1) + k^-(1) - k^+(2)]\cdot x\} \qquad (5.14)$$

and vanish in the spatial integration unless the resonance condition is satisfied.

The F mode is absent in an incompressible, nonrelativistic magnetofluid, as are dynamical three-mode couplings involving Alfvén waves in the Lagrangian description [3]. Dynamical three-mode couplings emerge in the nonrelativistic case when the assumption of incompressibility is relaxed. Furthermore, in the nonrelativistic Lagrangian formalism of [3] the longitudinal component of $\boldsymbol{\xi}$ excited during a collision between A modes 1, 2 has a wave vector $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and is expressed indirectly in terms of the initial and final transverse wave displacements through the incompressibility condition $J_3 = 1$. By contrast, in the relativistic case the additional freedom of choosing a time slicing allows one to pick longitudinal components separately for the two incident waves, with the result that the propagating interaction component $\boldsymbol{\xi}_I$ is purely transverse.

[7] Photon splitting is, however, of limited importance in this second, plasma-dominated regime: if the magnetic field is strong enough to induce a significant splitting rate ($B > 10^{12}$ G) then the dielectric tensor is dominated by the plasma (rather than by vacuum polarization) only at very high particle densities, so high that photons lose energy predominantly via the Compton recoil rather than by splitting [20].

To demonstrate that the simplified Lagrangian (5.7) does give the correct dynamics, we consider once again the collision between two A modes. Substituting Eq. (4.11) for $\partial_{\tau,z_0}\xi^{t,z}$ and expressing the derivatives ∂_τ and ∂_{z_0} in terms of light-cone derivatives $\partial_\pm$, the third order terms in Eq. (5.7) become

$$S_3' = -4\int d^4x_0\{\partial_+\boldsymbol{\xi}\cdot[(\partial_-\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}]+\partial_-\boldsymbol{\xi}\cdot[(\partial_+\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}]\}. \tag{5.15}$$

As for the second order terms in S' (whose variation gives rise to the kinetic terms for the interaction component $\boldsymbol{\xi}_I$), one can self-consistently choose the gauge $\xi_I^0=0$, since ξ_I^μ is the linear superposition of modes that asymptotically are A modes and F modes. In this gauge, $\partial_{z_0}\xi_I^3=-\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_{I,\perp}$, and the second order terms become

$$S_2' = -\int d^4x_0[4\partial_+\boldsymbol{\xi}_{I,\perp}\cdot\partial_-\boldsymbol{\xi}_{I,\perp}+(\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_{I,\perp})^2]. \tag{5.16}$$

Varying $S_2'+S_3'$ with respect to $\boldsymbol{\xi}_{I,\perp}$, one regains the equation of motion (4.12) for $\boldsymbol{\xi}_I$.

VI. SHEARED ALFVÉN CASCADE: SCALING

Now let us consider an ensemble of Alfvén waves in a box. The waves are injected at some outer scale and develop a range of wave numbers. In general, the shape of the power spectrum at a given wave number depends on whether the coupling between two Alfvén waves is *weak* or *strong*—that is, whether the fractional distortion suffered by a wave packet is small or of the order of unity. A similar effect occurs in a nonrelativistic MHD fluid, as has been analyzed in [3,6]. When the coupling is weak, conservation of energy and longitudinal momentum guarantee that each wave individually conserves ω and k_z during a collision [21,3], so collisions cause only $k_\perp$ to increase. In both weak and strong coupling regimes, the cascade is anisotropic with wave packets becoming increasingly concentrated in the dimensions perpendicular to $\mathbf{B}_0$. Previous attempts to derive the power spectrum of nonrelativistic MHD turbulence [22,23] did not properly account for the anisotropy of the cascade.

Collisions between A modes generate F modes as well as higher wave number A modes. This effect cannot be neglected in the relativistic regime, as it can in the nonrelativistic regime by imposing the assumption of incompressibility. Nonetheless, the three-mode coupling between two colliding F modes calculated at the beginning of Sec. V is smaller in amplitude by a factor $k_\parallel/k_\perp$ than the three-mode coupling between A modes calculated in Sec. IV. This means that in the large shear regime, the cascade proceeds almost entirely via collisions between A modes. The spectral index of the A modes (and F modeswith $k_F\sim k_{z,A}$) can be obtained by considering the A modes in isolation. Note, finally, that Alfvén wave emission by vibrating neutron stars is expected to be in the large shear regime, with $k_\perp/k_\parallel\sim c/v_\mu\sim 300$. Here, $v_\mu\sim 10^8$ cm s^{-1} is the characteristic shear-wave velocity in the crust [24].

The scaling we derive in the presence of three-mode couplings differs from that obtained by Ng and Bhattacharjee [5] and Goldreich and Sridhar [6], due in part to our different treatment of boundary conditions. The simplest way to excite A modes in the box is to move its boundaries. This implies that energy can be transported into the box from the medium outside. Self-consistency demands that energy also be transported out of the box; i.e., that the reflection coefficient R of waves incident on the boundary is less than unity. We assume for simplicity that $R\sim\frac{1}{2}$ and that the energy flow into the box balances the flow out of the box.[8] However, the time-averaged positions of the magnetic footpoints remain fixed, except for an initial transverse shift. An explicit realization of this model is a parallel series of boxes separated by partially reflecting membranes. By contrast, Goldreich and Sridhar [6] assume that the magnetic field lines are pinned rigidly at the walls of the box, with the turbulence being excited by internal body forces. This boundary condition forces 3-mode couplings (both dynamical and kinematic) to vanish near the outer scale, and in turn reduces the overall cascade rate.

