Self-gravitating fundamental strings

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We study the configuration of a typical highly excited string as one slowly increases the string coupling. The dominant interactions are the long-range dilaton and gravitational attraction. In four spacetime dimensions, the string slowly contracts from its initial (large) size until it approaches the string scale where it forms a black hole. In higher dimensions, the string stays large until the coupling reaches a critical value, and then it rapidly collapses to a black hole. The implications for the recently proposed correspondence principle are discussed. [S0556-2821(97)03824-1]

PACS number(s): 11.25.−w, 04.70.Dy

I. INTRODUCTION

We have recently formulated a *correspondence principle* which relates black holes and strings [1]. Developing ideas in [2], it was proposed that as one adiabatically decreases the string coupling g, a black hole makes a transition to a state of weakly coupled strings (and D-branes) with the same mass, charge and angular momentum as the black hole. For most black holes in string theory, namely, those without magnetic Neveu-Schwarz (NS) charges, the ratio of the horizon size to the fundamental string length $\alpha'^{1/2}$ decreases as one decreases g. The transition occurs when this ratio is of order 1. Before this point, the black hole is well defined as a solution of the low-energy supergravity theory; beyond this point, α' corrections become large and the metric near the horizon is no longer well defined.

By relating a black hole to a set of weakly coupled states, the correspondence principle provides a statistical description of black hole entropy. We have verified that the number of such weakly coupled states agrees with the Bekenstein-Hawking entropy in a wide variety of examples involving Ramond and electric NS charges in various dimensions. In contrast to the precise counting of states for extremal and near-extremal black holes [3,4], this method does not in general determine the numerical coefficient in the entropy since that would depend on the precise coupling at which the transition occurs. However, it applies to a much wider class of black holes and reproduces the correct functional dependence on the mass and charges. Other aspects of this correspondence principle have been investigated recently in [5].

In the present paper we would like to develop this further by considering the reverse process; we start with weakly coupled matter and increase the string coupling. We focus on the simplest case of a single highly excited string (in various dimensions). The physics of free highly excited strings has been much discussed in connection with the Hagedorn transition. One of our results is to include string interactions in

the behavior of the string. One might hope that this would ultimately lead to a better understanding of the Hagedorn transition, but it does not apply directly because we consider only a single isolated string.

Consider a string state at level $N \gg 1$, with mass $M^2 = 4N/\alpha'$. As we increase the string coupling, the effective Schwarzschild radius GM increases. It becomes of order the string scale when $g^2 N^{1/2} \sim 1$. This defines a critical coupling when the transition to a black hole can first occur:

$$g_c \sim N^{-1/4}. \tag{1.1}$$

Notice that $g_c \ll 1$ and is independent of the number of (noncompact) spatial dimensions. Conversely, if one starts with a Schwarzschild black hole and decreases the coupling, the horizon size will be of order the string scale when the coupling is given by Eq. (1.1), where $N \sim M_{bh}^2 \alpha'$. So this is the coupling at which the correspondence principle predicts the transition between black holes and strings.

However, a string at level N will form a black hole when $g \sim g_c$ only if it is confined to about the string scale. At zero coupling, the typical size of the string is much larger: $\ell = N^{1/4} \alpha'^{1/2}$. This follows from the random walk picture of the excited string [6], where it takes $N^{1/2}$ steps each of length $\alpha'^{1/2}$. The key question is what happens to this size as we increase g. Intuitively, one would expect gravitational and dilaton forces to cause the string state to shrink, but it is not obvious that it will shrink all the way to $\ell \sim \alpha'^{1/2}$ by the critical coupling (1.1). Surprisingly, it turns out that the behavior of the string as one increases g depends crucially on the number d of (noncompact) spatial dimensions. We will see that interactions remain negligible until the coupling is of order

$$g_0 \sim N^{(d-6)/8}. \tag{1.2}$$

As expected, the interactions always become important before the Schwarzschild radius reaches the original random walk size, which occurs at a coupling $g \sim N^{(d-4)/8}$.

