On-shell-improved lattice QCD with staggered fermions

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By using Symanzik's improvement program, we study on-shell-improved lattice QCD with staggered fermions. We find that there are as many as 15 independent lattice operators of dimension 6 (including both gauge and fermion operators) which must be added to the unimproved action to form an $O(a^2)$-improved action. Among them, the total number of dimension-6 gauge operators and fermion bilinears is 5. The other ten terms are four-fermion operators. At the tree level and tadpole-improved tree level, all ten four-fermion operators are absent. [S0556-2821(98)01201-6]

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I. INTRODUCTION

In recent years, there has been a surge in developing and applying improved actions for the numerical simulations of lattice QCD. Up to now, most research has focused on improvement of Wilson fermions in an effort to reduce the $O(a)$ cutoff effects in the simulations. On the other hand, the absence of $O(a)$ errors for the staggered fermion action [1,2] and the complexity of the staggered formalism mean that its $O(a^2)$ improvement has received little attention. Almost ten years ago, Naik proposed adding a third-nearest-neighbor term to the standard staggered fermion action to remove some $O(a^2)$ effects [3]. His study was based on the Dirac-Kähler equation, not on the standard staggered formulation. Although these two fermion formulations are the same in the free case, they are quite different when the gauge interactions are included and the difference is of the order of a^2. So the Naik term may not remove all $O(a^2)$ errors from the simulations by using staggered fermions. This statement was demonstrated by the recent numerical simulation from the MILC group [4,5].

One approach to improvement is to construct a perfect action [6]. The classical perfect action for free staggered fermions was already proposed in Ref. [7]. In this paper, we will apply Symanzik's improvement scheme [8] to staggered fermions and discuss its on-shell improvement[1] through $O(a^2)$. We will show that including only the Naik term in the improved staggered fermion action is not enough to remove all order a^2 errors from on-shell quantities. Meanwhile, because both the standard Wilson gauge action and the standard staggered fermion action have $O(a^2)$ errors, we must improve both of them at the same time. We will show that these two improvements are not independent, but connected by an isospectral transformation of the gauge fields. The recent calculations of the MILC and Bielefeld [10] groups can be easily explained by the result of our analysis.

This paper is organized as follows. In Sec. II, we will discuss the $O(a^2)$ improvement of the staggered fermion action by finding all linearly independent dimension-6 operators following Symanzik's scheme. In Sec. III, we will give the coefficients in the tree level up to order a^2. In Sec. IV, we will give the general isospectral transformation of fermion fields and gauge fields and give the form of the simplified on-shell-improved action. Section V is the conclusion. The computations of the tree-level coefficients are presented in two appendixes.

[1]The scheme of improved Hamiltonians for lattice QCD is not considered in this paper and is referred to Ref. [9].

II. POSSIBLE COUNTERTERMS

When Symanzik's improvement scheme is applied to construct an $O(a^2)$-improved lattice action, the first step is to find all dimension-6 operators which are scalars under the lattice symmetry group. These operators, treated as counterterms, are then added to the action to remove all $O(a^2)$ errors from physical quantities. Before doing that, we will introduce some notation which will simplify our presentation. For the transformation properties of staggered fermion fields, the reader is requested to consult Ref. [2], which we will refer to as paper I in the following.

A. Definitions and notation

The whole lattice can be viewed as being composed of elementary hypercubes consisting of 16 lattice sites. We will use x to label the individual lattice sites and y, which has only even coordinates, to label each hypercube. A site inside a hypercube is represented by a ''hypercubic vector'' A, whose components can only take the values of either 0 or 1. The relationship between these three vectors is given by

$$x=y+A. \tag{1}$$

The hypercubic fields are defined as

$$\chi_A(y)=\tfrac{1}{4}\chi(y+A), \tag{2a}$$

$$\overline{\chi_A}(y)=\tfrac{1}{4}\overline{\chi}(y+A). \tag{2b}$$

With the notation

$$\overline{\chi}\mathcal{M}(\mathcal{U})\chi=\sum_{y,AB}\overline{\chi_A}(y)\mathcal{M}(\mathcal{U})_{AB}\chi_B(y) \tag{2c}$$

and

$$\overline{(\gamma_S\otimes\xi_F)}_{AB}=\tfrac{1}{4}\mathrm{Tr}(\gamma_A^{\dagger}\gamma_S\gamma_B\gamma_F^{\dagger}), \tag{2d}$$

we can write the standard staggered fermion action in a compact form as

$$S_F=\bar{\chi}\left[\sum_{\mu}\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu+m\right]\chi. \tag{2e}$$

Furthermore, when discussing the fermion operators, we will use the following notation:

$$\not{\mathcal{D}}=\sum_{\mu}\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu, \tag{3a}$$

$$\mathcal{D}^2=\sum_{\mu}\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu, \tag{3b}$$

$$\mathcal{D}^3=\sum_{\mu}\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu, \tag{3c}$$

$$\mathcal{D}\not{\mathcal{D}}\mathcal{D}=\sum_{\mu\nu}\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu\overline{(\gamma_\nu\otimes I)}\mathcal{D}_\nu\overline{(\gamma_\mu\otimes I)}\mathcal{D}_\mu. \tag{3d}$$

B. Fermion bilinears

The lattice symmetry group of staggered fermion action [11,2] includes translation, reflection, rotation, charge conjugate, and a continuous $\mathrm{U}_V(1)$. When the mass parameter m is 0, there is a second continuous $\mathrm{U}_A(1)$ symmetry.

When $m=0$, we can identify the following five independent operators which are scalars under all symmetry transformations including $\mathrm{U}_A(1)$:

$$\mathcal{O}_1=\bar{\chi}\mathcal{D}^3\chi, \tag{4a}$$

$$\mathcal{O}_2=\bar{\chi}\tfrac{1}{2}(\mathcal{D}^2\not{\mathcal{D}}-\not{\mathcal{D}}\mathcal{D}^2)\chi, \tag{4b}$$

$$\mathcal{O}_3=\bar{\chi}\tfrac{1}{2}(\mathcal{D}^2\not{\mathcal{D}}+\not{\mathcal{D}}\mathcal{D}^2-2\not{\mathcal{D}}^3)\chi, \tag{4c}$$

$$\mathcal{O}_4=\bar{\chi}(\mathcal{D}^2\not{\mathcal{D}}+\not{\mathcal{D}}\mathcal{D}^2-2\mathcal{D}\not{\mathcal{D}}\mathcal{D})\chi, \tag{4d}$$

$$\mathcal{O}_5=\bar{\chi}\not{\mathcal{D}}^3\chi. \tag{4e}$$

For the case of nonzero fermion mass, the $\mathrm{U}_A(1)$ symmetry is violated, and there are two more allowed counterterms:

$$\mathcal{O}_6=m\bar{\chi}\not{\mathcal{D}}^2\chi, \tag{5a}$$

$$\mathcal{O}_7=m\bar{\chi}\mathcal{D}^2\chi. \tag{5b}$$

C. Four-fermion operators

When considering $O(a^2)$ corrections to the fermion action, we must examine not only dimension-6 operators bilinear in the fermion fields, but also four-fermion operators of dimension 6.

