# Illustrative example of how quark-hadron duality might work

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We discuss the issue of the local quark-hadron duality at high energies in two- and four-dimensional QCD. A mechanism of the dynamical realization of the quark-hadron duality in two-dimensional QCD in the limit of a large number of colors, $N_c \to \infty$ (the 't Hooft model), is considered. A similar mechanism of dynamical smearing may be relevant in four-dimensional QCD. Although particular details of our results are model dependent, the general features of the duality implementation conjectured previously get further support. [S0556-2821(98)04305-7]

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## I. INTRODUCTION

In recent years the focus of the applications of the operator product expansion (OPE) [1] has shifted towards processes with essentially Minkowskian kinematics. Perhaps the most well-known example is the theory of inclusive decays of heavy flavors (for a review see, e.g., Ref. [2]). This fact, as well as the increased demand for more accurate predictions, puts forward the study of the quark-hadron duality as an urgent task.

A detailed definition of the procedure which goes under the name of the quark-hadron duality (a key element of every calculation referring to Minkowskian quantities) was given in Refs. [2,3]. In a nutshell, a *truncated* OPE is analytically continued, term by term, from the Euclidean to the Minkowski domain.[1] A smooth quark curve obtained in this way is supposed to coincide at high energies (energy releases) with the actual hadronic cross section.

If duality is formulated in this way, it is perfectly obvious that at finite energies deviations from duality must exist. The difference between the measured physical cross section and a smooth OPE prediction will be referred to as an oscillating-exponential (duality violating) component. In Ref. [2] it was shown that if we knew the leading asymptotic behavior of the high order terms in the power series we could evaluate this component. Unfortunately, very little is known about this aspect of OPE, and we have to approach the problem from the other side—either by modeling the phenomenon [3] or by studying some general features of the appropriate spectral densities. One can also try to approach the problem purely phenomenologically. Recent work in this direction is reported in Refs. [5,6].

An illustrative spectral density, quite instructive in the studies of the issue of the quark-hadron duality, was suggested in [2],

$$\mathrm{Im}\ \Pi = \mathrm{const}\frac{N_c}{2}\sum_{n=1}^{\infty}\ \delta(\mathcal{E}-n), \tag{1}$$

where

$$\mathcal{E}=\frac{s}{\Lambda^2},$$

and from now on we will drop an inessential constant in front of the sum. The color factor $N_c$ is singled out for convenience. The imaginary part above represents, for positive values of $s$, a sum of infinitely narrow equidistant resonances, with equal residues. The distance between the resonances is $\Lambda^2$. It defines $\Pi(q^2)$ everywhere in the complex plane $q^2$, through the standard dispersion relation, up to an additive constant which can be adjusted arbitrarily. It is not difficult to see that the corresponding correlation function

$$\Pi(q^2)=-\frac{N_c}{2\pi}\left[\psi(\varepsilon)+\frac{1}{\varepsilon}\right], \tag{2}$$

where $\psi$ is the logarithmic derivative of Euler's $\Gamma$ function, and

$$\varepsilon=-\frac{q^2}{\Lambda^2}=-\mathcal{E}.$$

In the Minkowski domain $\mathcal{E}$ is positive, in the Euclidean domain $\varepsilon$ is positive. Then, the asymptotic expansion of $\Pi(q^2)$ in deep Euclidean domain is well known,

$$\Pi(q^2)\to-\frac{N_c}{2\pi}\left[\ln\varepsilon+\frac{1}{2\varepsilon}-\sum_{n=1}\ (-1)^{n-1}\frac{B_n\varepsilon^{-2n}}{2n}\right], \tag{3}$$

[1]Moreover, usually one deals with the practical version [4] of OPE, see Ref. [3] for further details.

 

where $B_n$ are the Bernoulli numbers. At large $n$ they grow factorially, as $B_n \sim (2n)!$ (see [7], page 23). We deal with the sign alternating series.

Although the spectral density (1) is admittedly a model, it was argued [2] that a similar factorial growth of the coefficients in the power (condensate) series is a general feature. The spectral density (1) may be relevant in the limit of the large number of colors, $N_c \to \infty$, when all mesons are infinitely narrow. This limit is not realistic, however. Moreover, in this limit the *local* quark-hadron duality, as we defined it, *never* takes place since even at high energies the hadronic spectral density never becomes smooth, even approximately. One can smear it by hand, of course, but then deviations from the local duality will be determined not only by the intrinsic hadronic dynamics, as is the case in the real world, but also by particular smearing procedure – they will depend on the weight function chosen for smearing, the interval of smearing, and so on. In the actual world the smearing occurs *dynamically*, since at high energies the resonance widths become non-negligible. The limits of $E \to \infty$ and $N_c \to \infty$ are not interchangeable.

Here we suggest and study more realistic (dynamically smeared) spectral densities compatible with all general properties of quantum chromodynamics. Starting from infinitely narrow resonances, as in Eq. (1), we introduce finite widths, ensuring smooth behavior. Technically, in the first part of the paper the problem of duality is analyzed in the two-dimensional 't Hooft model [8] (see, also, [9–13]). The quark confinement in this model is built-in. We then try to abstract general features of this solution, which may persist in QCD. In the second part of the paper an attempt is made to work out the same mechanism in four dimensions.

Dynamical ''smearing'' of the spectral densities occurring in QCD due to nonvanishing resonance widths, exhibits the same features of the high-energy behavior as was suggested in Refs. [2] and [3] on the basis of rather naive models, e.g., instanton models. Namely, the approach of the spectral density to the smooth limit (the deviation from duality) is exponential, with oscillations. This pattern seems to be general and may be considered now as a well-established model-independent fact. At the same time, the exponent determining the rate of the damping of the duality violations depends on details of the large distance dynamics. We were unable to find it from first principles, and had to settle for model-dependent determinations.

Let us note that several useful results on the relation between duality violations and the divergence of the power expansion were presented in Ref. [14], and we incorporate them.[2] The very idea of using the 't Hooft model as a theoretical laboratory adequate to the problem was formulated there. Moreover, it was noted [14] that the leading asymptotics of the high-order coefficients does not require the knowledge of the exact mass spectrum. Suffice it to know the leading term in the $n$ dependence, where $n$ is the radial excitation number, which immediately translates in the leading factorial behavior of the coefficients. In particular, the position of the low-lying states is irrelevant in the regime with the factorially growing coefficients. This is a specific feature of the asymptotic series.

The $\psi$ function model was originally suggested in Ref. [2] for the heavy-light quark systems. In Ref. [14] it was noted that it was more appropriate for the light-light quark systems since in the heavy-light systems the resonances are not expected to be equidistant. Straightforward quasiclassical estimates yield in this case that the meson energies (measured from the heavy quark mass) asymptotically scale as $\sqrt{n}$. (See also Ref. [15], where this scaling law is reproduced in a linear potential model.) In the present paper we further develop the $\psi$ function ansatz adapting it for the light-quark systems.

