Parton-hadron duality in QCD sum rules: Quantum-mechanical examples

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Motivated by recent work on three-point QCD sum rules in heavy quark physics, we use simple quantum mechanical models to study the basic issue of duality in three-point sum rules. We show that while in all of these models the duality in two-point sum rules works fine, the duality in three-point sum rules may be 100% violated, leading to completely unreliable predictions for the matrix elements in question. The implications for three-point QCD sum rules are discussed. A new estimate for the parameter λ_1 of HQET is given.
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I. INTRODUCTION

Recently there has been growing interest in the issue of parton-hadron duality in QCD [1–4]. This is due to both theoretical and experimental progress. On the one hand, we witness further improvements in theoretical methods of investigation of low energy properties of hadrons, such as the QCD sum rules and heavy quark effective theory. These methods enable one to calculate the hadronic properties directly from QCD, in a model independent way. These methods heavily rely on the validity of the parton hadron duality.

In addition to theoretical advances, one can now compare the predictions of these methods with the growing amount of the new experimental data. This leads naturally to renewed interest in the fundamentals of these methods, i.e., in the issue of how reliable they are.

One especially important application of the idea of the parton-hadron duality is the QCD sum rules [5] (see also Ref. [6] for review). For the past 20 years QCD sum rules have been widely and successfully used to predict masses (the so called two-point QCD sum rules) and the coupling constants (three-point QCD sum rules) of different hadrons and their decays.

The basic procedure in QCD sum rules is the following one: one calculates the physical quantity—the polarization operator of a certain number of currents in two ways. First, we calculate the polarization operator in terms of quarks and gluons, using asymptotic freedom. Then, we calculate the same polarization operator in terms of hadrons, using dispersion relations. One then equates the results from the parton model (the so called theoretical part of the QCD sum rule) with the sum over the hadron states (the so called phenomenological part of the sum rule). Usually, we are interested only in the properties of the lowest lying resonance. The contribution of the higher resonances created by the given currents is taken into account using the so called continuum model [7]. In other words, we approximate the hadron spectral density (i.e., the imaginary part of the polarization operator) by some smeared function. The standard approach is to assume that the corresponding smeared function is well approximated by the spectral density of the theoretical part of the sum rule. The assumed validity of this approach underlies all practical QCD sum rule calculations. It is exactly the issue of whether the parton hadron duality holds.

The issue of the parton hadron duality and the closely related issue of the right model for the continuum are not new. The subject was discussed in detail in the early paper by Shifman, Vainshtein and Zakharov [5], following the classical discussion by Poggio, Quinn and Weinberg [8] of the parton hadron duality in the case of the e^+-e^- annihilation.

Unfortunately, one cannot rigorously check the parton hadron duality directly in QCD. Although many arguments support duality, one cannot tackle the issue without the explicit theory of confinement. Consequently, one has to check the hadron parton duality (or, rather, its analogues) in simpler models. The existence (nonexistence) of duality in these models is a strong argument for (against) the parton hadron duality in real QCD.

While we are still unable to study the issue of duality directly in QCD, one can learn a lot by studying various exactly solvable models, the simplest of which are the quantum mechanical potential models. These models have been used to gain insight into the issue of the parton hadron duality in the case of two-point QCD sum rules [9,15,16].

The goal of the present paper is to study the analogue of the parton-hadron duality for three-point sum rules in the quantum-mechanical potential models. The issue has become epecially relevant recently, due to the extensive use of three-point QCD sum rules for the determination of different parameters of heavy quark physics (see e.g. the review [10]). In particular, it was found that different QCD sum rules lead to contradictory values of several fundamental parameters of heavy quark effective theory (HQET), like $\langle B|\vec{D}^2/(2m_B)|B\rangle$ [11,12]; these values, in turn, differ from the ones predicted on the basis of the analysis of the experimental data [13,14]. This requires us to go back and check once again the basic assumptions behind the QCD sum rule method.

Our present study confirms the old results [9,15,16] that state that the duality works excellently for the two-point sum rules. However, for the same models where the two-point duality works excellently, we shall see that we may encounter serious problems in the study of the three-point sum rules. Namely, we shall see that not only local duality, but also generalized duality (in the sense defined in Ref. [17]) are

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violated in the situation when the two-point sum rules work excellently.

We shall consider in this note three basic models:

(A) The harmonic oscillator. This is the potential model described by the potential

$$V(r)=\omega r^2/2. \tag{1.1}$$

(B) The linear oscillator. This model is the basis of the realistic potential models;

$$V(r)=\alpha r. \tag{1.2}$$

(C) The last model to consider is the linear potential perturbed by the Coulombic interaction that imitates the effects of the α_s corrections in the potential models:

$$V(r)=\alpha r-\beta/r. \tag{1.3}$$

The three-point sum rules for the harmonic oscillator were already studied in Refs. [17, 18]. (The numerical mistake made in Ref. [17] in the three-point sum rule for oscillator (but not, of course, for the QCD sum rules for the slope of Isgur-Wise function) was improved in Ref. [18]). In Ref. [17], it was shown that the duality may hold for the harmonic oscillator in generalized sense only: a one-dimensional integral of the phenomenological spectral density is dual to the corresponding integral of the theoretical part of the sum rule. However, later investigation [18] showed that even this duality does not take place, and the three-point sum rules for harmonic oscillator do not reproduce the true values. Here, we shall see that the harmonic oscillator case is not an exception but the general situation. The duality in three-point sum rules breaks down because the coupling signs become alternating. In all three models in question, the duality breaks for the sum rules determining ground state matrix elements of the following operators:

$$O_1\sim r^2, \quad O_3\sim r. \tag{1.4}$$

(Note the close analogy between the operator O_1 and the operator that determines the fundamental parameter of the HQET—the slope of the Isgur-Wise function.)

As a result of the duality breaking, the standard continuum model [7] does not describe the true spectral density for the first several resonances. So, the sum rules give answers that differ from the right ones (here, in quantum mechanics, we know, of course, the exact values) by 30–50 %. On the other hand, in the case of the kinetic energy operator $O_2\sim-\partial^2$, all nondiagonal transitions give positive contributions. The sum rules do work and the duality is not broken. Nevertheless, there is a big continuum contribution leading to large uncertainties in the predictions.

This paper is organized as follows. In Sec. II, we review the notion of duality and discuss the duality in the two-point sum rules for the potential models mentioned in the Introduction. In Sec. III, we study the duality in the three-point quantum mechanical sum rules. The duality fails for the operators $\hat{O}_1,\hat{O}_3$, and we trace the origin of its failure. On the other hand, for the operator $\hat{O}_2$ the duality holds, and we investigate the corresponding sum rules. In Sec. IV, we give our conclusions and discuss possible implications of our results for QCD. The details of the exact solution of the quantum mechanical models at hand are given in Appendix A, while the details of the calculations of the theoretical part of the sum rules are given in Appendix B.

II. DUALITY AND TWO-POINT SUM RULES

Let us recall now in more detail what do we mean by quark hadron duality in the case of the QCD sum rules. Consider the function $f(q^2)$ at some tensor structure in the polarization operator of two currents in the sum rule. One calculates f by means of the operator product expansion (OPE) in the Euclidean domain of momenta $q^2\leqslant 0$:

$$f(q^2)\sim a_0+a_1/q^4+\cdots . \tag{2.1}$$

Here the coefficient a_0 corresponds to the perturbation theory and the coefficients a_i, $i\geqslant 1$ correspond to the matrix elements of the relevant operators over the QCD vacuum. (Possibly, the coefficients a_0,a_i depend logarithmically on q^2). The function f is a smooth function of its argument q^2. Its imaginary part, denoted $\sigma_{\mathbf{p}}(s)$, is a smooth function as well. Another way to calculate f is to express it in terms of hadron properties by means of the dispersion relation:

$$f(q^2)=\int \frac{\mathrm{Im}\, f(s)}{(s-q^2)}\, ds\,. \tag{2.2}$$

The imaginary part $\sigma_h(s)\equiv \mathrm{Im}\, f(s)$ involved equals the sum of the delta functions over hadronic resonances contributing into the tensor structure in question:

$$\sigma_h(s)=\sum_n \beta_n^2\delta(s-m_n^2). \tag{2.3}$$

Now, the parton-hadron duality assumes that, starting at some threshold s_1, the integrals of the hadronic spectral density $\sigma_h(s)$ and the partonic one, $\sigma_p(s)$, with exponential weights coincide:

$$\int_{s_1}^{\infty}\sigma_{\mathbf{p}}(s)e^{-s/M^2}ds\sim\int_{s_1}^{\infty}\sigma_h(s)e^{-s/M^2}ds. \tag{2.4}$$

The minimal s_1 satisfying this equation is called the duality threshold. For QCD sum rules to work, s_1 must lie between the masses of the ground state and the first excited state contributing to f. In other words, the hadron spectral density, averaged over some interval of s, must be approximately equal to the spectral density calculated in the parton model. Of course, this is true for the exact spectral densities. However, our $\sigma_p(s)$ is only a part of the exact spectral density, namely the part corresponding to the first several terms in the OPE, analytically continued to Minkowsky space.

Once we know what the duality means in QCD, let us consider its quantum mechanical analogue. The quantum mechanical analogue of the polarization operator of two currents in QCD is the time dependent Green function:

$$S_0(0,T)=\sum_n |\psi_n(0)|^2e^{-E_nT}. \tag{2.5}$$

Here n runs over the S-states only. This function describes the propagation from the point (0,0) to the point (0,T) in the Euclidean time. One can calculate the function S_0 in two ways. First, we can calculate it as a power series in T for short times. The leading term in such a series is an analogue of the parton model contribution in QCD; the higher terms imitate the matrix elements of operators in the operator product expansion. Second, one can use the explicit formula (2.5) and calculate S_0 as a sum over the hadron states. In practice, we are interested in the properties of the ground state. So we represent

$$S_0(0,T)=|\psi_0(0)|^2 e^{-E_0 T}+\int_{E_c}^{\infty} dE\,\sigma_h(E)e^{-ET}, \quad (2.6)$$

where $\sigma_h(E)$ is the exact spectral density,

$$\sigma_h(E)=\sum_{n=1} \delta(E-E_n)|\psi_n(0)|^2. \quad (2.7)$$

Note that the leading term in the perturbation expansion of the $S_0(T)$ can also be represented as the integral of the spectral density $\sigma_{\mathbf{p}}(E)$ times e^{-ET}. Then the quantum mechanical duality means that, after average over some energy interval,

$$\sigma_h(E)\sim\sigma_{\mathbf{p}}(E), \quad E\geqslant E_c\,. \quad (2.8)$$

Equivalently, the integrals of the spectral functions times e^{-ET} are approximately equal as functions of T. The integral of the exact spectral density in the right-hand side (RHS) can be also calculated as the difference between the exact Green function $S_0(T)$ and the known exact expression for the ground state. Define

$$C_0(T)=\frac{S_0(0,T)-|\psi_0(0)|^2 e^{-E_0 T}}{|\psi_0(0)|^2 e^{-E_0 T}}, \quad (2.9)$$

and

$$C_{\mathrm{p}}(T)=\frac{\int_{E_c}^{\infty} dE\,\sigma_{\mathbf{p}}(E)e^{-ET}}{|\psi_0(0)|^2 e^{-E_0 T}}. \quad (2.10)$$