We assume that the turbulence is injected with some characteristic wavelength $2\pi/k_z$ smaller by a factor N than the size L of the box. When the boundaries move, this is directly related to the characteristic frequency of the motions. The k_z power spectrum is then cutoff below a wave number $2\pi N/L$ in the absence of a long term, secular displacement of the magnetic footpoints. By contrast, the transverse positions of the field lines are assumed in [6] to undergo a random walk along the background magnetic field, with this random walk being forced to vanish at the boundaries. This leads to a much broader power spectrum that is cut off only at $k_z=2\pi/L$. These differing assumptions about the excitation mechanism lead to significant differences in the cascade scaling, because the k_z power spectrum is conserved during weak collisions between Alfvén waves. Redistribution of power occurs neither during the dynamic interactions (which preserve k_z as long as the coupling $k_\perp\xi$ is weak), nor during the kinematic interactions (which preserve the k_z power spectrum even while modifying the $k_\perp$ power spectrum). Such redistribution would occur if the sides of the box underwent some sort of gradual plastic creep that caused a secular displacement of the field lines. In the application to strong-B neutron stars, this is a second-order effect which we neglect.

A. Strength of the mode coupling

The shape of the power spectrum depends on the *order* of the interaction between colliding Alfvén wave packets $\xi_+(z_-,\mathbf{x}_\perp)$ and $\xi_-(z_+,\mathbf{x}_\perp)$, as well as the strength of the coupling $k_\perp\xi$. This is best seen by integrating the time evolution equation (4.14) for $\boldsymbol{\xi}_I$ over $z_+=z+t$ at fixed $z_-=z-t$. In the temporal gauge (4.8), (4.9) only the dyanamical interactions between A modes survive. The change in ξ_+ integrated over the collision is equal to the asymptotic value of ξ_I at z and t large compared to the collision coordinates

[8] As is appropriate for the boundary between the magnetosphere and surface of a neutron star with a very strong magnetic field $B\sim 10^{14}-10^{15}$ G [24].

$$\delta\boldsymbol{\xi}'_+(z_-,\mathbf{x}_\perp)=\frac{1}{2}\,(\partial_z-\partial_t)\boldsymbol{\xi}_I(z,t,\mathbf{x}_\perp)\quad (z,t\to\infty), \tag{6.1}$$

namely,

$$\delta\boldsymbol{\xi}'_+(z_-,\mathbf{x}_\perp) = -\frac{1}{2}\int dz_+\boldsymbol{\nabla}_\perp[\partial_-\xi_+(z_-,\mathbf{x}_\perp)\cdot\partial_+\xi_-(z_+,\mathbf{x}_\perp)] = -\frac{1}{2}\,\boldsymbol{\nabla}_\perp[\partial_-\xi_+(z_-,\mathbf{x}_\perp)\cdot\Delta\xi_-(\mathbf{x}_\perp)]. \tag{6.2}$$

[Following the discussion in Sec. IV, this new mode can be regarded as a current-free A mode $(k_\perp=0)$, or as an F mode propagating along $\mathbf{B}_0$.] This depends on the transverse displacement of the background magnetic field lines across the wave packet

$$\Delta\boldsymbol{\xi}_-(\mathbf{x}_\perp)=\int_{-\infty}^{\infty}dz_+\partial_+\boldsymbol{\xi}_-(z_+,\mathbf{x}_\perp) \tag{6.3}$$

in a similar manner to the nonrelativistic case [5].

In sum, a left-moving wave $\boldsymbol{\xi}_+$ suffers a fractional distortion

$$\frac{|\delta\boldsymbol{\xi}'_+|}{|\boldsymbol{\xi}'_+|}\sim|\mathbf{k}_\perp^+ +\mathbf{k}_\perp^-|\Delta\xi_- \tag{6.4}$$

when $\Delta\boldsymbol{\xi}_-\neq 0$ and the coupling is weak. This corresponds to the three-mode reaction $A^+(1)+A^-(1)\to A^+(2)+A^-(1)$. Defining the dimensionless coupling parameter

$$\zeta_-\equiv|\mathbf{k}_\perp^+ +\mathbf{k}_\perp^-||\boldsymbol{\xi}_-|\sim\left(\frac{|\mathbf{k}_\perp^+ +\mathbf{k}_\perp^-|}{|k_z^-|}\right)\left(\frac{\delta B_-}{B_0}\right), \tag{6.5}$$

the condition for weak coupling can be written as

$$\frac{|\delta\xi'_+|}{|\xi'_+|}\sim\zeta_-\,\frac{|\Delta\xi_-|}{\xi_-}. \tag{6.6}$$

The distortion of the right-moving wave takes a similar form. When $\Delta\xi_\pm$ is small, the four-mode coupling $A^+(1)+A^-(1)\to A^+(2)+A^-(2)$ dominates, and

$$\frac{|\delta\xi'_+|}{|\xi'_+|}\sim\zeta_-^2. \tag{6.7}$$

This result may be obtained by substituting $\Delta\boldsymbol{\xi}'_+\sim(\mathbf{k}_\perp^+ +\mathbf{k}_\perp^-)(\xi'_+\cdot\xi_-)$ into the right side of (4.12).

Now let us estimate the net distortion suffered by an Alfvén wave packet as it crosses a box of size L. The three-mode distortion (6.6) depends linearly on the net transverse shift $\Delta\xi(L,\mathbf{k}_\perp)$ of the magnetic field lines across the box (which we fourier decompose in the transverse dimensions). This component of the distortion therefore depends crucially on the boundary conditions at the edge of the box. It does not vanish if the walls of the box transmit wave energy, whereas it does vanish for the rigid boundary conditions assumed in [6].

