Quantum bubble nucleation beyond the WKB approximation: Resummation of vacuum bubble diagrams

Hiroshi Suzuki* and Hirofumi Yasuta[†]

Department of Physics, Ibaraki University, Mito 310, Japan

(Received 5 September 1997; published 21 January 1998)

On the basis of Borel resummation, we propose a systematical improvement of the bounce calculus of the quantum bubble nucleation rate. We study a metastable super-renormalizable field theory, D-dimensional, $O(N)$ symmetric ϕ^4 model ($D<4$) with an attractive interaction. The validity of our proposal is tested in $D=1$ (quantum mechanics) by using the perturbation series of ground state energy to high orders. We also present a result in $D=2$, based on an explicit calculation of vacuum bubble diagrams to five loop orders. [S0556-2821(98)06204-3]

I. INTRODUCTION

In this paper, we propose a new approach for the tunneling phenomenon in a metastable super-renormalizable field theory, aiming at a systematical improvement of the bounce (or instanton) calculus [1,2]. We shall study a $(D-1)$-dimensional system ($D<4$) whose Hamiltonian is defined by

$$H=\int d^{D-1}x\left[\frac{c^2}{2}\,\pi^2+\frac{1}{2}\,(\partial_x\phi)^2+\frac{1}{2}\,m^2\phi^2\right.$$
$$\left.-\frac{1}{4!}\,g(\phi^2)^2\right],\quad g>0\,,\tag{1.1}$$

where ϕ is an N component real scalar field ($\phi^2\equiv\phi\cdot\phi$) and π is its conjugate momentum. The Hamiltonian (1.1) may be regarded as a Ginsburg-Landau-like effective theory in which the classical time evolution of the order parameter ϕ is determined by the first term. Our present approach therefore might become relevant for tunneling problems in condensed matter physics.

We study an imaginary part of the vacuum energy density. Since the potential energy in (1.1) is not bounded from below, the quantum tunneling makes the naive ground state (vacuum) $\phi=0$ metastable. Therefore, physical quantities have to be defined by an analytic continuation from a negative g. In particular, the continuation produces an imaginary part of the vacuum energy density Im $\mathcal{E}$, which can be regarded as a total decay width per unit volume of the quasivacuum,[1] i.e., $\Gamma=-2$ Im $\mathcal{E}/\hbar$.

The standard method for calculating such a tunneling amplitude in field theory is the bounce calculus [1,2]. The existence of the bounce solution, an extreme of the Euclidean action with one negative Gaussian eigenvalue, signifies a decay of the vacuum due to quantum tunneling [1,2]. The lead-

ing bounce approximation furthermore gives a quantitatively reliable estimation of the tunneling amplitude in the weak coupling region $g\ll 1$.

The systematical evaluation of higher order corrections to the leading bounce result, however, is difficult. One has to consider ''interactions'' among bounces and an integration over the quasicollective coordinates. Simultaneously, perturbative corrections around bounces have to be taken into account. One also has to resolve the ''mixing'' of those two effects. Even if this difficult task could be done, such an ''instanton expansion'' is likely to be an asymptotic expansion, thus we do not expect the quantitative validity in the strong coupling region.

We tackle this issue, i.e., a *systematical quantitative* improvement of bounce calculus, from a completely different viewpoint. Namely we utilize an information of the *conventional* perturbation series around the *naive* vacuum to evaluate the tunneling rate. This should sound strange because usually the quantum tunneling is regarded as a nonperturbative phenomenon. However, extensive studies on the large order behavior of the perturbation series [4] have revealed an intrinsic connection between the quantum tunneling and the nature of perturbation series. This connection is the backbone of our approach. Technically, we utilize the Borel resummation method [4] for extracting the tunneling rate from the perturbation series: Singularities of the Borel transform are expected to reproduce an imaginary part of the vacuum energy density.

In this approach based on the conventional perturbation expansion, every step of calculation (such as the renormalization) are well-understood and, in principle, the order of approximation can be systematically increased. To our knowledge, this kind of approach to tunneling phenomena in quantum mechanics was initiated in [5] on the basis of another kind of resummation method. See [6] and references therein. Our present proposal can be regarded as the natural generalization of [6] to quantum field theory.

We will take the ''natural unit'' $\hbar=c=1$ in what follows.