Below the functions $C_0(T)$ and $C_{\mathrm{p}}(T)$ are called ''continuum'' functions. If the parton hadron duality holds, these two functions must approximate one another for sufficiently small T (corresponding to sufficiently large E). Once duality is established, one can write the sum rule to determine the ground state parameters. In order to obtain the sum rule we simply rewrite equation (2.6) differently:

$$|\psi_0(0)|^2 e^{-E_0 T}=S_0(0,T)-\int_{E_c}^{\infty} dE\,\sigma_{\mathbf{p}}(E)e^{-ET}. \quad (2.11)$$

Here, for S_0 we use the power expansion, which can be obtained in perturbation theory. By fitting the RHS of equation (2.11) to exponent one is able to determine both ground state energy E_0 and its residue $|\psi_0(0)|^2$. The ground state energy can be easily obtained by first taking logarithm of (2.6) and then differentiating with respect to time T. Since our method is an approximate one, we have to introduce a notion of fiducial region. By fiducial domain we mean a window in T where two following conditions hold simultaneously. The first one is a control over the power expansion of S_0. Usually one demands for the last kept term in the expansion to saturate less than 30% of the whole expression. This way the upper edge of the window is determined. The second condition is a ground state dominance. This requirement is needed in order to suppress the relative contribution of the exited states. This condition determines a lowest edge of the window. Practically the contribution of all exited states [the integral term in (2.11)] is required to be less than 30%. The fiducial domain corresponds to the region where two asymptotics (small T and large T) matches. The important fact is that the sum rule (2.11) are essentially threshold dependent. Usually we do not know the value of the threshold parameter E_c except the general point that it should lie somewhat below the energy of the first exited state (which we do not know too). However, the standard philosophy of sum rules is to seek for a region (in E_c) where E_c dependence is small. The sum rule is then called stable. The variation of the result with E_c produces an error, which is unavoidable in the sum rule method. Two-point sum rules discussed below happen to be very stable with respect to the threshold variation. We depict the sum rules with only one optimal value for the threshold parameters, which we define by the best fit to exact known results. In all models, the optimal values appear to be very close to the guesses typical in practice, e.g., the midpoint between the lowest observed states.

Let us illustrate the quantum mechanical duality for two-point sum rules for the (A), (B), (C) models discussed in Introduction [9,15–18]. Although similar discussions are already present in the literature, we shall also consider these models for the sake of completeness and as simple illustrations of more complicated cases of Sec. III.

Consider the harmonic oscillator first. We use the dimensionless units $\omega=1$ and $2m=1$. The left, theoretical part of the sum rule can be represented as a perturbation series:

$$S_0^{\mathrm{har}}(0,T)=\left(\frac{1}{4\pi T}\right)^{3/2}\left(1-\frac{1}{4}T^2+\frac{19}{480}T^4-\frac{691}{120960}T^6+\cdots\right). \quad (2.12)$$

The corresponding spectral function is

$$\sigma_{\mathbf{p}}(E)=\frac{1}{4\pi^2}\sqrt{E}. \quad (2.13)$$

Consider now the right hand side of the sum rule. The spectral density $\sigma_h(E)$ can be represented as a sum of delta functions. For sufficiently high energies the summation can be approximately substituted by the integration [9,17]. Using $E_n=3/2+2n$ and $|\psi_n(0)|^2=(1/2\pi)^{3/2}[(2n+1)!!/2^n n!]$, one obtains

$$\sigma_h^{\mathrm{har}}(E)\sim\sum_n \left(\frac{2}{\pi}\right)^{1/2}\frac{E_n^{1/2}}{(2\pi)^{3/2}}\,\delta(E-E_n)$$
$$\approx\int\left(\frac{2}{\pi}\right)^{1/2}E_n^{1/2}\left(\frac{1}{2\pi}\right)^{3/2}\delta(E-E_n)\,\frac{dE_n}{2}=\frac{1}{4\pi^2}\sqrt{E}. \quad (2.14)$$

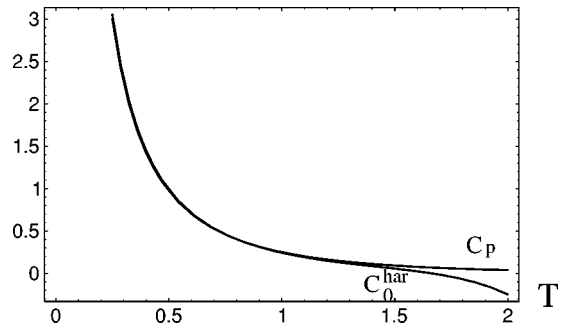

FIG. 1. Harmonic oscillator. Duality for the two-point function. Continuum functions $C_0^{\rm har}$ and $C_{\mathbf{p}}$ are plotted vs T. The energy threshold $E_c=2.6$.

The latter is just the theoretical spectral density (2.13). We see that for sufficiently high energies the duality holds indeed.

We can make more detailed estimates, relevant for the sum rules. Define the continuum functions $C_0^h(T)$ and $C_{\rm p}^h(T)$ (2.9), (2.10).

These two functions approximate each other very well for $T\leqslant 1.7$, (see Fig. 1, where the two functions are depicted for the optimum value of $E_c=2.6$). Once duality is established, one can write the sum rule to determine the ground state parameters. The fiducial region in this sum rule must, of course, be inside the region where the duality holds. Straightforward calculations (see Refs. [9] for details) show that this is true indeed, and the resulting values coincide with the exact ones very accurately.

Consider now the case of the linear oscillator. We use units where $\alpha=1$. The LHS of the sum rule is given by the asymptotic expansion of the propagator in perturbation theory. The corresponding Green function was obtained in Ref. [21] (see also Appendix B):

$$S_0^L(0,T)=\frac{1}{(4\pi T)^{3/2}}\left(1-\frac{\sqrt{\pi}}{2}T^{3/2}+\frac{5}{12}T^3+\cdots\right), \tag{2.15}$$

whereas the corresponding parton spectral function is given by Eq. (2.13). Here and below, the index L denotes the linear oscillator problem.

Pass now to the right-hand side (RHS) of the sum rule. Proceeding in the same spirit as for the harmonic oscillator, the local duality for high energies can be established. We use some exact results on the problem, collected in Appendix A. The square of the wave-function $|\psi_n^L(0)|^2$ equals $1/(4\pi)$. The large n asymptotic behavior of the energy levels $E_n^L=(3/2\pi n)^{2/3}$. The level density $\sigma_h^L(n)$ is

$$\sigma_h^L(n)\sim\frac{1}{4\pi}\sum_k\,\delta(k-n)\approx\frac{1}{4\pi}\int\delta(k-n)dn=\frac{1}{4\pi}. \tag{2.16}$$

The energy density is $\sigma_h^L(E)=\sigma_h^L(n)\,(\partial n/\partial E)$, which coincides with the ''bare'' spectral function (2.13).

The duality for sufficiently high energies established, we turn to the sum rule. The appropriate ''continuum'' functions $C_0^L(T)$ and $C_{\mathbf{p}}(T)$ of (2.9) and (2.10) are depicted in Fig. 2 for the optimal energy threshold $E_c=3.4$. The fit is perfect

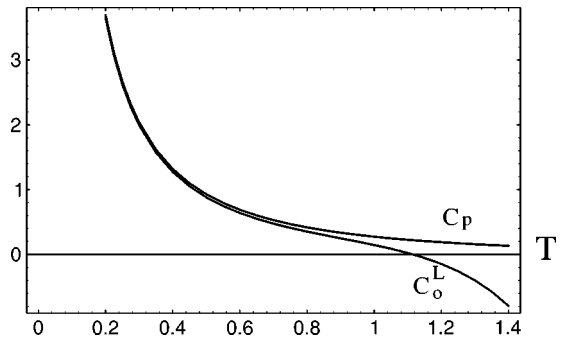

FIG. 2. Linear oscillator. Duality for the two-point function. Continuum functions C_0^L and $C_{\mathbf{p}}$ are plotted vs T. The energy threshold $E_c=3.4$.

and we can plot the sum rule for the ground state energy (Fig. 3). One obtains $E_0^L\approx(2.35\pm0.05)$, while the exact value is $E_{0\rm exact}^L=2.338$. We see the sum rules really do work.

Our last example of two-point functions, is C model with its ''linear+Coulomb'' potential. The model and its numeric solution are described in Appendix A, while the perturbative expansion of the propagator is presented in Appendix B. Below, numerical solutions of the model will be referred to as exact ones:

$$S_0^{\rm lc}(0,T)=\frac{1}{(4\pi T)^{3/2}}\left(1+b_0\sqrt{\pi}T^{1/2}+b_0^2\frac{\pi^2}{6}T-b_0\frac{3}{2}T^2-\frac{\pi^2}{2}T^{3/2}+\frac{5}{12}T^3+\cdots\right). \tag{2.17}$$

The parameter $b_0=0.57$ is defined in Appendix A. In QCD, parton spectral density acquires corrections due to α_s terms in the operator product expansion (OPE). In C model, the Coulomb interaction imitates the role of these α_s terms. The parton spectral function (2.13) is modified:

$$\sigma_{\mathbf{p}}^{\rm lc}(E)=\frac{1}{4\pi^2}E^{1/2}+\frac{b_0}{8\pi}+\frac{b_0^2}{48}E^{-1/2}. \tag{2.18}$$

Unfortunately, contrary to the previous examples we do not know exact solutions of the problem. However, it is natural to believe that the duality holds as in above cases, although we were not able to prove it explicitly.

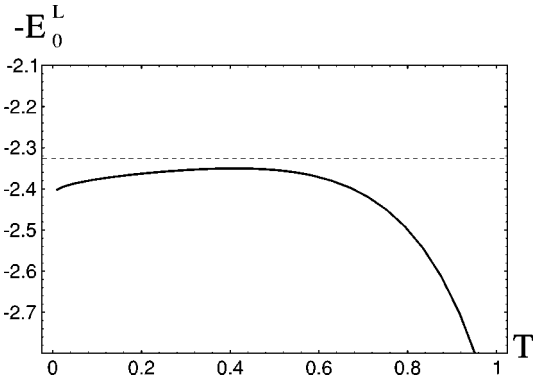

FIG. 3. Linear oscillator. Sum rule for the ground state energy. The dashed line corresponds to the exact value $E_{0\rm exact}^L=2.338$. The energy threshold $E_c=3.4$.