With our boundary conditions, the net transverse shift of the field lines across the box is comparable in amplitude to the instantaneous displacement at any point in the box

$$\Delta\xi(L,k_\perp)\sim\xi(k_\perp). \tag{6.8}$$

Here, $\xi(k_\perp)\sim k_z^{-1}\delta B(k_\perp)/B_0$. The net three-mode distortion of the wavepacket as it crosses the box is then

$$\frac{\delta\xi'}{\xi'}\sim k_\perp\xi. \tag{6.9}$$

The three-mode distortions are substantially uncorrelated between successive crossings of the box, because the shape of the wave packet is modified at each reflection. In a steady state, energy transmitted out of the box at wave number $k_\perp$ is replenished by the injection of equal energy at the same wave number. The wave $\delta\xi$ then grows as a random walk, and the damping time is

$$\frac{t_{\rm damp}}{L}\sim(k_\perp\xi)^{-2}. \tag{6.10}$$

This compares with the four-mode distortion, which is $\sim[(k_\perp/k_z)(\delta B/B_0)]^2\sim(k_\perp\xi)^2$ over one (parallel) wavelength and accumulates in a random walk of

$$N=\frac{k_zL}{2\pi} \tag{6.11}$$

collisions across the box,

$$\left(\frac{\delta\xi'}{\xi'}\right)^2\sim N(k_\perp\xi)^4. \tag{6.12}$$

The corresponding damping time is

$$\frac{t_{\rm damp}}{L}\sim\frac{1}{N(k_\perp\xi)^4}. \tag{6.13}$$

B. Cascade scaling

Notice that the three-mode coupling dominates in strength at the outer scale, unless the waves are strongly coupled ($\zeta\equiv k_\perp\xi\sim 1$) or N is very large. Assuming that this is not the case, the constancy of the energy flux (per unit volume and time) implies that

$$\frac{(k_\perp\xi)^2}{N}\,(\delta B)^2k_z\sim\text{const}, \tag{6.14}$$

which in turn implies

$$\frac{\xi(k_\perp)}{\xi_0}\sim\frac{\Delta\xi(k_\perp)}{\Delta\xi_0}\sim\left(\frac{k_\perp}{k_{\perp 0}}\right)^{-1/2} \tag{6.15}$$

while the cascade is weakly coupled and k_z, $N\sim$const. (Here the subscript 0 refers to the outer scale.) The three-mode and four-mode couplings become comparable in strength where $(k_\perp\xi)^2\sim N^{-1}$, which corresponds to

$$\frac{k_\perp^\star}{k_{\perp 0}}\sim N^{-1}\zeta_0^{-2}>1. \tag{6.16}$$

Here $\zeta_0 \equiv k_{\perp 0}\xi_0$ is the coupling parameter at the outer scale. At wave number $k_\perp^\star$, the four-mode coupling is still weak,

$$[k_\perp^\star \xi(k_\perp^\star)]^2 \sim N^{-1}. \tag{6.17}$$

At higher wave numbers, this weak cascade steepens into one dominated by the dynamic four-mode couplings, because the cumulative effect of those couplings grows as the waves transit the box.[9] Constant energy flux now implies $(k_\perp \xi)^4 (\delta B)^2 k_z \sim \text{const}$, and we recover the scaling derived in [3] for nonrelativistic Alfvén turbulence

$$\frac{\xi(k_\perp)}{\xi(k_\perp^\star)} \sim \left(\frac{k_\perp}{k_\perp^\star}\right)^{-2/3}. \tag{6.18}$$

This can be rewritten, using Eq. (6.16), as

$$\frac{\xi(k_\perp)}{\xi_0} \sim N^{-1/6} \zeta_0^{-1/3} \left(\frac{k_\perp}{k_{\perp 0}}\right)^{-2/3}. \tag{6.19}$$

We therefore have $\zeta \propto k_\perp^{1/3}$ and the cascade must eventually become strongly turbulent $(\zeta \to 1)$.

At still higher wave numbers,

$$k_\perp > k_\perp^{\star\star} = k_{\perp 0} \zeta_0^{-2} N^{1/2}, \tag{6.20}$$

we will assume that the strong cascade is critically balanced with the cascade time being of order the wave period and $\zeta = 1$ (see [25] for a detailed justification of this in the nonrelativistic case). Again assuming a constant rate of energy cascade, we find that

$$\frac{\xi(k_\perp)}{\xi(k_\perp^{\star\star})} \sim \left(\frac{k_\perp}{k_\perp^{\star\star}}\right)^{-1} \tag{6.21}$$

and

$$\frac{\omega}{\omega_0} = \frac{k_z}{k_{z0}} = \frac{\zeta_0^{4/3}}{N^{1/3}} \left(\frac{k_\perp}{k_{\perp 0}}\right)^{2/3}. \tag{6.22}$$

The corresponding scaling of the magnetic perturbation is

$$\frac{\delta B}{B} = \frac{\zeta_0^{1/3}}{N^{1/3}} \frac{\delta B_0}{B} \left(\frac{k_\perp}{k_{\perp 0}}\right)^{-1/3}. \tag{6.23}$$

Notice that

$$\frac{k_\perp^{\star\star}}{k_\perp^\star} \sim N^{3/2}. \tag{6.24}$$

We can summarize as follows. When the three-mode coupling between colliding Alfvén waves discovered by Ng and Bhattacharjee [5] is taken into account, the cascade divides into three regimes: an outer, weakly coupled cascade driven by dynamical three-mode interactions [with scaling (6.15) at $k_{\perp 0} < k_\perp < k_\perp^\star$]; an intermediate, weakly coupled cascade driven by dynamical four-mode interactions [with scaling (6.18) at $k_\perp^\star < k_\perp < k_\perp^{\star\star}$]; and an inner, strongly coupled cascade [with scaling (6.21)–(6.23) at $k_\perp > k_\perp^{\star\star}$]. When $N \sim 1$ the outer weak cascade blends directly into the inner strong cascade.

We emphasize that these results apply also to *nonrelativistic* magnetofluids. In their analysis of the nonrelativistic incompressible case, Goldreich and Sridhar [6] have also found three cascade regimes, but with dynamical 4-mode couplings dominating at low wave number, and kinematic 3-mode (and higher) couplings dominating at intermediate wave number. The basic reason for these differing conclusions lies in the treatment of boundary conditions and the assumed k_z power spectrum, as discussed above.