II. BOUNCE CALCULUS

Let us first recapitulate the result of bounce calculus [1,2] of the tunneling amplitude. It gives an important information

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*Email address: hsuzuki@mito.ipc.ibaraki.ac.jp

[†]Email address: yasuta@mito.ipc.ibaraki.ac.jp

[1]One can find a detailed account on this point in [3].

on the nature of a Borel singularity and, simultaneously, provides the ''standard'' with which our results can be compared.

The Lorentzian action corresponding to Hamiltonian (1.1) is given by

$$S[\phi] = \int d^D x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g(\phi^2)^2 \right]$$
$$\equiv i S_E[\phi], \tag{2.1}$$

where we have defined the Euclidean action $S_E[\phi]$ (the time coordinate of Euclidean spacetime is defined by $x^D = it$). We also assume an appropriate counter term $S_{\text{count}}[\phi]$ to remove the ultraviolet (UV) divergences:

$$S_{\text{count}}[\phi] = \int d^D x \left[\frac{N}{2} \int \frac{d^D k}{i(2\pi)^D} \ln(m^2 - k^2 - i\epsilon) + \cdots \right]. \tag{2.2}$$

The structure of abbreviated terms depends on the spacetime dimension D. See Eq. (5.1) as the example for $D = 2$.

We repeat the procedure in [2] to yield the imaginary part of the vacuum energy density for $g \ll 1$,

$$[\text{Im } \widetilde{\mathcal{E}}(\widetilde{g})]_{\text{bounce}} = -A_N C_{D,N} \left(\frac{S_0}{2\pi\widetilde{g}} \right)^{(D+N-1)/2} e^{-S_0/\widetilde{g}}, \tag{2.3}$$

where dimensionless combinations $\widetilde{\mathcal{E}} \equiv \mathcal{E}/m^D$ and $\widetilde{g} \equiv g/m^{4-D}$ have been introduced.

We briefly explain how the various factors in (2.3) emerge. One expands the Euclidean functional integral around the spherically symmetric bounce $\phi_c(r)$, which is a solution of the Euclidean equation of motion, $-\Delta \phi_c(r) + m^2 \phi_c(r) - g \phi_c(r)^3/3! = 0$. The spherically symmetric bounce has the least action [7] whose value is numerically given by[2] $(S_E[\phi_c] \equiv S_0/\widetilde{g})$

$$S_0 = \frac{3}{2} I_4 = \begin{cases} 8 & \text{for } D = 1, \\ 35.10269 & \text{for } D = 2, \\ 113.38351 & \text{for } D = 3. \end{cases} \tag{2.4}$$

Gaussian integrations around the bounce except the zero modes give rise to the coefficient $C_{D,N}$:

$$C_{D,N} = e^{iS_{\text{count}}[\phi_c]} m^{-(D+N-1)} (4-D)^{(N-1)/2}$$
$$\times |\det'(-\Delta + m^2 - g\phi_c^2/2)|^{-1/2}$$
$$\times [\det'(-\Delta + m^2 - g\phi_c^2/3!)]^{-(N-1)/2}. \tag{2.5}$$

The determinant can be evaluated analytically in $D = 1$ and numerically in $D = 2$ and $D = 3$ [8]:

$$C_{D,N} = \left[\bar{D}_R(1) \left(\frac{DI_4/4}{I_6 - I_4} \right)^D \right]^{-1/2} [\bar{D}_R(1/3)/4]^{-(N-1)/2}$$
$$= \begin{cases} 2\sqrt{3} \times (2\sqrt{3})^{N-1} & \text{for } D = 1, \\ 0.3503 \times (1.652)^{N-1} & \text{for } D = 2, \\ 10.189 \times (1.6569)^{N-1} & \text{for } D = 3. \end{cases} \tag{2.6}$$

Since the quadratic operator $-\Delta + m^2 - g\phi_c^2/2$ has one negative eigenvalue [2,7,8], the square root of the eigenvalue produces a factor $\pm i$. A physical requirement that we are concerned with a decaying process specified the branch in (2.3). The Jacobian from a zero mode to a collective coordinate is $\sqrt{S_0/(2\pi\widetilde{g})}$. Because there are $D+N-1$ collective coordinates (the spacetime position and the direction of bounce in N dimensional internal space), the power in (2.3) is resulted. The integration over the bounce direction in the internal space gives a factor A_N, the half area of $N-1$ dimensional unit sphere, $A_N \equiv \pi^{N/2}/\Gamma(N/2)$. Finally the integration over the position of bounce produces a factor, the system volume times the time period, which is divided to give the energy density: This completes our quick review of the bounce result (2.3).

