Degenerate domain wall solutions in supersymmetric theories

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A family of degenerate domain wall configurations, partially preserving supersymmetry, is discussed in a generalized Wess-Zumino model with two scalar superfields. We establish some general features inherent to the models with continuously degenerate domain walls. For instance, for purely real trajectories additional ''integrals of motion'' exist. The solution for the profile of the scalar fields for any wall belonging to the family is found in quadratures for an arbitrary ratio of the coupling constants. For a special value of this ratio the solution family is obtained explicitly in terms of elementary functions. We also discuss the threshold amplitudes for multiparticle production generated by these solutions. Unexpected nullifications of the threshold amplitudes are found. [S0556-2821(98)00506-2]

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I. INTRODUCTION

The existence of several degenerate supersymmetric vacua is a generic phenomenon in supersymmetric theories with scalar superfields. Moreover, in many instances the vacuum manifold consists of several isolated points. Thus, the possibility arises of domain wall configurations interpolating between these vacua.

It has been recently shown [1,2] that some of the domain wall configurations in (3+1)-dimensional theories possess distinct supersymmetric properties.

(i) They generate a central extension of the $N=1$ superalgebra. The wall tension is proportional to the central charge. Because of the nonrenormalization theorem for the central charge, this implies that the wall energy density is exactly calculable; it is not renormalized by the loop corrections.

(ii) They preserve two out of four original supercharges (''$N=1/2$ supersymmetry'') corresponding to the minimal supersymmetry in the (2+1)-dimensional space tangential to the wall.

(iii) The profile of the fields across these walls satisfies first order differential equations analogous to the Bogomol'nyi-Prasad-Sommerfeld (BPS) equations [3] (the so-called ''BPS-saturated walls,'' or ''BPS walls''). These equations were called the creek equations [4] because of a mechanical interpretation.[1]

A nonvanishing central charge of the $N=1$ superalgebra exists for any field configuration interpolating between two distinct vacua [4]. Not every such configuration is BPS saturated, however. Those domain walls that are BPS saturated possess peculiar features following from point (ii) above. The BPS domain walls have interesting properties even in the simplest theories, e.g., in the Wess-Zumino model with one chiral superfield. They become even more remarkable in the theories with two or more chiral superfields interacting with each other.

In the theories with K scalar superfields, generically, there are 2^K degenerate vacuum states and, correspondingly, there are at least 2^{K-1} (2^K-1) domain wall types [9]. In Ref. [10] it was shown that, quite typically, some of these domain walls turn out to be continuously degenerate. Collective coordinates exist correspondingly to a continuous deformation of the internal structure of the wall. Varying these coordinates we change the wall structure, leaving the wall energy density intact.

In this work we investigate this phenomenon, both in general aspect and in some simple examples. It will be argued that the continuous degeneracy is related to the existence of additional ''integrals of motion.'' We will explicitly find such an integral in a particular two-field model. Using this result, it becomes possible to obtain a generic family of the domain wall solutions in this two-field model. For arbitrary values of the coupling constants the solution is given in terms of quadratures. For some specific values a closed-form solution in terms of elementary functions exists. We take advantage of this explicit solution to extract consequences for the high-order multiparticle amplitudes at threshold.

The supersymmetric vacua are determined by the extrema of the superpotential $W(\phi_k)$,

$$\partial W/\partial\phi_k=0,\quad k=1,\ldots,K. \tag{1}$$

The general form of the BPS-saturation equations for a static wall in which the fields ϕ_i depend only on the coordinate z is

$$\partial_z\phi_k=\frac{\partial W^\dagger}{\partial\phi_k^\dagger}e^{i\alpha}, \tag{2}$$

where α is a constant (z independent) complex phase. Let us assume that two solutions of Eq. (1) are found, $\{\phi\}_1$ and

[1]Previously the creek equations were considered in different contexts in Refs. [5–8].

$\{\phi\}_2$, where the braces denote a set of all scalar fields ϕ_k. Denote the corresponding values of the superpotential by

$$W_1 \equiv W(\{\phi\}_1), \quad W_2 \equiv W(\{\phi\}_2). \tag{3}$$

Without loss of generality one can assume that $W_1=0$ and W_2 is real and positive (this can be always achieved by appropriate transformations of the superpotential). Then the phase α in Eq. (2) can be set equal to zero. If, additionally, the superpotential is real for real values of the superfields, to which we limit our investigation, then Eq. (2) takes the form

$$\partial_z \phi_k = \frac{\partial W}{\partial \phi_k}. \tag{4}$$

Now, if one interprets z as ''time,'' the latter equations have a simple mechanical interpretation [4]: they describe the flow of a very viscous fluid, whose inertia can be neglected, in the potential relief $-W(\phi_k)$ from one extremum $-W_1$ of the relief, along a gradient line, to a lower extremum $-W_2$. (Obviously, reflection of the z direction is possible, in which case the flow is described by the ''potential'' W rather than $-W$. Instead of the reversal of the z axis, one can view this as a shift by π of the phase α in Eq. (2). For definiteness we use the mechanical analogy within the conventions of Eq. (4).) From the fluid flow analogy it is clear that the necessary condition for the existence of a solution of Eq. (4) is that the extrema W_1 and W_2 be of different height. One can take advantage of a rich intuition one has in connection with the mechanical motion of this type. To make this analogy more graphic we will sometimes denote the derivative over z as

$$\partial_z \phi \to \dot{\phi}$$

in the cases where there is no menace of confusion. Correspondingly, the quantities conserved along the given trajectory will be referred to as integrals of motion. One integral of motion, energy, is well known of course; it is universal and has nothing to do with the specific trajectories under consideration. We will be interested in additional integrals of motion, specific to the creek equations.

