(Anti-)evaporation of Schwarzschild–de Sitter black holes

Raphael Bousso*
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom and Department of Physics, Stanford University, Stanford, California 94305-4060

Stephen W. Hawking†
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

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We study the quantum evolution of black holes immersed in a de Sitter background space. For black holes whose size is comparable to that of the cosmological horizon, this process differs significantly from the evaporation of asymptotically flat black holes. Our model includes the one-loop effective action in the s-wave and large N approximation. Black holes of the maximal mass are in equilibrium. Unexpectedly, we find that nearly maximal quantum Schwarzschild–de Sitter black holes anti-evaporate. However, there is a different perturbative mode that leads to evaporation. We show that this mode will always be excited when a pair of cosmological holes nucleates. [S0556-2821(98)00606-7]

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I. INTRODUCTION

Of the effects expected of a quantum theory of gravity, black hole radiance [1] plays a particularly significant role. So far, however, mostly asymptotically flat black holes have been considered. In this work, we investigate a qualitatively different black hole spacetime, in which the black hole is in a radiative equilibrium with a cosmological horizon.

The evaporation of black holes has been studied using two-dimensional toy models, in which one-loop quantum effects were included [2–4]. We have recently shown how to implement quantum effects in a more realistic class of two-dimensional models, which includes the important case of dimensionally reduced general relativity [5]. The result we obtained for the trace anomaly of a dilaton-coupled scalar field will be used here to study the evaporation of cosmological black holes.

We shall consider the Schwarzschild–de Sitter family of black holes. The size of these black holes varies between zero and the size of the cosmological horizon. If the black hole is much smaller than the cosmological horizon, the effect of the radiation coming from the cosmological horizon is negligible, and one would expect the evaporation to be similar to that of Schwarzschild black holes. Therefore we shall not be interested in this case. Instead, we wish to investigate the quantum evolution of nearly degenerate Schwarzschild–de Sitter black holes. The degenerate solution, in which the black hole has the maximum size, is called the Nariai solution [6]. In this solution the two horizons have the same size and the same temperature. Therefore they will be in thermal equilibrium. Intuitively, one would expect any slight perturbation of the geometry to cause the black hole to become hotter than the background. Thus, one may suspect the thermal equilibrium of the Nariai solution to be unstable. The initial stages of such a runaway would be an interesting and novel quantum gravitational effect quite different from the evaporation of an asymptotically flat black hole. In this paper we will investigate whether, and how, an instability develops in a two-dimensional model derived from four-dimensional general relativity. We include quantum effects at the one-loop level.

The paper is structured as follows: In Sec. II we review the Schwarzschild–de Sitter solutions and the Nariai limit. We discuss the qualitative expectations for the evaporation of degenerate black holes, which motivate our one-loop study. The two-dimensional model corresponding to this physical situation is presented in Sec. III, and the equations of motion are derived. In Sec. IV the stability of the quantum Nariai solution under different types of perturbations is investigated. We find, quite unexpectedly, that the Schwarzschild–de Sitter solution is stable, but we also identify an unstable mode. Finally, the no-boundary condition is applied in Sec. V to study the stability of spontaneously nucleated cosmological black holes.

II. COSMOLOGICAL BLACK HOLES

A. Metric

The neutral, static, spherically symmetric solutions of the Einstein equation with a cosmological constant Λ are given by the Schwarzschild–de Sitter metric

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where

$$V(r) = 1 - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2, \tag{2.2}$$

*Electronic address: bousso1@stanford.edu
†Electronic address: s.w.hawking@damtp.cam.ac.uk