In our approach, we do not need the explicit value of the determinant factor (2.6). It will be used merely for a comparison. On the other hand, the value of the bounce action (2.4) and the number of collective coordinates are important because they tell the position and the strength of the nearest Borel singularity to the origin of Borel plane.[3] They can be easily obtained. One only has to solve a one dimensional differential equation to find the bounce action, and a symmetrical consideration fixes the number of corrective coordinates.

III. RESUMMATION OF VACUUM BUBBLES

The leading bounce result (2.3) is reliable for the weak coupling region $\widetilde{g} \ll 1$. In this section, we present our proposal which is expected to work even in the strong coupling region.

We start with the *conventional* perturbative expansion of the vacuum energy density, namely a sum of the vacuum bubble diagrams,

$$\widetilde{\mathcal{E}}(\widetilde{g}) \sim \sum_{n=0}^{\infty} c_n \widetilde{g}^n, \tag{3.1}$$

where we have assumed an appropriate renormalization which makes c_ns finite. For super-renormalizable cases $D < 4$, only first several c_ns are UV divergent.

From the perturbation series (3.1), we construct the Borel (more precisely Borel-Leroy) transform:

$$B(z) \equiv \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+(D+N)/2)} z^n. \tag{3.2}$$

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[2]The numbers, I_4, I_6, $\bar{D}_R(1)$ and $\bar{D}_R(1/3)$ are defined in [8]: Eq. (58) of [8] should be read as $\Delta_T^\perp = (4-D)\bar{D}(1/3)/4$ for a general D.

[3]However, if one has perturbative coefficients to *sufficiently* higher orders, even those information from bounce calculus may be deduced from the perturbation series. See [6].

The argument of gamma function in the denominator has been chosen so that a singularity of the Borel transform nearest to the origin becomes a square root branch point [see (3.5)].

When the coupling constant g is *negative* in (1.1) and (2.1), namely with a bounded potential, the proof of the Borel summability of Green's functions [9] may be generalized to the vacuum energy density. With our definition (3.2), the Borel summability is expressed as

$$\widetilde{\mathcal{E}}(\widetilde{g}) = \frac{1}{(-\widetilde{g})^{(D+N)/2}} \int_0^\infty dz\ e^{z/\widetilde{g}} z^{(D+N)/2-1} B(-z).$$

$$(3.3)$$

Now, we assume that the Borel transform $B(z)$ is analytic on the complex z plane and possible singularities of $B(z)$ exist only on the positive real axis. Precisely the same assumption on the Borel transform of Green's functions has been made in the perturbation approach to the critical phenomena [11,4]. We also assume the behavior of $B(z)$ at $z=\infty$ is moderate enough and the convergence of the integral (3.3) is determined by the exponential factor.

Under these assumptions, we may rotate the integration contour in (3.3) as $z \to e^{\mp\pi i}z$ as it goes along the upper (lower) side of the positive real axis. This operation gives rise to the analytic continuation for $g>0$,

$$\widetilde{\mathcal{E}}(\widetilde{g}) = \frac{1}{\widetilde{g}^{(D+N)/2}} \int_0^\infty dz\ e^{-z/\widetilde{g}} z^{(D+N)/2-1} B(z \pm i\varepsilon),$$

$$(3.4)$$

which is our basic equation.

With (3.4), the leading bounce result (2.3) of the imaginary part for $\widetilde{g} \ll 1$ implies that there exists a singularity of Borel transform at $z=S_0$ [4]. Because of the structure of (3.4), it is the nearest singularity to the origin. A comparison of (2.3) with (3.4) shows that the singularity is a square root branch point:

$$B(z) = -\frac{1}{\sqrt{\pi}} A_N C_{D,N} \frac{S_0^{1/2}}{(2\pi)^{(D+N-1)/2}} (S_0-z)^{-1/2} + \cdots .$$

$$(3.5)$$

When substituted in (3.4), the branch cut (3.5) reproduces the imaginary part (2.3) for $\widetilde{g} \ll 1$. The sign of the singularity (3.5) itself is not determined solely from the comparison of (2.3) with (3.4) because there are two possible choices of the integration contour (upper side or lower side of the positive real axis). As a consistency test, we note that the expansion of (3.5) with respect to z yields negative c_ns in view of (3.2), which is nothing but the large order behavior of the perturbation series [4]. As we will see below, this is a correct property of actual perturbation coefficients when we follow the $-i\epsilon$ prescription. This consideration also shows that we have to take the *upper* contour to obtain a *negative* imaginary part, which is the physical one.