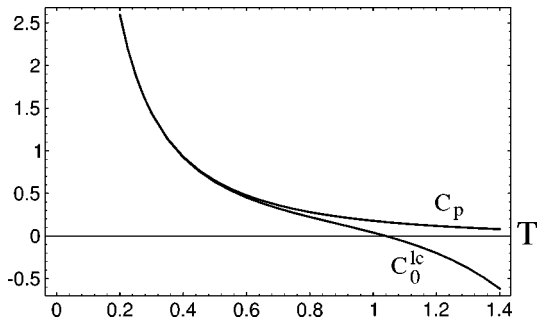

FIG. 4. Linear + Coulomb potential. Duality for the two-point function. Continuum functions $C_0^{\rm lc}$ and $C_{\mathbf{p}}^{\rm lc}$ are plotted vs T. The energy threshold $E_c=2.9$.

Following the procedure described above, the continuum functions $C_0^{\rm lc}(T)$ and $C_{\mathbf{p}}^{\rm lc}(T)$ (2.9) and (2.10) were computed and plotted for the optimal threshold $E_c=2.9$ (Fig. 4). Within the window $(0.2\leqslant T\leqslant 0.9)$, the duality is valid and we can study the sum rules. Figure 5 presents the sum rule for the ground state energy. The sum rule result $E_0^{\rm lc}=1.90\pm 0.05$ matches well the exact one $E_{0\rm exact}^{\rm lc}=1.83$.

III. DUALITY IN THREE-POINT SUM RULES

The goal of the present section is to study the issue of the duality in the three-point sum rules. While we have seen in the previous chapter that the duality holds for the two-point sum rules (in quantum mechanics at least), the situation for the three-point sum rules is clearly more complicated. Indeed, let us recall the general procedure of the analysis of three-point sum rules in QCD [19]. One considers the function $f(q_1^2,q_2^2)$ at the appropriate tensor structure of the polarization operator of three currents. One can calculate this function in two ways: using operator product expansion (the theoretical part of the sum rule) and saturating by resonances (the phenomenological part of the sum rule). For simplicity we shall restrict ourselves here by transitions between the same hadron under the action of some current. Then, after the Borel transformation in variables q_1^2,q_2^2 the phenomenological part of the sum rule can be represented as

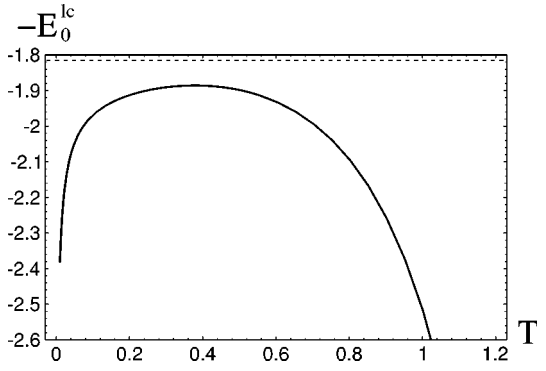

FIG. 5. Linear + Coulomb potential. Sum rule for the ground state energy. The dashed line corresponds to the exact value $E_{0\rm exact}^{\rm lc}=1.828$. The energy threshold $E_c=2.9$.

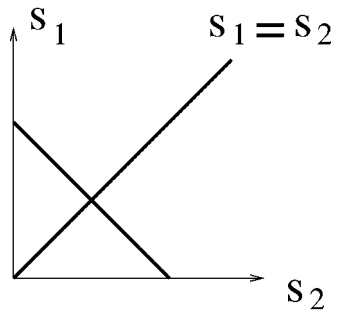

FIG. 6. (s_1,s_2) plane. Direction orthogonal to the diagonal is a direction of the integration for the generalized duality.

$$f(M_1^2,M_2^2)=g\beta_0^2 e^{-m^2(1/M_1^2+1/M_2^2)} + \int ds_1 ds_2 \sigma_h(s_1,s_2) e^{-s_1/M_1^2-s_2/M_2^2}. \tag{3.1}$$

Here g is the relevant coupling constant, β_0^2 is the square of the residue of the lowest lying resonance created by the current, m is its mass and M_1^2,M_2^2 are the relevant Borel parameters. Local duality means that the latter integral, taken over some part of the (s_1,s_2) plane is well approximated by the corresponding integral of the imaginary part $\sigma_{\mathbf{p}}(s_1,s_2)$ of the theoretical part of the sum rule (calculated using Wilson OPE). Even if the local duality does not hold, for the three-point sum rules it is possible to have the generalized duality. In fact, if there are sign alternating transition, it was argued in Ref. [17] that it may be senseless to speak about local duality. The parton model density is likely to be concentrated in the narrow area around the diagonal of the (s_1,s_2) plane, while the hadron density is spread over the whole plane. It was shown in Ref. [17] that in this case only ''generalized'' duality makes sense: the partonic spectral density, integrated in the direction orthogonal to the diagonal is approximately equal to the hadron spectral density, integrated in the same direction:

$$f(M^2)=g\beta_0^2 e^{-m_0^2(1/M^2)}+\int_{s_0}^{\infty} ds\sigma_h(s)e^{-s/M^2}. \tag{3.2}$$

Here $\sigma_h(s)$ is given by the integral

$$\sigma_h(s)=\int dA\sigma_h(s_1,s_2), \tag{3.3}$$

$A=(s_1-s_2)/2$. The parameter s_0 is the continuum threshold (see Fig. 6). The Borel parameters are $M^2=M_1^2=M_2^2$ and we stick to the symmetric point. We can define the parton spectral density exactly in the same way:

$$f(M^2)=\int \sigma_{\mathbf{p}}(s)e^{-s/M^2}ds. \tag{3.4}$$

Only for such sum rule there is a hope that the duality (defined in this generalized sense) is not violated.

Consider now the quantum mechanical analogue of the sum rule (3.2). The analogue of the polarization operator of three currents in quantum mechanics is the function [17]:

$$S_i(\tau_1,\tau_2)=\int d^3r K(0,\tau_1+\tau_2,r,\tau_1)\hat{O}_i(r)K(r,\tau_1,0,0)$$
$$=\int d^3r\sum_l e^{-E_l\tau_2}\psi_l(0)\psi_l^*(r)\hat{O}_i(r)$$
$$\times\sum_n e^{-E_n\tau_1}\psi_n(r)\psi_n^*(0). \quad (3.5)$$

Here $K(r,\tau_1,0,0)$ is the amplitude of the quark propagation from the point (0,0) to the point (r,τ_1) in the Euclidean time. At the point (r,τ_1) the operator $\hat{O}_i$ is inserted.

The vacuum expectation value (VEV) of the operator $\hat{O}_i$ is defined as

$$\langle 0|\hat{O}_i|0\rangle=\int d^3r\psi_0^*(r)\hat{O}_i(r)\psi_0(r). \quad (3.6)$$

The corresponding sum rule can be written as

$$S_i(\tau_1,\tau_2)=|\psi_0(0)|^2\langle 0|\hat{O}_i|0\rangle e^{-E_0(\tau_1+\tau_2)}$$
$$+\int_{s_0}^{\infty}ds_1ds_2\sigma_h(s_1,s_2)e^{-s_1\tau_1-s_2\tau_2}. \quad (3.7)$$

Here, $\sigma_h(s_1,s_2)$ is the exact spectral density. By $\sigma_{\mathbf{p}}(s_1,s_2)$ we denote the theoretical spectral function obtained, as for the two-point functions, from the T-expansion of S_i. From above it is clear that we must study the symmetric sum rules, i.e., $\tau_1=\tau_2=T/2$ (see Ref. [17] for details). We need to compare the $\sigma_{\mathbf{p}}(s)$ integrated with the weight e^{-sT}, with the corresponding integral of $\sigma_h(s)$. Here $s=(s_1+s_2)/2$, and $\sigma_{h,p}(s)$ are the spectral densities obtained from the spectral densities $\sigma_{h,p}(s_1,s_2)$ after the integration over the variable $A=(s_1-s_2)/2$ in the same way as in eq. (3.3).

Let us now consider the sum rules and duality for three models considered above and for the operators:

$$\hat{O}_1(r)=r^2/6;\quad \hat{O}_2(r)=-\partial^2;\quad \hat{O}_3(r)=r.$$

A. Harmonic oscillator

Since the operator $\hat{O}_3$ has no analogue in QCD, the sum rules only for two operators $\hat{O}_1$ and $\hat{O}_2$ will be discussed. These sum rules were already investigated in Refs. [17, 18]. For the sake of completeness their analysis is included and extended:

$$S_1^{\rm har}(T/2,T/2)=\frac{1}{32\pi^{3/2}}\frac{1}{T^{1/2}}\left(1-\frac{1}{3}T^2+\frac{44}{640}T^4-\frac{692}{60480}T^6+\cdots\right);$$

$$S_2^{\rm har}(T/2,T/2)=\frac{3/2}{(4\pi)^{3/2}}\frac{1}{T^{5/2}}\left(1-\frac{1}{6}T^2+\frac{5}{288}T^4+\cdots\right). \quad (3.8)$$

The corresponding parton spectral density is

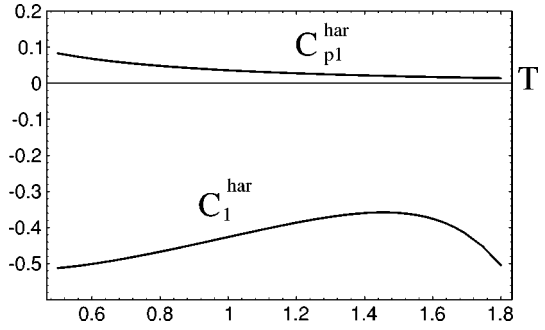

FIG. 7. Harmonic oscillator. Duality for the three-point function. Insertion of the operator $\hat{O}_1$. Continuum functions $C_1^{\rm har}$ and $C_{\rm p1}^{\rm har}$ are plotted vs T. The energy threshold $E_c=2$.

$$\sigma_{\rm p1}^{\rm har}(E)=\frac{1}{32\pi^2}E^{-1/2};$$

$$\sigma_{\rm p2}^{\rm har}(E)=\frac{3/2}{(4\pi)^{3/2}}\left(\frac{4}{3\sqrt{\pi}}E^{3/2}-\frac{1}{6\sqrt{\pi}}E^{-1/2}\right). \quad (3.9)$$