The surface energy density of the BPS wall is given by the difference of the superpotential at the two extrema:

$$\varepsilon = 2(W_2 - W_1). \tag{5}$$

Although at least some of the results to be reported below seem to be generic, we will make no attempt at formulating them in full generality. Instead, we will focus on a specific instructive example: a two-field model [9,10] with the scalar superfields Φ and X and the superpotential

$$W(\Phi,X) = \frac{m^2}{\lambda}\Phi - \frac{1}{3}\lambda\Phi^3 - \alpha\Phi X^2. \tag{6}$$

Here m is a mass parameter and λ and α are dimensionless coupling constants. It is assumed that the phases of the fields and W are adjusted in such a way that all parameters in the superpotential are real and positive. Only occasionally we will digress to more general models. Below the lowest component of Φ and X will be denoted by ϕ and χ, respectively.

Clearly, the superpotential (6) is not the most general form of the superpotential in the renormalizable (3+1)-dimensional models, even given the freedom of redefining the fields. It contains, however, sufficient features for a discussion of the nontrivial properties of the domain walls we are interested in.[2] The model with the superpotential (6) will be referred to below as a minimal two-field model. In terms of the scalar components ϕ and χ of the respective superfields Φ and X, the ''potential relief'' $-W$ has its maximum $-W_1$ at $\phi=-m/\lambda$, $\chi=0$ and the minimum $-W_2$ at $\phi=m/\lambda$, $\chi=0$. It also has two saddle points of equal height at $\phi=0$, $\chi=\pm m/\sqrt{\lambda\alpha}$. The BPS-saturated walls exist connecting the maximum and the minimum, and also connecting either of the saddle points with the maximum or the minimum. Moreover, all these BPS-saturated configurations belong to one and the same family of solutions, corresponding to the flow from the maximum to the minimum with different starting conditions [10]. All trajectories from the family are real.

This is an ideal setting for establishing the existence of additional integrals of motion. The one relevant to the model (6) is obtained in an explicit form. Certainly, given the additional constraint, one can readily reconstruct the full family of solutions in quadratures. Further simplifications arise (a) for arbitrary values of λ, α and the vanishing value of the integral of motion; (b) for $\lambda/\alpha=4$ and arbitrary value of the integral of motion. As a matter of fact, the first case was treated in Ref. [10] where it was found that

$$\phi(z) = \frac{m}{\lambda}\tanh\left(\frac{2\alpha}{\lambda}mz\right),$$

$$\chi(z) = \pm\frac{m}{\sqrt{\lambda\alpha}}\sqrt{1-\frac{2\alpha}{\lambda}}\,\mathrm{sech}\left(\frac{2\alpha}{\lambda}mz\right). \tag{7}$$

Until now, this was the only nontrivial solution for one specific configuration in the degenerate family that was explicitly known, apart from the trivial standard wall with $\chi=0$. In Sec. III we construct the solutions for all configurations belonging to this family. For an arbitrary ratio of the coupling constants $\rho\equiv\lambda/\alpha$, this solution is semi-explicit in the sense that the trajectory in the field space is found in terms of elementary functions, while the dependence of the fields on z, although expressed in quadratures, is not found analytically. An explicit dependence on z can be found in terms of elementary functions for two special values of ρ: $\rho=1$ and $\rho=4$. The case of $\rho=1$ is, however, trivial since after a $\pi/4$ rotation in the space of the fields (ϕ,χ) the model reduces to two fields not interacting with each other. The remaining case of $\rho=4$ is quite nontrivial and provides a whole family of new domain wall solutions.

In four dimensions the requirement of renormalizability restricts the form of the superpotential: it must be polynomial in fields of at most third order. If we dimensionally reduce the theory to two dimensions, the choice becomes infinitely

[2] An equivalent model was also considered in Ref. [11], and a similar model of higher order in the fields in Ref. [12], in connection with different problems.

richer. Any analytic function of Φ and X can serve as a superpotential, without spoiling the renormalizability of the two-dimensional model. We briefly discuss the issue of the continuous degeneracy of the soliton solutions in this setting (Sec. II).

The domain wall configurations, when viewed as depending on the Euclidean time τ rather than on the spatial coordinate z, are known [13] to be the generating functions for amplitudes of multiple production of bosons at threshold by field operators. For the model discussed here, these are the amplitudes for production of arbitrary numbers n_ϕ of the ϕ bosons and k_χ of the χ bosons at the corresponding thresholds,

$$A^{\phi}_{nk} \equiv \langle n_\phi k_\chi | \phi(0) | 0 \rangle \quad \text{and} \quad A^{\chi}_{nk} \equiv \langle n_\phi k_\chi | \chi(0) | 0 \rangle.$$

In Sec. IV we use the relation between the domain wall profile and the threshold production amplitudes in a twofold way: to point out a constraint on the solutions in the degenerate family stemming from the fact that they generate the same set of amplitudes and to find the multiboson amplitudes at the tree level explicitly in the case of $\rho=4$ where the explicit form of the solutions is available.

II. ADDITIONAL INTEGRALS OF MOTION

To begin with, we will discuss the occurrence of an additional integral of motion in the simplest example (6). In this model the creek equations have the form

$$\dot{\phi} = \frac{\partial W}{\partial \phi} = \frac{m^2}{\lambda} - \lambda \phi^2 - \alpha \chi^2,$$

$$\dot{\chi} = \frac{\partial W}{\partial \chi} = -2\alpha\phi\chi. \tag{8}$$

Let us introduce a ''dual'' function

$$\widetilde{W}(\Phi,X) = X^{-\rho}\left(\frac{m^2}{\lambda^2} - \Phi^2 - \frac{1}{\rho-2}X^2\right), \tag{9}$$

where

$$\rho \equiv \frac{\lambda}{\alpha}.$$

The meaning of the word ''dual'' will become clear shortly. Equation (9) assumes that the parameter $\rho \neq 2$. The case $\rho=2$ is special and has to be treated separately.