Basically we construct the Borel transform (3.2) from the actual perturbation series c_n and substitute it in (3.4) for extracting the ''nonperturbative'' information. However the radius of convergence of the series (3.2) is finite ($=S_0$) due to the singularity (3.5) and we have to analytically continue

the series (3.2) outside the convergence circle to perform the Borel integration (3.4). This is an impossible task without knowing all the perturbative coefficients. However, as is well known, this difficulty can be avoided by the conformal mapping technique [10]. We thus introduce a new variable λ by

$$z = 4S_0 \frac{\lambda}{(1+\lambda)^2}.$$

$$(3.6)$$

The point is that the convergence circle of the series (3.2) in terms of λ is now a unit circle, within which the whole cut z plane is mapped. In particular, the real axis $z>S_0$ is mapped on the circle $|\lambda|=1$. Therefore we may use a finite order truncation of the series of λ in the Borel integration (3.4) order by order. Then we may expect the sequence converges to (3.4) (as was the case in quantum mechanics [6]).

In terms of λ, the Borel transform (3.2) is expressed as

$$B(z) = \sum_{k=0}^\infty d_k \lambda^k,$$

$$d_k \equiv \sum_{n=0}^k (-1)^{k-n} \frac{\Gamma(k+n)(4S_0)^n}{(k-n)!\Gamma(2n)\Gamma(n+(D+N)/2)} c_n .$$

$$(3.7)$$

Combining (3.4), (3.6), and (3.7), and parametrizing the unit circle by $\lambda=e^{i\theta}$, we find the Pth order approximation of the imaginary part,

$$[\mathrm{Im}\ \widetilde{\mathcal{E}}(g)]_P = \left(\frac{S_0}{\widetilde{g}}\right)^{(D+N)/2} \int_0^\pi d\theta\ \exp\left(-\frac{S_0}{\widetilde{g}} \frac{1}{\cos^2 \theta/2}\right)$$

$$\times \frac{\sin \theta/2}{\cos^{D+N+1} \theta/2} \sum_{k=0}^P d_k \sin k\theta.$$

$$(3.8)$$

Note that this is *solely* expressed by the first P perturbative coefficients c_n and the value of bounce action S_0.

Our formula (3.8) relates the value of vacuum bubble diagrams c_n and an imaginary part of the vacuum energy density. In quantum field theory, usually one never seriously computes the vacuum bubble diagrams because one should be free to adjust the origin of a *real* part of the energy density and thus it has no direct physical meaning. On the other hand, the *imaginary* part of the vacuum energy density has a definite physical meaning as the decay width of the quasi-vacuum. Therefore the latter should not be modified under a change of convention for the former. How can this general consideration and the formula (3.8), which relates those two quantities, be reconciled?

In fact, there is no real contradiction. For example, suppose that we perform a finite renormalization of the vacuum energy density to eliminate the first Q perturbation coefficients. (Note that we cannot take $Q=\infty$ because a simple

sum of the perturbation series is diverging.) Then the new Borel transform would be defined by

$$B(z)_{\text{new}} \equiv \sum_{n=Q+1}^{\infty} \frac{c_n}{\Gamma(n+(D+N)/2)} z^n$$

$$= B(z) - \sum_{n=0}^{Q} \frac{c_n}{\Gamma(n+(D+N)/2)} z^n. \quad (3.9)$$

Note that the last *finite* series (i.e., polynomial) does not develop a singularity and thus cannot contribute to any imaginary part via the Borel integral. Therefore the imaginary part, as it should be, is invariant under such a change of

the origin of the energy density. Contrary to its peculiar looking, the formula (3.8) is workable in this way.

