$g=2$ as a gauge condition

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Charged matter spin-1 fields enjoy a nonelectromagnetic gauge symmetry when interacting with vacuum electromagnetism, provided their gyromagnetic ratio is 2. [S0556-2821(98)03604-2]

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It is agreed that the natural value for the gyromagnetic ratio g of an elementary charged particle coupling to the electromagnetic field $F_{\mu\nu}$ is $g=2$ (in the absence of small radiative corrections), so that the Bargmann-Michel-Telegdi [1] equation of motion for the spin vector S_μ takes its simplest form

$$\frac{\mathrm{d}S_\mu}{\mathrm{d}\tau}=\frac{e}{m}F_{\mu\nu}S^\nu.$$

Further reasons for this choice can be given:

(1) Within particle physics theories, Weinberg [2] has shown that the $g=2$ value must hold in the tree approximation in order that scattering amplitudes possess good high-energy behavior. Furthermore, Ferrara, Porrati, and Telegdi [3] have implemented this requirement of good high-energy behavior in a Lagrangian framework, and regained $g=2$.

(2) Spin-1/2 charged leptons carry $g=2$, and this confirms the view that they are ''elementary'' particles. The only known higher-spin, elementary charged particle—the W boson—possesses a gyromagnetic ratio consistent with $g=2$. This is of course in agreement with the ''standard model'' where electromagnetism is fitted into a non-Abelian gauge group, which involves nonminimal electromagnetic coupling that results in $g=2$ [4].

In this Brief Report there is offered yet another reason for preferring $g=2$ for charged vector mesons: The kinetic term in a manifestly Lorentz-invariant Lagrange density for spin-1 fields possesses a (nonelectromagnetic) gauge invariance that removes redundant field degrees of freedom, which do not propagate in the vacuum. This gauge invariance is preserved when the fields couple to external, *vacuum* electromagnetism ($\partial^\mu F_{\mu\nu}=0$) provided $g=2$.

This simple observation is easily demonstrated. Consider the Lagrange density for free complex vector fields W_μ:

$$\mathcal{L}_0=-\frac{1}{2}|\partial_\mu W_\nu-\partial_\nu W_\mu|^2. \tag{1}$$

[In the following, we consider massless fields, or alternatively in the case of massive fields the discussion concerns the kinetic (derivative) portion of the Lagrange density.] Clearly $\mathcal{L}_0$ possess the nonelectromagnetic gauge invariance

$$W_\mu\to W_\mu+\partial_\mu\xi \tag{2}$$

where ξ is complex. W_μ is coupled to electromagnetism by replacing derivatives with covariant ones,

$$\partial_\mu W_\nu\to D_\mu W_\nu\equiv(\partial_\mu+ieA_\mu)W_\nu, \tag{3}$$

and allowing a further nonminimal interaction:

$$\mathcal{L}=-\frac{1}{2}G^{*\mu\nu}G_{\mu\nu}+ie(g-1)F^{\mu\nu}W^*_\mu W_\nu \tag{4}$$

$$G_{\mu\nu}\equiv D_\mu W_\nu-D_\nu W_\mu. \tag{5}$$

The nonminimal interaction ensures that charged vector particles carry gyromagnetic ratio g. When the electromagnetic field is source-free and $g=2$, one verifies that $\mathcal{L}$ remains invariant (up to total derivative terms) against the nonelectromagnetic gauge transformation (2), provided the derivatives are replaced by covariant ones:

$$W_\mu\to W_\mu+D_\mu\xi. \tag{6}$$

As is well known, for massive fields with $g=2$ the transversality condition $D^\mu W_\mu=0$ follows from the Euler-Lagrange field equation, while in the massless case that condition can still be imposed thanks to the nonelectromagnetic gauge symmetry (6).

A similar situation holds for interactions with non-Abelian gauge potentials. When the vector meson fields form (in general) a complex multiplet W^i_μ, which transforms under non-Abelian gauge transformations with the unitary representation matrices U^{ij},

$$W^i_\mu\to(U^{-1})^{ij}W^j_\mu, \tag{7}$$

whose anti-Hermitian generators are T_a, $[T_a,T_b]=f_{abc}T_c$, then the gauge covariant derivative is

$$(D_\mu W_\nu)^i=\partial_\mu W^i_\nu+A^{ij}_\mu W^j_\nu \tag{8}$$

where the gauge potential is an element of the Lie algebra in this representation

$$A^{ij}_\mu=A^a_\mu T^{ij}_a. \tag{9}$$

The Lagrange density for these fields reads (apart from a possible mass term)

$$\mathcal{L}=-\frac{1}{2}(G^*_{\mu\nu})^i(G^{\mu\nu})^i+(g-1)(W^{\mu*})^iF^{ij}_{\mu\nu}(W^\nu)^j. \tag{10}$$

A nonminimal coupling to the gauge field strength is present, and one verifies that the transformation

$$W_\mu \to W_\mu + D_\mu \Theta \tag{11}$$

changes $\mathcal{L}$ only by total derivative terms when $g=2$ and the gauge fields are sourceless $(D^\mu F_{\mu\nu}=0)$.