The dual function has the property

$$\frac{\partial \widetilde{W}}{\partial \phi_i} = \varepsilon_{ij} S(\phi_i) \frac{\partial W}{\partial \phi_j}, \tag{10}$$

where $\phi_{1,2}$ stands for $\{\phi,\chi\}$, ε_{ij} is the antisymmetric tensor of the second rank, and S is some scalar function. In the model at hand

$$S(\Phi,X) = \frac{1}{\alpha}\frac{1}{X^{\rho+1}}. \tag{11}$$

Now, if z is interpreted as ''time,'' the dual function $\widetilde{W}$ is conserved along the trajectory. Indeed,

$$\dot{\widetilde{W}} = \frac{\partial \widetilde{W}}{\partial \phi_i}\dot{\phi}_i = \frac{\partial \widetilde{W}}{\partial \phi_i}\frac{\partial W^\dagger}{\partial \phi_i^\dagger}. \tag{12}$$

For real solutions Eq. (10) implies that the right-hand side of Eq. (12) vanishes.

Therefore, each particular real trajectory connecting the stationary points 1 and 2 of the superpotential is characterized by the value of the dual function on this trajectory. More exactly, dual functions conserved along the trajectory can be introduced for all solutions $\phi_k(z)$ with the constant, i.e., z independent, phases of the fields ϕ_k. By an appropriate redefinition of the fields, we can obviously return to the real solutions.

In the general case of nonminimal models, the superpotential (restricted to real values of the superfields) is characterized by the gradient lines and the level lines. The latter correspond to fixed values of W. Two nets of lines — gradient and level — are locally orthogonal to each other. The level lines of the dual function are the gradient lines of the superpotential, while the gradient lines of the dual function are the level lines of the superpotential. From this graphic interpretation it is intuitively clear that a dual function $\widetilde{W}$ must exist for every W, although, unlike the minimal two-field model, it is not always possible to find them analytically. The points where W (restricted to the real values of the scalar fields) develops maxima or minima are the singular points of $\widetilde{W}$. The saddle points of W are the saddle points of $\widetilde{W}$.

A. Solitons in (1+1)-dimensions

If in four dimensions the choice of the superpotential is severely restricted by the requirement of renormalizabilty (only polynomials which are at most cubic are allowed), in two dimensions any superpotential leads to a sensible quantum theory. If one takes a generalized Wess-Zumino model in four dimensions, with arbitrary number of fields and an arbitrary superpotential, and dimensionally reduces it to two dimensions, one arrives at $N=2$ supersymmetry in two dimensions. In two dimensions the domain walls become solitons — localized field configurations with finite energy. After quantization they are to be viewed as particles. The $N=2$ supersymmetric theories with scalar superfields in two dimensions were extensively studied in Ref. [7]. The case which is of most interest to us, a continuous family of degenerate solitons, seemingly was not discussed in this work.

Since we are not limited now to polynomial superpotentials we can consider whole families of models. Consider for definiteness two-field models. Let us start from a certain model with a superpotential W. One then can consider any other model with a superpotential $W_{\rm NEW} = f(W)$ where f is an arbitrary function. If in the original model the dual function is known, it remains the same for the whole family. Indeed, Eq. (10) implies that

$$\frac{\partial \widetilde{W}}{\partial \phi_i} = \varepsilon_{ij} S_{\rm NEW} \frac{\partial W_{\rm NEW}}{\partial \phi_j}, \quad S_{\rm NEW} = S(df/dW)^{-1}. \tag{13}$$

In other words, $\widetilde{W}$ remains the integral of motion for real trajectories in any model belonging to the given family.

The minimal two-field family, dimensionally reduced to $D=2$, presents a simplest example where continuously degenerate soliton solutions exist. Now one can easily provide a plethora of other interesting examples. For instance, one can consider superpotentials which are bounded from above and from below for real values of the superfields. In such models typically every soliton will appear as a member of a degenerate family of solitons, the degeneracy being unrelated to any external symmetry. Generalizations of the sine-Gordon model fall into this category. Consider, for example, the superpotential

$$W=-\sin\Phi-\sin X-\alpha(\sin\Phi)(\sin X), \tag{14}$$

where α is a dimensionless parameter, and all dimensional parameters are set equal to unity. This superpotential is periodic in Φ and X; for $\alpha=0$ it describes two decoupled fields (each of them presents a supergeneralization of the sine-Gordon model). If $\alpha\neq 0$, the fields Φ and X start interacting with each other. Inside the periodicity domain $0\leqslant\Phi,X\leqslant 2\pi$ the relief of the superpotential W is qualitatively similar to that of the minimal two-field model: $-W$ has one maximum at $\Phi=X=\pi/2$, one minimum at $\Phi=X=3\pi/2$, and two saddle points at $\Phi=\pi/2,X=3\pi/2$ and $\Phi=3\pi/2,X=\pi/2$ (at least for small values of α). Any real trajectory (out of a continuous family of trajectories) starting at the maximum leads to a minimum. The only exceptions are two trajectories leading to the saddle points (the exceptional, or basic solitons). The masses of two exceptional solitons are $4+4\alpha$ and $4-4\alpha$, respectively. Continuously degenerate solitons are bound states of two basic solitons, with mass 8, i.e., the binding energy exactly vanishes.

Unlike the minimal two-field model, all solutions in the model (14) have finite masses; there are no trajectories leading to abysses.

The degeneracy is not lifted due to quantum corrections. This suggests that in every model with continuously degenerate solitons there should exist a dual description where the exceptional solitons (comprising the degenerate ones) appear as decoupled (i.e., not interacting with each other) particles from the very beginning. This issue deserves further investigation.

III. SOLUTION FOR THE DEGENERATE WALLS

Now we return to the minimal two-wall models (6). Our task is to find the family of the wall trajectories (i.e., ϕ versus χ for every given wall solution and every allowed value of $\widetilde{W}$). Next we will find the wall solutions themselves, i.e., $\phi(z)$ and $\chi(z)$.