Using hypercubic coordinates, we can connect the spin and flavor indices in staggered four-fermion operators in combinations of the form

$$[\bar{\chi}\mathcal{M}(\mathcal{U})\chi]^2=\sum_{y}\sum_{AB}\overline{\chi_A}(y)\mathcal{M}(\mathcal{U})_{AB}\chi_B(y)\times\sum_{CD}\overline{\chi_C}(y)\mathcal{M}(\mathcal{U})_{CD}\chi_D(y). \tag{6}$$

However, the color indices in such an operator might be combined in four ways:

$$\overline{\chi_a}\delta_{aa'}(\mathcal{M}(\mathcal{U})\chi)_{a'}\overline{\chi_b}\delta_{bb'}(\mathcal{M}(\mathcal{U})\chi)_{b'}, \tag{7a}$$

$$\overline{\chi_a}\delta_{ab'}(\mathcal{M}(\mathcal{U})\chi)_{a'}\overline{\chi_b}\delta_{ba'}(\mathcal{M}(\mathcal{U})\chi)_{b'}, \tag{7b}$$

$$\overline{\chi_a}t^i_{aa'}(\mathcal{M}(\mathcal{U})\chi)_{a'}\overline{\chi_b}t^i_{ba'}(\mathcal{M}(\mathcal{U})\chi)_{b'}, \tag{7c}$$

$$\overline{\chi_a}t^i_{ab'}(\mathcal{M}(\mathcal{U})\chi)_{a'}\overline{\chi_b}t^i_{ba'}(\mathcal{M}(\mathcal{U})\chi)_{b'}, \tag{7d}$$

with

$$t^i=\frac{\lambda^i}{2}, \tag{8}$$

where λ^i are the SU(3) Gell-Mann matrices, and as usual, the repetition of the indices a, a', b, b', and i means summation. Because of the completeness relation of the matrices λ^i,

$$\sum_{i=1}^{8}\lambda^i_{aa'}\lambda^i_{bb'}=2\,\delta_{ab'}\delta_{ba'}-\frac{2}{3}\,\delta_{aa'}\delta_{bb'}, \tag{9}$$

the operators with the form of Eqs. (7a) and (7b) can be expressed as linear combinations of operators with the form of Eqs. (7c) and (7d). Furthermore, the operators with the form of Eq. (7d) can be expressed in terms of the operators with the form of Eq. (7c) by making a Fierz transformation. Hence we need only consider the operators with the form of Eq. (7c).[2]

For convenience, we will not write out the links explicitly in the remaining part of this section unless there would otherwise be confusion. After applying the staggered fermion symmetry transformation including rotation, reflection, charge conjugate, and the continuous $\mathrm{U}_V(1)\times\mathrm{U}_A(1)$, we found there are 18 operators which are invariant:

$$\mathcal{F}_1=[\bar{\chi}t^a\overline{(I\otimes I)}\chi]^2-[\bar{\chi}t^a\overline{(\gamma_5\otimes\xi_5)}\chi]^2+\sum_{\mu}\{[\bar{\chi}t^a\overline{(\gamma_\mu\otimes\xi_\mu)}\chi]^2-[\bar{\chi}t^a\overline{(\gamma_{5\mu}\otimes\xi_{5\mu})}\chi]^2\}, \tag{10a}$$

[2]If additional, explicit flavors of staggered fermions are introduced, Fierz symmetry cannot be used a second time, and so we will need to introduce both flavor adjoint and singlet fermion bilinears, effectively doubling the number of flavor-singlet, four-fermion operators that must be considered.

$$\mathcal{F}_2=[\overline{\chi t^a}\overline{(I\otimes I)}\chi]^2-[\overline{\chi t^a}\overline{(\gamma_5\otimes\xi_5)}\chi]^2 -\sum_{\mu}\{[\overline{\chi t^a}\overline{(\gamma_\mu\otimes\xi_\mu)}\chi]^2-[\overline{\chi t^a}\overline{(\gamma_{5\mu}\otimes\xi_{5\mu})}\chi]^2\}, \tag{10b}$$

$$\mathcal{F}_3=\sum_{\mu}[\overline{\chi t^a}\overline{(\gamma_{5\mu}\otimes I)}\chi]^2, \tag{10c}$$

$$\mathcal{F}_4=\sum_{\mu}[\overline{\chi t^a}\overline{(\gamma_\mu\otimes\xi_5)}\chi]^2, \tag{10d}$$

$$\mathcal{F}_5=\sum_{\mu\neq\nu\neq\lambda}[\overline{\chi t^a}\overline{(\gamma_{\mu\nu}\otimes\xi_\lambda)}\chi]^2, \tag{10e}$$

$$\mathcal{F}_6=\sum_{\mu\neq\nu}[\overline{\chi t^a}\overline{(\gamma_{\mu\nu}\otimes\xi_{5\nu})}\chi]^2, \tag{10f}$$

$$\mathcal{F}_7=\sum_{\mu\neq\nu\neq\lambda}[\overline{\chi t^a}\overline{(\gamma_\mu\otimes\xi_{\nu\lambda})}\chi]^2, \tag{10g}$$

$$\mathcal{F}_8=\sum_{\mu\neq\nu}[\overline{\chi t^a}\overline{(\gamma_{5\mu}\otimes\xi_{\mu\nu})}\chi]^2, \tag{10h}$$

$$\mathcal{F}_9=\sum_{\mu}[\overline{\chi t^a}\overline{(I\otimes\xi_{5\mu})}\chi]^2, \tag{10i}$$

$$\mathcal{F}_{10}=\sum_{\mu}[\overline{\chi t^a}\overline{(\gamma_5\otimes\xi_\mu)}\chi]^2, \tag{10j}$$

$$\mathcal{F}_{11}=\sum_{\mu}[\overline{\chi t^a}\overline{(\gamma_\mu\otimes I)}\chi]^2, \tag{10k}$$

$$\mathcal{F}_{12}=\sum_{\mu\neq\nu\neq\lambda}[\overline{\chi t^a}\overline{(\gamma_{\mu\nu}\otimes\xi_{5\lambda})}\chi]^2, \tag{10l}$$

$$\mathcal{F}_{13}=\sum_{\mu}[\overline{\chi t^a}\overline{(I\otimes\xi_\mu)}\chi]^2, \tag{10m}$$

$$\mathcal{F}_{14}=\sum_{\mu\neq\nu\neq\lambda}[\overline{\chi t^a}\overline{(\gamma_{5\mu}\otimes\xi_{\mu\lambda})}\chi]^2, \tag{10n}$$

$$\mathcal{F}_{15}=\sum_{\mu}[\overline{\chi t^a}\overline{(\gamma_{5\mu}\otimes\xi_5)}\chi]^2, \tag{10o}$$

$$\mathcal{F}_{16}=\sum_{\mu\neq\nu}[\overline{\chi t^a}\overline{(\gamma_{\mu\nu}\otimes\xi_\nu)}\chi]^2, \tag{10p}$$

$$\mathcal{F}_{17}=\sum_{\mu}[\overline{\chi t^a}\overline{(\gamma_5\otimes\xi_{5\mu})}\chi]^2, \tag{10q}$$

$$\mathcal{F}_{18}=\sum_{\mu\neq\nu}[\overline{\chi t^a}\overline{(\gamma_\mu\otimes\xi_{\mu\nu})}\chi]^2. \tag{10r}$$

These operators are not invariant under translation. However, the additional terms generated by translations are of dimension 7. For any operator listed above, we can combine it with some higher dimensional operator so that the new operator is invariant under translation. This new operator differs from the old one by a dimension-7 term, and both of them have the same continuum form. Therefore the translation symmetry does not reduce the number of invariant operators here.