Additionally, let us note that in three-dimensional space-time and with vector fields in the adjoint representation, which is real, there exists another term that is invariant (apart from a total derivative) against (11). The relevant Lagrange density reads

$$\mathcal{L} = -\frac{1}{4}(G_{\mu\nu})^a (G^{\mu\nu})^a + f_{abc}(W^\mu)^a (W^\nu)^b F^c_{\mu\nu} + m\varepsilon^{\mu\alpha\beta} W^a_\mu F^a_{\alpha\beta}. \tag{12}$$

Please observe that thanks to the Bianchi identity satisfied by $F_{\alpha\beta}$:$D_\mu \varepsilon^{\mu\alpha\beta} F_{\alpha\beta}=0$, the last term possesses the symmetry (11) for arbitrary field strengths, not only sourceless ones. The strength parameter m carries dimensions of mass and the term has been posited previously in a gauge- and parity-invariant mass generation mechanism for a gauge theory, where also the gauge transformation (11) was introduced [5].

The spin-2 case does not exhibit exactly the same behavior as above, yet something similar, but less direct, does hold. Without interactions, but with a mass term, the spin-2 equation of motion for a symmetric, second-rank tensor $h_{\mu\nu}$ can be taken as

$$m^2\left(h_{\mu\nu} + \frac{1}{2} g_{\mu\nu} h\right) = -\Box^2 h_{\mu\nu} + \partial_\mu h_\nu + \partial_\nu h_\mu - \partial_\mu \partial_\nu h$$

$$h_\nu \equiv \partial^\mu h_{\mu\nu}, \quad h \equiv h^\mu_\mu \tag{13}$$

and the right side (massless part) enjoys the nonelectromagnetic gauge invariance

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \theta_\nu + \partial_\nu \theta_\mu. \tag{14}$$

Electromagnetic interactions can be included by promoting all derivatives to covariant ones, ordered in a specific fashion, and adding a nonminimal interaction. Thus one has

$$m^2\left(h_{\mu\nu} + \frac{1}{2} g_{\mu\nu} h\right) = -D^2 h_{\mu\nu} + D_\mu h_\nu + D_\nu h_\mu - \frac{1}{2}(D_\mu D_\nu + D_\nu D_\mu) h - ieg(F_{\mu\alpha} h^\alpha_\nu + F_{\nu\alpha} h^a_\mu) \tag{15}$$

where now $h_\nu = D^\mu h_{\mu\nu}$. With the ordering chosen above, the gyromagnetic ratio is g. Upon calculating the response of the right side to the gauge covariant version of the substitution (14),

$$h_{\mu\nu} \to h_{\mu\nu} + D_\mu \theta_\nu + D_\nu \theta_\mu, \tag{16}$$

one finds, with constant $F_{\mu\nu}$ $(\partial_\alpha F_{\mu\nu}=0)$,

$$\Delta_{\mu\nu} = ie(2-g)(F_{\mu\alpha} D^\alpha \theta_\nu + F_{\nu\alpha} D^\alpha \theta_\mu) - ie(1+g) \times (F_{\mu\alpha} D_\nu \theta^\alpha + F_{\nu\alpha} D_\mu \theta^\alpha) \tag{17}$$

so that for *no* value of g is invariance regained. [Note that for the ''minimal'' value [6] $g=\frac{3}{2}$,

$$\Delta_{\mu\nu} = ie\tfrac{3}{2}(F_{\mu\alpha}(D^\alpha \theta_\nu - D_\nu \theta^\alpha) + F_{\nu\alpha}(D^\alpha \theta_\mu - D_\mu \theta_\alpha))$$

the nonvanishing response involves the antisymmetric combination $D_\alpha \theta_\beta - D_\beta \theta_\alpha$, while the definition of the transformation in (16) makes use of the symmetric combination $D_\alpha \theta_\beta + D_\beta \theta_\alpha$.] However, one may improve the situation by the following (rather artificial) consideration. Note that (16) implies that h_μ transforms as

$$h_\nu \to D^2 \theta_\nu + D^\mu D_\nu \theta_\mu \tag{18}$$

that is, the ordering of the noncommuting covariant derivatives is inherited from previous definitions. But if we view h_μ as an independent quantity, we can prescribe an arbitrary ordering in (18), which is equivalent to modifying (18) by a multiple of $F_{\nu\mu}\theta^\mu$:

$$h_\nu \to D^2 \theta_\nu + D^\mu D_\nu \theta_\mu + iecF_{\nu\mu}\theta^\mu = D^2 \theta_\nu + (1-c) D^\mu D_\nu \theta_\mu + cD_\nu D^\mu \theta_\mu. \tag{19}$$

Then the change in the kinetic part of the equation of motion (15) becomes

$$\Delta_{\mu\nu} = ie(2-g)(F_{\mu\alpha} D^\alpha \theta_\nu + F_{\nu\alpha} D^\alpha \theta_\mu) - ie(1+g-c) \times (F_{\mu\alpha} D_\nu \theta^\alpha + F_{\nu\alpha} D_\mu \theta^\alpha) \tag{20}$$

so that for the unique choice $g=2$ the nonelectromagnetic gauge invariance is maintained in the presence of constant fields, as long as the ordering in (19) is taken with $c=3$. It remains an open question whether a more fundamental/natural reason can be found for this *ad hoc* ordering prescription.

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