C. Damping time

These results have important implications for the damping rate of the wave turbulence at the outer scale. Let us suppose that waves are suddenly excited (for example, when the crust of a neutron star fractures, sending an Alfvén pulse in the star's magnetosphere). Then the damping time at the outer scale is

$$\frac{t_{\rm damp}}{L} \sim \min\left[\frac{1}{(k_\perp \xi)^2}, \frac{1}{N(k_\perp \xi)^4}\right]. \tag{6.25}$$

In the presence only of four-mode couplings, $t_{\rm damp}$ increases rapidly with decreasing wave amplitude (or shear); whereas the dependence is much weaker when the three-mode coupling is properly included.

VII. APPLICATION TO ASTROPHYSICAL X-RAY SOURCES

Strongly magnetized neutron stars and accreting black holes are plausible astrophysical sources of relativistic (or mildly relativistic) Alfvén turbulence. For example, a large pulse of Alfvén radiation is emitted when the rigid crust of a neutron star fractures, or when a magnetic flux tube rises buoyantly out of an accretion disk close to the last stable orbit surrounding a black hole. In such a situation, the rate of turbulent energy release L_w can be very high, so that the corresponding compactness within a radius ℓ

$$\ell_w = \frac{L_w \sigma_T}{4\pi m_e c^3 \ell} \tag{7.1}$$

[9] P. Goldreich (private communication) has noted that one should in principle include kinematic couplings between waves that arise due to mixing between the Lagrangian and Eulerian positions of the field lines [6]. At fourth order, these kinematic couplings would be larger in magnitude than the resonant four-mode couplings considered here (by a factor $N^{1/2}$) if the positions of the field lines were free to random walk across the box. However, as discussed in the introduction to this section, this possibility is eliminated by our assumption that the *time-averaged* positions of the field lines remain fixed at each side of the box (following an initial Alfvén impulse).

exceeds unity.[10]

This has three important consequences. First, the charges needed to support the fluctuating magnetic field at the outer scale themselves generate a surprisingly large scattering depth; second, the turbulence can be damped effectively by Compton scattering of the ambient radiation field (''Compton drag''); and, third, a turbulent cascade raises the damping rate even further by increasing the minimum charge density needed to support the fluctuating magnetic field.

This minimum charge density can readily be converted to a Thomson scattering optical depth

$$\tau_T = \sigma_T n_e \ell \sim \frac{\sigma_T \delta B(k_\perp \ell)}{4\pi e}. \qquad (7.2)$$

If the wave energy is sufficient to power a soft γ repeater (SGR) burst (energy $E_B \gtrsim 10^{41}$ erg) then

$$\tau_T \sim 0.06\left(\frac{k_\perp R}{300}\right)\left(\frac{E_B}{10^{41}\ \mathrm{erg}}\right)^{1/2}\left(\frac{\ell}{10\ \mathrm{km}}\right)^{-3/2}. \qquad (7.3)$$

Here we relate δB to the total wave energy E_B and confinement volume ℓ^3 and estimate $k_\perp/k_z \sim c/c_s$, where $c_s \sim 10^8\ \mathrm{cm\,s^{-1}}$ is the shear wave velocity deep in the crust.

Much larger scattering depths are generated by the cascade, since the current density scales with wave number as

$$j \sim k_\perp \delta B \propto k_\perp^{1/2}; k_\perp^{1/3}; k_\perp^{2/3}, \qquad (7.4)$$

in the outer (three-mode dominated) weak cascade (6.15); the intermediate (four-mode dominated) weak cascade (6.18); and the strong cascade (6.23), respectively. The inner scale of the cascade lies at a wave number $k_z(i)$ where the cascade is strongly coupled, under a wide range of conditions. This implies that $k_\perp \delta B \sim k_z B$ and, assuming that the waves are marginally charge starved at the inner scale, the scattering depth is related in a simple manner to $k_z(i)$,

$$k_z(i)\ell \sim \frac{3}{2}\,\frac{\tau_T(\ell)}{\alpha_{em}}\left(\frac{B}{B_{\mathrm{QED}}}\right)^{-1}. \qquad (7.5)$$

Here α_{em} is the fine-structure constant and $B_{\mathrm{QED}} = m_e^2 c^3/e\hbar = 4.4\times 10^{13}$ G. This has possible applications not only to SGR's but also to stellar-mass BH coronae. At this wave number, resonant wave-particle interactions are absent as long as

$$\frac{k_z(i)}{eB/m_e c^2} = \frac{3}{2}\left(\frac{\ell}{\ell_{m_e}}\right)^{-1}\frac{\tau_T}{\alpha_{em}}\left(\frac{B}{B_{\mathrm{QED}}}\right)^{-2} \ll 1. \qquad (7.6)$$

Here $\ell_{m_e} = \hbar/m_e c = 3.8\times 10^{-11}$ cm is the Compton wavelength of the electron. This condition is easily satified in the magnetosphere of a neutron star, and is marginally satisfied in the magnetic corona of a black hole accretion disk.

[10]This quantity equals the optical depth to electron scattering along a radial path, in the case where the plasma is composed of mildly relativistic electrons (and positrons). In this section we retain factors of c for clarity.

There are, in sum, three principal modes of MHD wave damping in highly compact astrophysical sources: first, adiabatic expansion (e.g., leakage onto open magnetic field lines); second, Compton drag at low τ_T in a dense photon field; and, third, a nonlinear cascade to high wave number. When the cascade dominates, the resulting radiative signature depends strongly on the scattering depth generated self-consistently by the waves.