The issue of duality in the 't Hooft model was touched upon, in an applied aspect (weak decays of heavy quarks), in the recent publication [6]. We do not comment on this work now, since a detailed analysis of this problem is under way and will be reported elsewhere.

The paper is organized as follows. Section II briefly recalls the basic strategy of the OPE-based calculations. Here we outline the main elements of the analysis to be presented below. In Sect. III we first discuss the 't Hooft model in the leading $1/N_c$ approximation. Then we calculate the resonance widths and evaluate the polarization operator of the two scalar currents in taking into account $1/N_c$ corrections. In Sec. IV we present qualitative arguments why the implementation of duality obtained in the 't Hooft model must be also relevant, at a qualitative level, in four-dimensional QCD. Section V gives conclusions and outlines possible physical applications of our results.

## II. OPE-BASED STRATEGY, SUM RULES, AND DUALITY

The issue of ''deviations from duality'' caused much confusion in the recent literature. Therefore, to begin with, we remind the reader the basic strategy and explain, in clear terms, where the deviations can occur and where there can be no deviations.

First of all, all calculations based on the operator product expansion are carried out in the Euclidean domain. Only away from the physical cuts this procedure is well defined. Any calculation consists of several crucial elements: identification of the operators which can appear in the expansion, separation of hard virtual momenta (higher than a normalization point $\mu$) from soft virtual momenta (lower than $\mu$) and, finally, calculation of the expansion coefficients in front of relevant operators. The latter is carried out in terms of quarks and gluons. That is why the normalization point $\mu$ must be chosen sufficiently high, and the calculation must be done in the Euclidean domain. Even if in some problems calculations are conveniently presented in such a way as if they were done in the Minkowski domain, actually the corresponding results must be understood as an analytic continuation.

The connection between the Euclidean predictions and measurable quantities is established via dispersion relations. In this way one can get certain sum rules. A large variety of them is offered on the market. If the Euclidean quantity is appropriately chosen, the (Euclidean) operator product expansion *converges*. The best-known example of such an appropriate choice is provided by the Shifman-Vainshtain-

[2]We strongly disagree, however, with some particular calculations and expressions presented in this paper.

Zakharov (SVZ) sum rules [16] obtained by virtue of the Borel transformation of the dispersion representation. Consider, for instance, the model of Ref. [2], see Eq. (2). The $1/Q^2$ expansion of $\Pi(q^2)$ in the Euclidean domain is factorially divergent. At the same time, the $1/M^2$ expansion of the Borel-transformed quantity has a finite radius of convergence. Indeed, the SVZ sum rule in the case at hand has the form

$$\frac{N_c\Lambda^2}{2\pi}\frac{e^{-\Lambda^2/M^2}}{1-e^{-\Lambda^2/M^2}}=\frac{1}{\pi}\int e^{-s/M^2}\mathrm{Im}\,\Pi(s)ds, \qquad (4)$$

(in the left-hand side $\Lambda^2$ plays the role of the inverse slope of the Regge trajectory). Assume that the left-hand side was calculated theoretically, using OPE, as an expansion in $1/M^2$. The domain of convergence of the power expansion is determined by the position of the nearest singularity in the complex $M^2$ plane. It is quite obvious that the expansion converges at

$$|M^2|>\frac{\Lambda^2}{2\pi}. \qquad (5)$$

Not only the radius of convergence is finite, due to the factor $2\pi$ in the denominator the domain of convergence extends to quite low values of $M^2$. If $\Lambda^2\sim 2$ GeV$^2$, the power series is convergent at as low values of $M^2$ as 0.4 GeV$^2$. This fact was empirically observed long ago [16].

If we consider the SVZ sum rule inside the convergence domain of OPE [see Eq. (5)], and aim at predicting the exponential integral on the right-hand side of Eq. (4) *per se*, it is meaningless to speak about deviations from OPE. In this formulation of the problem *there are no deviations*.

The problem of deviations arises when we try to predict the spectral density Im $\Pi$ point-by point, at large $s$, or certain integrals of Im $\Pi$, not directly reducible to ''good'' sum rules of the type (4). Certainly, if we assume that Im $\Pi$ is smooth starting from some boundary value $s_0$, Eq. (4) allows us to predict Im $\Pi(s)$ at $s>s_0$ unambiguously. If an oscillating component is allowed, however, one can always invent such a wild oscillating function, which, being integrated with the exponential weight, gives a contribution on the left-hand side of the sum rule (4) less than the last term of the power expansion retained, no matter how small this last term is. This component is referred to as duality violating. Clearly, quantum chromodynamics admits only very specific oscillating-exponential components in the spectral densities, if at all. The question is what particular oscillating-exponential functions are allowed by QCD dynamics. We first try to answer this question in the simpler context of the 't Hooft model, and then pass to discussion of QCD.

## III. THE 't HOOFT MODEL

The aim of this section is to study the quark-hadron duality in the 't Hooft model. We shall first briefly review the 't Hooft model in the $N_c\to\infty$ limit and discuss $1/N_c$ corrections in this model. Our original contribution is calculating the resonance widths for high excitations. Then, using these widths, we suggest an ansatz for the asymptotic behavior of the polarization operator of two scalar currents, based on the Breit-Wigner approximation. This resonance-saturated polarization operator will be referred to as phenomenological. We will confront it with the truncated power expansion. The difference between these two expressions gives an idea of the duality violation.

### A. The 't Hooft model: Generalities

Two-dimensional QCD is described by the Lagrangian

$$L=-\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}+\overline{q^f}(i\not{D}-m_f)q^f, \qquad (6)$$

where $f$ is the flavor index. Since the multiflavor aspect is irrelevant for our problem, we shall consider, for simplicity, one flavor; correspondingly, the index $f$ will be omitted hereafter. If the gauge coupling of the theory is $g$, it is convenient to introduce an effective coupling

$$\overline{g^2}=g^2N_c, \qquad (7)$$

which stays constant in the limit $N_c\to\infty$. The coupling constant $g$ has dimension of mass; it sets the scale for all dimensional quantities in the chiral limit, i.e., $m\to 0$. We introduce the scale

$$\mu^2=\frac{\overline{g^2}}{\pi}=\frac{g^2N_c}{\pi}, \qquad (8)$$

and measure all quantities in these units, e.g., the quark mass

$$\gamma^2=m^2/\mu^2,$$

while the mass of the $n$th meson

$$\mu_n^2=m_n^2/\mu^2,$$

and so on.[3]

As was shown by 't Hooft, the model with the Lagrangian (6) is exactly solvable in the limit $N_c\to\infty$. The bound state spectrum includes an infinite number of bound states whose masses lie on an almost linear trajectory. The properties of these bound states are described by the 't Hooft equation

$$\mu_n^2\phi_n(x)=\frac{(\gamma^2-1)\phi_n(x)}{x(1-x)}-\int_0^1\frac{\phi_n(y)dy}{(x-y)^2}. \qquad (9)$$

Here the integral is understood as a ''principal value,''

$$\frac{1}{p^2}=\lim_{\epsilon\to 0}\frac{1}{2}\left[\frac{1}{(p+i\epsilon)^2}+\frac{1}{(p-i\epsilon)^2}\right].$$

Moreover, $x$ is the momentum fraction of the meson carried by the antiquark (in the infinite momentum frame), while $1-x$ is that of the quark; $\phi_n(x)$ is the wave function of the $n$th bound state.

