Effective theories of coupled classical and quantum variables from decoherent histories: A new approach to the back reaction problem

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We use the decoherent histories approach to quantum theory to derive the form of an effective theory describing the coupling of classical and quantum variables. The derivation is carried out for a system consisting of a large particle coupled to a small particle with the important additional feature that the large particle is also coupled to a thermal environment producing the decoherence necessary for classicality. The effective theory is obtained by tracing out both the environment and the small particle variables. It consists of a formula for the probabilities of a set of histories of the large particle, and depends on the dynamics and initial quantum state of the small particle. It has the form of an almost classical particle coupled to a stochastic variable whose probabilities are determined by a formula very similar to that given by quantum measurement theory for continuous measurements of the small particle's position. The effective theory gives intuitively sensible answers when the small particle is in a superposition of localized states (unlike the simple mean field approach of coupling to the expectation values of the small system). The derived effective theory suggests a form of the semiclassical theory even when the quantum theory of the large system is not known, as is the case, for example, when a classical gravitational field is coupled to a quantized matter field, thus offering a new approach to the back reaction problem. [S0556-2821(97)04822-4]

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I. INTRODUCTION

What happens when a classical system interacts with a quantum system in a nontrivial superposition state? Quantum field theory in curved spacetime is an example of a number of situations where one would like to know the answer this question. There, the effect of the quantized matter field on the classical gravitational field is often assessed using the semiclassical Einstein equations [1,2]

$$G_{\mu\nu}=8\pi G\langle T_{\mu\nu}\rangle. \tag{1.1}$$

The left-hand side is the Einstein tensor of the classical metric field $g_{\mu\nu}$, and the right-hand side is the expectation value of the energy-momentum tensor of a quantum field.

Although we do not yet have a complete, workable theory of quantum gravity required to derive an equation such as Eq. (1.1), on general grounds it is clear that it is unlikely to be valid unless the fluctuations in $T_{\mu\nu}$ are small [3–5]. Indeed, Eq. (1.1) fails to give intuitively sensible results when the matter field is in a superposition of localized states [6,7]. It is by no means obvious, however, that we have to resort to quantum gravity to accommodate nontrivial matter states. This leads one to ask whether there exists a semiclassical theory with a much wider range of validity than Eq. (1.1), which gives intuitively reasonable results for nontrivial superposition states for the matter field.

The object of the present paper is to derive the form of an effective theory of coupled classical and quantum variables, in some simple models where the quantum theory of the entire system is known. From there, we can then make a reasonable postulate as to the form such a theory might take even when the quantum theory of the variables treated classically is not known. Of course, a number of previous authors have attempted either to derive or postulate the form of theories of coupled classical and quantum variables [8]. What is perhaps missing from most of these earlier approaches is an adequate characterization of what it means for one of the subsystems to be effectively classical. Here, we will work in the context of the decoherent histories approach to quantum theory [9–12], where a thorough characterization of what it means to be classical has been undertaken. This issue is an involved one, but simply, the system must be described by a decoherent set of histories consisting of the same type of variables at each moment of time whose probabilities are strongly peaked about classical equations of motion.

It should be stressed that we do not expect to derive a consistent theory describing the coupling of *fundamentally* classical variables to quantum variables. Rather, we are looking for the form of an effective theory in which variables which are the classical descendents of a (perhaps unknown) quantum theory couple to quantum variables. In contrast to fundamentally classical variables, classical descendents of quantum variables always suffer a certain amount of imprecision, partly due to their quantum fluctuations, but largely as a result of the coarse graining required for decoherence and, hence, to render them effectively classical. This imprecision feeds into the quantum variables they couple to and, as we shall see, confers some useful features.

In this paper, we will concentrate on some simple models in nonrelativistic quantum theory. To motivate the discussion, consider the following system. Suppose we have a large (''to be classical'') particle with coordinates X linearly coupled to a small particle with coordinates x. Let the action be

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$$S=\int dt\left(\frac{1}{2}M\dot{X}^2+\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-\lambda Xx\right). \quad (1.2)$$

Hence the equations of motion are

$$M\ddot{X}+\lambda x=0, \quad (1.3)$$

$$m\ddot{x}+m\omega^2x+\lambda X=0. \quad (1.4)$$

A naive semiclassical approach (the mean field approach), on which Eq. (1.1) is based, involves considering the equation

$$M\ddot{X}+\lambda\langle x\rangle=0 \quad (1.5)$$

together with the Schrödinger equation for the state of the small system with $X(t)$ as an external classical source. However, as stated above, we do not expect Eq. (1.5) to have a very wide range of validity.