A. Wave damping in an optically thick plasma: application to cosmological γ-ray bursts

First let us consider wave damping in a plasma that is very optically thick to scattering. What dissipative process determines the inner wave number $k_z(i)$ in this regime? On the outer scale l, the photon and electron fluids are effectively coupled by Compton scattering, but at sufficiently high wave number the scattering depth $\tau_T(\xi)$ across the wave displacement ξ becomes small enough that the moving electron fluid suffers significant drag. Since almost all the inertia is carried by the background magnetic field, the time scale for Compton drag is obtained by dividing the drag force per unit volume $\frac{4}{3}\sigma_T n_e (v_e/c)^2 U_\gamma c$ into the energy density of the waves

$$t_C = \frac{(\delta B)^2}{(4/3)\sigma_T n_e (v_e/c)^2 U_\gamma c}. \qquad (7.7)$$

Here, $(v_e/c)^2 = (\delta B/B)^2$ is the mean square oscillatory speed of the electrons. Comparing with the cascade rate k_z of strongly coupled Alfvén waves, this is

$$c t_C k_z = \frac{3}{2}\left(\frac{B^2/2}{U_\gamma}\right)\frac{(\delta B/B)_{k_z}}{\tau_T[\xi(k_z)]}. \qquad (7.8)$$

Compton drag begins to be effective at a wave number where the photon diffusion time $\sim[\tau_T(\xi)]\xi$ across the wave displacement ξ becomes less than k_z^{-1}. This corresponds to $[\tau_T(\xi)]^{-1} \sim (\delta B/B)_{k_z}$ and

$$c t_C k_z \sim \frac{(\delta B)^2_{k_z}}{U_\gamma} \qquad (7.9)$$

at the appropriate wave number. Notice that $\delta B_{k_z}/\delta B_0 \sim N^{-1/2}\zeta_0^2 (k_z/k_{z0})^{-1/2} \ll 1$ [from Eqs. (6.22) and (6.23)]. We conclude that the cascade is cutoff effectively by Compton drag when the photon density has increased to a fraction

$$\frac{U_\gamma}{(\delta B_0)^2} \sim \frac{(k_{\perp 0}\xi_0)^2}{N} \qquad (7.10)$$

of the MHD wave pressure at the outer scale. It is only during the earliest phase of such an optically thick cascade (or at low optical depths in the presence of a weak background photon source) that the cascade continues to a high wave number where the waves becomes charge starved.

This process should provide effective heating at extremely high scattering depths. For example, a cosmological γ-ray burst sources involve energy release at a rate $L_\gamma \sim 10^{51}\ \mathrm{erg\,s^{-1}}$, which in the most plausible models occurs within a radius less than $\sim$10–100 km.

The corresponding compactness is enormous $\ell_\gamma \sim 10^{15}$. Moreover, if the source contains a neutron star or neutron torus (orbiting a black hole), then this energy must be transported by ordered Poynting flux (a wound-up magnetic field) to avoid excessive baryon loading of the outflow. This can be achieved if the source has a rotation period $\sim 10^{-3}$ sec, and a poloidal magnetic field of strength $\sim 10^{15}$ G [26–29]. Dynamo action naturally generates magnetic fields of this strength in a variety of triggering models, including accretion-induced collapse of a white dwarf, binary neutron star and neutron star-black hole mergers, and failed Type Ib supernovae [26,30,29]. The alternative process of neutrino annihilation into pairs $\nu + \overline{\nu} \to e^+ + e^-$ ([31], and references therein) induces a matter outflow that is larger than the tolerable value by a factor $\sim 10^6$ for $L_{e^\pm} \sim 10^{51}$ erg s^{-1} [32].

The ratio of photon luminosity L_γ to (ordered) Poynting luminosity L_P at the base of the wind is a key parameter in models for the spectrum involving Compton up scattering by hotspots in an expanding relativistic MHD wind [29,32]. The mean photon energy emerging from the flow is $\langle E_\gamma \rangle \sim L_P / N_\gamma$, when the baryon loading lies at the critical value where Comptonization is effective and adiabatic losses are small. Near the base of the flow, the photon gas is very close to black body, and so $\langle E_\gamma \rangle$ is directly related to the effective temperature $T_{\rm eff}^4 \simeq L_\gamma / \sigma_{SB} 4\pi R^2 c$ at the light cylinder,

$$\langle E_\gamma \rangle \sim T_{\rm eff} = 0.8 \left(\frac{L_\gamma}{10^{50}} \right)^{1/4} \left(\frac{P}{10^{-3}\ \ \mathrm{s}} \right)^{-1/2} \ \mathrm{MeV}. \tag{7.11}$$

This is remarkably close to the observed range of spectral break energies, after allowing for cosmological redshift.

The loading of the outflow by baryon rest energy is tolerably small only if $L_\gamma < 10^{-2} L_P$ at the neutrinosphere. In other words, a key requirement of this model is that the wind be reheated from $L_\gamma \ll L_P$ to $L_\gamma \sim L_P$ well outside the neutrinosphere. This is plausibly accomplished by the sort of MHD cascade just described. The time scale for convection and/or differential rotation in the source is comparable to the $\sim 10^{-3}$ rotation period [26], which implies that the footpoints of the external poloidal magnetic field move around rapidly enough to excite strong turbulence in the external field. The resulting photon luminosity should be comparable to the wave luminosity (when the wave coupling $k_\perp \xi \gtrsim 1$) so that

$$\frac{L_\gamma}{L_P} \sim \left(\frac{\delta B}{B} \right)^2 \tag{7.12}$$

(these quantities being evaluated at the light cylinder).

B. Leakage from neutron star magnetospheres

Alfvén waves injected into the magnetosphere of a neutron star can either damp near the surface of the star, or escape the region of closed magnetic field lines, thereby driving a relativistic outflow. Waves trapped near the stellar surface can undergo a turbulent cascade. We examine these two damping mechanisms in succession, and indicate the ranges of wave luminosities over which each dominates.