In order to check duality let us, as for the two-point functions, Eqs. (2.9) and (2.10), define the appropriate continuum functions:

$$C_i(T)=\frac{S_i(T/2,T/2)-|\psi_0(0)|^2e^{-E_0T}\langle 0|\hat{O}_i|0\rangle}{|\psi_0(0)|^2e^{-E_0T}\langle 0|\hat{O}_i|0\rangle};$$

$$C_{\rm pi}(T)=\frac{\int_{E_c}^{\infty}dE\sigma_{\mathbf{p}}(E)e^{-ET}}{|\psi_0(0)|^2e^{-E_0T}\langle 0|\hat{O}_i|0\rangle}. \quad (3.10)$$

We shall use here the exact answers for the harmonic oscillator:

$$E_0^{\rm har}=3/2;\quad |\psi_0^{\rm har}(0)|^2=1/(2\pi)^{3/2};$$

$$\langle 0|\hat{O}_1|0\rangle_{\rm har}=1/3E_0^{\rm har};\quad \langle 0|\hat{O}_2|0\rangle_{\rm har}=1/2E_0^{\rm har}. \quad (3.11)$$

Let us consider the sum rules for the matrix elements $\langle 0|\hat{O}_1|0\rangle_{\rm har}$ and $\langle 0|\hat{O}_2|0\rangle_{\rm har}$. For the operator $\hat{O}_1$ both functions $C_1^{\rm har}$ and $C_{\rm p1}^{\rm har}$ are depicted for the energy threshold $E_c=2$ (Fig. 7). An important fact is immediately noticeable. The true continuum is negative and cannot be approximated by any positive asymptotics. Hence, no duality persists up to the first exited state. Note also that although the continuum contribution to the theoretical part of the sum rule is almost negligible (less than 5%), the real contribution of the exited states is significant and is about 40% of the ground state. Our choice of $E_c=2$ was motivated by standard guess—it is a midpoint between two observed states $E_0=3/2$ and $(E_0+E_1)/2=5/2$. The displayed picture is not sensitive to the threshold variation and the duality is broken for any threshold parameter.

Let us illustrate how duality breaking becomes fatal for the sum rule. The sum rule is obtained by transforming the continuum in equation (3.7) to the LHS and then dividing the expression by the two-point sum rule [Eq. (2.11)]:

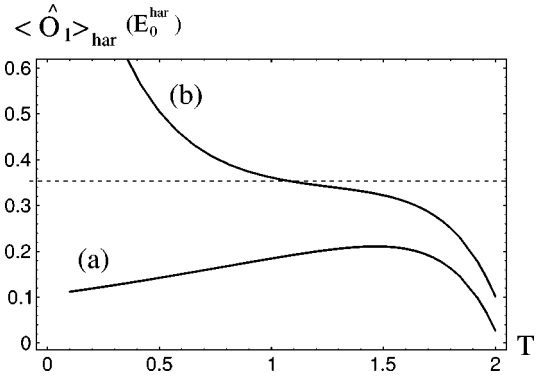

FIG. 8. Harmonic oscillator. (a) Three-point sum rules for VEV of the operator $\hat{O}_1$. The energy threshold $E_c=2$. (b) Three-point sum rules for VEV of the operator $\hat{O}_1$ with N=3 explicitly taken resonances. The energy threshold $E_c^N=5$. The dashed line corresponds to the exact value $\langle 0|\hat{O}_1|0\rangle_{\text{exact}}^{\text{har}}=1/3E_0^{\text{har}}$

$$\langle 0|\hat{O}_i|0\rangle=\frac{S_i(T/2,T/2)-\int_{E_c}^{\infty}dE\sigma_{\mathbf{pi}}(E)e^{-ET}}{S_0(T)-\int_{E_c^0}^{\infty}dE\sigma_{\mathbf{p0}}(E)e^{-ET}}. \quad (3.12)$$

Here S_i and S_0 are obtained by perturbation theory. Like in the previous case of two-point functions we have to determine a fiducial region. The same conditions of the ground state dominance and the control over the power expansions are applied. Of course, for the three-point sum rule (3.12) we obtain two fiducial domains. One corresponds to the numenator (three-point part) and another—to the denominator (two-point part). The final window is then obtained in the matching region. In some three-point sum rules discussed below continuum contributions are not small. Thus, the standard prescription of the method (requirement for the continuum to be less than 30%) may lead to a situation when the window almost shrinks to a point. In such cases we increase the bound up to 50%. An important notice is that three-point sum rules (3.12) depend on two (in general independent) threshold parameters E_c^0 and E_c. While E_c^0 has to be fitted by the corresponding two-point sum rule, E_c is a varying parameter of the three-point sum rules. In practice, one usually takes both thresholds equal. Below we present some arguments showing that in reality E_c is likely to be less than its two-point partner E_c^0. Like for the operator $\hat{O}_1$ (see above), in all three-point functions, which display the duality breaking, we take for the energy threshold E_c the value, which is somewhat close to midpoint between two lowest observed states. In all these cases, the sum rules appear to be almost nonsensitive to the threshold variation and our main conclusions on duality violation remain to be valid.

Figure 8(a) shows the sum rule for $\langle 0|\hat{O}_1|0\rangle_{\text{har}}$ with the energy threshold $E_c=2.5$. Within the window $(0.5\leqslant T\leqslant 1.6)$ the answer given by the sum rule is about 45% off from the exact one (3.11). Thus, the sum rule leads to completely wrong prediction. In the Ref. [18], it was argued that this failure is due to nondiagonal transitions, which are negative and numerically large. These transitions are not sufficiently suppressed and they produce a strong influence on the sum rule. Sign alternating nature of the exact spectral density $\sigma_h(E)$ leads to the duality breaking at high energies. To

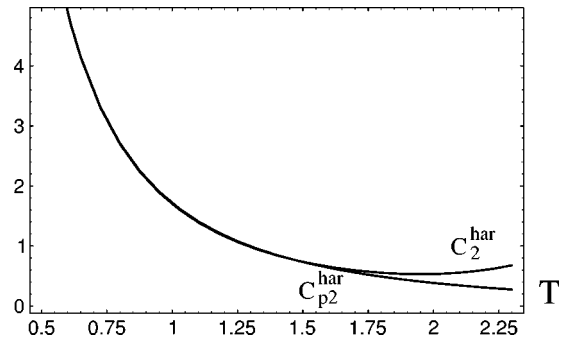

FIG. 9. Harmonic oscillator. Duality for the three-point function. Insertion of the operator $\hat{O}_2$. Continuum functions C_2^{har} and $C_{\text{p2}}^{\text{har}}$ are plotted vs T. The energy threshold $E_c=2$.

illustrate the point, we include in the sum rule explicitly a few low lying resonances. Correspondingly, the continuum threshold E_c rises:

$$\int_{E_c}^{\infty}dE\sigma_{\mathbf{pi}}(E)e^{-ET}\rightarrow\sum_{k=1}^{N}a_k^i e^{-\widetilde{E}_k T}+\int_{E_c^N}^{\infty}dE\sigma_{\mathbf{pi}}(E)e^{-ET}. \quad (3.13)$$

Here, k runs over a number N of the first low lying resonances in equation (3.5). The residues are denoted by a_k^i, while the resonances are ordered by their energy levels $\widetilde{E}_k$. For the case of the harmonic oscillator exact analytic expressions for the energy levels $\widetilde{E}_k$ and the residues of the interest a_k^i are known [17]. The energy $\widetilde{E}_k=3/2+k$:

$$a_k^1=\frac{1}{(2\pi)^{3/2}}\frac{4k+3}{6}\frac{1}{2^{2k}}\frac{(2k+1)!}{(k!)^2}, \quad k \text{ even};$$

$$a_k^1=(-1)\frac{1}{(2\pi)^{3/2}}\frac{2k+3}{3}\frac{1}{2^{2k}}\frac{(2k+1)!}{(k!)^2}, \quad k \text{ odd}.$$

The sign alternating nature of the exact spectral function is clearly observed.

We expect the energy threshold E_c^N to be of order $\widetilde{E}_N$—the energy of the last explicitly taken resonance. Sum rule with continuum of the form (3.13) is depicted for $N=3$ on Fig. 8(b) $(E_c^N=5)$. The desired plateau is clearly restored.

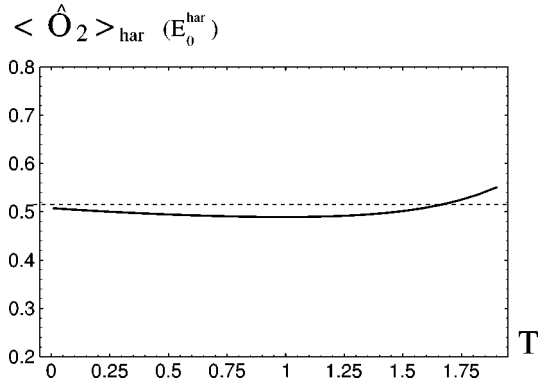

FIG. 10. Harmonic oscillator. Three-point sum rules for VEV of the operator $\hat{O}_2$. The dashed line corresponds to the exact value $\langle 0|\hat{O}_2|0\rangle_{\text{exact}}^{\text{har}}=1/2E_0^{\text{har}}$. The energy threshold $E_c=2$.

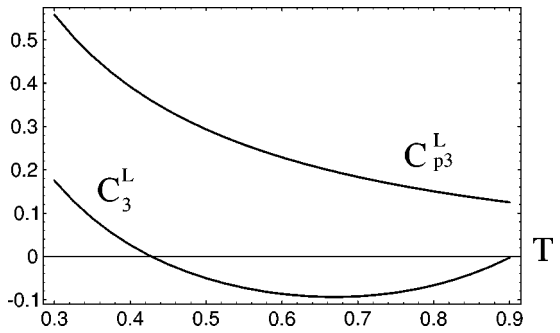

FIG. 11. Linear oscillator. Duality for the three-point function. Insertion of the operator $\hat{O}_3$. Continuum functions C_3^L and $C_{\rm p3}^L$ are plotted vs T. The energy threshold $E_c=3.3$.

The situation with the operator $\hat{O}_2$ strongly differs from the picture described above. Despite the fact that the nondiagonal transitions are not vanishing, they are of the same sign as the diagonal. Here we use the equation of motion to obtain residues a_k^2:

$$a_k^2=\frac{\widetilde{E}_k}{(2\pi)^{3/2}}-\frac{3}{2}a_k^1, \quad k \text{ even}; \quad a_k^2=(-1)\frac{3}{2}a_k^1, \quad k \text{ odd}.$$

(The factor 3/2 in front of a_k^1 is due to definition of the operator $\hat{O}_1$.)