Physically, when a large, classical particle interacts with a small quantum system, the large particle in some sense ''measures'' the position of the small system at each moment of time and then evolves according to the measured value. The probability for the large particle to measure a particular value of x will be determined by the quantum state of the small system and there will generally be nonzero probabilities for a wide range of different values of x. There is no reason why the average value $\langle x\rangle$ is the one that will almost always be measured, unless the coupling is very weak or the distribution of x is strongly peaked about $\langle x\rangle$. Therefore, what we expect in general is an ensemble of trajectories for the large particle, with a probability for each trajectory determined by the quantum state of the small particle. In this paper, we will derive a scheme of this type, using the decoherent histories approach, in a class of simple models.

We mention in passing that it is possible to proceed differently from this point and directly write down a phenomenological scheme for the coupling of classical and quantum variables using continous quantum measurement theory [13–17]. Such an approach was considered in Ref. [18]. The idea is that, in Eq. (1.3), X is treated as a classical variable and x is replaced by a classical stochastic variable $\overline{x(t)}$, the probability for which is given by a standard construction of quantum measurement theory:

$$\begin{aligned}p[\overline{x(t)}]=&\int \mathcal{D}x\,\mathcal{D}y\,\rho_0^B(x_0,y_0)\exp\left(-\int dt\,\frac{(x-\overline{x})^2}{2\sigma_1^2}\right.\\&\left.-\int dt\,\frac{(y-\overline{x})^2}{2\sigma_1^2}\right)\\&\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-\lambda xX\right)\right]\\&\times\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-\lambda yX\right)\right],\end{aligned} \quad (1.6)$$

where $\rho_0^B(x_0,y_0)$ is the initial density matrix of the small quantum system. Therefore, the scheme is to solve the equations of motion for the large particle with $\overline{x(t)}$ regarded as a classical source, and then the probability distribution on the trajectories $X(t)$ is that implied by the probability (1.6).

The formula (1.6) contains an arbitrary parameter σ_1 representing the imprecision in the continuous measurement. A reasonable estimate as to its value can be made by appealing to the fact that, as stated above, X is not fundamentally classical, but a classical descendent of a quantum variable. It therefore has intrinsic imprecision, which limits the precision with which it can carry out ''measurements'' of the small particle. The parameter σ_1 ought therefore to be approximately determined given the size of the fluctuations in X and the nature of the coupling between X and x [18].

The scheme we derive in this paper turns out to be very closely related to the phenomenological scheme presented in Ref. [18] (although it is not exactly the same). Furthermore, it yields a definite value for the parameter σ_1.

We will use the decoherent histories approach to quantum theory [9–12,19]. In this approach, the primary focus is on the probabilities for a set of histories of a closed system:

$$p(\alpha)=\mathrm{Tr}[P_{\alpha_n}(t_n)\cdots P_{\alpha_2}(t_2)P_{\alpha_1}(t_1)\rho P_{\alpha_1}(t_1)\cdots P_{\alpha_n}(t_n)]. \quad (1.7)$$

Here, the $P_\alpha(t)$ are projection operators in the Heisenberg picture:

$$P_\alpha(t)=e^{iHt/\hbar}P_\alpha e^{-iHt/\hbar}. \quad (1.8)$$

They are exhaustive and exclusive, which means, respectively,

$$\sum_\alpha P_\alpha=1, \quad P_\alpha P_\beta=\delta_{\alpha\beta}P_\alpha. \quad (1.9)$$

The projection operators describe the possible properties the system may have at each moment of time. In this paper we are mainly interested in histories characterized by imprecisely specified positions, in which case the projectors have the form

$$P_\alpha=\int_\alpha dx|x\rangle\langle x|, \quad (1.10)$$

where the integral is over some interval on the real axis labeled by α. In practice, it is often more convenient to work with so-called Gaussian projectors

$$P_{\overline{x}}=\frac{1}{(2\pi\sigma^2)^{1/2}}\int_{-\infty}^{\infty}dx\,\exp\left(-\frac{(x-\overline{x})^2}{2\sigma^2}\right), \quad (1.11)$$

which are only approximately exclusive.

Probabilities generally cannot be assigned to sets of histories unless there is negligible interference between them. The measure of the interference between any pair of histories is the decoherence functional:

$$\begin{aligned}&D(\alpha,\alpha')\\&\quad=\mathrm{Tr}[P_{\alpha_n}(t_n)\cdots P_{\alpha_2}(t_2)P_{\alpha_1}(t_1)\rho P_{\alpha_1'}(t_1)\cdots P_{\alpha_n'}(t_n)].\end{aligned} \quad (1.12)$$

When the condition of (approximate) decoherence is satisfied,

$$D(\alpha,\alpha')\approx 0 \quad \text{for} \quad \alpha\neq\alpha', \tag{1.13}$$

the interference between histories is negligible and probabilities obeying the probability sum rules may be assigned using the formula (1.7). The decoherence condition is typically only satisfied for histories that are coarse grained, i.e., histories for which the projections ask only very limited questions.