After adding some higher dimensional terms, we can make the 18 four-fermion operators listed above invariant under translation and rewrite them in terms of the fields $\chi(x)$ and $\overline{\chi(x)}$. First,

$$\mathcal{F}_1=\sum_{x,a}\overline{\chi(x)}t^a\chi(x)\sum_{e}\overline{\chi(x+e)}t^a\chi(x+e), \tag{11}$$

where the sum over e is a sum over the 8 possible lattice displacements of length ''1.'' Second,

$$\mathcal{F}_2=\sum_{x,a}\overline{\chi(x)}t^a\chi(x)\sum_{v}\overline{\chi(x+v)}t^a\chi(x+v), \tag{12}$$

where the sum over v is over the 32 possible lattice displacements of length ''$\sqrt{3}$.'' Next,

$$\mathcal{F}_i=\sum_{x,a}\sum_{\mu}\mathcal{C}^a_\mu(x)\frac{1}{256}\sum_{c}w(c)\eta_5(c)P^{(i)}_\mu(c)\mathcal{C}^a_\mu(x+c), \qquad i=3,\ldots,10. \tag{13}$$

This equation contains a number of new elements which we will now define. The sum over c is a sum over the 81 displacements with coordinates $c_\mu=-1,0,1$. The weight is

$$w(c)=\prod_{\mu=1}^{4}(2-|c_\mu|). \tag{14}$$

The fermion bilinear operator $\mathcal{C}^a_\mu(x)$ is given by

$$\mathcal{C}^a_\mu(x)=\overline{\chi(x)}t^a\sum_{v\perp\hat{\mu}}\chi(x+v), \tag{15}$$

where the sum is over the 8 possible lattice displacements of length ''$\sqrt{3}$'' which are perpendicular to $\hat{\mu}$ direction. The phase factors $P^{(i)}_\mu(c)$ are defined by

$$P^{(3)}_\mu(c)=\eta_\mu(c),\quad P^{(4)}_\mu(c)=\varepsilon(c)\eta_\mu(c),$$

$$P^{(5)}_\mu(c)=\varepsilon(c)\tau_\mu(c)\eta_\mu(c),\quad P^{(6)}_\mu(c)=\tau_\mu(c)\eta_\mu(c),$$

$$P^{(7)}_\mu(c)=\tau_\mu(c)\zeta_\mu(c),\quad P^{(8)}_\mu(c)=\varepsilon(c)\tau_\mu(c)\zeta_\mu(c),$$

$$P^{(9)}_\mu(c)=\varepsilon(c)\zeta_\mu(c),\quad P^{(10)}_\mu(c)=\zeta_\mu(c), \tag{16}$$

where

$$\tau_\mu(c)=\frac{1}{3}\sum_{\nu\neq\mu}(-1)^{c_\nu},$$

$$\eta_\mu(c)=(-1)^{c_1+\cdots+c_{\mu-1}},$$

$$\zeta_\mu(c)=(-1)^{c_{\mu+1}+\cdots+c_4},$$

$$\varepsilon(c)=(-1)^{c_1+\cdots+c_4},$$

$$\eta_5(c)=\prod_{\mu=1}^{4}\eta_\mu(c). \tag{17}$$

The remaining 8 operators can be written as

$$\mathcal{F}_i=\sum_{x,a}\sum_\mu \mathcal{B}_\mu^a(x)\frac{1}{256}\sum_c w(c)P_\mu^{(i)}(c)\mathcal{B}_\mu^a(x+c),$$

$$i=11,\ldots,18, \tag{18}$$

with the fermion bilinear operator

$$\mathcal{B}_\mu^a(x)=\tfrac{1}{2}[\overline{\chi(x)}t^a\chi(x+\hat{\mu})+\overline{\chi(x)}t^a\chi(x-\hat{\mu})], \tag{19}$$

and the phase factors are given by

$$P_\mu^{(11)}(c)=\eta_\mu(c),\quad P_\mu^{(12)}(c)=\tau_\mu(c)\eta_\mu(c),$$

$$P_\mu^{(13)}(c)=\zeta_\mu(c),\quad P_\mu^{(14)}(c)=\tau_\mu(c)\zeta_\mu(c),$$

$$P_\mu^{(15)}(c)=\varepsilon(c)\eta_\mu(c),\quad P_\mu^{(16)}(c)=\varepsilon(c)\tau_\mu(c)\eta_\mu(c),$$

$$P_\mu^{(17)}(c)=\varepsilon(c)\zeta_\mu(c),\quad P_\mu^{(18)}(c)=\varepsilon(c)\tau_\mu(c)\zeta_\mu(c).$$

We have now discussed all dimension-6 fermion operators which are invariant under the lattice symmetry group. Therefore, we can write down a suitable $O(a^2)$-improved staggered fermion action as

$$S_F=\overline{\chi}(\not{D}+m)\chi+a^2\sum_{i=1}^{7}b_i(g_0^2,ma)\mathcal{O}_i$$

$$+a^2\sum_{i=1}^{18}b_i'(g_0^2,ma)\mathcal{F}_i. \tag{20}$$

The reality of the action requires that b_2 be imaginary and that all other b' and b be real.

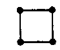
D. Gauge fields

The Symanzik improvement of the gauge theory action was studied more than a decade ago [12,13]. It was found that there are three independent six-link products which must be added to the original Wilson action to form an $O(a^2)$-improved gauge action. The improved gauge action can be written as

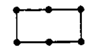
$$S_G[U]=\sum_{i=0}^{3}c_i(g_2^0)\mathcal{L}_i, \tag{21}$$

where the link products $\mathcal{L}_i$ are defined as

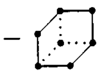
$$\mathcal{L}_0=\frac{\beta}{3}\sum_x \mathrm{Re\,Tr}\Big\langle 1- \quad \Big\rangle, \tag{22a}$$



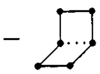
$$\mathcal{L}_1=\frac{\beta}{3}\sum_x \mathrm{Re\,Tr}\Big\langle 1- \quad \Big\rangle, \tag{22b}$$



$$\mathcal{L}_2=\frac{\beta}{3}\sum_x \mathrm{Re\,Tr}\Big\langle 1- \quad \Big\rangle, \tag{22c}$$



$$\mathcal{L}_3=\frac{\beta}{3}\sum_x \mathrm{Re\,Tr}\Big\langle 1- \quad \Big\rangle, \tag{22d}$$



where $\langle\,\rangle$ implies an average over orientations. The four c_i's satisfy the normalization condition

$$c_0(g_0^2)+8c_1(g_0^2)+8c_2(g_0^2)+16c_3(g_0^2)=1. \tag{23}$$

For on-shell improved pure gauge theory, it was shown that we can set $c_3(g_0^2)$ to zero by a change of field variable in the path integral. However, we have to be careful when we discuss an improved action which includes the quarks, because the change of gauge field variable will also have an impact on the fermion action. We will discuss this issue in the latter part of this paper when we discuss the isospectral transformations.

III. TREE-LEVEL IMPROVEMENT

A natural way to do the tree-level improvement is to expand the lattice action to order a^2 and to adjust the coefficients b_i so that the difference from the continuum Lagrangian is of order of a^3. This also improves the free propagator through order of a^2.

Define the gauge-covariant hypercubic fermion fields as

$$\varphi_A(y)=\mathcal{U}_A(y)\chi_A(y), \tag{24a}$$

$$\overline{\varphi_A}(y)=\overline{\chi_A}(y)\mathcal{U}_A^\dagger(y), \tag{24b}$$

where $\mathcal{U}_A(y)$ is the average of link products along the shortest paths from y to $y+A$. For the classical continuum limit of the standard staggered fermion action (see Appendix A), we find

$$S_F=\int_y\sum_{AB}\overline{\varphi_A}(y)\Bigg\{\sum_\mu\overline{(\gamma_\mu\otimes I)}_{AB}D_\mu+m\overline{(I\otimes I)}_{AB}$$
$$+a\sum_\mu\left[ig_0\sum_\lambda A_\lambda\overline{(\gamma_\mu\otimes I)}_{AB}F_{\lambda\mu}-\overline{(\gamma_5\otimes\xi_{5\mu})}_{AB}D_\mu^2\right]$$
$$+\frac{2a^2}{3}\sum_\mu\overline{(\gamma_\mu\otimes I)}_{AB}D_\mu^3$$
$$+\frac{i}{2}g_0a^2\Bigg[\sum_{\mu\nu\lambda}A_\lambda A_\nu\overline{(\gamma_\mu\otimes I)}_{AB}[D_\nu,F_{\lambda\mu}]$$
$$-\sum_{\mu\lambda}A_\lambda\overline{(\gamma_5\otimes\xi_{5\mu})}_{AB}([D_\mu,F_{\lambda\mu}]+3F_{\lambda\mu}D_\mu)\Bigg]\Bigg\}\varphi_B(y)$$
$$+O(a^3), \tag{25}$$

where D_μ is the continuum covariant derivative and $F_{\mu\nu}$ is the continuum field strength.