The integral equation (9) must be solved with the boundary conditions

[3]Normalization of the coupling constant $g$ coincides with that adopted in Refs. [8] and [10], but differs from that in Ref. [13].

$$\phi_n(x)\to\begin{cases} x^\beta, & x\to 0,\\ (1-x)^\beta, & x\to 1.\end{cases} \tag{10}$$

Here $\beta$ is the smallest (in the absolute value) root of the equation:

$$\pi\beta\cot(\pi\beta)=1-\gamma^2. \tag{11}$$

Below we shall be interested in the massless case, $\gamma=0$. In the massless limit, the lowest-lying state (the ''pion'') can be found from Eq. (9) analytically. Indeed, the solution $\phi_0=\text{const}$, corresponding to $\mu_0=0$, obviously goes through. We are interested, however, in highly excited states, $n\gg 1$.

In the original paper [8] 't Hooft suggested the following approximation for $\phi_n(x)$ at large $n$, $x$ not too close to 0 and 1:

$$\phi_n(x)=\sqrt{2}\sin(n\pi x). \tag{12}$$

Recent calculations [17], exploiting a new and improved numerical procedure (the so-called spline method), show that for the massless case a better approximation for large $n$ is

$$\phi_n(x)=\sqrt{2}\cos(\pi n x). \tag{13}$$

This formula works very well numerically everywhere except the very end points $x=0,1$, and a slight $x$-dependent shift when $x\sim 0.5$.

Note that the wave functions (13) satisfy the proper boundary conditions for the massless case, which corresponds to $\beta$ equal to zero and, hence,

$$\phi_n(0)=C,\phi_n(1)=PC. \tag{14}$$

Here $P=1$ for the states with the even parity and $P=-1$ for the states with the odd parity, and the constant $C\neq 0$.

The mass spectrum in the massless case was found by 't Hooft. The asymptotic behavior of $\mu_n^2$ is

$$\mu_n^2=\pi^2 n\{1+O[\ln(n)/n]+\cdots\}. \tag{15}$$

Note that the mass formula (15) does not depend on the choice of the wave function, Eq. (13) or the 't Hooft choice, at least in the leading in $n$ approximation [17].

### B. The meson widths

As was already mentioned, in the limit $N_c\to\infty$ the bound states in the 't Hooft model are stable, their widths vanish. However, once one takes into account the leading $1/N_c$ correction, the resonances begin to decay. In the first order in $1/N_c$ expansion there are only two-particle decays $a\to b+c$. The relevant coupling constants $g_{abc}$ are given by the following formula [10–12,17,18]:

$$g_{abc}=\mu^2\sqrt{\frac{\pi}{N_c}}[1-(-1)^{(\sigma_a+\sigma_b+\sigma_c)}](f^+_{abc}+f^-_{abc}). \tag{16}$$

Here $\sigma_a$ is the parity of the $a$th resonance. The constants $f^\pm_{abc}$ are determined from the following expressions:

$$f^\pm_{abc}=\frac{1}{1-\omega_\pm}\int_0^{\omega_\pm}\phi_a(x)\phi_b(x/\omega_\pm)\Phi_c\left(\frac{x-\omega_\pm}{1-\omega_\pm}\right) - \frac{1}{\omega_\pm}\int_{\omega_\pm}^1\phi_a(x)\Phi_b(x/\omega_\pm)\phi_c\left(\frac{x-\omega_\pm}{1-\omega_\pm}\right), \tag{17}$$

where $\omega_\pm$ are two roots of the algebraic equation corresponding to the mass-shell condition

$$m_a^2=\frac{m_b^2}{\omega}+\frac{m_c^2}{(1-\omega)}. \tag{18}$$

The function $\Phi_a(x)$ is defined as

$$\Phi_a(x)=\int_0^1 dy\,\frac{\phi_a(y)}{(x-y)^2}.$$

Using the above expressions for the decay couplings one can readily calculate the resonance widths in the leading $1/N_c$ approximation. They are given by

$$\Gamma_a=\frac{1}{8m_a}\sum_b\sum_c\frac{g^2_{abc}}{\sqrt{I(m_a,m_b,m_c)}}, \tag{19}$$

where $I$ is the standard ''triangular'' function

$$I(m_a,m_b,m_c)=\frac{1}{4}[m_a^2-(m_b+m_c)^2][m_a^2-(m_b-m_c)^2]. \tag{20}$$

The sum in Eq. (19) runs over all mesons $b$ and $c$ with the constraint $m_b+m_c<m_a$.

Our task was to establish the asymptotic behavior of the widths, as a function of the excitation number, at large values of $n$, in the leading $1/N_c$ approximation (all widths are proportional to $1/N_c$; the excitation number $n$ will be temporarily called $a$ in this section). In order to find the widths we first computed analytically, using the wave functions (13), the overlap integrals (17). The answer can be expressed via the integral sine and cosine functions and was obtained using the REDUCE program. Since it is very bulky it seems unreasonable to present here the final expression [19]. After computing the overlap integral we performed numerically summation over all possible $b$ and $c$ in Eq. (19) for $a$ up to 500. The result for the widths exhibits a remarkable pattern. The widths of the individual levels oscillate near a smooth square-root curve, see Fig. 1. This figure shows the width of the $a$th state vs $a$, up to $a=500$. The result of averaging over the interval of 20 resonances is depicted in Fig. 2. We see that the curve of the averaged resonance widths $\Gamma(a)\equiv\Gamma_a$ is very well approximated by the function

$$\Gamma(a)=\frac{A\mu}{\pi^2 N_c}\sqrt{a}[1+\mathcal{O}(1/a)], \tag{21}$$

where the parameter $\mu$ is introduced in Eq. (8), and $A$ is a constant which will be given below.