Four spacetime dimensions ($d=3$) yields perhaps the simplest behavior. In this case, $g_0 < g_c$ so as one increases

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the coupling, the interactions first become important and cause ℓ to decrease. In fact, we find that

$$\ell \sim \frac{\alpha'^{1/2}}{g^2 N^{1/2}} \qquad (1.3)$$

so that as g increases from g_0 to g_c, ℓ smoothly contracts from the random walk size to the string scale. For $d=4$, $g_0 = g_c$ so as one increases the coupling the string remains large until g approaches this value, and then it collapses to form a black hole. For $d=5$, $g_0 > g_c$. This leads to a form of hysteresis. If we start with a typical highly excited string state and increase g, it will remain large until $g \sim g_0$, at which point it collapses into a black hole. If we now decrease the coupling, the black hole remains a good description until $g \sim g_c$ at which point it turns into an excited string state.[1] For $d=6$, typical excited string states remain large until $g \sim 1$ when other strong coupling effects are likely to become important. The cases $d > 6$ can probably be analyzed by passing to a weakly coupled dual description.

We will derive the above results in the next two sections using a thermal scalar formalism [7], which has been applied previously to try to understand the critical behavior near the Hagedorn transition. (See [8] for another approach to include string interactions.) However, first we discuss their implications for the correspondence principle. At first sight, the fact that typical string states do not evolve into black holes at the critical coupling g_c in $d > 4$ seems to contradict both the explanation of black hole entropy and the assumed reversibility of the adiabatic change in g. However, this is not the case. The resolution, which was mentioned briefly in [1], is that as one decreases g, a higher-dimensional black hole becomes a highly excited string but in an *atypical* state. It must still be long, with a length of order $N^{1/2}\alpha'^{1/2} \sim M_{\mathrm{bh}}\alpha'$ since we would expect of order half of its energy to be in the form of string tension. But rather than a random walk, it is constrained to lie in a volume roughly set by the string scale. This is plausible because the correspondence principle should still hold if the black hole is placed in a box only slightly larger than its own size, which near g_c is the string length.

Are there enough of these atypical states to account for the black hole entropy? For random-walking strings the log of the number of states should be the number of steps, $N^{1/2}$ times a numerical constant. This is indeed the entropy of highly excited strings. But this should also hold for the random walk constrained to lie in a small volume. Compare random walks on an infinite two-dimensional square grid and a small grid, say two squares by two. The first walk has four choices at each step and an entropy $N^{1/2}\ln 4$. The second has 4, 3, or 2 at each step, depending on whether the path is in

the interior, at an edge, or at a corner, and so the entropy will be $N^{1/2}\ln O(3)$. The numerical coefficient is outside the accuracy of the correspondence principle in any event.[2] The net result is that the black hole evolves to a typical excited string state only in three and four spatial dimensions. But in all dimensions, *the reversible adiabatic transition at $g \sim g_c$ is between black holes and long but compact strings.*

The string states associated with large random walks should also contribute to the Bekenstein-Hawking entropy when they form a black hole at larger values of the coupling, but this is a small correction. For a given level N, a black hole which forms at $g = g_0$ has a larger mass in Planck units than a black hole which forms at[3] $g = g_c < g_0$. The dominant contribution to the entropy of this larger black hole comes from compact strings with $N' > N$ which make the transition when $g \sim g_c \sim N'^{-1/4}$. For example, in $d=5$, a string at level N forms a black hole at $g \sim g_0 \sim N^{-1/8}$ with mass

$$M_{\mathrm{bh}} \sim \frac{N^{1/2} g_0^{1/2}}{l_p} \sim \frac{N^{7/16}}{l_p}, \qquad (1.4)$$

where l_p is the Planck length. If we now decrease the coupling to $g \sim g_c$, the black hole will form an excited string with mass

$$\frac{N'^{1/2}}{\alpha'^{1/2}} \sim \frac{N^{7/16}}{(g_c \alpha')^{1/2}} \qquad (1.5)$$

which implies $N' = N^{7/6}$.

In the next section we review the properties of highly excited free strings, using the thermal scalar formalism. In section three we include interactions by first considering the string in a fixed metric and dilaton background, and then requiring that the background satisfy the equations of motion with the typical excited string as source. The appendix includes some details of the calculation of the stress energy tensor of the string.