IV. QUANTUM MECHANICS

We can extensively test the validity of our master formula (3.8) in quantum mechanics $D=1$. In this case, perturbative coefficients of the vacuum energy, namely the ground state energy, to very high orders are available. The most efficient way for computing them is the recursion formula method [12]. By generalizing it to the $O(N)$ symmetric model, we have computed c_n to $n=50$. The first several coefficients are[4]

$$c_0 = \frac{N}{2}, \quad c_1 = -\frac{N(N+2)}{96}, \quad c_2 = -\frac{N(N+2)(2N+5)}{4608},$$

$$c_3 = -\frac{N(N+2)(8N^2+43N+60)}{221184},$$

$$c_4 = -\frac{N(N+2)(168N^3+1437N^2+4270N+4420)}{42467328},$$

$$c_5 = -\frac{N(N+2)(1024N^4+12277N^3+57668N^2+126128N+108480)}{2038431744}. \quad (4.1)$$

The exact complex quasiground state energy is also available by a numerical diagonalization of the Hamiltonian in a (unconventional) Hilbert space with a rotated boundary condition (see, for example [6]). Therefore we can compare our formula (3.8) and the bounce result (2.3) with the exact value of the imaginary part.

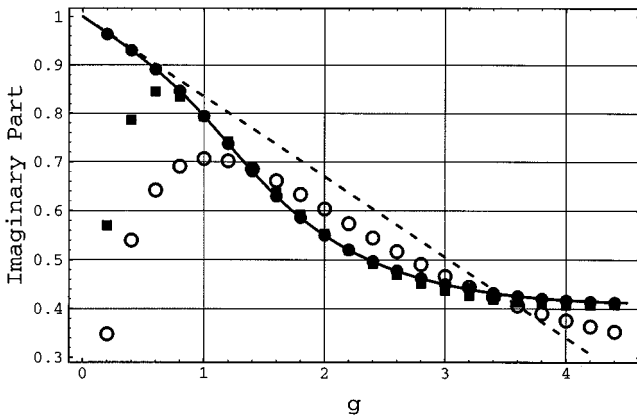In Fig. 1, we have plotted the result of (3.8) with $N=1$

FIG. 1. Ratio of an imaginary part of the quasiground state energy to the leading bounce result in $D=1$ and $N=1$ case. The solid line is the exact numerical value. Eq. (3.8) with $P=4$ (circle), $P=5$ (filled square), and $P=15$ (filled circle), is plotted. The broken line is the leading bounce result with the two loop correction (4.2).

and $P=4$, $P=5$, and $P=15$, respectively.[5] The solid line is the exact value obtained by the numerical diagonalization of the Hamiltonian. The imaginary part is normalized by the leading bounce result (2.3). The broken line, which is depicted for a comparison, is the bounce result including two loop radiative corrections [13],

$$[\text{Im } \widetilde{\mathcal{E}}(g)]_{\text{bounce plus two loops}}$$

$$= -A_N C_{1,N} \left(\frac{S_0}{2\pi\tilde{g}}\right)^{N/2} e^{-S_0/\tilde{g}} \left(1 - \frac{21N^2+54N+20}{576}\tilde{g}\right). \quad (4.2)$$

We see an excellent convergence of (3.8) to the exact value and, as was announced, the proposal in fact gives rise to the improvement of bounce calculus. The plotted range of the coupling constant was determined by a criterion that the real part of ground state energy (which can also be computed numerically) is lower than the potential barrier. Therefore the plotted range may be regarded as the quantum tunneling (not classically sliding) region. The agreement with the exact value is better for $\tilde{g}$ *larger* but this is physically reasonable: When $\tilde{g}$ becomes smaller, the potential barrier becomes

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[4]It is worthwhile to note that all the coefficients (except c_0) are proportional to $N+2$; this holds in an arbitrary dimension D. This can be proven by noting that the partition function for $N=-2$ can be expressed by a single component fermionic system with a trivial interaction $(\bar{\psi}\psi)^2 \equiv 0$.
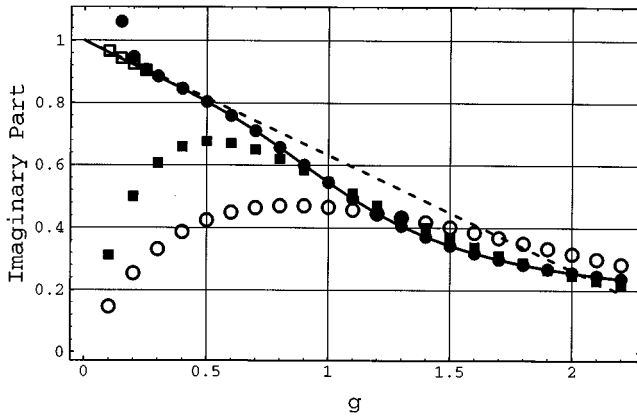
[5]This result has already been reported in [6].