$d\Omega^2$ is the metric on a unit two-sphere and μ is a mass parameter. For $0<\mu<\frac{1}{3}\Lambda^{-1/2}$, V has two positive roots r_c and r_b, corresponding to the cosmological and the black hole horizons, respectively. The limit where $\mu\to 0$ corresponds to the de Sitter solution. In the limit $\mu\to\frac{1}{3}\Lambda^{-1/2}$ the size of the black hole horizon approaches the size of the cosmological horizon, and the above coordinates become inappropriate, since $V(r)\to 0$ between the two horizons. Following Ginsparg and Perry [7], we write

$$9\mu^2\Lambda=1-3\epsilon^2,\quad 0\leqslant\epsilon\ll 1. \tag{2.3}$$

Then the degenerate case corresponds to $\epsilon\to 0$. We define new time and radial coordinates ψ and χ by

$$\tau=\frac{1}{\epsilon\sqrt{\Lambda}}\psi,\quad r=\frac{1}{\sqrt{\Lambda}}\left[1-\epsilon\cos\chi-\frac{1}{6}\epsilon^2\right]. \tag{2.4}$$

In these coordinates the black hole horizon corresponds to $\chi=0$ and the cosmological horizon to $\chi=\pi$. The new metric obtained from the transformations is, to first order in ϵ,

$$ds^2=-\frac{1}{\Lambda}\left(1+\frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi d\psi^2+\frac{1}{\Lambda}\left(1-\frac{2}{3}\epsilon\cos\chi\right)d\chi^2+\frac{1}{\Lambda}(1-2\epsilon\cos\chi)d\Omega_2^2. \tag{2.5}$$

This metric describes Schwarzschild–de Sitter solutions of nearly maximal black hole size.

In these coordinates the topology of the spacelike sections of Schwarzschild–de Sitter becomes manifest: $S^1\times S^2$. In general, the radius, r, of the two-spheres varies along the S^1 coordinate, χ, with the minimal (maximal) two-sphere corresponding to the black hole (cosmological) horizon. In the degenerate case, the two-spheres all have the same radius.

B. Thermodynamics

The surface gravities of the two horizons are given by [8]

$$\kappa_{c,b}=\sqrt{\Lambda}\left(1\mp\frac{2}{3}\epsilon\right)+O(\epsilon^2), \tag{2.6}$$

where the upper (lower) sign is for the cosmological (black hole) horizon. In the degenerate case, the two horizons have the same surface gravity and, since $T=\kappa/2\pi$, the same temperature. They are in thermal equilibrium; one could say that the black hole loses as much energy due to evaporation as it gains due to the incoming radiation from the cosmological horizon. Away from thermal equilibrium, for nearly degenerate Schwarzschild–de Sitter black holes, one could make the simplifying assumption that the horizons still radiate thermally, with temperatures proportional to their surface gravities. This would lead one to expect an instability: By Eq. (2.6), the black hole will be hotter than the cosmological horizon, and will therefore suffer a net loss of radiation energy. To investigate this suspected instability, a two-dimensional model is constructed below, in which one-loop terms are included.

III. TWO-DIMENSIONAL MODEL

The four-dimensional Lorentzian Einstein-Hilbert action with a cosmological constant is

$$S=\frac{1}{16\pi}\int d^4x(-g^{\mathrm{IV}})^{1/2}\left[R^{\mathrm{IV}}-2\Lambda-\frac{1}{2}\sum_{i=1}^{N}(\nabla^{\mathrm{IV}}f_i)^2\right], \tag{3.1}$$

where R^{IV} and g^{IV} are the four-dimensional Ricci scalar and metric determinant, and the f_i are scalar fields which will carry the quantum radiation.

We shall consider only spherically symmetric fields and quantum fluctuations. Thus, we make a spherically symmetric metric ansatz

$$ds^2=e^{2\rho}(-dt^2+dx^2)+e^{-2\phi}d\Omega^2, \tag{3.2}$$

where the remaining two-dimensional metric has been written in conformal gauge; x is the coordinate on the one-sphere and has a period of 2π. Now the spherical coordinates can be integrated out, and the action is reduced to

$$S=\frac{1}{16\pi}\int d^2x(-g)^{1/2}e^{-2\phi}\left[R+2(\nabla\phi)^2+2e^{2\phi}-2\Lambda-\sum_{i=1}^{N}(\nabla f_i)^2\right], \tag{3.3}$$

where the gravitational coupling has been rescaled into the standard form. Note that the scalar fields have acquired an exponential coupling to the dilaton in the dimensional reduction. In order to take quantum effects into account, we will find the classical solutions to the action $S+W^*$. W^* is the scale-dependent part of the one-loop effective action for dilaton coupled scalars, which we derived in a recent paper [5]:

$$W^*=-\frac{1}{48\pi}\int d^2x(-g)^{1/2}\left[\frac{1}{2}R\frac{1}{\Box}R-6(\nabla\phi)^2\frac{1}{\Box}R-2\phi R\right]. \tag{3.4}$$

The $(\nabla\phi)^2$ term will be neglected; we justify this neglect at an appropriate place below.