A number of recent papers have used the decoherent histories approach to discuss the emergence of classical behavior in simple particle models [10,20–23]. The decoherent histories approach is perhaps the most useful approach to this problem primarily for the following reason. When we say, in the context of quantum theory, that a particle exhibits almost classical behavior, we mean that the probability that it is found at a sequence of imprecisely specified positions at a sequence of times exists and, furthermore, that this probability is peaked about classical equations of motion [24,10]. Hence, to talk about classical properties of a point particle, we need to talk about the histories of imprecisely specified positions.

A commonly used coarse-graining procedure to ensure that histories of position are decoherent is to couple to a thermal environment. We therefore consider projections at each moment of time of the form

$$P_\alpha = P^A_\alpha \otimes I^{\mathcal{E}}, \tag{1.14}$$

where P^A_α denotes imprecise position projections for the particle and $I^{\mathcal{E}}$ denotes the identity on the environment. Using this basic setup, a number of recent papers have shown that for a thermal environment of sufficiently high temperature there exist decoherent histories of imprecisely specified position [10,23]. Furthermore, the probabilities for histories are then strongly peaked about classical equations of motion with dissipation, with thermal fluctuations about them [10,21]. If the particle is sufficiently massive, the effect of the thermal fluctuations is very small, and its behavior may therefore be said to be effectively classical.

Given, therefore, this characterization of what it means for a particle to be effectively classical, we may now turn to the main question we are interested in, which is to determine the form of the effective equations of motion when the classical particle is coupled to a small quantum particle. It should be clear that it is very easy to set up this problem in the decoherent histories approach. We quite simply couple a small particle in an arbitrary initial state to the case considered above. The closed system we consider therefore consists of a large particle (A) coupled to a small particle (B). The large particle is also coupled to a thermal environment ($\mathcal{E}$). The projections at each moment of time are therefore of the form

$$P_\alpha = P^A_\alpha \otimes I^B \otimes I^{\mathcal{E}}, \tag{1.15}$$

where I^B denotes the identity for the small particle. We again expect decoherence of the histories. The main thing we are interested in is the probability distribution for the histories of the large particle: We expect it to be similar to the case above, but modified in a way depending on the dynamics of the small particle and also its initial state (which we leave arbitrary).

Note that, since we are only interested in the effective equations of motion for the large particle A, we do not consider projections onto the properties of the small particle B. The histories of the small particle therefore do not need to be decoherent (indeed, the interesting case is that in which they might exhibit quantum behavior), and it is for this reason that we do not couple the small particle to the environment.

As we shall see, it is easy to set up the expression for the probability for histories of the large particle. The main issue is to express the result in a useful and recognizable form. We shall show that the effective equations of motion have the form of the classical equations of motion coupling X to x, but with the small particle variables x replaced by a stochastic c number $\overline{x(t)}$. Moreover, the probability distribution for $\overline{x(t)}$ is given by a formula bearing a close resemblance to the probability for a continuous position measurement in continuous quantum measurement theory.

The majority of our results are described in Sec. II, where we consider the simple linear model described above, linearly coupled to a thermal environment. We compute the probabilities for histories of the large particle. It has the form of a stochastic theory in which a classical variable X is coupled to a stochastic variable $\overline{x(t)}$ with a probability distribution for $\overline{x(t)}$. The distribution of $\overline{x(t)}$, in this simple linear model, essentially reduces to a Wigner function on the initial phase space data of the small particle (although smeared over a large region of phase space, so that it is positive). We discuss some properties of the scheme and show that the naive semiclassical approximation is recovered in the limit of very weak coupling. We also show that if the small particle is initially in a superposition of localized states, the large particle ''sees'' one or other of the localized states, and not the mean position of the entire state.

In Sec. III we demonstrate the connection with quantum theory of continuous measurements. We show that the probability distribution for $\overline{x(t)}$ is closely related to the formula for continuous quantum measurements, Eq. (1.6), and discuss the connection with the phenomenological scheme of Ref. [18].

The generalization to nonlinear systems with nontrivial couplings is straightforward and is considered in Sec. IV. Couplings involving the energy of the small particle are considered in Sec. V. We summarize and conclude in Sec. VI.