Since the square-root law (21) for the (averaged) widths is valid in such a large interval of the excitation numbers and turns out to be so accurate, it seems plausible that this for-

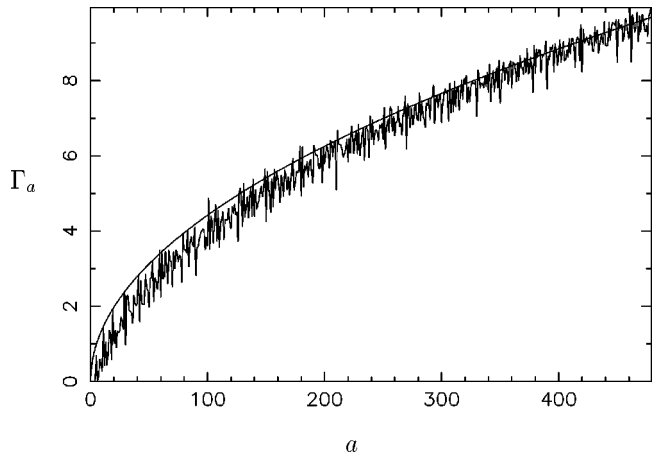


FIG. 1. The width of the $a$th state (in units of $\mu\pi^{-2}N_c^{-1}$) as a function of $a$ up to $a=500$.

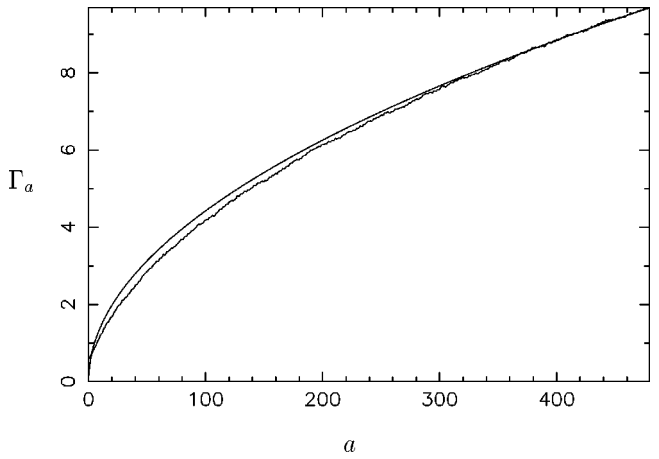


FIG. 2. Smeared widths in the same interval of $a$ and the square-root fit, Eq. (21).

mula could be obtained analytically. This is an interesting question by itself, especially in four-dimensional QCD. Unfortunately, we were unable to find exact analytic solution so far. Some qualitative arguments in favor of the exact square root dependence are discussed in Sec. IV. The numerical value of the constant $A$ is

$$A=0.44\pm 0.05. \tag{22}$$

Below this result for the (averaged) widths will be used for determining the asymptotic behavior of the polarization operator.[4]

Since the $1/N_c$ result for the decay width grows with $a$, one may worry about the $1/N_c^2$ corrections. If they grew with $a$ sufficiently fast this could invalidate Eq. (21) in the interval of the excitation numbers we are interested in, namely, $a=\text{const}\ N_c$, where the constant above can be numerically large, but it does not scale with $N_c$. Using quasiclassical arguments (see Sec. IV) one can show that the actual $1/N_c$ expansion parameter in Eq. (21) is $\sqrt{a}/N_c$. This means that at $a=\text{const}\ N_c$ corrections $O(1/N_c^2)$ and higher are negligible.

Concluding this section let us note that the same square-root was reported previously in Ref. [18]. We failed to reproduce the arguments of this work leading to the square-root law, however. What is important is that the constant analogous to $A$ in Ref. [18] is claimed to be proportional to $1/\sqrt{m}$, and, thus, blows up for massless quarks. This poses perplexing questions. The coincidence looks completely accidental.

### C. The Breit-Wigner approximation: An ansatz for the polarization operator

Once we had found the resonance widths, we can calculate the polarization operator in the Breit-Wigner approximation. Here we shall consider the most interesting case of the polarization operator of the two scalar currents. Let us start from this polarization operator in the $N_c\to\infty$ limit [10,20].

Define the two-point function of the scalar currents $j=\overline{q}q$:

$$\Pi(q^2)=i\int d^2x e^{iqx}\langle 0|T\{j(x),j(0)\}|0\rangle. \tag{23}$$

In the $N_c\to\infty$ limit $\Pi$ is given by [10]

$$\Pi(q^2)=-\sum_{n=0}^{n=\infty}\frac{g_n^2}{q^2-m_n^2+i\epsilon}. \tag{24}$$

Here the constants $g_n$ are the current residues

$$\langle 0|j(x)|n\rangle=g_n. \tag{25}$$

Note that the residues vanish for even $n$ [10], so that the sum in Eq. (24) runs over odd $n$ only. Below we shall be interested in the behavior of $\Pi(q^2)$ for large $|q^2|\gg\mu^2$. This behavior is dominated by the terms in the sum (24) with large $n\gg 1$ [10,14,20]. In order to calculate this sum explicitly we then need the large $n$ behavior of $g_n$. It was determined in the same classical paper [10] from the requirement of the compatibility of the expansion (24) and the perturbation theory asymptotics in the $Q^2\to\infty$ limit,

$$\Pi(q^2)\to-\frac{N_c}{2\pi}\ln\frac{Q^2}{\mu^2},\quad Q^2\equiv -q^2. \tag{26}$$

The coefficients $g_n$ for sufficiently large odd $n$ must be independent of $n$ and are equal to

$$g_n^2=N_c\pi\mu^2. \tag{27}$$

Then, taking into account the linear $n$ dependence of mass squared one can approximate the polarization operator (23) for sufficiently large $|q^2|$ by the $\psi$ function,

$$\Pi(q^2)-\Pi(0)=-\frac{N_c}{2\pi}\psi(\sigma),\quad \sigma=\frac{Q^2}{2\pi^2\mu^2}+\frac{1}{2}. \tag{28}$$

[4]Let us note that if one calculates the widths with the wave functions (12), the square-root behavior of Eq. (21) is intact, but the value of the constant $A$ is different, $A'\sim 0.007$, i.e. $\sim 50$ times smaller. This fact indicates that the square-root law is not sensitive to the precise form of the wave functions, while the value of the coefficient $A$ is.

We hasten to emphasize again that this formula is not supposed to work at nonasymptotic values of $Q^2$. For instance, it does not contain the massless ''pion.'' Moreover, by shifting a little bit the masses and residues of the low-lying resonances we let $\Pi(q^2)$ ''breathe'' at small $Q^2$ without changing the asymptotic behavior.

What will happen if we take into account finite widths of the resonances? To answer this question we calculate $\Pi(Q^2)$ in the Breit-Wigner approximation. (The continuation of the Breit-Wigner formula in the complex plane, away from the resonance position, is not unambiguous. We choose a specific continuation leading to proper analytic properties of the polarization operator, see below.) The inverse propagator of the $n$th bound state can be written as

$$D_n^{-1}(q^2) = -[q^2 - m_n^2 + \Sigma(q^2)], \tag{29}$$

where $\Sigma(q^2)$ a function of order $1/N_c$ reflecting the possibility of the transitions $a \to bc \to a$. This function is known at $q^2 = m_n^2$:

$$\text{Im}\ \Sigma(q^2 = m_n^2) = m_n \Gamma_n = \frac{A m_n^2}{\pi^3 N_c}. \tag{30}$$

Here we used Eqs. (13) and (19). Now we can write

$$D_n^{-1} = Q^2 \left( 1 - \frac{A}{\pi^4 N_c} \ln \frac{Q^2}{\Lambda^2} \right) + m_n^2. \tag{31}$$

It is easy to see that (i) at $q^2 = m_n^2$ Eq. (30) is satisfied; (ii) the pole is shifted to an unphysical sheet, so that on the physical sheet there are no singularities except the cut at positive real $q^2$. The property (i) is quite obvious. Let us comment on the property (ii).