Following Hayward [9], we render this action local by introducing an independent scalar field Z which mimics the trace anomaly. The f fields have the classical solution $f_i=0$ and can be integrated out. Thus we obtain the action

$$S=\frac{1}{16\pi}\int d^2x(-g)^{1/2}\left[\left(e^{-2\phi}+\frac{\kappa}{2}(Z+w\phi)\right)R-\frac{\kappa}{4}(\nabla Z)^2+2+2e^{-2\phi}(\nabla\phi)^2-2e^{-2\phi}\Lambda\right], \tag{3.5}$$

where

$$\kappa\equiv\frac{2N}{3}. \tag{3.6}$$

There is some debate about the coefficient of the ϕR term in the effective action. Our result [5] corresponds to the choice $w=2$; the Russo-Susskind-Thorlacius (RST) coefficient [3] corresponds to $w=1$, and the result of Nojiri and Odintsov [10] can be represented by choosing $w=-6$. In Ref. [9], probably erroneously, $w=-1$ was chosen. We take the large N limit, in which the quantum fluctuations of the metric are dominated by the quantum fluctuations of the N scalars; thus, $\kappa \gg 1$. In addition, for quantum corrections to be small we assume that $b \equiv \kappa\Lambda \ll 1$. To first order in b, we shall find that the behavior of the system is independent of w.

For compactness of notation, we denote differentiation with respect to $t(x)$ by an overdot (a prime). Further, we define, for any functions f and g,

$$\partial f \partial g \equiv -\dot{f}\dot{g} + f'g', \quad \partial^2 g \equiv -\ddot{g} + g'', \tag{3.7}$$

and

$$\delta f \delta g \equiv \dot{f}\dot{g} + f'g', \; \delta^2 g \equiv \ddot{g} + g''. \tag{3.8}$$

Variation with respect to ρ, ϕ and Z leads to the following equations of motion:

$$-\left(1-\frac{w\kappa}{4}e^{2\phi}\right)\partial^2\phi + 2(\partial\phi)^2 + \frac{\kappa}{4}e^{2\phi}\partial^2 Z + e^{2\rho+2\phi}(\Lambda e^{-2\phi}-1)=0, \tag{3.9}$$

$$\left(1-\frac{w\kappa}{4}e^{2\phi}\right)\partial^2\rho - \partial^2\phi + (\partial\phi)^2 + \Lambda e^{2\rho}=0, \tag{3.10}$$

$$\partial^2 Z - 2\partial^2\rho = 0. \tag{3.11}$$

There are two equations of constraint:

$$\left(1-\frac{w\kappa}{4}e^{2\phi}\right)(\delta^2\phi - 2\delta\phi\delta\rho) - (\delta\phi)^2 = \frac{\kappa}{8}e^{2\phi}[(\delta Z)^2 + 2\delta^2 Z - 4\delta Z\delta\rho], \tag{3.12}$$

$$\left(1-\frac{w\kappa}{4}e^{2\phi}\right)(\dot{\phi}' - \dot{\rho}\phi' - \rho'\dot{\phi}) - \dot{\phi}\phi' = \frac{\kappa}{8}e^{2\phi}[\dot{Z}Z' + 2\dot{Z}' - 2(\dot{\rho}Z' + \rho'\dot{Z})]. \tag{3.13}$$