Alfvén waves propagate along open magnetic field lines, and their energy is lost from the star. Even if the waves are first injected on closed magnetic field lines (as in the soft γ repeater model of [20]), they will couple to internal shear waves which transport energy throughout the crust [24]. Since the waves take many wave periods to leak out (the fraction of open field lines being small), one can approximate the wave amplitude $\delta B_\star$ as being constant over the surface of the star,[11] and estimate the Alfvén wave luminosity as

$$L_A \sim 2\pi \theta_{\rm open}^2 R_\star^2 \frac{(\delta B_\star)^2}{8\pi} c. \tag{7.13}$$

Here $\theta_{\rm open}$ is the polar angle of the last open magnetic field line, and we approximate the external field near the star as a dipole with polar flux density $B_\star$.

The width of the bundle of open field lines is determined by the rotation period P, in the usual manner, when the Alfvén wave pressure is very small [33]:

$$\theta_{\rm open}^2 = \frac{2\pi R_\star}{cP} \equiv \frac{R_\star}{R_{lc}}. \tag{7.14}$$

However, above a critical Alfvén wave pressure a larger fraction of field lines are forced open by the pressure of Alfvén waves. To estimate this fraction, one balances the dipole magnetic pressure with the wave pressure

$$\frac{L_A}{4\pi R_A^2 c} \sim \frac{B^2(R_A)}{8\pi}, \tag{7.15}$$

and then notes that

$$\theta_{\rm open}^2 \sim \left(\frac{R_A}{R_\star} \right)^{-1} \sim \left(\frac{2L_A}{B_\star^2 R_\star^2 c} \right)^{1/4}. \tag{7.16}$$

Since $\theta_{\rm open}$ depends implicitly on L_A, one can combine Eqs. (7.13) and (7.16) to obtain

$$\theta_{\rm open}^2 \sim \left(\frac{\delta B_\star}{B_\star} \right)^{2/3} \tag{7.17}$$

and

$$L_A \sim 0.2 \left(\frac{\delta B_\star}{B_\star} \right)^{8/3} B_\star^2 R_\star^2 c. \tag{7.18}$$

In this regime, the neutron star loses energy primarily by Alfvén wave and particle emission, and *not* by rotational torques.

Nonetheless, the usual magnetic dipole energy loss formula for a rotating neutron star is modified, because the the poloidal field scales as

$$B(R) \sim B(R_A) \left(\frac{R}{R_A} \right)^{-2} \quad (R_A < R < R_{lc}) \tag{7.19}$$

---

[11]Because the transmission coefficient between crust and magnetosphere depends on the strength and orientation of the surface field $B_\star$, $\delta B_\star$ will in fact vary by a numerical factor of the order of unity.

in between the Alfvén radius and the light cylinder. This increases the field strength at the light cylinder over the usual value. The rotational energy loss rate also increases to

$$L_{\rm sd}\sim 0.1B^2(R_{lc})R_{lc}^2c\sim 0.1L_A(\Omega R_A/c)^2 \quad (R_{lc}>R_A), \tag{7.20}$$

where $\Omega=2\pi/P$ and the normalization is estimated from the standard magnetic dipole formula (e.g., [33]). Although $L_{\rm sd}$ remains smaller than L_A, the rotation period decreases *exponentially* with time in this regime

$$\frac{d\ \ln\Omega}{dt}\sim -0.1\frac{L_A}{I}\left(\frac{R_A}{c}\right)^2. \tag{7.21}$$

It has been suggested that the halo of nonthermal radio-emitting particles surrounding SGR 1806-20 [34] is powered by such an Alfvén wave-driven wind [20,35]. This model therefore suggests that the rotation period of SGR 1806-20 may be far longer than the 8 sec period of SGR 0526-66 (the source of the very bright burst on March 5, 1979, which is not surrounded by any detected plerion).

The Alfvén waves may also be released in a short period of time. In particular, the initial hard spike of the March 5, 1979 superburst which appears to have been an expanding relativistic fireball [20,36]. However, the lack of spectral evolution in the softer repeat bursts emitted by the SGR sources argues against models (e.g., [37]) in which the photons are emitted from a relativistic outflow as opposed to a trapped plasma.

C. Turbulent cascade in neutron star magnetospheres

Alfvén waves trapped on closed magnetic field lines will cascade to high wave number, via the three-mode and four-mode couplings considered in Sec. VI. The wave damping rate is a stronger function of wave amplitude than is the leakage rate (which depends on the amplitude only through $\theta_{\rm open}$). The main uncertainty in estimating this damping rate involves the form of the Alfvén wave packets. If the waves are injected directly by a sudden irreversible horizontal displacement of the crust, then the magnetic field lines undergo a net shift $\Delta\xi\sim k_\parallel^{-1}(\delta B/B_0)$ across each wave packet. The total cascade luminosity within a confinement volume $\sim(4\pi/3)R^3$ is

$$L_{\rm cas}\sim\zeta_0^2\,\frac{(\delta B)^2}{8\pi}\,\nu_0\,\frac{4\pi}{3}\,R^3. \tag{7.22}$$

Here δB is related to the wave amplitude at the stellar surface by $\delta B\sim\delta B_\star(B/B_\star)^{1/2}=\delta B_\star(R/R_\star)^{-3/2}$. The wave frequency at the outer scale is ν_0 and the strength parameter at the outer scale is

$$\zeta_0(R)=\frac{\delta B}{B}\frac{k_\perp}{k_\parallel}=\zeta_0(R_\star) \tag{7.23}$$

since $k_\perp(R)=k_\perp(R_\star)(B/B_\star)^{1/2}$. Thus,

$$L_{\rm cas}\sim\frac{1}{6}\left(\frac{\delta B_\star}{B_\star}\right)^4\left(\frac{k_\perp(R_\star)}{k_\parallel}\right)^2\left(\frac{\nu_0R_\star}{c}\right)B_\star^2R_\star^2c. \tag{7.24}$$

Note that the stronger dependence on $\delta B_\star$ than L_A [Eq. (7.18)], as well as the additional dependence on $k_\perp/k_\parallel$. A plausible value for this last parameter is ~300 [24].