II. HIGHLY EXCITED FREE STRINGS

We are interested in the properties of a typical string state of mass $M \gg \alpha'^{-1/2}$, given by the microcanonical ensemble. However, it is easier to calculate in the canonical ensemble, and so we will do this and then solve for the mass in terms of the temperature. Consider the one-string expectation value of some quantity X,

$$\langle X \rangle = Z^{-1}\mathrm{Tr}(Xe^{-\beta H}), \quad Z = \mathrm{Tr}(e^{-\beta H}). \qquad (2.1)$$

As is well known, there is a limiting (Hagedorn) temperature beyond which the traces diverge [9]. This divergence is due to the exponential rise in the density of states $n(M) \sim e^{\beta_H M}$ where the inverse Hagedorn temperature is of order the

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[1] For $g_c < g < g_0$, there is a very small probability that the large string will undergo a fluctuation to a small volume and become a black hole. There is also a very small probability that the black hole will Hawking radiate a large string. In addition, both the long string and black hole slowly lose mass by radiating light particles. Since we are ignoring these effects, our adiabatic change in g should not be so slow that the long string and black hole become unstable.

[2] Note, however, that this coefficient appears in the exponent in the number of states, so the actual number of constrained random walks is much less than unconstrained walks.

[3] They, of course, have the same mass in string units, but it is the black hole mass in Planck units which remains constant as g is varied.

string scale $\beta_H \sim \alpha'^{1/2}$. The critical behavior as $\beta \to \beta_H$ is governed by strings with $M \gg \alpha'^{-1/2}$, and so the properties of these strings can be extracted from the critical behavior.

This critical behavior can be described by an effective field theory of a single complex scalar field in one fewer spacetime dimension [7]. This can be understood as follows. The string partition function can be calculated from a path integral in Euclidean time with period β. Let us make a Euclidean rotation so that instead we are considering the zero-temperature behavior with a spatial dimension compactified. The Hagedorn singularity then appears at a critical compactification radius. Such a singularity must arise from a field becoming massless. In this case it is a scalar of winding number one which becomes tachyonic for $\beta < \beta_H$,[4]

$$m^2(\beta) = \frac{\beta^2 - \beta_H^2}{4\pi^2 \alpha'^2}. \tag{2.2}$$

The critical behavior of the free string partition function is thus given by the thermal scalar path integral

$$\mathbf{Z} = \int [d\chi] \, e^{-S_\chi}, \tag{2.3}$$

where

$$S_\chi = \beta \int d^d x [\partial_i \chi^* \partial^i \chi + m^2(\beta) \chi^* \chi], \tag{2.4}$$

and d is the number of spatial dimensions. The field χ has winding number one and χ^* has winding number minus 1. Equation (2.3) is the full multistring partition function; the single-string partition function is $Z = \ln \mathbf{Z}$. The physical meaning of the thermal scalar has been a source of confusion. It has no apparent dynamical significance, but is useful in determining the static properties of highly excited strings.

As an example, let us review Brandenberger and Vafa's use of the thermal scalar to calculate the density of states [10]. The log of the path integral is

$$Z = -\sum_a \ln \lambda_a, \tag{2.5}$$

where the λ_a are the eigenvalues of $-\nabla^2 + m^2(\beta)$. When the sizes of the spatial dimensions are small compared to $m(\beta)^{-1}$, the splitting of the λ_a is large compared to the lowest eigenvalue

$$\lambda_1 = m^2(\beta) \tag{2.6}$$

and this eigenvalue dominates the critical behavior

$$Z_c(\beta) \approx -\ln \lambda_1 \approx -\ln(\beta - \beta_H). \tag{2.7}$$

This determines the density of states $n(M)$ for large mass:

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[4]This tachyon is present even in supersymmetric string theories, because the thermal boundary conditions imply that spinors are antiperiodic, which breaks supersymmetry.

$$n(M) = \frac{e^{\beta_H M}}{M}, \tag{2.8}$$

where

$$Z(\beta) = \int_0^\infty dM \, e^{-\beta M} n(M). \tag{2.9}$$

(Note that we are using M for the string mass and m for the thermal scalar mass.)

When d spatial dimensions are larger than $m(\beta)^{-1}$,

$$Z(\beta) \approx -V \int \frac{d^d k}{(2\pi)^d} \ln[k^2 + m^2(\beta)]$$

$$\approx V \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{dM}{M} e^{-\beta M + \beta_H M - 2\pi^2 \alpha'^2 k^2 M / \beta_H}$$

$$= \int_0^\infty dM \, e^{-\beta M} n(M) \tag{2.10}$$

for

$$n(M) = V \frac{\beta_H^{d/2}}{(4\pi^2 \alpha')^d} \frac{e^{\beta_H M}}{M^{1+d/2}}. \tag{2.11}$$

We are ignoring divergences at $M \to 0$, which are ultraviolet from the point of view of the effective field theory, but which relate to the uninteresting light strings.