A. General case: arbitrary ratio $\rho\equiv\lambda/\alpha$

It is convenient to introduce dimensionless field variables f and h as

$$\phi=\frac{m}{\lambda}f,\qquad \chi=\frac{m}{\sqrt{\lambda\alpha}}h,$$

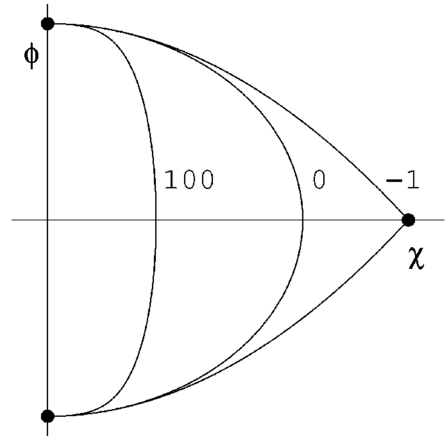

FIG. 1. Few trajectories in the field space from the family of degenerate solutions (at $\rho=4$). The numbers represent the values of the constant C for the corresponding trajectory. The heavy dots show the positions of the vacuum states.

and to set the mass parameter m to one (it can be restored, if needed, from dimension). The BPS-saturation equation (4) then take the form

$$\frac{df}{dz}=1-f^2-h^2,$$

$$\frac{dh}{dz}=-\frac{2}{\rho}fh. \tag{15}$$

By eliminating the variable z from these equations, one finds the equation for the trajectory in the (f,h) plane,

$$\frac{df}{dh}=-\frac{\rho}{2}\frac{1-f^2-h^2}{fh}. \tag{16}$$

The general solution of this equation can be written as

$$f^2=1-\frac{\rho h^2}{\rho-2}-Ch^{\rho}, \tag{17}$$

where C is an integration constant. It is connected with the integral of motion, Eq. (9), by a simple proportionality relation,

$$\widetilde{W}=\left(\frac{m}{\lambda}\right)^{2-\rho}\rho^{-\rho/2}C.$$

The full trajectory runs from the point $(f,h)=(-1,0)$ in the ''distant past'' (i.e., $z=-\infty$) to the point $(f,h)=(1,0)$ in the ''distant future'' (i.e., $z=\infty$). The relation (17) determines it piecewise: from $(f,h)=(-1,0)$ to $(f,h)=(0,h_{\max})$ and from $(0,h_{\max})$ to $(1,0)$, where $h_{\max}$ is the maximal amplitude of h on the trajectory (see Fig. 1). For the given value of C the value $h_{\max}$ is obviously determined from the solution of Eq. (17) for h with f being set to zero,

$$\frac{\rho}{\rho-2}h_{\max}^2+Ch_{\max}^{\rho}=1. \tag{18}$$

Using the symmetry under $h\leftrightarrow -h$, it is sufficient to discuss only the trajectories with positive h.

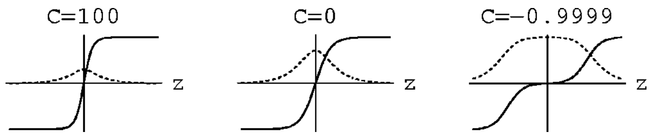

FIG. 2. The profile of the fields ϕ (solid) and χ (dashed) as a function of the spatial coordinate z for three values of C $(\rho=4)$. It is seen that at C approaching the critical value $C_*=-1$, the profile separates into two walls.

The freedom in C is in fact limited by the condition that the fields remain real along the whole trajectory. In this connection it should be noted that the BPS-saturation equation (2) is intrinsically nonanalytic, thus the trajectory found for the real values of the fields cannot be continued to the complex values of the fields. The requirement of the real trajectories translates into the requirement that $h_{\max}$ is real positive and varies in the interval $[0,1]$. An elementary inspection of Eq. (18) yields the allowed domain for the constant C,

$$C_* \leqslant C \leqslant +\infty, \quad C_*=\frac{2}{2-\rho}.$$

One can readily see that the trajectory with $C=+\infty$ has $h(z)\equiv 0$ and, thus, reduces to the well-known wall solution in the one-field theory, $f(z)=\tanh z$. On the other hand, the trajectory with C equal to the critical value $C=C_*$ has $h_{\max}=1$. Thus, it in fact describes two infinitely separated domain walls: one interpolating between $(f,h)=(-1,0)$ and $(0,1)$ (the saddle point of W) and the other between $(0,1)$ and $(1,0)$. The configurations with the intermediate values of C interpolate between these two extremes; in a sense, they can be viewed as the solutions with the latter two walls at a finite separation (see Fig. 2). Remarkably, the degeneracy of the energy of the $C>C_*$ solutions implies that the latter two walls do not interact with each other.

If $\rho>2$, the point $C=0$ belongs to the allowed interval. For $C=0$ (i.e., $\tilde{W}=0$) the trajectory takes an algebraic form. One can immediately find an explicit domain wall solution, see Eq. (7). As a matter of fact, it was obtained previously [10].

It should be noted that the value $\rho=2$ presents a special case because of the singularity in Eq. (17) at $\rho=2$. This singularity can easily be resolved, however, either by considering the limiting procedure in Eq. (17), or by solving the Eq. (16) separately for $\rho=2$. The result is that the relation (17) at $\rho=2$ reduces to

$$f^2=1+h^2(\ln h^2-\tilde{C}), \tag{19}$$

where $\tilde{C}$ is another integration constant bound by the condition $1\leqslant \tilde{C}\leqslant +\infty$.

The relation (17) gives the trajectory of the field configuration in the (f,h) plane (see Fig. 1). In order to find the coordinate dependence of each of the fields, one can substitute f as found from Eq. (17) into the second of Eqs. (15) and then obtain the solution in an implicit form, ''in quadratures''

$$z=-\frac{\rho}{2}\int \frac{dh}{hf(h)}. \tag{20}$$

B. Explicit solution for $\rho=4$

For arbitrary values of ρ there is no algebraic expression for the integral in Eq. (20) in terms of known functions. The exceptional cases are

$$\rho=\frac{1}{2},\frac{2}{3},6, \quad \text{and} \quad 8,$$

when the integral is expressed in terms of elliptic functions, and

$$\rho=1 \quad \text{and} \quad \rho=4,$$

when the integral is elementary.