From Eq. (3.11), it follows that

$$Z = 2\rho + \eta, \tag{3.14}$$

where η satisfies

$$\partial^2\eta = 0. \tag{3.15}$$

The remaining freedom in η can be used to satisfy the constraint equations for any choice of ρ, $\dot{\rho}$, ϕ and $\dot{\phi}$ on an initial spacelike section. This can be seen most easily by decomposing the fields and the constraint equations into Fourier modes on the initial S^1. By solving for the second term on the right hand side of Eq. (3.12), and by using Eqs. (3.14) and (3.15), the first constraint yields one algebraic equation for each Fourier coefficient of η. Similarly, the second constraint yields one algebraic equation for the time derivative of each Fourier coefficient of η. If the initial slice was non-compact, this argument would suffice. Here it must be verified, however, that η and $\dot{\eta}$ will have a period of 2π. The problem reduces to the question of whether the two constant mode constraint equations can be satisfied. Indeed, while for each oscillatory mode of η there are two degrees of freedom (the Fourier coefficient and its time derivative), the second time derivative of the constant mode coefficient, $\ddot{\eta}_0$, must vanish by Eq. (3.15). Thus there is only one degree of freedom, $\dot{\eta}_0$, for the two constant mode equations. However, since we have introduced no odd modes (i.e., modes of the form $\sin kx$) in the perturbation of ϕ, none of the fields will contain any odd modes. Since each term in Eq. (3.13) contains exactly one spatial derivative, each term will be odd. Therefore all even mode components of the second constraint vanish identically. In particular the constant mode component will thus be automatically satisfied. Then the freedom in $\dot{\eta}_0$ can be used to satisfy the constant mode component of the remaining constraint, Eq. (3.12), through the first[1] term on the right hand side.

IV. PERTURBATIVE STABILITY

A. Perturbation ansatz

With the model developed above we can describe the quantum behavior of a cosmological black hole of maximal mass under perturbations. The Nariai solution is still characterized by the constancy of the two-sphere radius, $e^{-\phi}$. Because of quantum corrections, this radius will no longer be exactly $\Lambda^{-1/2}$. Instead, the solution is given by

$$e^{2\rho} = \frac{1}{\Lambda_1}\frac{1}{\cos^2 t}, \; e^{2\phi} = \Lambda_2, \tag{4.1}$$

where

$$\frac{1}{\Lambda_1} = \frac{1}{8\Lambda}[4-(w+2)b + \sqrt{16-8(w-2)b+(w+2)^2b^2}], \tag{4.2}$$

$$\Lambda_2 = \frac{1}{2w\kappa}[4+(w+2)b - \sqrt{16-8(w-2)b+(w+2)^2b^2}]. \tag{4.3}$$

Expanding to first order in b, one obtains

$$\frac{1}{\Lambda_1} \approx \frac{1}{\Lambda}\left(1-\frac{wb}{4}\right), \tag{4.4}$$

$$\Lambda_2 \approx \Lambda\left(1-\frac{b}{2}\right). \tag{4.5}$$

[1]Note that $\dot{\eta}_0$ can thus be purely imaginary, as indeed it will be for the Nariai solution, signaling a negative energy density of the quantum field.

Let us now perturb this solution so that the two-sphere radius, $e^{-\phi}$, varies slightly along the one-sphere coordinate, x:

$$e^{2\phi}=\Lambda_2[1+2\epsilon\sigma(t)\cos x], \tag{4.6}$$

where we take $\epsilon \ll 1$. We will call σ the *metric perturbation*. A similar perturbation could be introduced for $e^{2\rho}$, but it does not enter the equation of motion for σ at first order in ϵ. This equation is obtained by eliminating $\partial^2 Z$ and $\partial^2\rho$ from Eq. (3.9) using Eqs. (3.11) and (3.10), and inserting the above perturbation ansatz. This yields

$$\frac{\ddot{\sigma}}{\sigma}=\frac{a}{\cos^2 t}-1, \tag{4.7}$$

where

$$a\equiv\frac{2\sqrt{16-8(w-2)b+(w+2)^2b^2}}{4-wb} \tag{4.8}$$

To first order in b, one finds that

$$a\approx 2+b, \tag{4.9}$$

which means that w and, therefore, the ϕR term in the effective action play no role in the horizon dynamics at this level of approximation. This is also the right place to discuss why the term $\sqrt{-g}(\nabla\phi)^2(1/\Box)R$ in the effective action can be neglected. In conformal coordinates this term is proportional to $(\partial\phi)^2\rho$. Thus, in the ρ-equation of motion, Eq. (3.9), it will lead to a $(\partial\phi)^2$ term, which is of second order in ϵ and can be neglected. In the ϕ equation of motion, Eq. (3.10), it yields terms proportional to κ that are of first order in ϵ. They will enter the equation of motion for σ via the $\kappa e^{2\phi}\partial^2 Z$ term in Eq. (3.10). Thus they will be of second order in b and can be dropped. The neglect of the $\log \mu^2$ term [5] can be justified in the same way.