The thermal scalar also makes precise the random walk picture of the highly excited string: in a first-quantized description, the χ path integral is just the sum over random walks. Consider for example the number of string states passing through the origin and a second point x. This is given by the thermal scalar path integral as

$$\langle \chi^* \chi(x) \chi^* \chi(0) \rangle \sim e^{-2|x|m(\beta)}. \tag{2.12}$$

In the random walk picture, a string of energy M is described by a Gaussian whose width is proportional to $M^{1/2}$. Averaging over the thermal ensemble (only the exponential in the density of states is relevant) then gives

$$\int_0^\infty dM e^{-x^2 C/M} e^{-(\beta - \beta_H) M} \sim e^{-2|x| \sqrt{C(\beta - \beta_H)}}. \tag{2.13}$$

Indeed this has the same x and β dependence as the path integral result (2.12), and determines $C = \beta_H / 2\pi^2 \alpha'^2$. The size of the random walk is then $l^2 = M/2C$. Since also $l \approx m(\beta)^{-1}$, the mass depends on the temperature as $M \propto m(\beta)^{-2} \propto (\beta - \beta_H)^{-1}$.

The random walk picture also provides a simple explanation for the prefactors in the density of states (2.8) and (2.11). The naive exponential count of the states of a random walk overcounts by the length of the walk, since it is irrelevant where along the loop the walk starts—hence the factor M^{-1} in the density (2.8). In a large volume there is an additional overcounting by the volume of the walk $O(M^{d/2})$, because only walks where the end coincides with the beginning are allowed.

III. HIGHLY EXCITED STRINGS WITH SELF-INTERACTION

We now wish to see how interactions modify the behavior of a typical highly excited string. Since the string state is large compared to the string scale, the most important interactions will be the long-ranged ones due to exchange of gravitons and dilatons. The statistical mechanics of random walks with self-interactions is the subject of polymer physics, and the scaling arguments we will make are similar to the methods used in that subject [11]. However, the case of a polymer with a long-range attractive interaction has not previously arisen.

Note that we are considering the self-interaction of a single string, not the harder problem of the effect of interactions on the full thermal ensemble at the Hagedorn transition. In particular, there is no Jeans instability even though gravity will be important. We will study the effect of interactions in a mean field approximation. We first determine the behavior a highly excited string in a fixed metric and dilaton background, and then require that the background solve the field equations with the typical string as source.

Consider a static dilaton Φ and static string metric analytically continued to imaginary time: $ds^2 = G_{\tau\tau} d\tau^2 + G_{ij} dx^i dx^j$. The thermal scalar action in this background is

$$S_\chi = \beta \int d^d x \sqrt{G} e^{-2\Phi} \left(G^{ij} \partial_i \chi^* \partial_j \chi + \frac{\beta^2 G_{\tau\tau} - \beta_H^2}{4\pi^2 \alpha'^2} \chi^* \chi \right).$$
(3.1)

The explicit factor of $G_{\tau\tau}$ is from the proper length of the winding string. The $\tau\tau$ component of the metric also appears in $\sqrt{G}$ since this action can be obtained by dimensional reduction from a $d+1$ action. The effective field theory of the low-energy degrees of freedom also includes the graviton-dilaton action

$$-\frac{\beta}{2\kappa^2} \int d^d x \sqrt{G} e^{-2\Phi} (R + 4 G^{ij} \partial_i \Phi \partial_j \Phi).$$
(3.2)

Note that we are not interested in the full quantum field theory, which would generate the full thermal ensemble. Rather we want the single-string partition function, corresponding to one random walk and so exactly one χ loop. This can be written as a field theory by adding an index $a = 1, \ldots, n$ to χ and taking the $n \to 0$ limit, but we will not use this formalism.

For weak fields, the interactions between the dilaton and thermal scalar in Eq. (3.1) are suppressed by derivatives or $\beta - \beta_H$. This is not true for the metric, due to the explicit factor of $G_{\tau\tau}$.[5] This cubic interaction is proportional to the dimensionless string coupling g. Other string interactions such as the exchange of massive string excitations, or a splitting-joining interaction of the long string, require the random walk to intersect itself (or come within the string length), and so give rise to a quartic interaction of the thermal scalar. The exchange is order g^2 and the splitting-joining

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[5]This is the string metric, so in the Einstein metric there are both gravitational and dilaton forces.

of order g, but the quartic interaction is less relevant than the cubic gravitational interaction and can be neglected. Thus the dominant interaction is simply the gravitational attraction of one part of the string on another.