The elliptic cases are rather cumbersome, while the case of $\rho=1$ is trivial: at $\rho=1$ the model considered here describes two fields $\tilde{\phi}=(\phi+\chi)/\sqrt{2}$ and $\tilde{\chi}=(\chi-\phi)/\sqrt{2}$ that do not interact with each other. For these reasons we pursue here the explicit solution only for the exceptional case of $\rho=4$.

If we choose this specific value, $\rho=4$, in Eqs. (17) and (20), we readily find the explicit expressions for the fields f and h versus z,

$$f(z)=\frac{u^2-C-1}{(u+1)^2+C}, \quad h^2(z)=\frac{2u}{(u+1)^2+C}, \tag{21}$$

where $u=\exp(z-z_0)$ and z_0 is an arbitrary shift of the coordinate z. [Clearly, the solution can be centered at $z=0$ so that $f(0)=0$, if z_0 is chosen as $2z_0=-\ln(C+1)$.] For $\rho=4$ the integration constant C is bounded by the condition

$$-1\leqslant C\leqslant +\infty.$$

For completeness we also present the same solution in terms of the fields $\phi(z)$ and $\chi(z)$, with the normalization factors restored (Fig. 2),

$$\phi(z)=\frac{m}{\lambda}\,\frac{e^{2m(z-z_0)}-C-1}{(e^{m(z-z_0)}+1)^2+C},$$

$$\chi(z)=\pm\frac{m}{\lambda}\,\frac{2\sqrt{2}e^{m(z-z_0)/2}}{\sqrt{(e^{m(z-z_0)}+1)^2+C}}. \tag{22}$$

The constant C plays the role of a collective coordinate. The occurrence of another collective coordinate, z_0, is a trivial consequence of the fact that the wall solution spontaneously breaks the translational invariance of the original model in the z direction.

At the same time, C is unrelated to spontaneous breaking of any symmetry of the model. It reflects the fact that the symmetry of the particular solutions at hand is higher than that of the model *per se*. As far as we know, Eq. (22) presents the first explicit example of a soliton family in the renormalizable ϕ^4 theory with a continuous degeneracy due to continuous deformations of the soliton structure.

One can readily derive from the explicit solution above the limiting cases discussed in Sec. II A. At the critical value $C=-1$, the solution in Eq. (21) degenerates into

$$f=\frac{u}{u+2},\quad h^2=\frac{2}{u+2} \tag{23}$$

and describes the domain wall connecting the vacua $(f,h)=(0,1)$ and $(1,0)$. In order to get the wall connecting $(f,h)=(-1,0)$ with $(0,1)$ one can use the symmetry under $f\leftrightarrow -f$ and reverse the sign of f in the first equation in Eq. (23). In order to recover the one-field solution in the limit $C\to+\infty$, one has to accordingly adjust the coordinate shift z_0 as $\exp(-2z_0)=C+1$. Then in the limit $C\to+\infty$ one gets $h=0$ and $f=\tanh z$.

IV. THRESHOLD MULTIPARTICLE AMPLITUDES

In this section we will take advantage of the explicit wall family solution found above in order to extract certain predictions for the high-order behavior of the multiparticle amplitudes at thresholds. The corresponding analysis for the one-field Wess-Zumino model was carried out in Ref. [4]. In the one-field model there is little distinction with the nonsupersymetric case (for a review see Ref. [14]). The two-field model is much more interesting since it reveals a new pattern.

In what follows we will need to know that in the vacuum with $\phi=m/\lambda$ and $\chi=0$ (in which our wall trajectory ends) the mass of the ϕ quantum is equal to $2m$, while the mass of the χ quantum is equal to $2m/\rho$. In the case $\rho=4$ to be analyzed below, the mass of the χ quantum is equal to $m/2$. The same is valid for the vacuum $\phi=-m/\lambda,\chi=0$, from which the trajectories originate.

A. An overview of the formalism

The solutions to the field equations, in particular the domain wall solutions, are directly related to multiparticle amplitudes, by virtue of the formalism developed by Brown [13] (for a more recent review see Ref. [14]). Being adapted to the present problem of two fields the formalism is constructed as follows. Consider for definiteness the amplitude $\langle n,k|\phi(0)|0\rangle$ describing the production by the field operator $\phi(x)$ of a multiparticle state consisting of n on-shell bosons of the field ϕ with 4-momenta p_a $(a=1,\dots,n)$ and k on-shell bosons of the field χ with 4-momenta p_b $(b=1,\dots,k)$ in a vacuum $|0\rangle$ of the theory. According to the standard reduction formula, this amplitude is expressed through the response of the system to external sources $\rho_\phi(x)$ and $\rho_\chi(x)$, coupled to the corresponding fields as $\rho_\phi\phi+\rho_\chi\chi$ in the Lagrangian,

$$\langle n,k|\phi(0)|0\rangle=\left[\prod_{a=1}^{n}\lim_{p_a^2\to m_\phi^2}\int d^4x_a e^{ip_a x_a}(m_\phi^2-p_a^2)\frac{\delta}{\delta\rho_\phi(x_a)}\right]\times\left[\prod_{b=1}^{k}\lim_{p_b^2\to m_\chi^2}\int d^4x_b e^{ip_b x_b}(m_\chi^2-p_b^2)\frac{\delta}{\delta\rho_\chi(x_b)}\right]\langle 0_{out}|\phi(x)|0_{in}\rangle^{\rho_\phi,\rho_\chi}\big|_{\rho_\phi=0,\ \rho_\chi=0}, \tag{24}$$

where m_ϕ and m_χ are the masses of the respective bosons in the vacuum considered.