Thus, the exact spectral function is always positive. This fact will be shown to be crucial for the duality to hold. In order to check the duality, the continuum functions $C_2^{\rm har}$ and $C_{\rm p2}^{\rm har}$, Eq. (3.10), are depicted (Fig. 9) for the optimal energy threshold $E_c=2$. Both functions match excellently and the duality is established. Consequently, the sum rule for VEV of the kinetic energy operator can be investigated (Fig. 10). An important remark is in order. In the case at hand, continuum dominates in the sum rule (it saturates more than 50%) and the window is practically absent. The obtained sum rule displays strong sensitivity to the continuum threshold. Thus, such a behavior of the sum rule is much like the one obtained in QCD [12]. Nevertheless, fitting the energy threshold, the exact result (3.11) can be easily reproduced: $\langle 0|\hat{O}_2|0\rangle_{\rm har}=0.5$ at $E_c=2$. At this point we disagree with the conclusions of Ref. [18] on the sum rule failure. In this paper, the continuum threshold was taken the same as for the two-point function $E_c=2.5$. However, in the three-point function at hand, $E=2.5$ is the energy level of the first nondiagonal transition state. Consequently, for the sum rule the value of the energy threshold eventually has to be taken lower.

B. Linear oscillator

With the same emphasise on duality, let us investigate the three-point functions for the linear potential. The operator $\hat{O}_3$ plays now a role of the potential and it is the virial partner of the operator $\hat{O}_2$. Since no exact propagator is known, asymptotic expansions of the three-point functions [Eq. (3.5)] are obtained perturbatively. Details of this computations are presented in Appendix B:

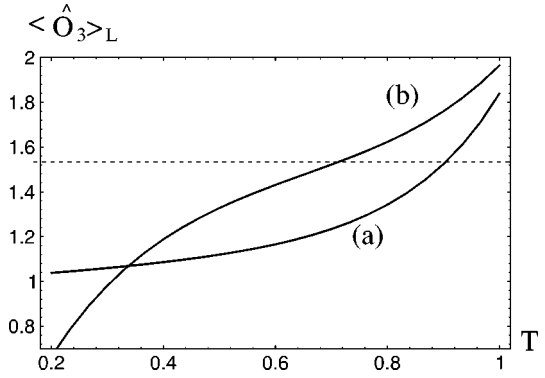

FIG. 12. Linear oscillator. (a) Three-point sum rules for VEV of the operator $\hat{O}_3$. The energy threshold $E_c=3.3$. (b) Three-point sum rules for VEV of the operator $\hat{O}_3$ with $N=11$ explicitly taken resonances. The energy threshold $E_c^N=7$. The dashed line corresponds to the exact value $\langle 0|\hat{O}_3|0\rangle_{\rm exact}^L=1.559$.

$$S_1^L(T/2,T/2)=\frac{1}{32\pi^{3/2}}\frac{1}{T^{1/2}}\left(1-\left(\sqrt{\pi}-\frac{4}{3\sqrt{\pi}}\right)T^{3/2}+\left(\frac{247}{128}-\frac{16\sqrt{2}}{15}\right)T^3+\cdots\right);$$

$$S_2^L(T/2,T/2)=\frac{3}{16\pi^{3/2}}\frac{1}{T^{5/2}}\left(1-\frac{4}{3\sqrt{\pi}}T^{3/2}+\left(\frac{52\sqrt{2}}{45}-\frac{1664}{1440}\right)T^3+\cdots\right);$$

$$S_3^L(T/2,T/2)=\frac{1}{4\pi^2}\frac{1}{T}\left(1-\left(\sqrt{2\pi}-\frac{7\sqrt{\pi}}{8}\right)T^{3/2}+\left(\frac{35}{48}-\frac{5\pi}{64}\right)T^3+\cdots\right). \tag{3.14}$$

The corresponding parton spectral densities are

$$\sigma_{\rm p1}^L(E)=\frac{1}{32\pi^2}E^{-1/2};$$

$$\sigma_{\rm p2}^L(E)=\frac{3}{16\pi^{3/2}}\left(\frac{4}{3\sqrt{\pi}}E^{3/2}-\frac{4}{3\sqrt{\pi}}\right);$$

$$\sigma_{\rm p3}^L(E)=\frac{1}{4\pi^2}. \tag{3.15}$$

The general picture with the three-point sum rules for the linear oscillator is very similar to the one of the harmonic oscillator. We start from the operator $\hat{O}_3$ and check the duality first. Figure 11 shows the continuum functions C_3^L and $C_{\rm p3}^L$ [Eq. (3.10)]; $E_c=3.3$. Again, the true continuum is mostly negative and cannot be represented by the asymptotics. A best fit would be reached in the ''no continuum'' approximation. Consider now the sum rule for the VEV of the operator. The window for the linear oscillator is moved to the left: $0.3\leqslant T\leqslant 0.7$. Comparing to the exact numerical

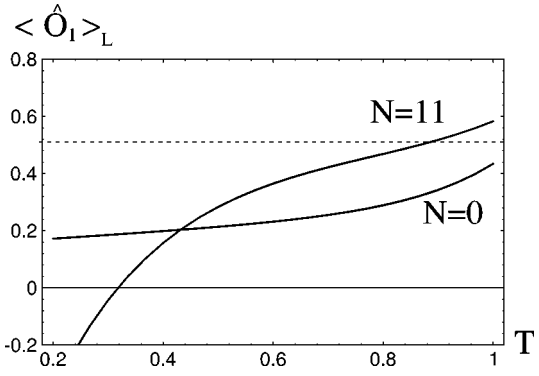

FIG. 13. Linear oscillator. Three-point sum rules for VEV of the operator $\hat{O}_1$ and the sum rule with $N=11$ explicitly taken resonances. The dashed line corresponds to the exact value $\langle 0|\hat{O}_1|0\rangle^L_{\text{exact}}=0.486$. The energy thresholds $E_c=3.3$, $E^N_c=7$.

result $\langle 0|\hat{O}_3|0\rangle^L_{\text{exact}}=1.559$, one $\langle 0|\hat{O}_3|0\rangle^L=1.1\pm 0.1$ yield by the sum rule [Fig. 12(a)] is 35% smaller. We account here for the situation, when the positive diagonal transitions almost cancel the negative nondiagonal matrix elements. This results in the fact that the large number of resonances must be explicitly taken in Eq. (3.13) in order for the sum rule to be saturated. Figure 12(b) presents the sum rule at $N=11$ (The numerical values for the residues a_k are given in the Tables II and III). The energy threshold is $E_c=7$ that lies between third and fourth energy levels.

Consider now the operator $\hat{O}_1$. The sum rule displays the same problem with duality as for the harmonic oscillator. No duality persists up to the first exited state. Figure 13 shows the sum rule for VEV of the operator together with an improved continuum model (3.13). The plateau is restored when $N=11$ transitions are taken explicitly.

Let us turn now to the three-point function with $\hat{O}_2$ operator inserted. The corresponding hadron spectral function is positive. As it was argued above, positive spectral functions do not cause duality breaking. The case at hand confirms this statement. Although the sum rule strongly depends on the continuum threshold parameter, it, nevertheless, yields the correct value (Fig. 14); $\langle 0|\hat{O}_2|0\rangle^L=0.81\pm 0.01$, compared to the exact $\langle 0|\hat{O}_2|0\rangle^L_{\text{exact}}=0.779$. The optimal energy threshold parameter is $E_c=2.8$, which is significantly below the threshold parameter corresponding to the two-point sum rule.

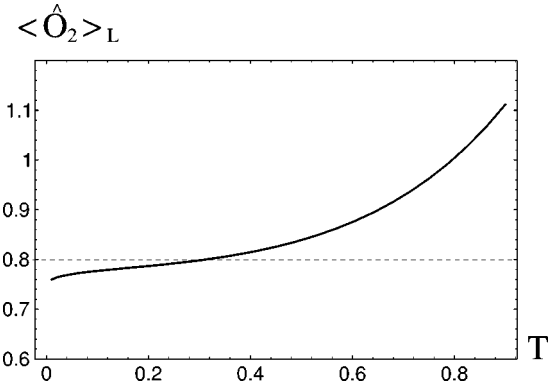

FIG. 14. Linear oscillator. Three-point sum rules for VEV of the operator $\hat{O}_2$. The dashed line corresponds to the exact value $\langle 0|\hat{O}_2|0\rangle^L_{\text{exact}}=0.779$. The energy threshold $E_c=2.8$.

As a common property of the three-point functions with sign changing spectral functions, we see that a large amount of resonances must be taken into account explicitly. In other words, no duality is valid in the low energy area. This effect is due to the fact, which was already mentioned. The window moves to the left, where exited states are not sufficiently suppressed.

C. Linear + Coulomb model

Coulomb term added to the potential improves a bit the situation slightly throwing out the window to the right. However, the general picture of the duality breaking still persists. We now present our results for the three-point sum-rules in ''linear + Coulomb'' potential. The three-point functions (3.5) are obtained by perturbation (see Appendix B):

$$S^{\text{lc}}_1(T/2,T/2)=\frac{1}{32\pi^{3/2}}\frac{1}{T^{1/2}}\left(1+b_0\frac{4(\pi-1)}{3\sqrt{\pi}}T^{1/2}+b_0^2 1.358T\right.$$
$$-b_0 1.179T^2-\left(\sqrt{\pi}-\frac{4}{3\sqrt{\pi}}\right)T^{3/2}$$
$$\left.+\left(\frac{247}{128}-\frac{16\sqrt{2}}{15}\right)T^3\right);$$

$$S^{\text{lc}}_2(T/2,T/2)=\frac{3}{16\pi^{3/2}}\frac{1}{T^{5/2}}\left(1+b_0\frac{4+2\pi}{3\sqrt{\pi}}T^{1/2}+b_0^2 0.985T\right.$$
$$-b_0 0.89T^2-\frac{4}{3\sqrt{\pi}}T^{3/2}$$
$$\left.+\left(\frac{52\sqrt{2}}{45}-\frac{1664}{1440}\right)T^3\right);$$

$$S^{\text{lc}}_3(T/2,T/2)=\frac{1}{4\pi^2}\frac{1}{T}\left(1+b_0 1.679T^{1/2}+b_0^2 1.477T\right.$$
$$-b_0 1.179T^2-\left(\sqrt{2\pi}-\frac{7\sqrt{\pi}}{8}\right)T^{3/2}$$
$$\left.+\left(\frac{35}{48}-\frac{5\pi}{64}\right)T^3\right). \qquad (3.16)$$

The corresponding spectral densities are

$$\sigma^{\text{lc}}_{\text{p1}}(E)=\frac{1}{32\pi^2}E^{-1/2};$$

$$\sigma^{\text{lc}}_{\text{p2}}(E)=\frac{3}{16\pi^{3/2}}\left(\frac{4}{3\sqrt{\pi}}E^{3/2}+b_0\frac{4+2\pi}{3\sqrt{\pi}}E+b_0^2 1.111E^{1/2}\right.$$
$$\left.-\frac{4}{3\sqrt{\pi}}-b_0 0.50E^{-1/2}\right);$$