We can make a simple estimate for when this interaction will be important. The critical dimension for a cubic interaction is $d = 6$. The coupling is relevant for $d < 6$, so we can anticipate that the effect of gravity will be greater in lower dimensions. This is consistent with the fact that the gravitational potential falls off more rapidly in higher dimensions. A cubic coupling constant has units of length$^{(d-6)/2}$ so the effective dimensionless coupling is

$$g m^{(d-6)/2} \sim g (\beta - \beta_H)^{(d-6)/4} \sim g M^{(6-d)/4}, \quad (3.3)$$

temporarily omitting factors of α' to make the dependences clearer. Thus if we hold $N \sim M^2 \alpha'$ large and fixed and increase g from zero, the interaction becomes important at

$$g_0 \sim N^{(d-6)/8} \quad (3.4)$$

for $d < 6$. Recall that string scale black holes are formed when $g \sim g_c \sim N^{-1/4}$. For $d = 3$, we have $g_0 < g_c$ so the interactions modify the free string behavior in the weakly coupled regime. For $d = 4$, $g_0 \sim g_c$ so the interactions become important at the same scale where the localized strings become black holes. For $d = 5$, $g_c < g_0$ so the interactions become important in the regime where the free strings are metastable.

In an attractive potential, $G_{\tau\tau} < 1$ with $G_{\tau\tau} \to 1$ at infinity, one expects the following effect. Locally, the effective temperature $G_{\tau\tau}^{-1/2} \beta^{-1}$ is increased, and the string can access more states than at temperature β^{-1}. We would therefore expect the random walk to be concentrated in the region of smallest $G_{\tau\tau}$, and the critical temperature β_C^{-1} to be reduced relative to β_H^{-1}. This is the case, at least when the potential has a bound state. The operator

$$-(\nabla_\mu - 2\Phi_{,\mu}) \nabla^\mu + \frac{\beta^2 (G_{\tau\tau} - 1)}{4\pi^2 \alpha'^2} + \frac{\beta^2 - \beta_H^2}{4\pi^2 \alpha'^2} \quad (3.5)$$

then has lowest eigenvalue λ_1 less than the flat space value $(\beta^2 - \beta_H^2)/4\pi^2 \alpha'^2$. As β decreases from above, this eigenvalue then vanishes at $\beta_C > \beta_H$. The density of states then has the same form as in the small volume case above, but with β_C replacing β_H,

$$n(M) = \frac{e^{\beta_C M}}{M}. \quad (3.6)$$

The bound state wave function gives the shape of the random walk.

The bound state picture gives a simple interpretation of the coupling g_0. The condition that the operator (3.5) have a bound state is roughly

$$\ell^2 V \gtrsim 1, \quad (3.7)$$

where ℓ is the range of the potential and V its depth. Taking the gravitational potential of a long string at its random walk radius $\ell = N^{1/4} \alpha'^{1/2}$, one has

$$\ell^2 V \sim G M \ell^{4-d} \sim g^2 N^{(6-d)/4}. \quad (3.8)$$

The bound state criterion (3.7) is then $g \gtrsim g_0$.

We now wish to require that the background satisfy the field equations with sources coming from the excited string. In the mean field approximation, we average these sources over all excited strings with the same mass:

$$R + 4\nabla^2\Phi - 4\nabla_\mu\Phi\nabla^\mu\Phi = 2\kappa^2\langle J \rangle,$$

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi = \kappa^2[e^{2\Phi}\langle T_{\mu\nu}\rangle + G_{\mu\nu}\langle J\rangle], \quad (3.9)$$

where J is the quantity in parentheses in the scalar action (3.1). It is shown in the Appendix that the sources are simply given by the classical expression evaluated at the bound state wave function χ, times an appropriate normalization. The wave function satisfies

$$\left\{ -(\nabla_\mu - 2\Phi_{,\mu})\nabla^\mu + \frac{\beta_C^2(G_{\tau\tau}-1)}{4\pi^2\alpha'^2} + \frac{\beta_C^2-\beta_H^2}{4\pi^2\alpha'^2} \right\}\chi = 0. \quad (3.10)$$