II. A SIMPLE LINEAR MODEL

We now compute the decoherence functional for a simple linear model. The model consists of a large free particle linearly coupled to a small harmonic oscillator [with action (1.2)], but the large particle is also coupled to a thermal bath. The large particle could start out in an arbitrary state, but we are assuming it is almost classical, and so it is most useful to start it out in a state with almost definite position and momentum. The near-classical behavior of the large particle is

assured by its coupling to the environment. The small particle starts out in an arbitrary initial state. We would like to know how the large classical particle responds to the presence of the small quantum particle in an arbitrary quantum state. More precisely, what is the effective description of the large particle, in terms of the quantum state of the small particle? In the decoherent histories approach we can quite simply calculate directly the probability that the large particle will take a particular trajectory.

A. Probabilities for histories

After tracing out the thermal bath modes, the decoherence functional for the model is

$$D[\overline{X,Y}]=\int \mathcal{D}X\ \mathcal{D}Y\ \mathcal{D}x\ \mathcal{D}y\ \rho_0^A(X_0,Y_0)\rho_0^B(x_0,y_0)\exp\left(-\int dt\ \frac{(X-\overline{X})^2}{2\sigma^2}-\int dt\ \frac{(Y-\overline{Y})^2}{2\sigma^2}\right)$$
$$\times\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}M\dot{X}^2-\frac{1}{2}M\dot{Y}^2\right)-D\int dt(X-Y)^2\right]\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-\lambda Xx\right)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-\lambda Yy\right)\right]. \tag{2.1}$$

This formula is an elementary generalization of similar ones used in Refs. [10, 25, 21]. The integration is over paths $X(t)$, $Y(t)$, $x(t)$, and $y(t)$ which fold into the initial density matrices $\rho^A(X_0,Y_0)$ and $\rho^B(x_0,y_0)$ at the initial time and at the final time $X=Y$ and $x=y$ are integrated over. We have used Gaussian projections of width σ to specify the trajectories of the large particle (although we did not need to do this—exact projections may have been used, but this is a bit more awkward [10]). The influence functional formalism of Feynman and Vernon has been used to handle the thermal bath [26,27]. The only remnant of this environment is the term proportional to $(X-Y)^2$, and the constant D is given by $D=2M\gamma kT/\hbar^2$. For simplicity, we are working in the limit of high temperature and negligible dissipation, but these restrictions are easily relaxed.

For macroscopic values of M, T, and γ, D is exceedingly large, thereby very effectively suppressing contributions from widely different values of X and Y. The coarse-graining scale of X and Y is set by the parameter σ, and hence the condition for approximate decoherence is $D>1/\sigma^2$ [23,25]. We are generally interested in histories which are maximally refined, that is, as fine grained as possible consistent with a given standard of approximate decoherence [10]. This means, in this case, that σ is taken to be as small as possible, which means that it is of order $D^{-1/2}$.

The probabilities for histories $\overline{X(t)}$, which may now be assigned, are given by the diagonal elements of Eq. (2.1). Introducing

$$Q=\tfrac{1}{2}(X+Y),\quad \xi_1=X-Y, \tag{2.2}$$

the integration over ξ_1 may be carried out, and the probabilities are

$$p[\overline{X(t)}]=\int \mathcal{D}Q\ \mathcal{D}x\ \mathcal{D}y\ W_0^A(M\dot{Q}_0,Q_0)\rho^B(x_0,y_0)\exp\left[-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}}\int dt\left(M\ddot{Q}+\frac{1}{2}\lambda(x+y)\right)^2\right]$$
$$\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-\lambda Qx\right)\right]\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-\lambda Qy\right)\right], \tag{2.3}$$

where $\widetilde{D}=D+1/(4\sigma^2)$. An integration by parts was performed, in the exponent, which picks up a boundary term $(-i/\hbar)M\dot{Q}(0)\xi_1(0)$ (recall that $\xi_1=0$ at the final time). The integration over $\xi_1(0)$ then effectively produces the Wigner transform W_0^A of the initial density matrix ρ_0^A [10,28].