Now let us define a new set of fermion field variables

$$\phi_A=\exp\left(-a\sum_\lambda A_\lambda\overline{D_\lambda}\right)\varphi_A, \tag{26a}$$

$$\overline{\phi_A}=\overline{\phi_A}\exp\left(-a\sum_\lambda A_\lambda\overleftarrow{\overline{D_\lambda}}\right), \tag{26b}$$

where $\overline{D}$ is defined in Eq. (A3a). This definition is an obvious extension to higher order in a of the order a transformation discussed in paper I chosen to remove extraneous terms of $O(a)$ which appear when the original staggered fermion action is written in terms of hypercubic variables. The continuum limit of the staggered fermion action in terms of the ϕ field can be written as

$$S_F=\int_y\sum_{AB}\overline{\phi_A}(y)\Bigg\{\sum_\mu\overline{(\gamma_\mu\otimes I)}_{AB}\left[D_\mu+\frac{a^2}{6}D_\mu^3\right]$$
$$+m\overline{(I\otimes I)}_{AB}\Bigg\}\phi_B(y)+O(a^3). \tag{27}$$

From this equation, it is easy to get the tree-level values of the coefficients occurring in Eq. (20). We obtain

$$b_1(0,ma)=-\tfrac{1}{6}, \tag{28}$$

and all other b_i are zero.

The coefficients in Eq. (21) were given in Refs. [12] and [13]. Their values are

$$c_0(0)=\tfrac{5}{3},\quad c_1(0)=-\tfrac{1}{12},\quad c_2(0)=c_3(0)=0. \tag{29}$$

Normally, one gets the tree-level coefficients from diagrams with only one vertex, i.e., from a direct expansion of the action in the lattice spacing a. In contrast, the four-fermion tree-level coefficients come from tree graphs with two vertices.[3] For staggered fermions, the high momentum gluon exchange gives rise to flavor-changing four-fermion terms. At the tree level, we get three nonzero terms (see Appendix B), the coefficients of which are

$$b'_{12}=\frac{g_0^2}{8}, \tag{30a}$$

$$b'_{13}=\frac{g_0^2}{24}, \tag{30b}$$

$$b'_{14}=\frac{g_0^2}{16}, \tag{30c}$$

and all other b'_i are zero. These three terms belong to a restricted class of operators which can be expressed as a product of two fermion bilinears, with each such bilinear composed of fields $\overline{\chi(x)}$ and $\chi(x')$ with a distance between x and x' of precisely *one* link. There are a total of eight such operators given in Eq. (18). The coefficients quoted above were computed using the naive fields specified by the standard staggered fermion action as expressed in Eq. (A1). As we will discuss in the next section, all the coefficients of the eight terms in Eq. (18) can be changed by a transformation of the field variables and a choice of fields can be found for which these coefficients are zero. Thus, the eight terms in Eq. (18), including the three terms quoted above, actually do not appear in the on-shell-improved action.

IV. ON-SHELL IMPROVEMENT

The on-shell-improved action is not unique. Given one improved action, we can obtain another one by a transformation of the fields. However, all these actions are equivalent because all such actions will give the same value for a specific on-shell quantity. Thus, we can choose to minimize the number of operators occurring in the on-shell-improved action by an appropriate definition of the field variables.

A. Isospectral transformation on fermion fields

To simplify the improved fermion action, Eq. (20), we consider the following transformation:

$$\chi\rightarrow(1+a^2\varepsilon_1m\mathcal{D}\!\!\!/+a^2\varepsilon_2\mathcal{D}\!\!\!/^2+a^2\varepsilon_3\mathcal{D}^2)\chi, \tag{31a}$$

$$\overline{\chi}\rightarrow\overline{\chi}(1+a^2\varepsilon'_1m\overleftarrow{\mathcal{D}\!\!\!/}+a^2\varepsilon'_2\overleftarrow{\mathcal{D}\!\!\!/}^2+a^2\varepsilon'_3\overleftarrow{\mathcal{D}}^2). \tag{31b}$$

This is the most general transformation of the fermion fields χ and $\overline{\chi}$ which preserves their transformation properties under the lattice symmetries. After rescaling the fermion fields and redefining the fermion mass parameter

$$m'=m[1-a^2m^2(\varepsilon_1-\varepsilon'_1)], \tag{32}$$

to the first order in the ε_i's and second order in a, we end up with the change of the action

$$\delta S_F=a^2[(\varepsilon'_3-\varepsilon_3)\mathcal{O}_2+(\varepsilon'_3+\varepsilon_3)\mathcal{O}_3+(\varepsilon'_2+\varepsilon_2+\varepsilon'_3+\varepsilon_3)\mathcal{O}_5$$
$$+(\varepsilon_1-\varepsilon'_1+\varepsilon_2+\varepsilon'_2)\mathcal{O}_6+(\varepsilon_3+\varepsilon'_3)\mathcal{O}_7]+O(a^3). \tag{33}$$

The reality of the transformed action requires that

[3]We thank G. P. Lepage for pointing out the existence of such tree-level contributions.

$$\varepsilon_3' = \varepsilon_3^*, \tag{34a}$$

$$\varepsilon_2 + \varepsilon_2' = \text{real}, \tag{34b}$$

and

$$\varepsilon_1 - \varepsilon_1' = \text{real}. \tag{34c}$$

Thus we can always choose appropriate values of the ε_i's and ε_i''s to make the coefficients b_2, b_3, b_5, and b_6 in Eq. (20) vanish after it is rewritten by the new field variables. For example,

$$\varepsilon_3' = -\tfrac{1}{2}(b_3 + b_2), \tag{35a}$$

$$\varepsilon_3 = -\tfrac{1}{2}(b_3 - b_2), \tag{35b}$$

$$\varepsilon_2' + \varepsilon_2 = b_3 - b_5, \tag{35c}$$

$$\varepsilon_1 - \varepsilon_1' = b_5 - b_3 - b_6. \tag{35d}$$

Notice that this argument is valid to any order of g_0^2. Hence, to all orders in perturbation theory, we can always choose

$$b_2(g_0^2,ma) = b_3(g_0^2,ma) = b_5(g_0^2,ma) = b_6(g_0^2,ma) = 0. \tag{36}$$

B. Isospectral transformation on gauge fields

The general form of the gauge field transformation which changes the action at $O(a^2)$ can be written as

$$U_\mu(x) \to U_\mu'(x) = \exp[\varepsilon X_\mu(x)] U_\mu(x). \tag{37}$$

$U_\mu'(x)$ and $U_\mu(x)$ must have the same transformation properties under lattice symmetry group. The general form of $X_\mu(x)$ has been given in Ref. [13] for the case in which $X_\mu(x)$ depends on only the gauge variables. It is the anti-Hermitian traceless part of another field $Y_\mu(x)$:

$$X_\mu(x) = Y_\mu(x) - Y_\mu(x)^\dagger - \frac{1}{N}\,\mathrm{Tr}[Y_\mu(x) - Y_\mu(x)^\dagger], \tag{38}$$

with $N=3$ for SU(3) gauge theory, and

$$\begin{aligned} Y_\mu(x) = \frac{1}{4}\sum_\nu \; [&U_\nu(x) U_\mu(x+\hat{\nu}) U_\nu^\dagger(x+\hat{\mu}) U_\mu^\dagger(x) \\ &- U_\mu(x) U_\nu^\dagger(x-\hat{\nu}+\hat{\mu}) U_\mu^\dagger(x-\hat{\nu}) U_\nu(x-\hat{\nu})]. \end{aligned} \tag{39}$$

Under the field transformation, Eq. (37), the path integral is invariant:

$$\begin{aligned} &\int [dU]\exp(S_F[U] + S_G[U]) \\ &\quad= \int [dU']\exp(S_F[U'] + S_G[U']) \\ &\quad= \int [dU](1 + \Delta J + \epsilon[\Delta S_G + \Delta S_F]) \\ &\qquad\times \exp(S_F[U] + S_G[U]) + O(\epsilon^2). \end{aligned} \tag{40}$$

The change of the fermion action in Eq. (20) is

$$\Delta S_F = a^2\,\frac{\epsilon}{2}\,\mathcal{O}_4 + O(a^3). \tag{41}$$

The Jacobian $1+\Delta J$ and the change of the gauge action ΔS_G will not generate new terms, but only change the coefficients c_i in $S_G[U]$. Lüscher and Weisz [13] discussed the changes of these coefficients in detail. Their results showed that we can choose an appropriate value of ϵ to set

$$c_3(g_0^2) = 0 \tag{42}$$

to all orders in perturbation theory. From Eq. (41), we see that if we want this to persist, the coefficient $b_4(g_0^2,ma)$ will not be zero in general. On the other hand, we can also set $b_4(g_0^2,ma)=0$; however, $c_3(g_0^2)$ will then in general be nonzero.