The low-energy field equations (3.9) are valid only when all derivatives are small compared to the string scale. Due to the explicit factors of α' in the eigenvalue equation (3.10) this requires the further approximation

$$\beta_C^2 - \beta_H^2 \ll 1,$$

$$h_{\tau\tau} \equiv G_{\tau\tau} - 1 \ll 1. \quad (3.11)$$

Thus we can linearize the equations for the background. In the usual Lorentz gauge, $R_{\mu\nu} = -\frac{1}{2}\partial^2 h_{\mu\nu}$. To leading order, $\langle J \rangle = 0$, so the dilaton equation becomes $R + 4\partial^2\Phi = 0$ with solution $\Phi = h_\mu{}^\mu/8$. The $R_{\tau\tau}$ equation reduces to Newton's law

$$\partial^i\partial_i h_{\tau\tau} = 2\kappa^2 M\chi^*\chi, \quad (3.12)$$

where we have imposed the normalization $\int d^d x \chi^*\chi = 1$. Solving this, the eigenvalue equation (3.10) becomes

$$-\partial^i\partial_i\chi(x) - \frac{\beta_H^2\kappa^2 M}{2\pi^2(d-2)\omega_{d-1}\alpha'^2}\chi(x)\int d^d x' \frac{\chi^*\chi(x')}{x_>^{d-2}}$$

$$= \frac{\beta_H(\beta_H-\beta_C)}{2\pi^2\alpha'^2}\chi(x), \quad (3.13)$$

where $x_>$ is the greater of $|x|$ and $|x'|$ and ω_{d-1} is the volume of the unit S_{d-1}.

Equation (3.13) is a nonlinear Schrödinger equation with attractive Coulomb self-interaction. The essential physics can be obtained by a scaling argument. Define a dimensionless string coupling

$$g^2 = \frac{\beta_H^2\kappa^2}{2\pi^2(d-2)\omega_{d-1}\alpha'^2} \quad (3.14)$$

and rescaled variables[6]

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[6]This preserves the normalization $\int d^d x \chi^*\chi = \int d^d y \psi^*\psi = 1$.

$$x = (g^2 M)^{1/(d-4)}y, \quad \chi(x) = (g^2 M)^{d/(8-2d)}\psi(y) \quad (3.15)$$

(setting aside temporarily the case $d=4$). The eigenvalue equation (3.13) becomes

$$-\partial_{y^i}\partial_{y^i}\psi(y) - \psi(y)\int d^d y' \frac{\psi^*\psi(y')}{y_>^{d-2}} = \zeta\psi(y), \quad (3.16)$$

where

$$\beta_H - \beta_C = \zeta\frac{2\pi^2\alpha'^2(g^2M)^{2/(4-d)}}{\beta_H}. \quad (3.17)$$

We are interested in the lowest bound state solution to this equation. Formally this can be found by minimizing

$$I = \int d^d y \partial_{y^i}\psi^*(y)\partial_{y^i}\psi(y)$$

$$-\frac{1}{2}\int d^d y \int d^d y' \frac{\psi^*\psi(y)\psi^*\psi(y')}{y_>^{d-2}} \quad (3.18)$$

subject to $\int d^d y \psi^*\psi = 1$. However, we need to be sure that this functional is bounded from below. If we rescale $\psi(y) \to \rho^{d/2}\psi(\rho y)$, the first (positive) term scales as ρ^2 and the second (negative) term scales as ρ^{d-2}. The negative term becomes arbitrarily large as $\rho \to \infty$ (for $d \geq 3$). For $d=3$ the positive term grows faster in this limit and so the variational principle predicts a lowest bound state. For $d \geq 5$, I can be arbitrarily negative and there is no state of lowest eigenvalue. For $d=4$ one can perform the scaling in the original variables (3.13). The two terms both scale as ρ^2 so the coupling does not scale out. For small coupling the kinetic term dominates and I is positive. Past a critical coupling the potential dominates and I can decrease without bound.