Alternatively, if the Alfvén wave packets are harmonic with negligible net shift, then $L_{\rm cas}\propto\zeta_0^4$ instead of ζ_0^2, and expression (7.24) is modified to

$$L_{\rm cas}\sim\frac{1}{6}\left(\frac{\delta B_\star}{B_\star}\right)^6\left(\frac{k_\perp(R_\star)}{k_\parallel}\right)^4\left(\frac{\nu_0R_\star}{c}\right)B_\star^2R_\star^2c. \tag{7.25}$$

ACKNOWLEDGMENTS

We thank Peter Goldreich for useful discussions. C. T. is supported by the Alfred P. Sloan Foundation, and by NASA ATP through Grant No. NAG5-3100. Part of the work described here was completed while the authors were visitors at the Institute for Theoretical Physics during the fall 1995 program on Astrophysical Gamma-Ray Sources. The ITP is supported by NSF Grant No. PHY 94-07194.

APPENDIX A

In this appendix we prove that the action principle in the Lagrangian formulation, discussed in Sec. II, gives the correct equations of motion. We begin by demonstrating that Eq. (2.11) for the dual of the electromagnetic field tensor

$$\tilde F^{\mu\nu}=\frac{1}{J_4}\frac{\partial x^\mu}{\partial x_0^\rho}\frac{\partial x^\nu}{\partial x_0^\sigma}\tilde F_0^{\rho\sigma} \tag{A1}$$

is indeed an integral of the constraint equations

$$\frac{\partial\tilde F^{\mu\nu}}{\partial x^\nu}=0. \tag{A2}$$

This is true for an arbitrary (nonstatic, inhomogeneous) unperturbed reference background $\tilde F_0^{\rho\sigma}$, provided it too satisfies the constraint equations in the unperturbed coordinates. Substituting into the constraint equations and changing to differentiation over the unperturbed coordinates

$$\frac{\partial x_0^\alpha}{\partial x^\nu}\frac{\partial}{\partial x_0^\alpha}\left[\frac{1}{J_4}\frac{\partial x^\mu}{\partial x_0^\rho}\frac{\partial x^\nu}{\partial x_0^\sigma}\tilde F_0^{\rho\sigma}\right]=0. \tag{A3}$$

Carrying out the differentiation

$$\left[\frac{-1}{J_4}\frac{\partial J_4}{\partial(\partial x^\gamma/\partial x_{0\delta})}\frac{\partial^2x^\gamma}{\partial x_0^\sigma\partial x_{0\delta}}+\frac{\partial x_0^\alpha}{\partial x^\nu}\frac{\partial^2x^\nu}{\partial x_0^\alpha\partial x_0^\sigma}\right]\frac{\partial x^\mu}{\partial x_0^\rho}\frac{\tilde F_0^{\rho\sigma}}{J_4}=0, \tag{A4}$$

where we have used the antisymmetry of $F_0^{\rho\sigma}$ and the background constraint equations to eliminate two terms. Because

$$\frac{\partial x_{0\delta}}{\partial x^\gamma}=\frac{1}{J_4}\frac{\partial J_4}{\partial(\partial x^\gamma/\partial x_{0\delta})}, \tag{A5}$$

the left-hand side of Eq. (A4) vanishes identically, thereby proving that the constraint equation is satisfied.

Note that Eq. (A1) is simply the usual coordinate transformation of a tensor, apart from the Jacobian factor. This implies that the pseudoscalar $\mathbf{E}\cdot\mathbf{B}=0$ in the perturbed fluid, provided it is zero in the reference background. Hence Eq. (A1) also enforces the MHD condition.

We now turn to the action principle:

$$S'=-\int d^4x\,\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}=\int d^4x\,\frac{1}{4}\,\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}. \quad \text{(A6)}$$

This may be written as an integral over the unperturbed coordinates

$$S'=\int d^4x_0\,\frac{J_4}{4}\,\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}\equiv\int d^4x_0 L. \quad \text{(A7)}$$

Extremizing this action then gives the Euler-Lagrange equations

$$\frac{\partial}{\partial x_0^\alpha}\left[\frac{\partial L}{\partial(\partial x^\mu/\partial x_0^\alpha)}\right]=0 \quad \text{(A8)}$$

or

$$\frac{\partial}{\partial x_0^\alpha}\left[\frac{1}{J_4}\,\tilde{F}_0^{\gamma\delta}\,\frac{\partial x_\mu}{\partial x_0^\gamma}\,\frac{\partial x_\nu}{\partial x_0^\delta}\,\tilde{F}_0^{\alpha\beta}\,\frac{\partial x^\nu}{\partial x_0^\beta}-\frac{\partial J_4}{\partial(\partial x^\mu/\partial x_0^\alpha)}\,\frac{1}{4}\,\tilde{F}^2\right]=0. \quad \text{(A9)}$$

It is straightforward to show [cf., Eq. (B1) below] that the Jacobian has the property

$$\frac{\partial}{\partial x_0^\alpha}\left[\frac{\partial J_4}{\partial(\partial x^\mu/\partial x_0^\alpha)}\right]=0. \quad \text{(A10)}$$

In addition, the antisymmetry of the field tensor and the background constraint equations imply

$$\frac{\partial}{\partial x_0^\alpha}\left(\tilde{F}_0^{\alpha\beta}\,\frac{\partial x^\nu}{\partial x_0^\beta}\right)=0. \quad \text{(A11)}$$