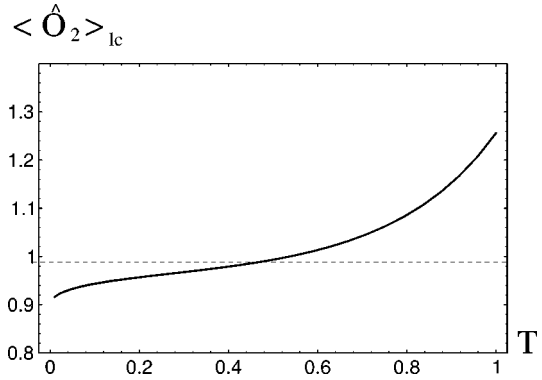

FIG. 15. Linear + Coulomb model. Three-point sum rules for VEV of the operator $\hat{O}_2$. The dashed line corresponds to the exact value $\langle 0|\hat{O}_2|0\rangle^L_{\rm exact}=0.972$. The energy threshold $E_c=2.4$.

$$\sigma^{\rm lc}_{\rm p3}(E)=\frac{1}{4\pi^2}(1+b_0 0.948E^{-1/2}). \tag{3.17}$$

The behavior of the sum rule for the kinetic energy operator is similar to ones of the harmonic and linear oscillators. Because of the positiveness of the exact spectral function no duality breaking is accounted for. Figure 15 shows the sum rule at the optimal energy threshold $E_c=2.4$. The exact value $\langle 0|\hat{O}_2|0\rangle^{\rm lc}_{\rm exact}=0.972$ and it matches well the one given by the sum rule: $\langle 0|\hat{O}_2|0\rangle^{\rm lc}=0.96\pm 0.02$.

Let us now consider the operators $\hat{O}_1$ and $\hat{O}_3$. Below we plot two graphs representing our results on the three-point functions with duality breaking. Figure 16 shows the usual sum rule and the sum rule with $N=5$ explicitly taken resonances ($E_c=5.5$) for VEV of the operator $\hat{O}_3$. Note that this operator is no more a virial partner of the operator $\hat{O}_2$. Taking five transitions explicitly we restore the plateau at the exact level $\langle 0|\hat{O}_3|0\rangle^{\rm lc}_{\rm exact}=1.401$. Figure 17 closes our analysis. It describes the sum rule and the sum rule with $N=5$ explicitly taken resonances for VEV of the operator $\hat{O}_1$; $\langle 0|\hat{O}_1|0\rangle^{\rm lc}_{\rm exact}=0.399$.

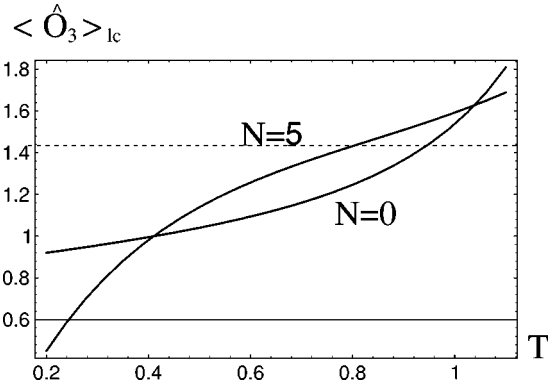

FIG. 16. Linear + Coulomb model. Three-point sum rules for VEV of the operator $\hat{O}_3$ and the sum rule with $N=5$ explicitly taken resonances. The dashed line corresponds to the exact value $\langle 0|\hat{O}_3|0\rangle^L_{\rm exact}=1.401$. The energy thresholds $E_c=2.9$, $E^N_c=5.5$.

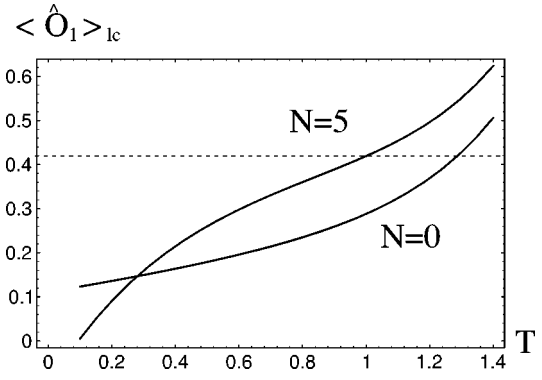

FIG. 17. Linear + Coulomb model. Three-point sum rules for VEV of the operator $\hat{O}_1$ and the sum rule with $N=5$ explicitly taken resonances. The dashed line corresponds to the exact value $\langle 0|\hat{O}_1|0\rangle^L_{\rm exact}=0.399$. The energy thresholds $E_c=2.9$, $E^N_c=5.5$.

IV. CONCLUSION

Motivated by the recent work on three-point sum rules in QCD, especially connected to heavy quark effective theory, we studied the two- and three-point sum rules in three different nonrelativistic quantum mechanical models with confining potentials. We have seen that though in all cases the two-point sum rules work perfectly well, the three-point sum rules may fail. Their predictions for the matrix elements of the operators $\hat{O}_{1,3}$ may differ by 30–50 % from the corresponding true values. The reason of the failure is the breakdown of duality. The theoretical spectral function is always smooth and positive. On the other hand, we have seen by explicit calculation that the ''phenomenological'' spectral density is wildly oscillating and even has a sign changing component due to the sign changing nondiagonal transitions. We have seen that, though the diagonal transitions between radial excitations have positive transition constants, the nondiagonal transitions always have negative sign. Consequently, duality does not work for the first several resonances, even if one understands the duality in a ''generalized'' sense (i.e., even after the integration in the direction orthogonal to the diagonal). The averaged ''hadron'' density strongly differs from the ''theoretical'' one. It seems that the duality starts to work for energies high enough. The corresponding threshold lies near the 3rd–4th resonance and depends on the model and on the operator. However, this is of no practical interest, because one cannot separate the leading resonance contribution in order to apply the sum rules. The standard continuum model does not work, and the smooth theoretical spectral density strongly differs from real one. For the two-point sum rules the whole situation is quite contrary: the duality does work in all known examples (Sec. II).

We have also seen that the absence of the continuum has no relation to the validity of the sum rule. In fact, small continuum contribution may arise from the mutual cancellation of positive (diagonal) and negative (nondiagonal) transitions. Such a behavior is displayed by the sum rules for the matrix elements of the operators r and r^2. Moreover, it seems that the absence of the continuum contribution is a general feature of the sum rules whose right hand side is contaminated by the sign changing transitions. On the other hand, sum rules for the matrix element of the operator $-\partial^2$

work sufficiently well, despite the fact that they are dominated by large continuum. The duality holds thanks to the positiveness of the spectral density. This property is very similar to that for the three-point sum rules for the transitions under the action of Hamiltonian, where the duality is also not violated [18].

One may conclude that the duality breaking is a general feature of the quantum mechanical analogue of the QCD sum rules in the case of sign changing nondiagonal transitions. Furthermore, our study suggests that the duality still holds if all transitions are positive. However, the continuum contribution to the sum rule is likely to be large and the result may heavily depend on the continuum threshold.

Our results were obtained for the nonrelativistic quantum mechanical models. It will be very interesting to check if our picture of the duality breaking still holds for the relativistic analogues of the models (A), (B), and (C).

Unfortunately, we do not know yet, what may be implications of our results in real QCD. Several conclusions, however, can be reached. First, one must be very careful in presence of the sign alternating transitions contributing to the polarization operators. Second, the smallness of the continuum contribution is not always of a good omen, and may occur due to complicated cancellations in the right hand side of the sum rule. Nevertheless, the situation in QCD may be considerably better. Borel transform in QCD may suppress nondiagonal transitions stronger than in quantum mechanics, and this may lead to the smallness of the contamination. Perhaps, the degree of contamination depends on the matrix element one calculates and the form of the sum rule chosen.

Certainly, further work is needed; in particular, it is desirable to investigate whether the complete relativistic calculation improves the situation with duality. Detail examination of whether in the three-point QCD and HQET sum rules there are indeed sign changing transitions is required. The problem of their relative suppression must be carefully studied.

Our work was essentially motivated by big discrepancies among values of the matrix element $\langle B|\vec{D}^2/(2m_B)|B\rangle$ obtained using different sum rules (see Refs. [11, 12]) Consequently, we work with simple nonrelativistic analogues for the potential models of B-mesons and study the sum rules for nonrelativistic analogues of the HQET operator $\vec{D}^2/(2m_B)$ (and operators related to $\vec{D}^2/(2m_B)$ by the virial theorem). Our results imply that the value for the matrix element of the operator $\vec{D}^2/(2m_B)$ from Ref. [11] is underestimated. On the other hand, the situation in quantum mechanics seems to be close to that considered in Ref. [12]. However, the strong dependence on the continuum threshold implies that the accuracy of the relevant sum rule may be quite low. Moreover, in the text, we argued that the energy threshold of the three-point sum rule should be taken lower (about 20%, according to our experience) than that of the corresponding two-point sum rule. The reason is that the contribution of the first nondiagonal transition to the polarization operator is suppressed by the factor $\exp[-(E_1+E_0)T/2]$, while the contributions of the first diagonal transition in the three-point sum rule as well as of the first excited state in the two-point sum rule are suppressed by $\exp[-E_1T]$. Consequently, we decided to reexamine the sum rule of Ref. [12]. In the latter, the energy threshold for the three-point sum rule ($\omega_0 \sim 1-1.2$ GeV) was taken exactly the same as for the two-point sum rule. We have seen above that this treatment of the sum rule may lead to a significant overestimation of the matrix element. We investigated the leading-order sum rule [Eqs. (3.10) and (3.12) of Ref. [12]] for the kinetic energy operator for various three-point thresholds $\omega_1 \sim 0.8$ GeV. The results of our analysis are depicted on Fig. 18. The sum rule yields the following value for the matrix element of the kinetic energy operator: $\langle B|\vec{D}^2/(2m_B)|B\rangle = -0.3 \pm 0.1$ GeV^2 compared to $\langle B|\vec{D}^2/(2m_B)|B\rangle = -0.6 \pm 0.1$ GeV^2 for $\omega_1 \sim 1$ GeV. The obtained value is in a good agreement with the ones obtained in Refs. [13, 14]. (Of course, this result must be considered not as a QCD sum rule prediction, but rather as an indication that there is no contradiction between Refs. [13, 14] and QCD sum rule approach.)