Let us first consider the case $d=3$. Since all constants have been scaled out of Eq. (3.16), we expect the lowest eigenvalue to be $\zeta_0 \sim O(-1)$ and the size of the bound state to be $O(1)$ in the y variable. In terms of the original variables, this gives

$$\beta_C - \beta_H \sim \frac{\alpha'^2 g^4 M^2}{\beta_H},$$

$$\ell \sim g^{-2}M^{-1}. \quad (3.19)$$

We can also express this in terms of the excitation level of the string. Because the redshift (3.11) is small, the mass-level relation is approximately as in the free case

$$M^2 = \frac{4}{\alpha'}N. \quad (3.20)$$

Thus the size (3.19) of the string state is of order

$$\ell \sim \frac{\alpha'^{1/2}}{g^2 N^{1/2}}. \quad (3.21)$$

This is one of our main results. It is nonperturbative in the coupling g, and is valid for $g_0 < g < g_c$. Since $g_0 \sim N^{-3/8}$ and $g_c \sim N^{-1/4}$, it shows that the size of a typical excited string in three spatial dimensions smoothly interpolates from the random walk size to the string scale as one slowly increases the coupling. For $g < g_0$, the interactions are negligible and the string remains at its random walk size. The result (3.21) is not applicable since there is no bound state. For $g > g_c$ the string forms a black hole. If one increases the coupling further, the black hole size will be fixed to be $N^{1/4}$ in Planck units, but grow as $g N^{1/4}$ in string units.

For $d=4$ and $d=5$, once the string coupling reaches g_0 the estimate (3.7) indicates that bound states form, but we have seen that the system becomes unstable: there are states of arbitrarily negative energy. We interpret this as saying that once the interaction becomes important the long string collapses all the way to a black hole. For $d=6$ the interaction is marginal and for $d>6$ it is irrelevant, but this does not mean that it can be neglected. These terms refer to the scaling if we hold g fixed and increase the length scale—that is, M. However, we are holding M fixed and increasing g. In this case, one always reaches the coupling g_0, where a bound state can form. Again, it is unstable and should collapse. For $d>6$, $g_0 \gg 1$ and the theory is out of the range of validity of the original theory. One can still discuss the evolution of the string by passing to a weakly coupled dual theory. The original fundamental string becomes a solitonic string with tension that increases as the dual coupling $\tilde{g} = 1/g$ is decreased. If this state does not decay, one might expect it to form a black hole when $\tilde{g} \sim 1/g_0$. However, the dual theory has much lighter degrees of freedom—long dual strings, for example—with much higher entropy at given mass. If the solitonic string rapidly decays into these dual strings, then a black hole will not form. This would imply that most excited states of strings in higher dimensions never form a black hole for any value of the string coupling. However, the decay of the solitonic string to the dual strings might be quite slow, because it is locally a Bogomol'nyi-Prasad-Sommerfeld (BPS) state: small loops must break off and contract for it to decay.

It is a pleasure to thank J. Cardy and A. Strominger for discussions. This work was supported in part by NSF Grant Nos. PHY94-07194, PHY95-07065, and PHY97-22022.

APPENDIX: CALCULATION OF $\langle T_{\mu\nu} \rangle$

In this appendix, we compute the mean value of the stress energy tensor among all string states with mass M. First we represent this tensor as a functional derivative of the string Hamiltonian:

$$T^{\mu\nu} = -\frac{2}{\sqrt{G}} \frac{\delta H}{\delta G_{\mu\nu}}. \tag{A1}$$

Its expectation value in a typical state of mass M is then

$$\langle T^{\mu\nu} \rangle = \frac{\text{Tr}\{T^{\mu\nu} \delta(H-M)\}}{\text{Tr}\{\delta(H-M)\}}$$

$$= \frac{2}{\sqrt{G}\,\text{Tr}\{\delta(H-M)\}} \frac{\delta}{\delta G_{\mu\nu}} \text{Tr}\{\theta(M-H)\}. \tag{A2}$$

Evaluating the traces using the density of states (3.6) gives

$$\langle T^{\mu\nu} \rangle \approx \frac{2}{\sqrt{G}} e^{-M\beta_C} \frac{\delta}{\delta G_{\mu\nu}} (\beta_C^{-1} e^{M\beta_C}) \approx \frac{M}{\beta_C^2 \sqrt{G}} \frac{\delta \beta_C^2}{\delta G_{\mu\nu}}, \tag{A3}$$

where these expressions are valid in the limit of large M. The critical temperature β_C was defined by $\lambda_1(G_{\mu\nu}, \beta_C) = 0$, so

$$\frac{\delta \beta_C^2}{\delta G_{\mu\nu}} = -\frac{\delta \lambda_1 / \delta G_{\mu\nu}}{\delta \lambda_1 / \delta \beta^2 |_{\beta = \beta_C}}. \tag{A4}$$