The classical response, i.e., the classical solution of the field equations in the presence of the sources generates, through Eq. (24), the tree-level amplitudes, which we will be mainly concerned with here. Moreover, as will be seen, the configurations, of the type of the domain walls, depending on only one variable, are related to multiparticle production exactly at the threshold, i.e., at the spatial momenta of the produced particles exactly equal to zero. In this situation it is sufficient to consider the response of the system in Eq. (24) to spatially uniform time-dependent sources,

$$\rho_\phi(t)=\rho_\phi(\omega_\phi)e^{i\omega_\phi t},\quad \rho_\chi(t)=\rho_\chi(\omega_\chi)e^{i\omega_\chi t},$$

and take in the very end the on-shell limit in Eq. (24) by tending ω_ϕ to m_ϕ and $\omega_\chi\to m_\chi$. The spatial integrals in Eq. (24) then give the normalization spatial volume, conventionally set to one, while the time dependence with the fixed functional form of the sources implies that the propagator factors and the functional derivatives enter in the combination

$$(m_\phi^2-p_a^2)\frac{\delta}{\delta\rho_\phi(x_a)}\to(m_\phi^2-\omega_\phi^2)\frac{\delta}{\delta\rho_\phi(t)}=\frac{\delta}{\delta a_\phi(t)},$$

$$(m_\chi^2-p_b^2)\frac{\delta}{\delta\rho_\chi(x_b)}\to(m_\chi^2-\omega_\chi^2)\frac{\delta}{\delta\rho_\chi(t)}=\frac{\delta}{\delta a_\chi(t)}, \tag{25}$$

where

$$a_\phi(t)=\frac{\rho_\phi(\omega_\phi)e^{i\omega_\phi t}}{m_\phi^2-i\epsilon-\omega_\phi^2},\quad a_\chi(t)=\frac{\rho_\chi(\omega_\chi)e^{i\omega_\chi t}}{m_\chi^2-i\epsilon-\omega_\chi^2} \tag{26}$$

coincide with the response of free fields to the external sources. For finite amplitudes of the sources the response is singular in the on-shell limit $\omega_\phi\to m_\phi$, $\omega_\chi\to m_\chi$. Therefore, following Brown [13], the amplitudes of the sources should be taken to zero in this limit, so that the factors $a_\phi(t)$ and $a_\chi(t)$ are finite:

$$a_\phi(t)\to a_\phi e^{im_\phi t},\quad a_\chi(t)\to a_\chi e^{im_\chi t}.$$

Thus, for the purpose of calculating the multiparticle amplitudes at the tree level, one looks for a solution of the classical field equations with no sources. The only information about the sources left from the above limiting procedure is that the sources drive only positive frequencies in the fields, thus the condition for the sought solution is that it should contain only the positive frequency part with all harmonics being multiples of $e^{im_\phi t}$ and $e^{im_\chi t}$. The latter condition is equivalent to requiring that the solution for the fields goes to the classical vacuum at infinity in the Euclidean time $\tau=\mathrm{Im}t\to+\infty$. The multiparticle amplitudes are then given by the derivatives of the solution $\phi(t)$,

$$A_{n,k}^{\phi}\equiv\langle n,k|\phi(0)|0\rangle=\left(\frac{\partial}{\partial a_{\phi}(t)}\right)^{n}\left(\frac{\partial}{\partial a_{\chi}(t)}\right)^{k}\phi(t)|_{a_{\phi}=0,a_{\chi}=0}. \tag{27}$$

Since the equations for the fields ϕ and χ are coupled, one simultaneously finds the solution for the field χ and, thus, the amplitudes for the multiboson production by χ,

$$A_{n,k}^{\chi}\equiv\langle n,k|\chi(0)|0\rangle=\left(\frac{\partial}{\partial a_{\phi}(t)}\right)^{n}\left(\frac{\partial}{\partial a_{\chi}(t)}\right)^{k}\chi(t)|_{a_{\phi}=0,a_{\chi}=0}. \tag{28}$$

The operational procedure for calculating the amplitudes is therefore as follows. First, one obtains the solution of the Euclidean classical field equations depending only on time τ and approaching, at $\tau\to+\infty$, the vacuum state (ϕ_0,χ_0) in which the scattering theory is considered. Then, the solution is expanded in the harmonics of $e^{-m_\phi\tau}$ and $e^{-m_\chi\tau}$,

$$\phi(\tau)=\sum_{n=0,k=0}^{\infty}F_{n,k}e^{-(nm_\phi+km_\chi)\tau},$$

$$\chi(\tau)=\sum_{n=0,k=0}^{\infty}H_{n,k}e^{-(nm_\phi+km_\chi)\tau}, \tag{29}$$

where $F_{n,k}$ and $H_{n,k}$ are the coefficients of the expansion.

Note that

$$F_{0,0}=\phi_0 \quad \text{and} \quad H_{0,0}=\chi_0,$$

while the coefficients of the appropriate first harmonics are identified as the described above factors a_ϕ and a_χ,

$$F_{1,0}=a_\phi, \quad H_{0,1}=a_\chi.$$

Then, according to Eqs. (27) and (28), the amplitudes are expressed as

$$A_{n,k}^{\phi}=n!k!\frac{F_{n,k}}{F_{1,0}^{n}H_{0,1}^{k}}, \quad A_{n,k}^{\chi}=n!k!\frac{H_{n,k}}{F_{1,0}^{n}H_{0,1}^{k}}. \tag{30}$$

Before closing this discussion of the general formalism, we would like to emphasize that the latter equations can be also viewed as a constraint on any solution approaching a vacuum state (ϕ_0,χ_0) at $\tau\to+\infty$. Namely, if up to the appropriate linear terms the fields behave as

$$\phi(\tau)=\phi_0+ae^{-m_\phi t}+\cdots, \quad \chi(\tau)=\chi_0+be^{-m_\chi t}+\cdots,$$

then the subsequent terms in the expansion of the fields are fully determined in terms of a, b and the fixed set of the amplitudes A^ϕ and A^χ,

$$F_{n,k}=\frac{A_{n,k}^{\phi}a^nb^k}{n!k!}, \quad H_{n,k}=\frac{A_{n,k}^{\chi}a^nb^k}{n!k!}. \tag{31}$$