The easiest way to demonstrate that there are no singularities on the physical sheet is as follows. Observe, that at our level of accuracy one can write, instead of Eq. (31),

$$D_n^{-1} = (z + m_n^2), \tag{32}$$

where

$$z = Q^2 (Q^2 / 2\pi^2 \mu^2)^{-A/\pi^4 N_c}. \tag{33}$$

Here the constant $\Lambda^2$ in Eq. (31) is adjusted in accordance with Eq. (35) below.

The physical sheet on the complex $Q^2$ plane [Fig. 3(a)] is mapped onto a sheet with a ''defect angle'' on the complex $z$ plane [Fig. 3(b)]. Going into the shaded area we pass to the unphysical sheets. Note that the pole of $D_n$ lies in the shaded area.

Assembling all pieces together we conclude that, with the resonance widths switched on, Eqs. (24) and (28) are substituted by

$$\Pi(Q^2) - \Pi(0) = \text{const} \times \sum D_n^{-1}(Q^2)$$

$$= -\frac{1}{1 - A/(\pi^4 N_c)} \frac{N_c}{2\pi} \psi(\tilde{\sigma}), \tag{34}$$

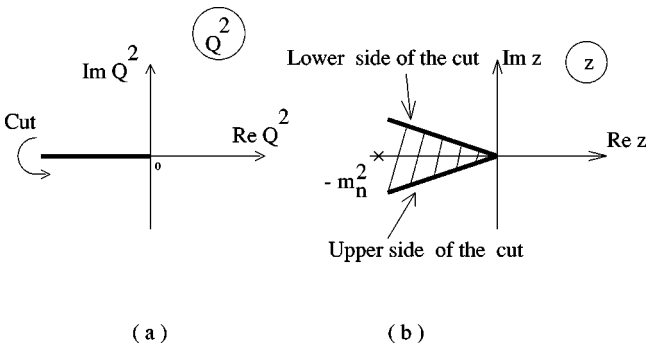


FIG. 3. Analytical structure of the polarization operator. (a) The polarization operator must be analytic everywhere in the complex $Q^2$ plane, except the cut running on the negative real semiaxis of $Q^2$ (positive real semiaxis of $q^2$). The imaginary part of the polarization operator must be positive at the upper side of the $q^2$ cut; (b) The mapping of the $Q^2$ plane onto the $z$ plane, Eq. (33). The physical sheet on the $Q^2$ plane corresponds to the $z$ plane with the shaded sector removed. The boundaries of the sector correspond to the lower and upper sides of the $q^2$ cut.

where

$$\tilde{\sigma} = \frac{z}{2\pi^2 \mu^2} + \frac{1}{2}. \tag{35}$$

The constant in front of $\psi(\tilde{\sigma})$ is adjusted in such a way as to leave intact the high $Q^2$ asymptotics, see Eq. (26). The term proportional to $A$ is clearly subleading in $1/N_c$. Strictly speaking, we should have omitted it at the level of accuracy accepted here.

By construction, all singularities of the polarization operator (34) are on the unphysical sheet. The discontinuity at the cut $q^2 \geqslant 0$ will be calculated below.

#### D. OPE and the asymptotics of the polarization operator

The expression (34) for the polarization operator is clearly not exact, since we have made a number of approximations. They do not affect, however, the large $Q^2$ asymptotics of $\Pi(Q^2)$. Neither the leading asymptotics of high orders in the $1/Q^2$ expansion depends on these approximations. Since we are interested in the high energy behavior, distortions introduced in $\Pi(Q^2)$ at finite $Q^2$ by the approximations made are not important. In particular, we will disregard the fact that low-order terms in the $1/Q^2$ expansion of $\Pi(Q^2)$ (''condensates'') may come out with ''wrong'' coefficients. By adjusting the positions and residues of a few lowest resonances we can always get any desirable coefficients for any given *finite* number of terms in the $1/Q^2$ expansion.

Let us first study the impact of the resonance widths on $\Pi(Q^2)$ in the Euclidean domain. As we saw, the effect due to the nonvanishing widths essentially reduces to the substitution of the variable $\sigma$ defined in Eq. (26) by $\tilde{\sigma}$, see Eq. (33). Therefore, the change in the asymptotic $1/Q^2$ expansion at large (Euclidean) values of $Q^2$ is rather insignificant. If at $N_c = \infty$ the $n$th term of the power expansion is $C_n (Q^2)^{-k_n}$, at the $1/N_c$ level it becomes

$$C_n \frac{1}{(Q^2)^{k_n(1-\alpha)}} \to C_n \frac{1}{(Q^2)^{k_n}} (1 + \alpha k_n \ln Q^2), \tag{36}$$

$$\alpha = \frac{A}{\pi^4 N_c}. \tag{37}$$

The second term on the right-hand side in Eq. (36) is a small $1/N_c$ correction which reminds us of a logarithmic anomalous dimension in QCD. Unlike the first term in Eq. (36), the logarithmic term develops an imaginary part at large positive real values of $q^2$. If we treated Eq. (36) in the framework of OPE (more exactly, in practical version [4]), then we would predict that the spectral density

$$\mathrm{Im}\Pi(s)|_{s\gg\mu^2} = \frac{N_c}{2}\left[1 - \sum_{n=1}^{n=n_0} C_n \frac{\alpha\pi k_n}{(-s)^{k_n}}\right], \tag{38}$$

where $s = q^2$ and $n_0$ is the highest term retained in practical OPE. By construction, the prediction for the spectral density obtained from practical OPE is smooth. As a matter of fact, all correction terms in Eq. (38) are suppressed by $1/N_c$, since $\alpha \sim 1/N_c$, and are numerically insignificant at large $N_c$.

Let us examine now the ''physical spectral density,'' i.e., the imaginary part at positive $q^2$ following directly from Eq. (34). Our task is to reveal an oscillating component, not suppressed by $1/N_c$.

In order to find the imaginary part analytically, it is convenient to use the reflection property of the $\psi$ function:

$$\psi(\tilde{\sigma}) = \psi(-\tilde{\sigma}) - \pi\cot(\pi\tilde{\sigma}) - 1/\tilde{\sigma}. \tag{39}$$

It is obvious that the polarization operator (34) is a sum of three terms, corresponding to three different terms on the right-hand side of Eq. (39). Moreover, it is evident that the first and the third terms are smooth functions of $q^2$. Their contribution to the imaginary part of the polarization operator for large $s$ corresponds to the smooth component obtained from OPE, see the correction terms in Eq. (38).