The derivatives of the eigenvalues are given by first-order perturbation theory:

$$\frac{e^{2\Phi}}{\sqrt{G}} \frac{\delta \lambda_1}{\delta G_{\tau\tau}} = \frac{\beta_C^2 \chi^* \chi}{4\pi^2 \alpha'^2} + \frac{1}{2} G^{\tau\tau} J,$$

$$\frac{e^{2\Phi}}{\sqrt{G}} \frac{\delta \lambda_1}{\delta G_{ij}} = -\nabla^{(i} \chi^* \nabla^{j)} \chi + \tfrac{1}{2} G^{ij} J,$$

$$\frac{\delta \lambda_1}{\delta \beta^2} = \int d^d x \sqrt{G}\, G_{\tau\tau} e^{-2\Phi} \frac{\chi^* \chi}{4\pi^2 \alpha'^2}, \tag{A5}$$

where χ is a solution to Eq. (3.10), and J is the quantity in parentheses in the action S_χ (3.1) evaluated on the bound state wave function. The resulting stress energy tensor is simply the variation of S_χ with respect to the metric, evaluated on the bound state wave function.

The stress energy tensor (A3) satisfies two important consistency checks. Since it does not include the stress energy of the dilaton field, it is not conserved by itself. Instead it satisfies

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 2 e^{-2\Phi} \langle J \rangle \nabla^\nu \Phi. \tag{A6}$$

This is required for the consistency of the field equations and Bianchi identities, and is a necessary check because the action S_χ is not manifestly invariant under time reparametrizations. It is also correctly normalized in the following sense. In a static spacetime, the total energy associated with the matter is

$$M_{\text{matter}} = \int_\Sigma T_{\mu\nu} \xi^\mu n^\nu d\Sigma, \tag{A7}$$

where ξ^μ is the timelike Killing vector, and the integral is over a static surface Σ with unit normal n^ν and proper volume $d\Sigma$. Using the above expression for $\langle T_{\mu\nu}\rangle$, we find[7]

$$M_{\text{matter}} = -\int \langle T_\tau{}^\tau\rangle \sqrt{G} d^d x = M. \tag{A8}$$

The total Arnowitt-Deser-Misner (ADM) mass of a static spacetime can be expressed similarly in terms of an integral of the Ricci tensor rather than the stress energy tensor. Assuming D spacetime dimensions, and using the Einstein metric, one has

$$M_{\text{ADM}} = \frac{D-2}{(D-3)\kappa^2}\int_\Sigma \widetilde{R}_{\mu\nu}\xi^\mu\widetilde{n}^\nu d\Sigma. \tag{A9}$$

This differs from M_{matter} since it also includes the gravitational binding energy. Rewriting this expression in terms of the string metric yields

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[7] Recall that $\sqrt{G}$ includes the $\tau\tau$ component of the metric.

$$M_{\text{ADM}} = \frac{D-2}{(D-3)\kappa^2}\int_\Sigma e^{-2\Phi}\bigg[R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi$$
$$+ \frac{2}{D-2}G_{\mu\nu}(\nabla^2\Phi - 2\nabla_\mu\Phi\nabla^\mu\Phi)\bigg]\xi^\mu n^\nu d\Sigma. \tag{A10}$$

Using the equations of motion (3.9) this becomes

$$M_{\text{ADM}} = \frac{D-2}{D-3}\int \left[\langle T_{\mu\nu}\rangle - G_{\mu\nu}\frac{\langle T_\alpha{}^\alpha\rangle}{D-2}\right]\xi^\mu n^\nu d\Sigma. \tag{A11}$$

For weak fields, one recovers $M_{\text{ADM}} = M$. In general, these two masses will not be equal, but even when a black hole is about to form, they will differ only by a factor of order unity.

In terms of the correspondence principle, if one starts at zero coupling with a (compact) string state of mass M, it will form a black hole at larger coupling with mass M_{ADM}. Since these two masses differ only by a numerical factor, the black hole entropy is reproduced (up to a similar factor) even if the black hole mass is equated with the string mass at zero coupling.

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