In order to find out the possible redundant four-fermion operators, we will generalize the argument given in Ref. [14]. The lattice action is rewritten in a concise form as

$$S = \sum_x \; [\overline{\chi} h_x(U)\chi + \mathrm{Tr} f_x(U)]. \tag{43}$$

Consider the small change of a link variable,

$$\delta U_\mu(x) = i\varepsilon_{\alpha\beta}(x+c)\mathrm{Re}[t^a U_\mu(x+c)]_{\alpha\beta} t^a U_\mu(x), \tag{44a}$$

$$\delta U_\mu^\dagger(x) = -i\varepsilon_{\alpha\beta}(x+c)\mathrm{Re}[t^a U_\mu(x+c)]_{\alpha\beta} U_\mu^\dagger(x) t^a, \tag{44b}$$

where ε is a real small number. The Jacobian differs from 1 in order ε^2. To the first order of ε, we get the following identity:

$$\begin{aligned} &\int dU_\mu(x) e^{S[\chi,\overline{\chi},U]} \\ &\quad\times\Bigg\{ \overline{\chi t^a} U_\mu(x)\,\frac{\partial h_x(U_\mu(x))}{\partial U_\mu(x)}\,\chi\varepsilon_{\alpha\beta}(x+c) \\ &\quad\times \mathrm{Re}[t^a U_\mu(x+c)]_{\alpha\beta} + \mathrm{Tr}\left[t^a U_\mu(x)\,\frac{\partial f_x(U_\mu(x))}{\partial U_\mu(x)}\right] \\ &\quad\times \varepsilon_{\alpha\beta}(x+c)\mathrm{Re}[t^a U_\mu(x+c)]_{\alpha\beta}\Bigg\} = 0. \end{aligned} \tag{45a}$$

Similarly, we get

$$\begin{aligned} &\int dU_\mu(x) e^{S[\chi,\overline{\chi},U]}\Bigg\{\overline{\chi t^a} U_\mu(x)\,\frac{\partial h_x(U_\mu(x))}{\partial U_\mu(x)} \\ &\quad\times \chi\varepsilon_{\alpha\beta}(x+c)\mathrm{Im}[t^a U_\mu(x+c)]_{\alpha\beta} \\ &\quad+ \left\{ \mathrm{Tr}\left[t^a U_\mu(x)\,\frac{\partial f_x(U_\mu(x))}{\partial U_\mu(x)}\right]\varepsilon_{\alpha\beta}(x+c) \right. \\ &\quad\times \mathrm{Im}[t^a U_\mu(x+c)]_{\alpha\beta}\Bigg\} = 0. \end{aligned} \tag{45b}$$

Combining the above two equations, we get

$$\begin{aligned}Q^{a}_{\alpha\beta}(\hat{\chi},\chi)=&\int dU_{\mu}(x)e^{S[\chi,\overline{\chi},U]}\\&\times\Bigg\{\overline{\chi t^{a}}U_{\mu}(x)\ \frac{\partial h_{x}(U_{\mu}(x))}{\partial U_{\mu}(x)}\ \chi\varepsilon_{\alpha\beta}(x+c)\\&\times[t^{a}U_{\mu}(x+c)]_{\alpha\beta}\\&+\mathrm{Tr}\left[t^{a}U_{\mu}(x)\ \frac{\partial f_{x}(U_{\mu}(x))}{\partial U_{\mu}(x)}\right]\varepsilon_{\alpha\beta}(x+c)\\&\times[t^{a}U_{\mu}(x+c)]_{\alpha\beta}\Bigg\}\\=&0.\end{aligned}\tag{46}$$

In the above equation, we replace $\varepsilon_{\alpha\beta}(x+c)$ by $a^{3}\overline{\chi^{\alpha}}(x+c)\chi^{\beta}(x+c+\mu)$, multiply it by a combined phase factor

$$\sum_{i=11}^{18}\epsilon_{i}'\,\eta_{\mu}(x)P^{i}_{\mu}(c),\tag{47}$$

and sum it on the hypercubic vector c. After substituting $h_{x}(U)$ and $f_{x}(U)$ by the actual staggered fermion action and the gauge action, we see that we can add the terms

$$\Delta S=\frac{a^{2}}{2}\sum_{i=11}^{18}\epsilon_{i}'\mathcal{F}_{i}+a^{2}\epsilon_{11}'\mathcal{O}_{4}+O(a^{3})\tag{48}$$

to the action without changing the path integral to the order of $O(a^{3})$. Notice that because of the identity

$$\sum_{c}\ \eta_{\mu}(c)P^{i}_{\mu}(c)=\delta_{i,11},\quad i=11,\ldots,18,\tag{49}$$

there is only one dimension-6 bilinear operator in Eq. (48). Therefore, we conclude that the four-fermion operators $\mathcal{F}_{11},\ldots,\mathcal{F}_{18}$, whose fermion bilinear operators consist of two sites separated by one link, are redundant and their coupling constants $b_{11}',\ldots,b_{18}'$ can be set to be zero. Again, the coefficient of operator $\mathcal{O}_{4}$ will get a change accordingly. Is it possible to get rid of some other four-fermion operators? From the above discussion, we conclude ''no.'' Because in the original staggered fermion action χ and $\overline{\chi}$ are separated by one link, there is no way to generate an operator like $\mathcal{F}_{1},\ldots,\mathcal{F}_{10}$ when we multiply that link variable by some Grassmann variables.

C. $O(a^2)$ on-shell-improved action

The $O(a^{2})$-improved action for lattice QCD can be written as

$$S_{\mathrm{QCD}}=S_{G}[U]+S_{F}[\chi,\overline{\chi},U],\tag{50}$$

where S_{F} is given by Eq. (20) and S_{G} is given by Eq. (21).

The operator $\mathcal{O}_{4}$ involves the lattice sites in two nearest-neighbor hypercubes. There exists another operator $\mathcal{O}_{4}'$ which only involves lattice sites inside one hypercube and is equivalent to $\mathcal{O}_{4}$ up to an operator of dimension 7. We can construct this operator by replacing the link variable $U_{\mu}(x)$ in the Dirac operator $\not{D}$ with a modified link (e.g., the MILC ''fat link'') $\mathcal{W}_{\mu}(x)U_{\mu}(x)$ [5]:

$$\begin{aligned}\mathcal{O}_{4}'=\sum_{x,\mu}\ \overline{\chi(x)}\ \frac{\eta_{\mu}(x)}{2a}\,&[\mathcal{W}_{\mu}(x)U_{\mu}(x)\chi(x+\hat{\mu})\\&-U^{\dagger}_{\mu}(x-\hat{\mu})\mathcal{W}^{\dagger}_{\mu}(x-\hat{\mu})(x-\hat{\mu})].\end{aligned}\tag{51}$$

This new operator $\mathcal{O}_{4}'$ obeys all the staggered fermion lattice symmetries. The factor $\mathcal{W}_{\mu}(x)$ has the form

$$\begin{aligned}\mathcal{W}_{\mu}(x)=\frac{1}{a^{2}}\sum_{\nu\neq\mu}&[U_{\nu}(x)U_{\mu}(x+\hat{\nu})U^{\dagger}_{\nu}(x+\hat{\mu})U^{\dagger}_{\mu}(x)\\&+U^{\dagger}_{\nu}(x-\hat{\nu})U_{\mu}(x-\hat{\nu})U_{\nu}(x+\hat{\mu}-\hat{\nu})U^{\dagger}_{\mu}(x)-2],\end{aligned}\tag{52}$$

and its continuum limit is $ig_{0}a[D_{\nu},F_{\nu\mu}]$. Thus $\mathcal{O}_{4}'$ has the same continuum limit as $\mathcal{O}_{4}$. Because it is simpler, we will choose $\mathcal{O}_{4}'$ instead of $\mathcal{O}_{4}$ to make our on-shell-improved action.