Furthermore, the absolute normalization of the coefficients a and b is rather a matter of convention. Indeed, under a shift of τ,

$$\tau\to\tau-\tau_0,$$

these coefficients change as

$$a\to ae^{m_\phi\tau_0}, \quad b\to be^{m_\chi\tau_0}.$$

Thus, the only parameter that distinguishes between essentially different solutions approaching the same vacuum state is the ratio

$$c=\frac{a}{b^{m_\phi/m_\chi}}.$$

Therefore, in a general two-field theory a family of solutions approaching a vacuum state at $\tau\to+\infty$ is parametrized by a single parameter c. This parameter is in one-to-one correspondence with the integration constant C, or the value of the dual function $\widetilde{W}$ on the trajectory. The coefficients of the expansion (29) are then fixed by the multiparticle amplitudes.

B. Multiparticle amplitudes in the supersymmetric model

The domain wall solutions discussed in Sec. II can be directly applied to calculating the multiparticle amplitudes. To this end one should consider the fields depending on the Euclidean time τ rather than on the spatial coordinate z. Since this amounts to a trivial relabeling of the variable, we retain here the notation z for the variable. We also use the notation $a=F_{1,0}$ and $b=H_{0,1}$. Every BPS-saturated solution from the family under consideration approaches at $z\to+\infty$ the vacuum at $(\phi,\chi)=(m/\lambda,0)$. Thus, this is the vacuum state in which the multiparticle amplitudes are generated by the solutions. Remember that the masses of the particles in this vacuum are expressed in terms of the parameters of the model as $m_\phi=2m$, $m_\chi=2(\alpha/\lambda)m=m_\phi/\rho$.

In the case of arbitrary ratio ρ one can use Eq. (17) to obtain a relation between the coefficients a, b and the constant C. Indeed, at $z\to+\infty$ the field $\chi(z)$ goes to zero as

$$\chi(z)=be^{-m_\chi z}+\cdots,$$

corresponding in the dimensionless variables to

$$h(z)=\frac{\sqrt{\lambda\alpha}b}{m}e^{-2z/\rho}.$$

The linear in e^{-2z} harmonics in $f(z)$ arises from the term in Eq. (17) with the constant C, from where one finds the coefficient a of the linear in $e^{-m_\phi z}$ harmonics in $\phi(z)$ $[\phi(z)=m/\lambda+ae^{-m_\phi z}+\cdots]$ as

$$a=-\frac{C}{2}\frac{m}{\lambda}\left(\frac{\sqrt{\lambda\alpha}b}{m}\right)^{\rho}. \tag{32}$$

In connection with the derivation of the latter relation, it should be noted that for $\rho>2$ the harmonics with $\exp(-m_\phi z)$ is not the leading one in the field $\phi(z)$ at large z because of the presence of the second harmonics with the mass of χ: $F_{0,2}\exp(-2m_\chi z)$. Thus, the presented derivation, strictly speaking, is justified only at $\rho<2$. However, the relation (32) is also applicable at $\rho>2$ since the fields and the coefficients of their expansion are analytic functions of the couplings, and the relation (32) can be analytically continued from $\rho<2$ to the domain $\rho>2$.

It can be also noted that the solution in Eq. (7) with $C=0$ has $a=0$, according to Eq. (32). Thus, it generates only the amplitudes of multiple production of the bosons of the field χ by either the operator $\chi(0)$ or $\phi(0)$. For this reason, it expands in the harmonics determined only by the mass of χ.

Furthermore, for a rational ratio ρ the masses m_ϕ and m_χ are also in a rational proportion. Thus if only the z dependence of the fields were known, there would be an ambiguity, at least in some harmonics, in separation between the production of the χ bosons and the ϕ bosons. However, this ambiguity is resolved if the dependence of the solution on C is known, by using Eq. (32), which shows that the constant C serves as a ''tag'' for a ϕ boson. The power of C in the given harmonics gives the number of the ϕ bosons in the amplitude generated by this harmonics.

We illustrate this method for our explicit solution in the case of $\rho=4$ and we also find explicitly the amplitudes $A^\phi_{n,k}$ and $A^\chi_{n,k}$ in this case.

Setting for definiteness $z_0=0$ in the explicit solution in Eq. (22), we find the coefficient b determining the rate of approach of the field $\chi(z)$ to its vacuum value (zero),

$$b=\frac{2\sqrt{2}m}{\lambda}. \tag{33}$$

Furthermore, using the relation (32) at $\rho=4$ we also obtain the coefficient a,

$$a=-2C\frac{m}{\lambda}. \tag{34}$$

We then expand the expressions for the fields in Eq. (22) in powers of C and, finally, each term of this expansion in powers of e^{-mz}. In this way we get

$$\phi(z)=\sum_{n=0,k=0}^{\infty}\mathcal{F}_{n,k}C^n e^{-(2n+k/2)mz},$$

$$\chi(z)=\sum_{n=0,k=0}^{\infty}\mathcal{H}_{n,k}C^n e^{-(2n+k/2)mz}, \tag{35}$$

where the coefficients $\mathcal{F}_{n,k}$ ($\mathcal{H}_{n,k}$) are nonzero for even (odd) k, as is expected from the symmetry of the model at hand. The explicit expressions for these coefficients are

$$\mathcal{F}_{n,k}=\frac{2m}{\lambda}(-1)^{n+k/2}\frac{(2n+k/2)!}{(2n)!(k/2)!},\quad (n+k>0),$$

$$\mathcal{H}_{n,k}=\frac{2\sqrt{2}m}{\lambda}(-1)^{n+(k-1)/2}\frac{\Gamma(n+1/2)[2n+(k-1)/2]!}{\sqrt{\pi}n!(2n)![(k-1)/2]!}. \tag{36}$$

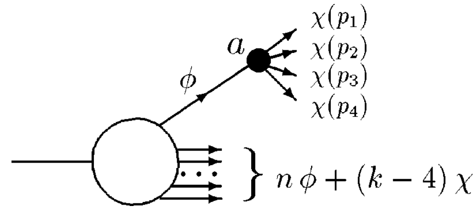

FIG. 3. A graph with four χ bosons originating from a single virtual ϕ, which is singular at threshold if $m_\phi/m_\chi=4$ unless the scattering amplitude a (filled circle) vanishes at threshold. Both the open and the filled circles represent the sum of tree graphs.