In order to study the oscillating-exponential component one must consider the second term

$$\mathrm{Im}\ \Pi(s) = \frac{N_c}{2}\mathrm{Im}\ \cot(\pi\tilde{\sigma}) = -\frac{N_c}{2}\frac{\sinh(2y)}{\cosh(2y) - \cos(2x)}, \tag{40}$$

where

$$x = \pi\ \mathrm{Re}\ \tilde{\sigma} \approx -\frac{s}{2\pi\mu^2}[1 - \alpha\ln(s)],$$

$$y = \pi\ \mathrm{Im}\ \tilde{\sigma} \approx -\frac{\alpha s}{2\mu^2}. \tag{41}$$

Taking into account only the leading in $N_c$ terms it is easy to rewrite the spectral density (40) as follows:

$$\mathrm{Im}\ \Pi(s) = \frac{N_c}{2}\frac{\sinh(\alpha s/\mu^2)}{\cosh(\alpha s/\mu^2) - \cos(s/\pi\mu^2)}. \tag{42}$$

At $\alpha s \ll \mu^2$ we are in the resonance zone (here and below we use the nomenclature of Ref. [3], see Sec. 5.2). In order to find the high energy behavior of the imaginary part of the polarization operator, we must go to the oscillation zone, to energies $\alpha s \gg \mu^2$. Then

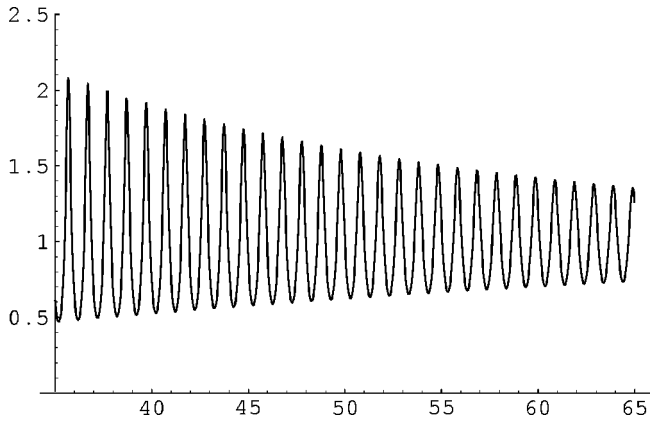


FIG. 4. The spectral density corresponding to Eq. (34) in the 't Hooft model, versus $s$ (in the units $2\pi^2\mu^2$). $N_c$ is put equal to 3, and the normalization factor is chosen in such a way that asymptotically the spectral density displayed must approach unity.

$$\mathrm{Im}\ \Pi(q^2) \to \frac{N_c}{2}\left[1 + 2\exp\left(-\frac{\alpha s}{\mu^2}\right)\cos\left(\frac{s}{\pi\mu^2}\right)\right]. \tag{43}$$

The unit term corresponds to the leading asymptotics (it reproduces the OPE prediction), while the second term is an oscillating-exponential component, missing in practical OPE.[5] It presents a deviation from duality we are hunting for (remember, by duality we understand a specific procedure formulated in Refs. [2,3]).

The oscillating component is suppressed by the damping exponential, $\exp(-\alpha s/\mu^2)$. A comment is in order regarding the exponent,

$$\frac{\alpha s}{\mu^2} \equiv \frac{A}{\pi^4 N_c}\frac{s}{\mu^2}. \tag{44}$$

This exponent determines the boundary between the oscillation and the resonance zones, $s_0 \sim \pi^4 N_c \mu^2$. This estimate is in accord with intuition. Indeed, the exponential suppression of the oscillations should start when the resonance width becomes larger than the distance between the neighboring resonances. Since $\Gamma_n \sim \mu\sqrt{n}/N_c\pi^3$ and the distance between the neighboring resonances $\Delta m_n \sim \pi\mu/\sqrt{n}$, this occurs at the excitations number $n \sim \pi^4 N_c$. Thus, the factor $1/N_c$ in Eq. (44) is obvious. What is more remarkable, is a large additional numerical suppression $\pi^4$ pushing the boundary to higher energies and making the exponential suppression weaker. It can be traced back to the numerical suppression of the width in Eq. (21) which, in turn, is due to a strong numerical suppression of the phase space. Because of this fact the damping of the oscillations occurs very slowly in the 't Hooft model. Figure 4 shows Im $\Pi$ in the window from the 35th to 65th oscillation. We see that even here the oscil-

[5]In terms of an index $\sigma$ introduced in Sec. V B of Ref. [3], Eq. (43) implies that $\sigma = 2$.

lation amplitude is quite sizable. Unlike other features, expected to be generic, this numerical aspect—a very slow rate of the oscillation damping—is specific to the 't Hooft model and is not expected to survive in four-dimensional QCD, since it is entirely due to ''abnormally'' narrow widths of the resonances in two-dimensional QCD. Semiquantitative estimates of the damping exponent in four-dimensional QCD will be given in Sec. IV.

In summary, for $q^2 \leqslant 0$ the polarization operator is given by the $\psi$ function of the positive real argument and is a smooth function with no singularities. It is well approximated by its asymptotic expansion. The latter is most conveniently written in terms of the variable $\tilde{\sigma}$ defined above [see Eq. (35)], for which it becomes the standard asymptotic expansion of the $\psi$ function. In the Minkowskian domain, starting from the scale $s_0 \sim \pi^4 \mu^2 N_c$, one can approximate the polarization operator by its smooth asymptotics (analytically continued from the expansion in the Euclidean domain) *plus* an exponentially decreasing and oscillating term (43).

## IV. FOUR-DIMENSIONAL QCD

Unlike in the 't Hooft model, we cannot solve four-dimensional QCD. At the qualitative level, however, we can apply the same ideas. The general pattern of the asymptotic behavior will be the same, but the numerical aspects look different.

Our goal in this section is to discuss the polarization operator of two vector currents:

$$\Pi_{\mu\nu}(q^2) = i \int \exp(iqx) \langle 0 | T\{j_\mu(x) j_\nu(0)\} | 0 \rangle d^4x. \quad (45)$$

Here $j_\mu(x)$ is the vector current:

$$j_\mu(x) = \overline{u}\, \gamma_\mu d(x). \quad (46)$$

Because of the conservation of the vector current the polarization operator can be represented as

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi(q^2). \quad (47)$$

Furthermore, $\Pi(q^2)$ is dimensionless, and is perfectly analogous to the polarization operator (21) we dealt with in the 't Hooft model. According to the standard wisdom of multicolor QCD, we expect that at $N_c = \infty$ and high energies $\Pi(q^2)$ is representable as a sum of equidistant infinitely narrow resonances, with a constant residue, i.e., we arrive at the same $\psi$ function. The dynamical smearing is again provided by the resonance widths (a $1/N_c$ effect). What is known about the widths of the highly excited states in four-dimensional QCD?