B. Horizon tracing

In order to describe the evolution of the black hole, one must know where the horizon is located. The condition for a horizon is $(\nabla\phi)^2=0$. Equation (4.6) yields

$$\frac{\partial\phi}{\partial t}=\epsilon\dot{\sigma}\cos x, \quad \frac{\partial\phi}{\partial x}=-\epsilon\sigma\sin x. \tag{4.10}$$

Therefore, the black hole and cosmological horizons are located at

$$x_{\mathrm{b}}(t)=\arctan\left|\frac{\dot{\sigma}}{\sigma}\right|, \quad x_{\mathrm{c}}(t)=\pi-x_{\mathrm{b}}(t). \tag{4.11}$$

To first order in ϵ, the size of the black hole horizon, r_{b}, is given by

$$r_{\mathrm{b}}(t)^{-2}=e^{2\phi[t,x_{\mathrm{b}}(t)]}=\Lambda_2[1+2\epsilon\delta(t)], \tag{4.12}$$

where we define the *horizon perturbation*

$$\delta\equiv\cos x_{\mathrm{b}}=\sigma\left(1+\frac{\dot{\sigma}^2}{\sigma^2}\right)^{-1/2}. \tag{4.13}$$

We will focus on the early time evolution of the black hole horizon; the treatment of the cosmological horizon is completely equivalent.

To obtain explicitly the evolution of the black hole horizon radius, $r_{\mathrm{b}}(t)$, one must solve Eq. (4.7) for $\sigma(t)$, and use the result in Eq. (4.13) to evaluate Eq. (4.12). If the horizon perturbation grows, the black hole is shrinking: This corresponds to evaporation. It will be shown below, however, that the behavior of $\delta(t)$ depends on the initial conditions chosen for the metric perturbation, σ_0 and $\dot{\sigma}_0$.

C. Classical evolution

As a first check, one can examine the classical case, $\kappa=0$. This has $a=2$, and Eq. (4.7) can be solved exactly. From the constraint equations, Eq. (3.12) and (3.13), it follows that

$$\dot{\sigma}=\sigma\tan t. \tag{4.14}$$

Therefore the appropriate boundary condition at $t=0$ is $\dot{\sigma}_0=0$. The solution is

$$\sigma(t)=\frac{\sigma_0}{\cos t}. \tag{4.15}$$

Then Eq. (4.13) yields

$$\delta(t)=\sigma_0=\text{const.} \tag{4.16}$$

Since the quantum fields are switched off, no evaporation takes place; the horizon size remains that of the initial perturbation. This simply describes the case of a static Schwarzschild–de Sitter solution of nearly maximal mass, as given in Eq. (2.6).

D. Quantum evolution

When we turn on the quantum radiation ($\kappa>0$) the constraints no longer fix the initial conditions on the metric perturbation. There will thus be two linearly independent types of initial perturbation. The first is the one we were forced to choose in the classical case: $\sigma_0>0$, $\dot{\sigma}_0=0$. It describes the spatial section of a quantum-corrected Schwarzschild–de Sitter solution of nearly maximal mass. Thus one might expect the black hole to evaporate. For $a>2$, Eq. (4.7) cannot be solved analytically. Since we are interested in the early stages of the evaporation process, however, it will suffice to solve for σ as a power series in t. Using Eq. (4.13) one finds that

$$\delta(t)=\sigma_0\left[1-\frac{1}{2}(a-1)(a-2)t^2+O(t^4)\right]\approx\sigma_0\left[1-\frac{1}{2}bt^2\right]. \tag{4.17}$$

The horizon perturbation shrinks from its initial value. Thus, the black hole size *increases*, and the black hole grows, at least initially, back towards the maximal radius. One could say that nearly maximal Schwarzschild–de Sitter black holes ‘‘anti-evaporate.’’