Hence the Euler-Lagrange equations become

$$\tilde{F}_0^{\alpha\beta}\,\frac{\partial x^\nu}{\partial x_0^\beta}\,\frac{\partial}{\partial x_0^\alpha}\left(\frac{1}{J_4}\,\tilde{F}_0^{\gamma\delta}\,\frac{\partial x_\mu}{\partial x_0^\gamma}\,\frac{\partial x_\nu}{\partial x_0^\delta}\right)-\frac{\partial J_4}{\partial(\partial x^\mu/\partial x_0^\alpha)}\,\frac{\partial}{\partial x_0^\alpha}\left(\frac{1}{4}\,\tilde{F}^2\right)=0. \quad \text{(A12)}$$

The two derivative operators may be written

$$\tilde{F}_0^{\alpha\beta}\,\frac{\partial x^\nu}{\partial x_0^\beta}\,\frac{\partial}{\partial x_0^\alpha}=\tilde{F}_0^{\alpha\beta}\,\frac{\partial x^\nu}{\partial x_0^\beta}\,\frac{\partial x^\epsilon}{\partial x_0^\alpha}\,\frac{\partial}{\partial x^\epsilon}=J_4\tilde{F}^{\epsilon\nu}\,\frac{\partial}{\partial x^\epsilon} \quad \text{(A13)}$$

and

$$\frac{\partial J_4}{\partial(\partial x^\mu/\partial x_0^\alpha)}\,\frac{\partial}{\partial x_0^\alpha}=J_4\,\frac{\partial}{\partial x^\mu}. \quad \text{(A14)}$$

Therefore

$$\tilde{F}^{\epsilon\nu}\,\frac{\partial\tilde{F}_{\mu\nu}}{\partial x^\epsilon}-\frac{1}{4}\,\frac{\partial\tilde{F}^2}{\partial x^\mu}=0. \quad \text{(A15)}$$

Rewriting the dual in terms of the field tensor and simplifying, we finally obtain the equation of motion

$$F_{\mu\nu}\,\frac{\partial F^{\rho\mu}}{\partial x^\rho}=0. \quad \text{(A16)}$$

The results obtained in this section follow more immediately from the alternative field transformation (2.16). The derivation will be left to the reader.

APPENDIX B

In this appendix we consider the interaction between 2 MHD waves, and derive the relation between the quantities $\partial\cdot\xi$, $(\xi\cdot\partial)\xi^\mu$ (involving derivatives with respect to the perturbed coordinates $x^\mu=x_0^\mu+\xi^\mu$ of the magnetofluid), and the analogous quantities $\partial_0\cdot\xi$, $(\xi\cdot\partial_0)\xi^\mu$ (involving derivatives with respect to the unperturbed coordinates x_0^μ).

In four dimensions, one has

$$\frac{\partial x_0^\mu}{\partial x_\nu}=\frac{1}{J_4}\,\frac{\partial J_4}{\partial(\partial x^\nu/\partial x_0^\mu)}=-\frac{1}{6J_4}\,\varepsilon^{\nu\alpha\beta\gamma}\varepsilon^{\mu\rho\sigma\tau}\,\frac{\partial x_\alpha}{\partial x_0^\rho}\,\frac{\partial x_\beta}{\partial x_0^\sigma}\,\frac{\partial x_\gamma}{\partial x_0^\tau}. \quad \text{(B1)}$$

This implies

$$\partial_\mu\xi^\mu=-\frac{1}{6}\,\varepsilon^{\mu\alpha\beta\gamma}\varepsilon^{\nu\rho\sigma\tau}\,\frac{\partial\xi_\mu}{\partial x_0^\nu}\,\frac{\partial x_\alpha}{\partial x_0^\rho}\,\frac{\partial x_\beta}{\partial x_0^\sigma}\,\frac{\partial x_\gamma}{\partial x_0^\tau} \quad \text{(B2)}$$

in a gauge where $J_4=1$. The last three derivatives can each be written as

$$\frac{\partial x_\alpha}{\partial x_0^\rho}=\eta_{\alpha\rho}+\frac{\partial\xi_\alpha}{\partial x_0^\rho}, \quad \text{(B3)}$$

and so Eq. (B2) can be expanded in powers of ξ. In the presence of 2 MHD fourier modes, the antisymmetry of $\varepsilon^{\mu\alpha\beta\gamma}$ forces all terms involving three or more factors of ξ to vanish. This implies

$$\partial_\mu\xi^\mu=\partial_{0,\mu}\xi^\mu+(\partial_{0,\mu}\xi^\mu)^2-(\partial_{0,\mu}\xi^\nu\partial_{0,\nu}\xi^\mu). \quad \text{(B4)}$$

Similarly one has

$$\xi^\mu\partial_\mu\xi^\delta=\xi^\mu\partial_{0,\mu}\xi^\delta+(\partial_{0,\nu}\xi^\nu)\xi^\mu\partial_{0,\mu}\xi^\delta-(\partial_{0,\mu}\xi^\nu)\xi^\nu\partial_{0,\nu}\xi^\delta. \quad \text{(B5)}$$

The simultaneous conditions

$$\partial_\mu\xi^\mu=(\xi^\mu\partial_\mu)\xi^\delta=0 \quad \text{(B6)}$$

are satisfied if

$$\partial_{0,\mu}\xi^\mu=(\xi^\mu\partial_{0,\mu})\xi^\delta=0. \quad \text{(B7)}$$

The addition of a third interaction term ξ_I (second order in the two interacting modes) does not change these relations. For example, in the case of two colliding A modes, the interaction wave ξ_I with time-evolution equation (4.14) satisfies $(\xi_\pm\cdot\partial)\xi_I=0$ in the gauge (B6) and (B7). Similarly, inspection of Eq. (B1) shows that the derivatives

$$\partial_\rho\xi^\mu=\partial_{\rho,0}\xi^\mu+O(\xi_\pm^4) \quad \text{(B8)}$$

remain equal to third order in $\xi_\pm$, even though the difference between $\mathbf{x}$ and $\mathbf{x}_0$ is first order in ξ.

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