In the case at hand we do not have in our disposal quantitative tools which would allow us to calculate the widths. Such a calculation could have been performed in a noncritical string theory, were this theory available. Unfortunately, the issue is not worked out,[6] and we have to resort to qualitative arguments. If the string-based picture of color confinement is indeed valid, one can hardly avoid the conclusion that the resonance widths must grow linearly with $m_n$ [22], much in the same way as in the 't Hooft model.

Let us remind the reader of the quasiclassical arguments of Ref. [22], showing that $\Gamma_n \sim m_n / N_c$ in four-dimensional QCD. When a highly excited meson state is created by a local source, it can be considered, quasiclassically, as a pair of (almost free) ultrarelativistic quarks; each of them with energy $m_n/2$. These quarks are created at the origin, and then fly back-to-back, creating behind them a flux tube of the chromoelectric field. The length of the tube $L \sim m_n / \Lambda^2$ where $\Lambda^2$ is the string tension. The decay probability is determined, to order $1/N_c$, by the probability of creating an extra quark-antiquark pair. Since the pair creation can happen anywhere inside the flux tube, it is natural to expect that

$$\Gamma_n \sim \frac{1}{N_c} L \Lambda^2 = \frac{B}{N_c} m_n, \quad (48)$$

where $B$ is a dimensionless coefficient of order 1.

Let us note in passing that the $1/N_c^2$ corrections due to creation of two quark pairs are of order $L^2/N_c^2$ within this picture. Since $L \sim m_n \sim \sqrt{n}$, the expansion parameter is $\sqrt{n}/N_c$.

Strictly speaking, the estimate (48) must be, rather, viewed as a lower bound, since the transverse fluctuations of the flux tube can increase the decay probability. However, most likely, these transverse fluctuations will materialize in the form of emission of glueballs, a subleading $1/N_c^2$ effect, which is not considered here. It is not fully clear what impact these fluctuations may have on the quark-antiquark pair production. It seems plausible that they only affect the numerical coefficient in Eq. (48), which is not calculated anyway. Given all naivete of the arguments and the estimate (48), at the present stage it is reasonable to accept it as a working hypothesis.

As was mentioned, in real QCD we expect $B \sim 1$, see estimates in Ref. [22] and below. Apart from this numerical difference, everything else is perfectly analogous to the consideration we have carried out in the 't Hooft model. Hence, we conclude that in four-dimensional QCD a viable model of the approach to asymptotics is provided by the same $\psi$ function:

$$\Pi(Q^2) - \Pi(0) = \text{const} \psi(\tilde{\sigma}), \quad (49)$$

where, in the case at hand,

$$\tilde{\sigma} = \left( \frac{Q^2}{\Delta m^2} \right)^{1 - B/N_c \pi} + C. \quad (50)$$

Here $C$ is a numerical constant correlated with the position of the lowest resonance,

$$m_0^2 = C \Delta m^2 [1 + O(1/N_c)], \quad \frac{\Gamma_0}{m_0} = \frac{B}{N_c} [1 + O(1/N_c)].$$

The normalization constant in front of the $\psi$ function in Eq. (49) will be chosen in the form

$$-\left(1 - \frac{B}{N_c \pi}\right)^{-1},$$

[6] Some attempts in this direction were reported in Ref. [21].

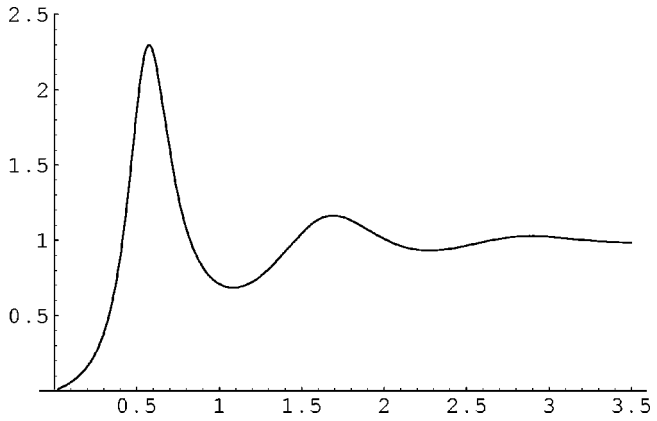


FIG. 5. The imaginary part of the polarization operator corresponding to Eq. (50) with $\Delta m^2=1$ GeV$^2$, $C=0.6$ and $B=0.78$. The number of colors is set equal to three. The horizontal axis presents $s$, in GeV$^2$.

so that the imaginary part at asymptotically high $s$ is automatically normalized to unity. The plot of Im $\Pi$ is shown on Fig. 5. Although fine details of this spectral density are somewhat distorted compared to the actual experimentally measured spectral densities in the $e^+e^-$ annihilation or $\tau$ decays (e.g., the $\rho$ meson width comes out $\sim 1.5$ larger than the experimental one) the qualitative similarity of our model with experiment is remarkable. To substantiate the point we display a plot of the spectral density in the vector isoscalar channel borrowed from Ref. [23] (Fig. 6).

The oscillation zone starts at $s_0 \sim \Delta m^2 N_c (2\pi B)^{-1}$. Above this boundary the asymptotic form of the oscillating exponential terms, analogous to Eq. (43), is given by

$$\mathrm{Im}\Pi(Q^2)=\mathrm{const}\left[1+2\exp\left(-\frac{2\pi B s}{N_c \Delta m^2}\right)\cos\left(\frac{2\pi s}{\Delta m^2}\right)\right]. \tag{51}$$

Both, the onset of the oscillation zone and the oscillation structure are in nice qualitative agreement with what we see on Fig. 6 presenting experimental data on $e^+e^-$ annihilation and $\tau$ decays.

In summary, our model spectral density (50), with the appropriate values of parameters, properly captures all important features which must be inherent to spectral densities in real QCD. First, the corresponding polarization operator is a sum of an infinite number of simple poles on the unphysical sheet, so that the correct analytical properties ensue automatically. The $1/Q^2$ expansion has the right structure. The spectral density following from Eqs. (49) and (50) is dynamically smeared by the resonance widths. Purely pictorially it closely resembles what is measured experimentally.