This is a surprising result, since intuitive thermodynamic arguments would have led to the opposite conclusion. The anti-evaporation can be understood in the following way. By specifying the metric perturbation, the radiation distribution of the Z field is implicitly fixed through the constraint equations (3.12) and (3.13). Our result shows that radiation is heading towards the black hole if the boundary condition $\sigma_0>0$, $\dot{\sigma}_0=0$ is chosen.

Let us now turn to the second type of initial metric perturbation: $\sigma_0=0$, $\dot{\sigma}_0>0$. Here the spatial geometry is unperturbed on the initial slice, but it is given a kind of ''push'' that corresponds to a perturbation in the radiation bath. Solving once again for σ with these boundary conditions, and using Eq. (4.13), one finds, for small t,

$$\delta(t)=\dot{\sigma}_0 t^2. \tag{4.18}$$

The horizon perturbation grows. This perturbation mode is unstable, and leads to evaporation.

We have seen that the radiation equilibrium of a Nariai universe displays unusual and non-trivial stability properties. The evolution of the black hole horizon depends crucially on the type of metric perturbation. Nevertheless, one may ask the question whether a cosmological black hole will typically evaporate or not. Unless Λ is very small, cosmological black holes cannot come into existence through classical gravitational collapse, since they exist in an exponentially expanding de Sitter background. The only natural way for them to appear is through the quantum process of pair creation [7]. This pair creation process can also occur in an inflationary universe, because of its similarity to de Sitter space [8,11,12]. The nucleation of a Lorentzian black hole spacetime is described as the analytic continuation of an appropriate complex solution of the Einstein equations, which satisfies the no boundary condition [13]. We will show below that the no boundary condition selects a particular linear combination of the two types of initial metric perturbation, thus allowing us to determine the fate of the black hole.

V. NO BOUNDARY CONDITION

To obtain the unperturbed Euclidean Nariai solution in conformal gauge, one performs the analytic continuation $t=i\tau$ in the Lorentzian solution, Eq. (4.1). This yields

$$(ds^{\mathrm{IV}})^2=e^{2\rho}(d\tau^2+dx^2)+e^{-2\phi}d\Omega^2 \tag{5.1}$$

and

$$e^{2\rho}=\frac{1}{\Lambda_1}\frac{1}{\cosh^2\tau}, \quad e^{2\phi}=\Lambda_2. \tag{5.2}$$

In four dimensions, this describes the product of two round two-spheres of slightly different radii, $\Lambda_1^{-1/2}$ and $\Lambda_2^{-1/2}$. The analytic continuation to a Lorentzian Nariai solution corresponds to a path in the τ plane, first along the real τ axis, from $\tau=-\infty$ to $\tau=0$, and then along the imaginary axis from $t=0$ to $t=\pi/2$. This can be visualized geometrically by cutting the first two sphere in half, and joining to it a Lorentzian $(1+1)$-dimensional de Sitter hyperboloid. Because the (τ,x) sphere has its north (south) pole at $\tau=\infty$ $(\tau=-\infty)$, it is convenient to rescale time:

$$\sin u=\frac{1}{\cosh\tau}, \tag{5.3}$$

or, equivalently,

$$\cos u=-\tanh\tau, \quad \cot u=-\sinh\tau, \; du=\frac{d\tau}{\cosh\tau}. \tag{5.4}$$

With the new time coordinate u, the solution takes the form

$$(ds^{\mathrm{IV}})^2=\frac{1}{\Lambda_1}(du^2+\sin^2 u\,dx^2)+\frac{1}{\Lambda_2}d\Omega^2. \tag{5.5}$$

Now the south pole lies at $u=0$, and the nucleation path runs to $u=\pi/2$, and then parallel to the imaginary axis $(u=\pi/2+iv)$ from $v=0$ to $v=\infty$.