FIG. 2. Same as Fig. 1, but for $N=2$. Eq. (3.8) with $P=30$ (square) is also plotted for a comparison.

FIG. 3. Same as Fig. 1, but with the first two perturbation coefficients are set by hand to zero, $c_0=c_1=0$.

higher and wider and thus the tunneling phenomena is difficult to be detected from a perturbative expansion around the potential origin.

Figure 2 is the same as Fig. 1, but for $N=2$. In this case, $P=15$ shows a small excess in the weak coupling region. However we verified that (3.8) eventually converges to the exact value, as indicated by the plots of $P=30$ (squares).

In both figures, the bounce calculus including two-loop corrections (the broken line) is giving a rather nice fitting of the exact value. Therefore one may wonder whether the imaginary part is almost saturated by the perturbation corrections around the bounce, or, the multibounce contribution is crucial in this coupling constant region. It is thus of interest to study how large the multibounce contribution is. We may estimate the two bounce contribution by studying the interaction between bounces and the quasicollective coordinate (e.g., separation between two bounces) integration [14]. The interaction is attractive when they have the same orientation and thus the two bounce contribution dominates the functional integral when the separation is small. However, since the notion of multibounce is meaningful only for a large separation, one may define the partition function by an analytic continuation from the negative g [14]. In this way, we find, for $N=1$,[6]

$$[\operatorname{Im} \widetilde{\mathcal{E}}(g)]_{\text{two bounces}}=\pm\frac{\pi}{2}[\operatorname{Im}\widetilde{\mathcal{E}}(g)]_{\text{bounce}}^2=\pm\frac{24}{g}e^{-16/\widetilde{g}},$$
(4.3)

which is 42% of the one bounce contribution for $\widetilde{g}=4$, the boundary of the tunneling region. Therefore, if one pursues the bounce calculus, the inclusion of multi-bounce contributions should be crucial in this coupling constant region.

From the excellent agreements in Figs. 1 and 2, therefore we conclude that our approach based on the perturbation series around the trivial vacuum is in fact taking into account the multi-bounce contributions. In other words, our assumption (3.4) contains a statement that an imaginary part of the

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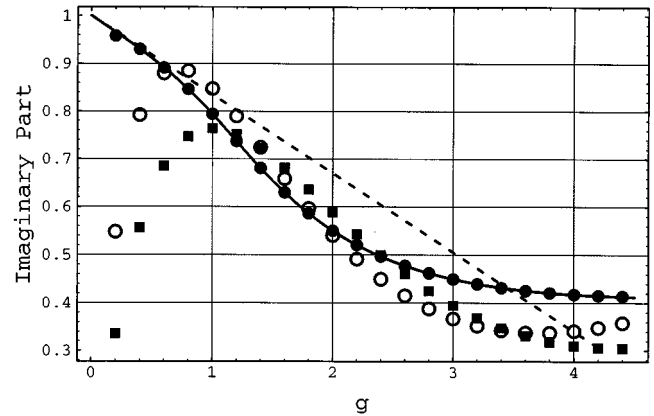[6]The sign of the contribution cannot be fixed by this prescription alone.

vacuum energy density is completely saturated by Borel singularities of the perturbation series around the *trivial* configuration.

As the final application of the quantum mechanical model, we may test our assertion made in (3.9) that a finite subtraction of the perturbation series does not affect the imaginary part when P is sufficiently large. Figure 3 is the same as Fig. 1 but the first two coefficients c_0 and c_1 are set to zero *by hand*. We see the same kind of convergence behavior even with this subtraction (except the intermediate oscillating behavior; note that $P=4$ is better than $P=5$) and $P=15$ reproduces almost the same result as Fig. 1.

V. TUNNELING ON LINE

After observing our proposal works quite well in quantum mechanics, let us try to apply our formula (3.8) to $D=2$, that represents a certain one dimensional system (line or wire). For $D=2$, we assume the following counter terms in (2.2):

$$S_{\text{count}}[\phi]=\int d^2x\left\{\frac{N}{2}\int\frac{d^2k}{i(2\pi)^2}\ln(m^2-k^2-i\epsilon)\right.$$
$$-\frac{1}{12}(N+2)g\int\frac{d^2k}{i(2\pi)^2}\frac{1}{m^2-k^2-i\epsilon}\phi^2$$
$$\left.+\frac{1}{24}N(N+2)g\left[\int\frac{d^2k}{i(2\pi)^2}\frac{1}{m^2-k^2-i\epsilon}\right]^2\right\},$$
(5.1)

where the first and third terms remove the zero-point energy to two loop orders, and the second is the counter term for the one loop self energy.[7] Since the present system is super-renormalizable, the counter terms (5.1) remove all the UV divergences to higher orders.