From the above discussion, we can see that one possible on-shell-improved action for lattice QCD can be constructed as

$$S^{(1)}_{\mathrm{QCD}}=\sum_{i=0}^{2}c_{i}(g_{0}^{2})\mathcal{L}_{i}+S^{(1)}_{F},\tag{53}$$

with

$$\begin{aligned}S^{(1)}_{F}=&\overline{\chi(}\not{D}+m)\chi+a^{2}b_{1}(g_{0}^{2},ma)\mathcal{O}_{1}+a^{2}b_{4}(g_{0}^{2},ma)\mathcal{O}_{4}'\\&+a^{2}b_{7}(g_{0}^{2},ma)\mathcal{O}_{7}+a^{2}\sum_{i=1}^{10}b_{i}'(g_{0}^{2},ma)\mathcal{F}_{i}.\end{aligned}\tag{54}$$

Another possible choice would be

$$S^{(2)}_{\mathrm{QCD}}=\sum_{i=0}^{3}c_{i}(g_{0}^{2})\mathcal{L}_{i}+S^{(2)}_{F},\tag{55}$$

with

$$\begin{aligned}S^{(2)}_{F}=&\overline{\chi(}\not{D}+m)\chi+a^{2}b_{1}(g_{0}^{2},ma)\mathcal{O}_{1}+a^{2}b_{7}(g_{0}^{2},ma)\mathcal{O}_{7}\\&+a^{2}\sum_{i=1}^{10}b_{i}'(g_{0}^{2},ma)\mathcal{F}_{i}.\end{aligned}\tag{56}$$

In either case, the tree-level-improved action is the same because both $b_{4}(0,ma)$ and $c_{3}(0)$ vanish.

D. Formulas arranged for lattice computation

For the operators appearing in Eqs. (53) or (55), the gauge part, given by Eq. (21), is already in a form handy for numerical simulation. However, the fermion bilinear operators in $S_F^{(1)}$ or $S_F^{(2)}$ are represented in terms of the hypercubic fields. In order to do a numerical calculation, it is convenient to rewrite them in terms of the original field variables χ and $\overline{\chi}$.

Let us first consider the four-fermion operators. These ten operators, described in Secs. II and IV, are already written in terms of the original variables χ and $\overline{\chi}$. However, they can be simulated using known Monte Carlo techniques only if auxiliary Yukawa fields are introduced so that these four-fermion operators can be written in a bilinear form. The resulting gauge-Yukawa fermion action will be quite complicated with Hermiticity properties that depend on the sign of the original four-fermion coefficients. Further, the positive-definite character of the staggered fermion action may be lost unless these new Yukawa terms possess some additional (staggered fermion) $\mathrm{U}_A(1)$ symmetry. Since these four-fermion terms are not present in the tree-level (or tadpole-improved tree-level) approximation, we will not consider this question further in this paper.

Only considering the fermion bilinear operators, the fermion action $S_F^{(1)}$ in Eq. (54) can be rewritten as

$$S_F^{(1)}=a^4\sum_{x,\mu}\overline{\chi(x)}\,\frac{\eta_\mu(x)}{2a}\,[\mathcal{U}_\mu(x)\chi(x+\hat\mu)-\mathcal{U}_\mu^\dagger(x-\hat\mu)\chi(x-\hat\mu)]+[1-\alpha_3(g_0^2,ma)]ma^4\sum_x\overline{\chi(x)}\chi(x)$$
$$+\alpha_3(g_0^2,ma)ma^4\sum_{x,\mu}\overline{\chi(x)}\,\tfrac{1}{2}\,[U(x,x+2\hat\mu)\chi(x+2\hat\mu)+U(x,x-2\hat\mu)\chi(x-2\hat\mu)]$$
$$-\alpha_1(g_0^2,ma)a^4\sum_{x,\mu}\overline{\chi(x)}\,\frac{\eta_\mu(x)}{6a}\,[U(x,x+3\hat\mu)\chi(x+3\hat\mu)-U(x,x-3\hat\mu)\chi(x-3\hat\mu)], \tag{57}$$

with

$$\mathcal{U}_\mu(x)=[1+\alpha_1(g_0^2,ma)-\alpha_2(g_0^2,ma)]U_\mu(x)+\alpha_2(g_0^2,ma)\widetilde{U}_\mu(x), \tag{58a}$$

$$\widetilde{U}_\mu(x)=\frac{1}{6}\sum_{\substack{\nu\\ \nu\neq\mu}}[U_\nu(x)U_\mu(x+\hat\nu)U_\nu^\dagger(x+\hat\mu)+U_\nu^\dagger(x-\hat\nu)U_\mu(x-\hat\nu)U_\nu(x-\hat\nu+\hat\mu)], \tag{58b}$$

and

$$U(x,x+2\hat\mu)=U_\mu(x)U_\mu(x+\hat\mu), \tag{59a}$$

$$U(x,x-2\hat\mu)=U_\mu^\dagger(x-\hat\mu)U_\mu^\dagger(x-2\hat\mu), \tag{59b}$$

$$U(x,x+3\hat\mu)=U_\mu(x)U_\mu(x+\hat\mu)U_\mu(x+2\hat\mu), \tag{59c}$$

$$U(x,x-3\hat\mu)=U_\mu^\dagger(x-\hat\mu)U_\mu^\dagger(x-2\hat\mu)U_\mu^\dagger(x-3\hat\mu). \tag{59d}$$

The three parameters α_1, α_2, and α_3 are real numbers and related to the parameters b_1, b_4, and b_7 of Eq. (54) by

$$\alpha_1=-\tfrac{3}{4}b_1, \tag{60a}$$

$$\alpha_2=b_4, \tag{60b}$$

$$\alpha_3=\tfrac{1}{2}b_7. \tag{60c}$$

The form of $S_F^{(2)}$, when rewritten using the fermion fields χ and $\overline{\chi}$, is the same as Eq. (57), but with $\alpha_2(g_0^2,ma)=0$:

$$F_F^{(2)}=S_F^{(1)}\big|_{\alpha_2(g_0^2,ma)=0}. \tag{61}$$

At the tree level, we have

$$\alpha_1(0,ma)=\tfrac{1}{8},\quad \alpha_2(0,ma)=\alpha_3(0,ma)=0. \tag{62}$$

One way to do a better job than tree-level improvement may be to use tadpole improvement. Following the work of Lepage and Mackenzie [15], we can replace all links $U_\mu(x)$ which appear in lattice operators by $(1/u_0)U_\mu(x)$ (Of course, the normalization of the coupling constant will be different.) The constant u_0 is the mean value of the link matrix and is measured in the simulation by the quantity $u_0=[\mathrm{Re}\langle\frac{1}{3}\,\mathrm{Tr}U_\square\rangle]^{1/4}$. The tadpole-improved tree-level coefficients are the same as above except that the parameter α_1 which appears in Eqs. (57) and (58a) must be replaced by two different parameters α_1^a and α_1^b. The value of the parameter α_1^a appearing in Eq. (57) is $\alpha_1=1/8u_0^2$, while the parameter α_1^b in Eq. (58a) is still 1/8.[4]

V. CONCLUSION

Using Symanzik's improvement program, we have discussed the $O(a^2)$ on-shell improvement of the staggered fermion action in a systematic way. Our first step was to find all dimension-6 lattice operators which are scalars under the lat-

[4]Thanks to Attila Mihaly for pointing this out.

tice symmetry group. We found that there are 5 linearly independent fermion bilinear operators that are invariant under all lattice symmetry transformations. When the mass parameter is not zero, the $U_A(1)$ symmetry of the staggered fermion is violated, and there are 2 more fermion bilinear operators that violate only this $U_A(1)$ symmetry and are proportional to the mass of the fermions. For staggered fermions, we observed that there are 18 independent four-fermion operators. Therefore, we have at most 25 fermion operators which can be added as counterterms to the standard staggered fermion action to remove all $O(a^2)$ errors from all physical quantities. Including the 3 independent dimension-6 gauge operators, we end up with 28 counterterms for the $O(a^2)$-improved lattice QCD with staggered fermions.