The perturbation of $e^{2\phi}$, Eq. (4.6), introduces the variable σ, which satisfies the Euclidean version of Eq. (4.7):

$$\sin^2 u\frac{d^2\sigma}{du^2}+\sin u\cos u\frac{d\sigma}{du}-(1-a\sin^2 u)\sigma=0. \tag{5.6}$$

In addition, the nature of the Euclidean geometry enforces the boundary condition that the perturbation vanish at the south pole:

$$\sigma(u=0)=0. \tag{5.7}$$

Otherwise, $e^{2\phi}$ would not be single valued, because the coordinate x degenerates at this point. This leaves $\dot{\sigma}$ as the only degree of freedom in the boundary conditions at $u=0$.

It will be useful to define the parameter c by the relation $c(c+1)\equiv a$. The classical case, $a=2$, corresponds to $c=1$; for small b, they receive the quantum corrections $a=2+b$ and $c=1+b/3$. With the boundary condition, Eq. (5.7), the equation of motion for σ, Eq. (5.6), can be solved exactly only for integer $c(a=2,6,12,20,...)$. The solution is of the form

$$\sigma(u)=\sum_{0\leqslant k<c/2}A_k\sin(c-2k)u, \tag{5.8}$$

with constants A_k. Even for non-integer c, however, this turns out to be a good approximation in the region $0\leqslant u\leqslant\pi/2$ of the (u,v) plane. Since we are interested in the case where $b\ll 1$, the sum in Eq. (5.8) contains only one term, and we use the approximation[2]

[2]Treating Eq. (5.6) perturbatively in b around $a=2$ leads to untractable integrals. Fortunately the guessed approximation in Eq. (5.9) turns out to be rather accurate, especially for late Lorentzian times v, which is the regime in which we claim our results to be valid. It is easy to check numerically that for sufficiently large $v(v>10)$, both the real and imaginary parts of Eq. (5.9) have a relative error $b/30$ or less. The result for the phase of the prefactor, Eq. (5.13), has a relative error of less than 10^{-4}, independently of b. Crucially, the exponential behavior at late Lorentzian times is reproduced perfectly, as the ratio

$$\frac{\partial\sigma/\partial v}{\sigma},$$

using the approximation, agrees with the numerical result to machine accuracy. Therefore the relative error in Eq. (5.15) is the same as in Eq. (5.9); in both equations it is located practically entirely in the magnitude of the prefactor. These statements hold for $0\leqslant b\leqslant 1$, which really is a wider interval than necessary.

$$\sigma(u)\approx\widetilde{A}\sin cu. \tag{5.9}$$

It is instructive to consider the classical case first. (Physically, this is questionable, since the no boundary condition violates the constraints at second order in ϵ.) For $b=0$, the solution $\sigma(u)=\widetilde{A}\sin u$ is exact. Along the Lorentzian line $(u=\pi/2+iv)$, this solution becomes $\sigma(v)=\widetilde{A}\cosh v$. By transforming back to the Lorentzian time variable t, one can check that this is the stable solution found in the previous section, with $\sigma_0=\widetilde{A}$, $\dot{\sigma}_0=0$. For real $\widetilde{A}$, it is real everywhere along the nucleation path. Thus, when the quantum fields are turned off, the Euclidean formalism predicts that the unstable mode will not be excited. This is a welcome result, since there are no fields that could transport energy from one horizon to another.

Once b is non-zero, however, it is easy to see that $\partial\sigma/\partial u$ no longer vanishes at the origin of Lorentzian time, $u=\pi/2$. This indicates that the unstable mode, $\dot{\sigma}_0\neq 0$, will be excited. Unfortunately, checking this is not entirely straightforward, because σ is no longer real everywhere along the nucleation path. One must impose the condition that σ and $\dot{\sigma}$ be real at late Lorentzian times. We will first show that this can be achieved by a suitable complex choice of A. One can then calculate the horizon perturbation, δ, from the real late-time evolution of the metric perturbation, σ, to demonstrate that evaporation takes place.