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[7]Therefore, our coupling constant g is not the physical four point scattering amplitude. The finite renormalization, which is required to translate the result into the one in terms of the physical coupling constant, might be performed by utilizing the resummed β function in [11].
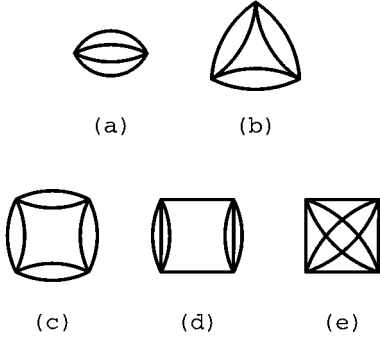
FIG. 4. Vacuum bubble diagrams with the counter terms (5.1) to five loop orders.

To find the lower order perturbative coefficients c_n in (3.1), we have explicitly calculated the vacuum bubble diagrams to five loop orders. With the counter terms (5.1), $c_0 = c_1 = 0$, and there are five diagrams to be evaluated (Fig. 4).

We have the following numbers: The diagram (a),

$$c_2 = -\frac{N(N+2)}{3} \times 8.833\ 8951 \times 10^{-5}. \qquad (5.2)$$

The diagram (b),

$$c_3 = -\frac{N(N+2)(N+8)}{27} \times 3.012\ 767\ 294 \times 10^{-6}. \qquad (5.3)$$

The diagrams (c), (d), and (e), respectively,

$$c_4 = -\frac{N(N+2)(N^2+6N+20)}{81} \times 5.657\ 478\ 653\ 058 \times 10^{-8}$$

$$-\frac{N(N+2)^2}{9} \times 1.006\ 825\ 50 \times 10^{-7}$$

$$-\frac{N(N+2)(5N+22)}{81} \times 2 \times 10^{-7}. \qquad (5.4)$$

In the above numbers, we have verified only the last digit contains the roundoff error due to numerical integrations. In the diagram (e), we used the analytical expression of one loop triangle diagram evaluated by [15]. The numerical error for the diagram (e) is much larger than others: However we verified that the final result in Fig. 5 is quite insensitive to the precise value of (e) by varying it within the estimated error. As the general property, all the diagrams have an overall minus sign according to the $-i\epsilon$ prescription. Notice that we are computing the energy density, instead of the effective action.

We could not compare the exact value of the energy density, although this task might be done by invoking a lattice formulation in a complex rotated functional space. Therefore it is not clear which range of $\widetilde{g}$ is the tunneling region in this case. As an *order of magnitude* estimation, we may consider the effective potential to two loops which behaves around the origin as
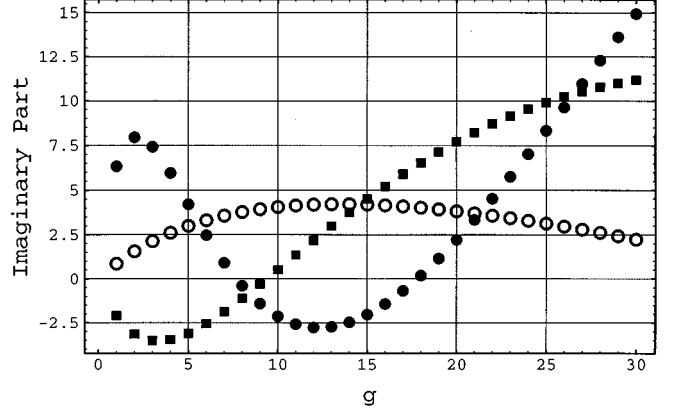

FIG. 5. Ratio of the imaginary part of the vacuum energy density computed by (3.8) to the leading bounce result in $D=2$ and $N=1$ case. $P=2$ (circle), $P=3$ (filled square), and $P=4$ (filled circle), are plotted.

$$V_{\text{eff}}[\phi] = \left[\frac{1}{2} - \frac{1}{3}(N+2) \times 6.18 \times 10^{-4}\widetilde{g}^{\,2}\right] m^2\phi^2 + \cdots, \qquad (5.5)$$

where the counter terms (5.1) have been taken into account. To the two loop order, the real part of the vacuum energy density vanishes, $c_0 = c_1 = 0$. Therefore we may regard the value of $\widetilde{g}$ which changes the sign of curvature of the effective potential at the origin as a measure of the tunneling region. This rough estimation indicates $0 < g \lesssim \sqrt{3/(N+2)} \times 30$ is the tunneling region.