For on-shell improvement, we can use the isospectral transformation of the field variables to eliminate all possible redundant operators. Including such field transformations, we concluded that we need at most 15 independent lattice operators of dimension 6 to construct the $O(a^2)$ on-shell-improved lattice QCD with staggered fermions. Ten of these are four-fermion operators, which are absent at the tree level and, hence, of the order of $O(g_0^4 a^2)$ at most. The other 5 are fermion bilinear operators and gauge operators and only two of them are nonzero at the tree level. Two possible improved actions are given by Eqs. (53) and (55).

Thus we found that the Naik term is not the only term in the improved staggered fermion action. It is worth emphasizing that to remove the $O(a^2)$ errors from lattice computation, we must use both an improved gauge action and an improved fermion action at the same time, not just one of them.

The recent numerical results from the MILC [4,5] and Bielefeld [10] groups are consistent with our analysis. Furthermore, in the free case, our result is the same as Naik's, as should be expected given the equivalence of free Dirac-Kähler and staggered fermions.

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I would like to thank Professor Norman H. Christ. Without our intensive discussions during all stages of this work, this paper could not have been finished. This research was supported by the U.S. Department of Energy under Grant No. DE-FG02-92 ER40699.

APPENDIX A: CLASSICAL CONTINUUM LIMIT OF STAGGERED FERMION ACTION

In this appendix, we will evaluate the continuum limit of the staggered fermion action,

$$S_F = a^4 \sum_x \overline{\chi(x)} \Bigg[\sum_\mu \eta_\mu(x) \frac{1}{2a} [U_\mu(x)\chi(x+\hat\mu) - U_\mu^\dagger(x-\hat\mu)\chi(x-\hat\mu)] + m\chi(x) \Bigg], \tag{A1}$$

through order a^2. As the first step in this derivation, we rewrite this action using the covariant hypercubic fermion fields defined in Eq. (24) in which $\mathcal{U}_A(y)$, the average of the link products along the shortest paths from site y to site $y+A$, is defined as

$$\mathcal{U}_A(y) = \frac{1}{4!} \sum_{P(\mu\nu\rho\sigma)} U_\mu(y)^{A_\mu} U_\nu(y+A_\mu\hat\mu)^{A_\nu} U_\rho(y+A_\mu\hat\mu + A_\nu\hat\nu)^{A_\rho} U_\sigma(y+A_\mu\hat\mu + A_\nu\hat\nu + A_\rho\hat\rho)^{A_\sigma}, \tag{A2}$$

where the summation is on all permutations of $(\mu\nu\rho\sigma)$. We define the hypercubic first- and second-order covariant derivatives as

$$\overline{D_\mu}\varphi_A(y) = \frac{1}{4a} [U_\mu(y)U_\mu(y+\hat\mu)\varphi_A(y+2\hat\mu) - U_\mu^\dagger(y-\hat\mu)U_\mu^\dagger(y-2\hat\mu)\varphi_A(y-2\hat\mu)], \tag{A3a}$$

$$\overline{\Delta_\mu}\varphi_A(y) = \frac{1}{4a^2} [U_\mu(y)U_\mu(y+\hat\mu)\varphi_A(y+2\hat\mu) + U_\mu^\dagger(y-\hat\mu)U_\mu^\dagger(y-2\hat\mu)\varphi_A(y-2\hat\mu) - 2\varphi_A(y)]. \tag{A3b}$$

Then, with no approximation, we can rewrite Eq. (A1) as

$$\begin{aligned} S_F = (2a)^4 \sum_y \sum_{AB} \sum_\mu \overline{\varphi_A}(y) \Bigg\{ & \overline{(\gamma_\mu\otimes I)}_{AB} \frac{1}{2} [w_\mu^{(1)}(y;AB) + w_\mu^{(2)}(y;AB)] \overline{D_\mu} - \overline{(\gamma_5\otimes\xi_{5\mu})}_{AB} \frac{1}{2} [w_\mu^{(1)}(y;AB) \\ & - w_\mu^{(2)}(y;AB)] \overline{D_\mu} + \overline{(\gamma_\mu\otimes I)}_{AB} \frac{a}{2} [w_\mu^{(1)}(y;AB) - w_\mu^{(2)}(y;AB)] \overline{\Delta_\mu} - \overline{(\gamma_5\otimes\xi_{5\mu})}_{AB} \frac{a}{2} [w_\mu^{(1)}(y;AB) + w_\mu^{(2)}(y;AB)] \overline{\Delta_\mu} \\ & + \overline{(\gamma_\mu\otimes I)}_{AB} \frac{1}{4a} [w_\mu^{(1)}(y;AB) + w_\mu^{(4)}(y;AB) - w_\mu^{(2)}(y;AB) - w_\mu^{(3)}(y;AB)] - \overline{(\gamma_5\otimes\xi_{5\mu})}_{AB} \frac{1}{4a} [w_\mu^{(1)}(y;AB) \\ & + w_\mu^{(2)}(y;AB) - w_\mu^{(3)}(y;AB) - w_\mu^{(4)}(y;AB)] \Bigg\} \varphi_B(y) + (2a)^4 \sum_y \sum_A m \overline{\varphi_A}(y) \varphi_A(y). \end{aligned} \tag{A4}$$

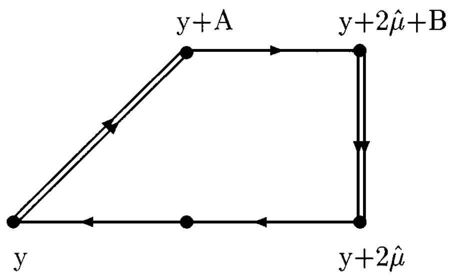

FIG. 1. A typical graph of $w_\mu^{(1)}(y;AB)$. It starts from y, through $y+A$, $y+2\mu+B$, $y+2\mu$, $y+\mu$, and then comes back to y. The two hypercubic vectors A and B satisfy the condition $A_\mu=1$, $B_\mu=0$.

The two hypercubic vectors A and B satisfy the delta function $\overline{\delta}(A+B+\hat{\mu})$, and the overbar means modulo 2. The four closed-loop link products are defined as

$$w_\mu^{(1)}(y;AB)=\delta_{A_\mu 1}\mathcal{U}_A(y)U_\mu(y+A)\mathcal{U}_B^\dagger(y+2\hat{\mu})\times U_\mu^\dagger(y+\hat{\mu})U_\mu^\dagger(y), \tag{A5a}$$

$$w_\mu^{(2)}(y;AB)=\delta_{A_\mu 0}\mathcal{U}_A(y)U_\mu^\dagger(y-2\hat{\mu}+B)\mathcal{U}_B^\dagger(y-2\hat{\mu})\times U_\mu(y-2\hat{\mu})U_\mu(y-\mu), \tag{A5b}$$

$$w_\mu^{(3)}(y;AB)=\delta_{A_\mu 1}\mathcal{U}_A(y)U_\mu^\dagger(y+B)\mathcal{U}_B^\dagger(y), \tag{A5c}$$

$$w_\mu^{(4)}(y;AB)=\delta_{A_\mu 0}\mathcal{U}_A(y)U_\mu(y+A)\mathcal{U}_B^\dagger(y). \tag{A5d}$$

If we represent $\mathcal{U}_A(y)$ by a doubled line which starts at site y and ends at site $y+A$, and $\mathcal{U}_A^\dagger(y)$ by a doubled line which starts at site $y+A$ and ends at site y, then some typical loops can be shown by four figures (Figs. 1–4).