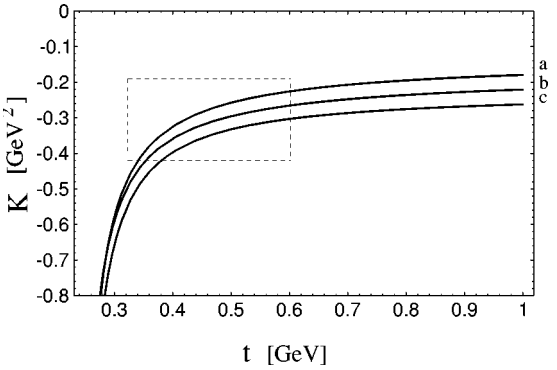

FIG. 18. The Ball and Braun sum rule to leading-order as a function of the Borel parameter t for different values of the continuum thresholds: (a) $\omega_0 = 1$ GeV, $\omega_1 = 0.7$ GeV; (b) $\omega_0 = 1.2$ GeV, $\omega_1 = 0.85$ GeV; (c) $\omega_0 = 1$ GeV, $\omega_1 = 0.8$ GeV. The dashed line indicates the working region.

Note added in proof. After this research was finished we learned that the quantum mechanical duality was studied in Ref. [24] for $S \to P$ transitions.

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APPENDIX A

In the first part of this appendix we present some known information about solutions of the three-dimensional Schrödinger equation in the case of linear potential. Three operators relevant to our study are defined. Their matrix elements are computed numerically and represented in tables. The sec-

TABLE I. The energy levels of quantum mechanical problem with linear potential.

E_0^L	E_1^L	E_2^2	E_3^L	E_4^L	E_5^L	E_6^L	E_7^L	E_8^L	E_9^L
2.338	4.088	5.521	6.787	7.944	9.023	10.040	11.009	11.936	12.829

TABLE II. Matrix elements of the operator $\hat{O}_1$ between states of linear potential problem.

$\langle i\|\hat{O}_1\|j\rangle_L$	0	1	2	3	4
0	0.486	0.427	−0.039	0.010	−0.004
1	0.427	1.485	0.950	−0.075	0.018
2	−0.039	0.950	2.709	1.556	−0.116
3	0.010	−0.075	1.556	4.094	2.223
4	−0.004	0.018	−0.116	2.223	5.610

TABLE III. Matrix elements of the operator $\hat{O}_3$ between states of linear potential problem.

$\langle i\|\hat{O}_3\|j\rangle_L$	0	1	2	3	4
0	1.559	0.653	−0.197	0.101	−0.064
1	0.653	2.725	0.974	−0.275	0.134
2	−0.197	0.974	3.680	1.248	−0.341
3	0.101	−0.275	1.248	4.524	1.493
4	−0.064	0.134	−0.341	1.493	5.296

ond part of this appendix is devoted to the same analysis for the linear + Coulomb potential.

1. Linear potential

We look for exact solutions of the three-dimensional Schrödinger equation:

$$[-\partial^2+\alpha r]\psi_n^L=\tilde{E}_n^L\psi_n^L. \tag{A1}$$

In the dimensionless variables,

$$E_n^L=\tilde{E}_n^L/\alpha^{2/3};\quad x=r\alpha^{1/3}.$$

Equation (A1) takes form

$$[-\partial^2+x]\psi_n^L=E_n^L\psi_n^L. \tag{A2}$$

Since the wave-function of the orbitally exited states are vanishing in the origin, we restrict our analysis to S-states only. The S-states of the equation (A2) are given by the normalized Airy functions:

$$\psi_n^L=\text{const}\times\text{Ai}(x-E_n)/x. \tag{A3}$$

Imposing condition of nonsingularity on the wave function in the origin, we obtain the discrete spectrum:

$$\text{Ai}(-E_n)=0.$$

Table I presents ten first energy levels of the problem. Taking into account the well-known fact (Ref. [20])

$$\text{Ai}(\xi\to-\infty)\sim\sin\left(\frac{2}{3}\xi^{2/3}+\pi/4\right),$$

we can determine a large n asymptotic behavior of the energy levels:

$$\frac{2}{3}(E_n)^{3/2}=n\pi,\quad n\to\infty. \tag{A4}$$

An important and very special property of the Airy functions is

$$|\psi_n^L(0)|^2=\frac{1}{4\pi}, \tag{A5}$$

and does not depend on n. However,

$$\psi_n^L(0)=(-1)^n\frac{1}{\sqrt{4\pi}}.$$

Below, we present the exact numerical results for the matrix elements of the following operators:

$$\hat{O}_1=x^2/6;\quad \hat{O}_2=-\partial^2;\quad \hat{O}_3=x.$$

The matrix elements are defined:

$$M_k^{ij}\equiv\langle i|\hat{O}_k|j\rangle\equiv\int d^3x\,\psi_i^{L*}(x)\hat{O}_k(x)\psi_j^L(x). \tag{A6}$$

Note that by the equation of motion

$$M_2^{ij}=E_i\delta^{ij}-M_3^{ij}. \tag{A7}$$

The lowest quarters of the matrices M_1 and M_3 age given by Tables II and III respectively.

2. Linear + Coulomb potential

$$[-\partial^2+V(r)]\psi_n^{\text{lc}}=\tilde{E}_n^{\text{lc}}\psi_n^{\text{lc}}. \tag{A8}$$

The potential $V(r)$ is taken of the form

$$V(r)=\alpha r-\frac{4}{3}\frac{\alpha_s(r)}{r};\quad \alpha_s(r)=\frac{2\pi}{9\ln(\delta+\gamma/r)}. \tag{A9}$$

Here, $\alpha_s(r)$ is a running coupling constant and it reflects the asymptotic freedom of the strong interaction. In order to pass to dimensionless variables the following change of variables is performed:

$$x=(2m\alpha)^{1/3}r;\quad E^{\text{lc}}=\tilde{E}_n^{\text{lc}}(2m/\alpha^2)^{1/3}.$$

In the new variables equation (A8) reads

$$[-\partial^2+x-b(x)/x]\psi_n^{\text{lc}}=E_n^{\text{lc}}\psi_n^{\text{lc}};$$

$$b(x)=\frac{8\pi(4m^2/\alpha)^{1/3}}{27\ln(\delta+\gamma(2m\alpha)^{1/3}/x)}. \tag{A10}$$

TABLE IV. The energy levels of quantum mechanical problem with linear + Coulomb potential.

E_0^{lc}	E_1^{lc}	E_2^{lc}	E_3^{lc}	E_4^{lc}	E_5^{lc}	E_6^{lc}	E_7^{lc}	E_8^{lc}	E_9^{lc}
1.828	3.745	5.245	6.551	7.735	8.833	9.865	10.845	11.783	12.684

In Appendix B we present a perturbative determination of the propagator for the given problem. Unfortunately, this procedure cannot be performed (at least analytically) if the running logarithm is present. Since we do not wish to introduce any uncertainty due to the logarithm factorization, we fix the running coupling constant α_s at some value. A special choice of the parameters is not important for our analysis, but to be concrete we choose some quasirealistic ones [22]: the string tension $\alpha=0.14$ GeV^2; $\delta=2$; $\gamma=1.87$ GeV^{-1}; for the constituent mass we take $m=0.35$ GeV. When related problems are investigated within QCD the running coupling constant α_s is usually of order 0.3. In our study α_s is set to be 0.28, which corresponds to some fixed point in space, namely $x_0=0.088$. In the text, a parameter b_0 is used and it is defined as $b_0 \equiv b(x_0)$ (A10).

In order to solve equation (A10), we consider the Coulomb potential as a perturbation. Solution of the nonperturbed problem was described above. The Hamilton (A10) is diagonalized in the basis of the wave functions (A3):

$$\psi_n^{\rm lc}(x)=\sum_{k=1}^{N_L} C_n^k \psi_k^L(x). \tag{A11}$$

In order to obtain the low energy spectrum it is sufficient to take into account only a few levels (here N_L denotes the number of levels). Table IV presents eigenvalues computed by the numerical diagonalization.

Products of the wave functions in the origin are

$$C_{nm}^{\rm lc} \equiv \psi_n^{\rm lc}(0)\psi_m^{\rm lc*}(0)=\frac{1}{4\pi}\sum_{k,k'}^{N_L} C_n^k C_m^{k'}(-1)^{k+k'}. \tag{A12}$$

Coefficients $C_{nm}^{\rm lc}$ appear to be slowly changing monotonic functions of the number of levels N_L taken into account in equations (A11) and (A12). In order to extract these coefficients $N_L=60$ was taken and then the results were extrapolated. To illustrate the point we plot $C_{00}^{\rm lc}$ as a function of N_L (Fig. 19). This function slowly approaches (as a power law) to its limiting value $C_{00}^{\rm lc}=1.83$. Table V is devoted to the coefficients $C_{nm}^{\rm lc}$.

As for the linear potential, we present tables of numerically calculated exact matrix elements (Tables VI, VII, and VIII).

APPENDIX B

In this appendix, we present a perturbative derivation of propagators and matrix elements of interest. The potential $V(r)=\alpha r-\beta/r$ is considered as a perturbation to the free particle motion. To study ''pure'' linear potential β must be set to zero.

The ''free'' propagator is given by

$$G_0(r,T,r',t')=\frac{1}{[4\pi(T-t')]^{3/2}}\exp\left[-\frac{(\vec{r}-\vec{r'})^2}{4(T-t')}\right]. \tag{B1}$$

Note that in our units $2m=1$. This factor can be always restored in final expressions. The exact propagator $G^{\rm ex}$ can be expressed in a formal perturbation series:

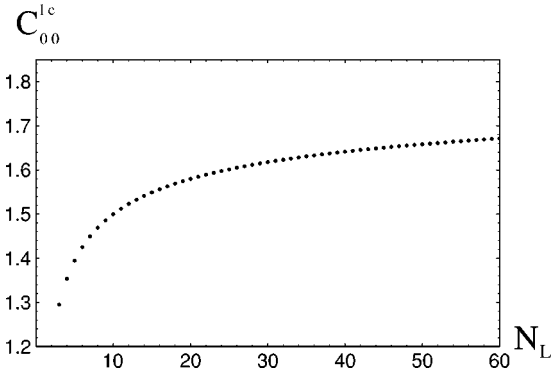

FIG. 19. $C_{00}^{\rm lc}$ as a function of N_L.

TABLE V. Coefficients $C_{nm}^{\rm lc}$.

$C_{nm}^{\rm lc}$	0	1	2	3	4
0	1.83	−1.66	−1.58	1.53	1.47
1	−1.66	1.50	1.44	−1.39	−1.35
2	−1.58	1.44	1.37	−1.34	−1.31
3	1.53	−1.39	−1.34	1.32	1.28
4	1.47	−1.35	−1.31	1.28	1.25

TABLE VI. Matrix elements of the operator $\hat{O}_1$ between states of linear + Coulomb potential problem.