In Fig. 5, we have plotted an imaginary part of the vacuum energy density (3.8) with $P=2$, $P=3$, and $P=4$. Again it is normalized by the bounce result (2.3). Unfortunately, it seems impossible to draw a definite conclusion from Fig. 5: We do not observe the convergence behavior and in some region even an unphysical result, the positive imaginary part, can be found. It is not clear whether this is due to the luck of orders of the perturbation series, or there exists a fundamental obstruction for our approach we did not encounter in quantum mechanics. (The nature of the nearest Borel singularity in the present case, $D=2$ and $N=1$, is the same as that of $N=2$ quantum mechanics, Fig. 2.) If we nevertheless take Fig. 5 at its face value, it is suggesting that the true tunneling amplitude is much *larger* than the leading bounce result in the strong coupling region. However, this would remain a speculation without having much higher order perturbation coefficients, which will clarify the real convergence property of our proposal.

VI. CONCLUSION

In this paper, we have proposed a new approach to the tunneling phenomena ''the decay of the false vacuum'' in a super-renormalizable field theory. Our approach utilizes the information of conventional perturbation series around the naive vacuum $\phi=0$. We have verified numerically that the singularities of the Borel transform reproduce an accurate tunneling rate in quantum mechanics ($D=1$). We have also presented a result in $D=2$, based on an explicit calculation of vacuum bubble diagrams to five loop orders. Unfortu-

nately the number of orders of the perturbation series is not sufficient to make definite conclusions on the convergence property and the true tunneling rate in this case. Only the higher order calculation will answer these questions.

Although we have only presented results for $D=1$ and $D=2$ in this paper, the formula (3.8) is waiting for the straightforward application to $D=3=1+2$ dimensional system (tunneling on plane). We expect a calculation of perturbative coefficients to six loops is tractable because the analytical structure of one loop diagrams in $D=3$ is rather simpler than that of $D=2$ [15]. As a consequence, definite conclusions on a convergence property of the method and on the true tunneling rate may be drawn. We hope to come back this problem in a near future.

Another possible test of our proposal is offered by an interesting field theoretical model in [17], for which both the bounce calculus and the perturbative calculation to very high orders are possible [18].

Finally we briefly comment on a possible generalization to the just renormalizable case, $D=4=1+3$. The first trouble one encounters is that all the vacuum bubble diagrams have an overall (i.e., not subdiagram) UV divergence. A simple renormalization which sets all the coefficients c_n zero order by order, appears meaningless.[8] Therefore a natural procedure is that one first introduces an UV cutoff while hoping the imaginary part (3.8) is finite and independent of the regularization for $P \rightarrow \infty$. Next we are worried about an emergence of the UV renormalon [16], another known source of the Borel singularity. Interestingly, it does not emerge in our present case because ϕ^4 model with a *negative g is* asymptotically free[9] (no Landau pole). Of course, since ϕ^4 model in $D=4$ is trivial when the UV cutoff is removed, the relation to the ''true'' imaginary part is not obvious. Presumably, our proposal for the four dimensional ϕ^4 model is meaningful only with an UV cutoff, but such an UV cutoff is naturally provided in condensed matter physics.

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[8]A similar situation occurs when one evaluates an imaginary part of the vacuum energy density due to the UV renormalon. See the third reference of [16].

[9]In *massless* theory, the infrared (IR) renormalon produces an imaginary part of the vacuum energy density: $[\text{Im}\,\mathcal{E}(g)]_{\text{IR renormalon}} = -1/(64\pi)\mu^4 \exp[-4/(b_0 g(\mu))]$, where μ is the renormalization point and $b_0 = N/[3(4\pi)^2]$ is the one loop coefficient of β function. This gives rise to a Borel singularity at $z = 4/b_0$.

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