From Eq. (5.9) one obtains the Lorentzian evolution of σ,

$$\sigma(v)=\widetilde{A}\sin c\left(\frac{\pi}{2}+iv\right) \tag{5.10}$$

$$=\widetilde{A}\left(\sin\frac{c\pi}{2}\cosh cv + i\cos\frac{c\pi}{2}\sinh cv\right). \tag{5.11}$$

For late Lorentzian times (i.e., large v), $\cosh cv\approx\sinh cv\approx e^{cv}/2$, and so the equation becomes

$$\sigma(v)\approx\frac{1}{2}\widetilde{A}(ie^{-ic\pi/2})e^{cv}. \tag{5.12}$$

This can be rendered purely real by choosing the complex constant $\widetilde{A}$ to be

$$\widetilde{A}=A(-ie^{ic\pi/2}), \tag{5.13}$$

where A is real.

Now we can return to the question of whether the Euclidean boundary condition leads to evaporation. After transforming the time coordinate, the expression for the growth of the horizon perturbation, Eq. (4.13), becomes

$$\delta(v)=\sigma\left[1+\cosh^2 v\left(\frac{\partial\sigma/\partial v}{\sigma}\right)^2\right]^{-1/2}. \tag{5.14}$$

The late time evolution is given by $\sigma(v)=(A/2)e^{cv}$. This yields, for large v,

$$\delta(v)\approx\frac{A}{2}e^{cv}(1+c^2e^{2v})^{-1/2}\approx\frac{A}{2c}\exp\left(\frac{b}{3}v\right). \tag{5.15}$$

This result confirms that pair created cosmological black holes will indeed evaporate. The magnitude of the horizon perturbation is proportional to the initial perturbation strength, A. The evaporation rate grows with $\kappa\Lambda$. This agrees with intuitive expectations, since κ measures the number of quantum fields that carry the radiation.

VI. SUMMARY

We have investigated the quantum stability of the Schwarzschild–de Sitter black holes of maximal mass, the Nariai solutions. From four-dimensional spherically symmetric general relativity with a cosmological constant and N minimally coupled scalar fields we obtained a two-dimensional model in which the scalars couple to the dilaton. The one-loop terms were included in the large N limit, and the action was made local by introducing a field Z which mimics the trace anomaly.

We found the quantum corrected Nariai solution and analyzed its behavior under perturbations away from degeneracy. There are two possible ways of specifying the initial conditions for a perturbation on a Lorentzian spacelike section. The first possibility is that the displacement away from the Nariai solution is non-zero, but its time derivative vanishes. This perturbation corresponds to nearly degenerate Schwarzschild–de Sitter space, and somewhat surprisingly, this perturbation is stable at least initially. The second possibility is a vanishing displacement and non-vanishing derivative. These initial conditions lead directly to evaporation. The different behavior of these two types of perturbations can be explained by the fact that the initial radiation distribution is implicitly specified by the initial conditions, through the constraint equations.

If neutral black holes nucleate spontaneously in pairs on a de Sitter background, the initial data will be constrained by the no boundary condition: It selects a linear combination of the two types of perturbations. By finding appropriate complex compact instanton solutions we showed that this condition leads to black hole evaporation. Thus neutral primordial black holes are unstable.

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1974).
[2] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D **45**, 1005 (1992).
[3] J. G. Russo, L. Susskind and L. Thorlacius, Phys. Lett. B **292**, 13 (1992).
[4] J. G. Russo, L. Susskind and L. Thorlacius, Phys. Rev. D **46**, 3444 (1992).
[5] R. Bousso and S. W. Hawking, Phys. Rev. Lett. **56**, 7788 (1997).
[6] H. Nariai, Sci. Rep. Res. Inst. Tohoku Univ. A **35**, 62 (1951).

[7] P. Ginsparg and M. J. Perry, Nucl. Phys. **B222**, 245 (1983).
[8] R. Bousso and S. W. Hawking, Phys. Rev. D **54**, 6312 (1996).
[9] J. D. Hayward, Phys. Rev. D **52**, 2239 (1995).
[10] S. Nojiri and S. D. Odintsov, Mod. Phys. Lett. A **112**, 2083 (1997).
[11] R. Bousso and S. W. Hawking, Phys. Rev. D **52**, 5659 (1995).
[12] R. Bousso, Phys. Rev. D **55**, 3614 (1997).
[13] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).