$\langle i\vert\hat{O}_1\vert j\rangle_{\rm lc}$	0	1	2	3	4
0	0.399	0.367	0.042	−0.013	−0.006
1	0.367	1.363	−0.874	0.079	0.021
2	0.042	−0.874	2.569	1.471	0.120
3	−0.013	0.079	1.471	3.942	−2.136
4	−0.006	0.021	0.120	−2.136	5.448

TABLE VII. Matrix elements of the operator $\hat{O}_2$ between states of linear + Coulomb potential problem.

$\langle i\vert\hat{O}_2\vert j\rangle_{\rm lc}$	0	1	2	3	4
0	0.972	−0.810	−0.338	0.216	0.162
1	−0.810	1.483	1.093	−0.390	−0.237
2	−0.338	1.093	1.934	−1.348	−0.445
3	0.216	−0.390	−1.348	2.342	1.582
4	0.162	−0.237	−0.445	1.582	2.718

TABLE VIII. Matrix elements of the operator $\hat{O}_3$ between states of linear + Coulomb potential problem.

$\langle i\vert\hat{O}_3\vert j\rangle_{\rm lc}$	0	1	2	3	4
0	1.401	0.607	0.197	−0.105	−0.068
1	0.607	2.615	−0.929	0.271	0.136
2	0.197	−0.929	3.591	1.206	0.336
3	−0.105	0.271	1.206	4.447	−1.454
4	−0.068	0.136	0.336	−1.454	5.227

$$G^{\rm ex}(r,T,r',t')=G_0(r,T,r',t')-\int d^3s\int_{t'}^{T}d\tau G_0(r,T,s,\tau)V(s)G_0(s,\tau,r',t')$$
$$+\int d^3s\int_{t'}^{T}d\tau\int d^3s'\int_{t'}^{\tau}d\tau' G_0(r,T,s,\tau)V(s)G_0(s,\tau,s',\tau')V(s')G_0(s',\tau',r',t')+\cdots . \quad \text{(B2)}$$

In the following analysis we retain only terms up to the second order in perturbation. All terms of the order $\alpha^3,\beta^3,\alpha\beta^2,\alpha^2\beta$ and higher will be systematically omitted. We denote:

$$G^{\rm ex}=G_0+\alpha G_1^L+\alpha^2G_2^L+\beta G_1^C+\beta^2G_2^C+2\alpha\beta G_2^{\rm LC};$$
$$G_1^L=\langle G_0|s|G_0\rangle;\quad G_2^L=\langle G_1^L|s|G_0\rangle;\quad G_1^C=\langle G_0|1/s|G_0\rangle;$$
$$G_2^C=\langle G_1^C|1/s|G_0\rangle;\quad G_2^{\rm LC}=\langle G_1^L|1/s|G_0\rangle. \quad \text{(B3)}$$

Here, brackets denote the integration over s and t defined in Eq. (B2). Unfortunately, these integrations cannot be done in closed form. However, the time integration can be performed. This is done with the aid of the following integrals (Ref. [23]):

$$\int_0^t d\tau\frac{1}{[(t-\tau)\tau]^{3/2}}\exp\left[-\frac{x^2}{(t-\tau)}-\frac{y^2}{\tau}\right]=\frac{\sqrt{\pi}}{t^{3/2}}\frac{x+y}{xy}\exp\left[-\frac{(x+y)^2}{t}\right];$$
$$\int_0^t d\tau\frac{1}{(t-\tau)^{3/2}\tau^{1/2}}\exp\left[-\frac{x^2}{(t-\tau)}-\frac{y^2}{\tau}\right]=\frac{\sqrt{\pi}}{x\sqrt{t}}\exp\left[-\frac{(x+y)^2}{t}\right].$$

After the time integration is removed, the angular integration can be easily performed as well. The remaining integrals over the radial components are of a nonreducible form, and we cannot proceed any further. Finally we arrive at the following expressions for the propagator components:

$$G_1^L(r,T,0,t')=-\frac{1}{(4\pi)^{3/2}}\frac{r}{2(T-t')^{1/2}}\left[\int_1^\infty dx e^{-x^2r^2/4(T-t')}+e^{-r^2/4(T-t')}\right];$$
$$G_2^L(r,T,0,t')=-\frac{1}{32\pi^{3/2}}\frac{1}{(T-t')^{1/2}}\int_0^\infty\frac{ds s^4}{r}\int_1^\infty dw\left[e^{-(r+s(1+w))^2/4(T-t')}-e^{-(|r-s|+ws)^2/4(T-t')}\right.$$
$$\left.+\int_1^\infty dy y(e^{-(r+s(1+wy))^2/4(T-t')}-e^{-(|r-s|+wys)^2/4(T-t')})\right];$$
$$G_1^C(r,T,0,t')=-\frac{1}{(4\pi)^{3/2}}\frac{1/r}{(T-t')^{1/2}}\int_0^\infty\frac{dx}{x}[e^{-r^2(1+2x)^2/4(T-t')}-e^{-r^2(|1-x|+x)^2/4(T-t')}];$$
$$G_2^C(r,T,0,t')=-\frac{1}{(4\pi)^{3/2}}\frac{1}{2(T-t')^{1/2}}\int_0^\infty\frac{ds}{r}\int_0^\infty\frac{dx}{x}\int_{|1-x|+x}^{1+2x}dw[e^{-(r+s(1+w))^2/4(T-t')}-e^{-(|r-s|+ws)^2/4(T-t')}];$$
$$G_2^{\rm LC}(r,T,0,t')=-\frac{1}{(4\pi)^{3/2}}\frac{1}{2(T-t')^{1/2}}\int_0^\infty\frac{ds s^2}{r}\int_1^\infty dw\left[\int_1^\infty dx x(e^{-(r+s(1+wx))^2/4(T-t')}-e^{-(|r-s|+wxs)^2/4(T-t')})\right.$$
$$\left.+e^{-(r+s(1+w))^2/4(T-t')}-e^{-(|r-s|+ws)^2/4(T-t')}\right].$$

In order to obtain equations (2.15) and (2.17), the limit $r\to 0$ is taken. At this limit the propagator components (B3) are completely calculated analytically and results of Ref. [21] are recovered.

Having in our disposal the exact propagator (B3), we can proceed in computing the matrix element of the operators $\hat{O}_i(r)$. We are interested only in the following amplitudes: the free propagation from the point (0,0), in the point $(r,T/2)$, where the operator $\hat{O}_i(r)$ is inserted, and then there is the free motion to the point $(0,T)$.

$$\langle G^{\rm ex}(0,T,r,T/2)|\hat{O}_i(r)|G^{\rm ex}(r,T/2,0,0)\rangle=\langle G_0|\hat{O}_i|G_0\rangle+2\alpha\langle G_1^L|\hat{O}_i|G_0\rangle+\alpha^2\langle G_1^L|\hat{O}_i|G_1^L\rangle+2\alpha^2\langle G_2^L|\hat{O}_i|G_0\rangle+2\beta\langle G_1^C|\hat{O}_i|G_0\rangle$$
$$+\beta^2\langle G_1^C|\hat{O}_i|G_1^C\rangle+2\beta^2\langle G_2^C|\hat{O}_i|G_0\rangle+2\alpha\beta\langle G_1^C|\hat{O}_i|G_1^L\rangle+2\alpha\beta\langle G_2^{\rm LC}|\hat{O}_i|G_0\rangle. \quad \text{(B4)}$$

In the equation (B4) only three-dimensional r integration is denoted by the brackets. The operator $\hat{O}_i(r)$ is one of the following operators:

$$\hat{O}_1(r)=r^2/6; \quad \hat{O}_2(r)=-\partial^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right); \quad \hat{O}_3(r)=r.$$

For any matrix element in (B4) r enters as a polynomial times Gaussian. Thus, the r integration can be easily done. All remaining integrals have a fractional form and can be computed analytically or numerically. Final results of these calculations are formulated in the text (3.14), (3.16).

[1] M. Shifman, in *Proceedings of Workshop on Continuous Advances in QCD*, edited by A. Smilga (World Scientific, Singapore, 1994), p. 249; in *Particles, Strings, and Cosmology*, Proceedings of the Joint Meeting of the International Symposium on Particles, Strings and Cosmology and the 19th Johns Hopkins Workshop on Current Problems in Particle Theory-PASCOS, Baltimore, Maryland, 1995, edited by F. Bagger (World Scientific, Singapore, 1996), hep-ph/9505289.

[2] B. Blok and T. Mannel, Mod. Phys. Lett. A **11**, 1263 (1996).

[3] B. Chibisov, R. D. Dikeman, M. Shifman, and N. Uraltsev, Int. J. Mod. Phys. A **12**, 2075 (1997).

[4] B. Blok, M. Shifman, and N. Uraltsev, Nucl. Phys. **B494**, 237 (1997).

[5] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[6] L. I. Reinders, H. R. Rubinstein, and S. Yazaki, Phys. Rep. **127**, 1 (1985).

[7] B. L. Ioffe, Nucl. Phys. **B188**, 317 (1981).

[8] E. C. Poggio, H. R. Quinn, and S. Weinberg, Phys. Rev. D **13**, 1958 (1976).

[9] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Sov. J. Nucl. Phys. **32**, 840 (1980); M. Shifman, *ibid.* **36**, 749 (1982).

[10] M. Neubert, Phys. Rep. **245**, 259 (1994).

[11] M. Neubert, Phys. Rev. D **46**, 1076 (1992); Phys. Lett. B **389**, 727 (1996).

[12] P. Ball and V. Braun, Phys. Rev. D **49**, 2472 (1994).

[13] M. Gremm, A. Kapustin, Z. Ligeti, and M. B. Wise, Phys. Rev. Lett. **77**, 20 (1996).

[14] V. Chernyak, Phys. Lett. B **387**, 173 (1996).

[15] J. B. Whitenton, B. Durand, and L. Durand, Phys. Rev. D **28**, 607 (1983).

[16] C. Quigg and J. Rosner, Phys. Rep. **56**, 167 (1979); R. A. Berttlmann, Acta Phys. Austriaca **53**, 305 (1981); J. S. Bell and R. A. Berttlmann, Nucl. Phys. **B177**, 218 (1981).

[17] B. Blok and M. Shifman, Phys. Rev. D **47**, 2949 (1993).

[18] I. Bigi, M. Shifman, and N. Uraltsev, hep-ph/9703290.

[19] V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. **115B**, 410 (1982); B. L. Ioffe and A. Smilga, Nucl. Phys. **B216**, 373 (1983).

[20] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1970).

[21] J. B. Whitenton, B. Durand, and L. Durand, Phys. Rev. D **28**, 597 (1983).

[22] M. R. Ahmady, R. R. Mendel, and J. D. Telman, Phys. Rev. D **55**, 419 (1997).

[23] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), p. 357.

[24] A. Magakyan and A. Khodjamirian, Sov. J. Nucl. Phys. **50**, 830 (1989).