To expand the links in powers of "a," we take advantage of the parallel transporter from x to $x+\hat{\mu}$ to define the gauge field $\mathcal{A}_\mu(x)$ by the path-ordered exponential

$$U_\mu(x)=P\ \exp\left\{ig_0a\int_0^1 d\tau\mathcal{A}_\mu(x+\tau\hat{\mu})\right\}. \tag{A6}$$

When $a\to 0$, we have

$$\overline{D_\mu}=D_\mu+\tfrac{2}{3}a^2D_\mu^3+O(a^3), \tag{A7a}$$

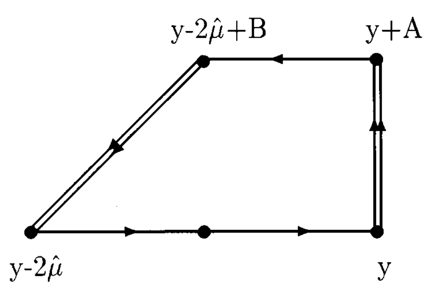

FIG. 2. A typical graph of $w_\mu^{(2)}(y;AB)$. It starts from y, through $y+A$, $y-2\mu+B$, $y-2\mu$, and then comes back to y. The two hypercubic vectors A and B satisfy the condition $A_\mu=0$, $B_\mu=1$.

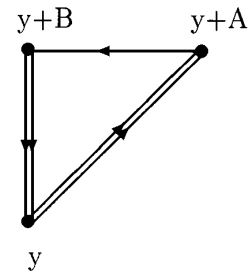

FIG. 3. A typical graph of $w_\mu^{(3)}(y;AB)$. It starts from y, through $y+A$, $y+B$, and then comes back to y. The two hypercubic vectors A and B satisfy the condition $A_\mu=1$, $B_\mu=0$.

$$\overline{\Delta_\mu}=D_\mu^2+O(a^2), \tag{A7b}$$

where $D_\mu=\partial_\mu+ig_0\mathcal{A}_\mu$ is the continuum covariant derivative. Expanding the w's in powers of "a," we get

$$\tfrac{1}{2}[w_\mu^{(1)}+w_\mu^{(2)}]=1+O(a^3), \tag{A8a}$$

$$\tfrac{1}{2}[w_\mu^{(1)}-w_\mu^{(2)}]=\tfrac{3}{2}ig_0a^2\sum_\nu A_\nu F_{\nu\mu}+O(a^3), \tag{A8b}$$

$$\frac{1}{4a}[w_\mu^{(1)}+w_\mu^{(4)}-w_\mu^{(2)}-w_\mu^{(3)}]=ig_0a\sum_\nu A_\nu F_{\nu\mu}+\tfrac{1}{2}ig_0a^2\sum_{\lambda\nu}A_\lambda A_\nu[D_\nu,F_{\lambda\mu}]+O(a^3), \tag{A8c}$$

$$\frac{1}{4a}[w_\mu^{(1)}+w_\mu^{(2)}-w_\mu^{(3)}-w_\mu^{(4)}]=\tfrac{1}{2}ig_0a^2\sum_\nu A_\nu[D_\mu,F_{\nu\mu}]+O(a^3). \tag{A8d}$$

Finally, we get the classical continuum limit of the staggered fermion action as Eq. (25).

APPENDIX B: CALCULATION OF THE TREE-LEVEL COEFFICIENTS OF FOUR-FERMION OPERATORS

We evaluate the amplitude represented by the graph shown in Fig. 5 for the case of vanishing external momenta, $p_i^\mu=0$. The amplitude is given by

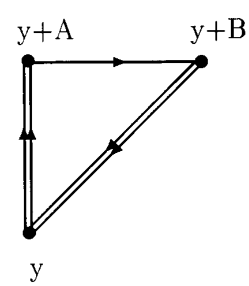

FIG. 4. A typical graph of $w_\mu^{(4)}(y;AB)$. It starts from y, through $y+A$, $y+B$, and then comes back to y. The two hypercubic vectors A and B satisfy the condition $A_\mu=0$, $B_\mu=1$.

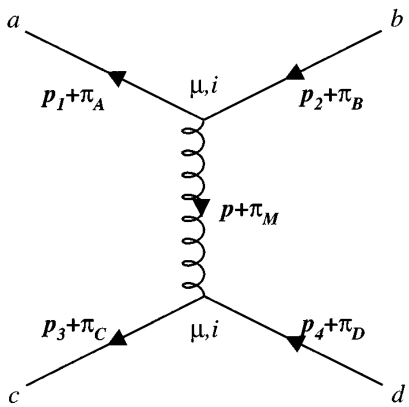

FIG. 5. The Feynman graph which generates four-fermion operators at the tree level.

$$K^{abcd}_{ABCD}=-g_0^2 t^i_{ab}t^i_{cd}\sum_{\mu}\sum_{M}{}' \frac{1}{\hat{\pi}_M^2} \times\cos[\tfrac{1}{2}(\pi_A+\pi_B)_\mu]\delta(A+B+M+\hat{\eta}_\mu) \times\cos[\tfrac{1}{2}(\pi_C+\pi_D)_\mu]\delta(C+D+M+\hat{\eta}_\mu), \tag{B1}$$

where a, b, c, and d are color indices, and $\hat{\eta}_\mu$ is a hypercubic vector whose ν's component is 1 only if $\nu<\mu$. Here π_M is the momentum propagated by the gluon and the quantity $\hat{\pi}_M^2$ is defined as

$$\hat{\pi}_M^2=\frac{4}{a^2}\sum_{\nu} \sin^2(M_\nu\pi/2). \tag{B2}$$

The primed summation on M is for all hypercubic vectors M with $M_\mu=0$.

After some algebra and using the notation of Ref. [16], we get

$$K^{abcd}_{ABCD}=-g_0^2 t^i_{ab}t^i_{cd}\left\{\frac{a^2}{4}\sum_{\mu\neq\nu\neq\lambda}(\overline{\overline{\gamma_{\mu\nu}\otimes\xi_{5\lambda}}})_{AB}(\overline{\overline{\gamma_{\mu\nu}\otimes\xi_{5\lambda}}})_{CD} +\frac{a^2}{8}\sum_{\mu\neq\nu\neq\lambda}(\overline{\overline{\gamma_{5\lambda}\otimes\xi_{\mu\nu}}})_{AB}(\overline{\overline{\gamma_{5\lambda}\otimes\xi_{\mu\nu}}})_{CD} +\frac{a^2}{12}\sum_{\mu}(\overline{\overline{I\otimes\xi_{\mu}}})_{AB}(\overline{\overline{I\otimes\xi_{\mu}}})_{CD}\right\}, \tag{B3}$$

with

$$(\overline{\overline{\gamma_S\otimes\xi_F}})_{AB}=\sum_{CD}\tfrac{1}{4}(-1)^{AC}\overline{(\gamma_S\otimes\xi_F)}_{CD}\tfrac{1}{4}(-1)^{DB}. \tag{B4}$$

Then the action differs from the continuum by a four-fermion term:

$$\Delta S=\frac{1}{2}\sum_{ABCD}\sum_{abcd} K^{abcd}_{ABCD}\overline{\chi^a}(\pi_A)\chi^b(\pi_B)\overline{\chi^c}(\pi_C)\chi^d(\pi_D). \tag{B5}$$

Written in terms of the hypercubic fields, we get

$$\Delta S=-\frac{g_0^2}{2}\left(\frac{a^2}{4}\mathcal{F}_{12}+\frac{a^2}{8}\mathcal{F}_{14}+\frac{a^2}{12}\mathcal{F}_{13}\right). \tag{B6}$$

The counterterm is the opposite of the above term. Hence we get the coefficients listed in Eq. (30).

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