The combinations $\mathcal{F}_{n,k}C^n$ and $\mathcal{H}_{n,k}C^n$ are identified as respectively the coefficients $F_{n,k}$ and $H_{n,k}$ in the general expansion of Eq. (29). The latter coefficients are related to the multiparticle amplitudes as given by Eq. (31). Using the explicit expressions in Eqs. (33) and (34) for b and a, one arrives at the relation between the amplitudes and the found coefficients $\mathcal{F}_{n,k}$ and $\mathcal{H}_{n,k}$. Namely,

$$A^\phi_{n,k}=(-1)^n n!k!\left(\frac{\lambda}{m}\right)^{n+k}\frac{\mathcal{F}_{n,k}}{2^{n+3k/2}}$$

$$=(-1)^{k/2}\left(\frac{\lambda}{m}\right)^{n+k-1}\frac{n!k!(2n+k/2)!}{2^{n+3k/2-1}(2n)!(k/2)!},$$

$$A^\chi_{n,k}=(-1)^n n!k!\left(\frac{\lambda}{m}\right)^{n+k}\frac{\mathcal{H}_{n,k}}{2^{n+3k/2}}$$

$$=(-1)^{(k-1)/2}\left(\frac{\lambda}{m}\right)^{n+k-1}$$

$$\times\frac{k!\Gamma(n+1/2)[2n+(k-1)/2]!}{\sqrt{\pi}2^{n+3(k-1)/2}(2n)![(k-1)/2]!}. \tag{37}$$

This concludes our calculation of the threshold amplitudes in the minimal two-field model.

C. New zeros

A remarkable property of our result is that the amplitudes are finite even though a state of four χ bosons at threshold is degenerate in energy with one ϕ boson. In other words, any graph, where four final χ bosons with the four-momenta p_1,p_2,p_3, and p_4 originate from a single line of ϕ (see Fig. 3), contains the factor

$$\frac{a_{1,4}(p_1,p_2,p_3,p_4)}{(p_1+p_2+p_3+p_4)^2-m_\phi^2}. \tag{38}$$

Here $a_{1,4}(p_1,p_2,p_3,p_4)$ is the conventional Feynman scattering amplitude for the process $\phi\to4\chi$. At threshold the denominator in Eq. (38) goes to zero, and the graph becomes singular unless the Feynman amplitude $a_{1,4}(p_1,p_2,p_3,p_4)$ also vanishes when all four momenta are at threshold. The latter cancellation indeed takes place in the model considered here. This can be seen by examining the amplitude $A^\phi_{1,4}$, which is exactly the threshold limit of the expression in Eq. (38). Indeed, the amplitudes A^ϕ and A^χ considered here are the matrix elements of the field operators in Eqs. (27) and

(28). These matrix elements have the propagators of the final on-shell bosons amputated, but the propagator of the incoming virtual field is not amputated and remains included in the corresponding amplitude A. The conventional Feynman scattering amplitude is thus obtained by multiplying the amplitude A by the inverse propagator of the incoming line. In the case of the process $\phi \to 4\chi$, the inverse propagator of the incoming ϕ is vanishing at threshold of four χ. Thus, the Feynman scattering amplitude also vanishes. Clearly, this cancellation can be also verified by an explicit calculation of the tree Feynman graphs.

This cancellation can be extended to a general case of an arbitrary even integer value of ρ, with the exception of $\rho=2$. In the latter case the exponential behavior of χ at $z \to +\infty$ generates a nonexponential dependence of ϕ through the logarithm in Eq. (19), which implies that in this case a resonance between the degenerate states does take place. For all other values of ρ the coefficients in the expansion of the type as in Eq. (29) can be constructed by iterations and are nonsingular. This means that for the values of ρ where the resonance could potentially occur, i.e. $\phi \to \rho\chi$ for even integer ρ, it actually does not take place due to vanishing of the corresponding Feynman scattering amplitude.

V. CONCLUSIONS

In summary, in a class of supersymmetric models with the continuously degenerate family of BPS domain walls (with real trajectories) an additional integral of motion is observed. The occurrence of this integral allowed us to find a generic solution from the family in quadratures, while for a specific ratio of the coupling constants the whole wall family is obtained in the closed form in terms of elementary functions. We then further utilize the result for deriving the multiparticle amplitudes at threshold in the minimal two-field Wess-Zumino model. The threshold amplitudes are calculated in a closed form for $\rho=4$. In the course of the calculation we have found an unexpected cancellation of the tree graphs for the Feynman amplitude of the process $\phi \to 4\chi$ at the threshold, due to which cancellation the multiparticle threshold amplitudes are finite. We also conclude that the same cancellation takes place for the process $\phi \to \rho\chi$ at arbitrary even integer ρ, except $\rho=2$. The relation of this cancellation to additional integrals of motion is yet to be studied. It can be also noted that the nullification of the amplitudes is somewhat reminiscent of the general property of nullification [15,16] for the on-shell processes $2 \to many$ at the threshold in scalar theories.

Interesting phenomena occur when the models at hand are dimensionally reduced to $D=2$. The two-dimensional theories thus obtained have extended supersymmetry,

$$N=2.$$

A continuous degeneracy of the soliton family (persisting with all quantum corrections included) reflecting the possibility of the continuous deformations of the solution profile can be seemingly interpreted in this case as the existence of decoupled ''basic solitons.'' Revealing these decoupled basic solitons in an explicit form and studying their properties is an obvious next problem to be dealt with in the given range of questions.

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