## V. CONCLUSIONS

In this paper we address the issue of the preasymptotic component of the spectral density not seen in practical OPE. This component oscillates, with the amplitude being exponentially damped. The approach to the problem is complementary to that of Refs. [2,3]. It is gratifying to observe that the general features of the overall picture come out the same

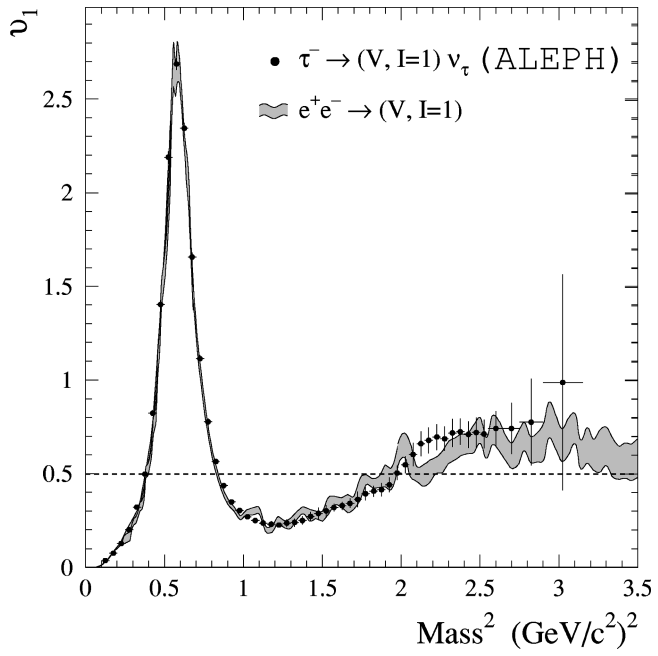


FIG. 6. The spectral density in the vector isoscalar channel measured in $e^+e^-$ annihilation and $\tau$ hadronic decays.

as in the previous analyses. The details are different. For instance, the instanton-based model discussed in Refs. [2,3] yields the interval between the successive oscillations growing with $s$, while our present result implies equidistant oscillations in the $s$ scale. In this aspect the instanton-based model is seemingly less realistic. Moreover, for obvious reasons it does not allow one to trace the proper $N_c$ dependence, while our present analysis does. Needless to say, it reproduces the desired regularity—the fact that the exponent governing the exponential damping of the oscillations is proportional to $1/N_c$.

The important lesson we draw is confirmation of a general pattern of the duality-violating component in the spectral densities at high energies inferred previously: exponential character, modulated by oscillations. Particular details are model dependent. The 't Hooft model is solvable, and all questions can be explicitly answered. In four-dimensional QCD it is possible to provide only educated guesses. Further efforts are needed to back them up by more solid calculations.

We believe that the suggested ansatz for the spectral density, Eqs. (49) and (50), gives a very good idea of the duality-violating contributions. It is compatible with all general principles of field theory, $1/N_c$ expansion of QCD, and folklore knowledge which is universally believed to be true.

Although our discussion was phrased in terms of the spectral densities, its implications are wider. In particular, it is quite probable that a component of the very same structure is present in the so-called hard quantities without OPE, e.g., thrust. Recently it was realized that the perturbative predictions for such quantities must be supplemented by $1/Q$ corrections (for a review see, e.g., [24]). If we are right, in the intermediate domain of moderate momentum transfers the oscillating component might be noticeable too.

An interesting question which deserves a new dedicated analysis (independently of our model of duality-violating

contributions) is the behavior of the resonance widths as a function of the excitation number at high $n$ in multicolor four-dimensional QCD. Another obvious direction of expansion is combining the $\psi$ function ansatz (49) with information on specific low-dimension condensates which might allow one to obtain a fully realistic description of the polarization operator in the entire complex plane.

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[1] K. Wilson, Phys. Rev. **179**, 1499 (1969); Phys. Rev. D **3**, 1818 (1971); *Proceedings of the 1971 International Symposium on Electron and Photon Interactions at High Energies*, edited by N. Mistry, New York, 1972 (unpublished), p. 115; K. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974); A. M. Polyakov (unpublished).

[2] M. Shifman, in *Continuous Advances in QCD*, Proceedings of the Workshop, Minneapolis, Minnesota, 1994, edited by A. Smilga (World Scientific, Singapore, 1994), p. 249, hep-ph/9405246; in *Particles, Strings and Cosmology*, Proceedings of the 19th Johns Hopkins Workshop, Baltimore, Maryland, 1995, edited by J. Bagger *et al.* (World Scientific, Singapore, 1996), p. 69, hep-ph/9505289.

[3] B. Chibisov, R. Dikeman, M. Shifman, and N. Uraltsev, Int. J. Mod. Phys. A **12**, 2075 (1977).

[4] V. Novikov, M. Shifman, A. Vainshtein, and Z. Zakharov, Nucl. Phys. **B249**, 445 (1985).

[5] B. Blok and T. Mannel, Mod. Phys. Lett. A **11**, 1263 (1996); B. Blok, M. Shifman, and N. Uraltsev, Nucl. Phys. **B494**, 237 (1997).

[6] B. Grinstein and R. Lebed, Phys. Rev. D **57**, 1366 (1998).

[7] Z.X. Wang and D.R. Guo, *Special Functions* (World Scientific, Singapore, 1989).

[8] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974) [Reprinted in G. 't Hooft, *Under the Spell of the Gauge Principle* (World Scientific, Singapore, 1994), p. 443]; see also Ref. [9] for a discussion of subtleties in the regularization of the 't Hooft model.

[9] F. Lenz, M. Thies, S. Levit, and K. Yazaki, Ann. Phys. (N.Y.) **208**, 1 (1991).

[10] C. Callan, N. Coote, and D. Gross, Phys. Rev. D **13**, 1649 (1976).

[11] M. Einhorn, Phys. Rev. D **14**, 3451 (1976).

[12] M. Einhorn, S. Nussinov, and E. Rabinovici, Phys. Rev. D **15**, 2282 (1977).

[13] I. Bars and M. Green, Phys. Rev. D **7**, 537 (1978).

[14] A. Zhitnitsky, Phys. Rev. D **53**, 5821 (1996).

[15] P. Colangelo, C. Dominguez, and G. Nardulli, Phys. Lett. B **409**, 417 (1997).

[16] M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. **B147**, 385 (1979).

[17] W. Krauth and M. Staudacher, Phys. Lett. B **388**, 808 (1997).

[18] M. Bishary, Ann. Phys. (N.Y.) **129**, 435 (1980).

[19] The full answer and the program used to generate it can be obtained from the authors via email.

[20] A. Zhitnitsky, Sov. J. Nucl. Phys. **43**, 999 (1986); **44**, 139 (1986).

[21] M. B. Green and G. Veneziano, Phys. Lett. **36B**, 477 (1971); A. Yu. Dubin, A. B. Kaidalov, and Yu. A. Simonov, Phys. Lett. B **323**, 41 (1994).

[22] A. Casher, N. Neurberger, and S. Nussinov, Phys. Rev. D **20**, 179 (1979).

[23] ALEPH Collaboration, R. Barate *et al.*, Z. Phys. C **76**, 15 (1997).

[24] V.M. Braun, Theoretical Uncertanties in the Determination of $\alpha_s$ from Hadronic Event Observables: The Impact of Nonperturbative Effects, Talk given at the Annual Meeting of Division of Particles and Fields of the APS (DPF'96), Minneapolis, MN, 1996, hep-ph/9610212.