Equation (2.3) may be written

$$p[\overline{X(t)}]=\int \mathcal{D}\overline{q}\ \mathcal{D}Q\ W_0^A(M\dot{Q}_0,Q_0)w_Q[\overline{q(t)}]\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}(1-\eta)}\int dt(M\ddot{Q}+\lambda\overline{q})^2\right), \tag{2.4}$$

where

$$w_Q[\overline{q(t)}]=\int \mathcal{D}x\ \mathcal{D}y\ \rho_0^B(x_0,y_0)$$
$$\times\exp\left[-\frac{\lambda^2}{4\hbar^2\widetilde{D}\eta}\int dt\left(\frac{(x+y)}{2}-\overline{q}\right)^2\right]$$
$$\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-\lambda Qx\right)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-\lambda Qy\right)\right]. \tag{2.5}$$

To achieve the decomposition (2.5), we have effectively deconvolved the second part of the Gaussian in Eq. (2.4), using the functional integral generalization of the formula

$$\exp[-(x-y)^2]=\int dz\ \exp\left(-\frac{(x-z)^2}{1-\eta}-\frac{(y-z)^2}{\eta}\right). \tag{2.6}$$

This deconvolution is, of course, not unique, and η is an arbitrary constant parametrizing this nonuniqueness [although clearly the total probability distribution (2.4) is independent of η]. This trick turns out to be useful for smearing the Wigner functions of each particle, thereby rendering them positive [29,28].

Written in the form (2.4) the probability distribution now has a reasonably natural interpretation. First of all recall that we are assuming that the Wigner function of the large particle is strongly peaked about particular values of Q_0 and $M\dot{Q}_0$. Hence, in the absence of the coupling to the small particle, Eq. (2.4) describes a probability distribution for the large particle strongly peaked about a single classical solution with prescribed initial conditions. The width of peaking about the classical solution is controlled by the factor $\hbar^2\widetilde{D}$, which is of order $M\gamma kT$, and this is typically very small for macroscopic values of M, γ, and T.

With the small particle coupled in, however, there is the integration over $\overline{q(t)}$ together with the weight function (2.5). In the next section, we will show that it is closely related to the probability distribution for continuously measuring the position $q(t)$ of the small particle. Equation (2.4) is therefore the sought-after result: It describes an ensemble of trajectories for the large particle with a weight depending on the initial conditions and dynamics of the small particle.

B. Weight function

The weight function (2.5) may be further evaluated as follows. Introduce $q=\frac{1}{2}(x+y)$ and $\xi_2=x-y$. Then the ξ_2 integration may be done with the result

$$w_Q[\overline{q(t)}]=\int \mathcal{D}q\ W_0^B(m\dot{q}_0,q_0)\exp\left(-\frac{\lambda^2}{4\hbar^2\widetilde{D}\eta}\times\int dt(q-\overline{q})^2\right)\delta[m\ddot{q}+m\omega^2q+\lambda Q], \tag{2.7}$$

where W_O^B is the initial Wigner function of the small particle. Now let

$$q(t)=q_0\cos\omega t+\frac{\dot{q}_0}{\omega}\sin\omega t+\lambda\int dt'G(t,t')Q(t')+\delta q(t), \tag{2.8}$$

where $G(t,t')$ is the Green function for the harmonic oscillator and $\delta q(0)=0=\delta\dot{q}(0)$. Then the δ functional in the functional integration in Eq. (2.7) becomes $\delta[m\delta\ddot{q}(t)]$, which implies, given the above initial conditions, that $\delta q(t)=0$. All that remain are two ordinary integrations over q_0 and $p_0=m\dot{q}_0$:

$$w_Q[\overline{q(t)}]=\int dp_0\ dq_0\ W_0^B(p_0,q_0)\exp\left[-\frac{\lambda^2}{4\hbar^2\widetilde{D}\eta}\times\int dt\left(q_0\cos\omega t+\frac{p_0}{m\omega}\sin\omega t+\lambda\int dt'G(t,t')Q(t')-\overline{q(t)}\right)^2\right]. \tag{2.9}$$

This shows that the weight function is in fact a Wigner function smeared over a region of phase space of size Δ, where

$$\Delta\sim\frac{\hbar^2\widetilde{D}m\omega^2}{\lambda^2}\sim\frac{M\gamma kT}{\lambda^2}m\omega^2. \tag{2.10}$$

The quantity $M\gamma kT/\lambda^2$ (divided by time) is a measure of the thermal fluctuations $(\Delta x)^2_{\rm th}$ in the position of the small particle induced by its coupling to the large particle. The quantity $\hbar/(m\omega)$ is representative of the quantum fluctuations $(\Delta x)^2_q$ of the small particle. For a wide range of choices of the parameters of the model, the induced thermal fluctuations are much larger than the quantum fluctuations, and it follows that $\Delta\gg\hbar$. (This will always be the case, for example, if the large particle is sufficiently massive.) This means, first of all, that as long as η is not too small, $w_Q[\overline{q(t)}]$ is positive, even though the Wigner function is not, since a smearing of the Wigner function over cells larger in size than about $\hbar$ yields a positive distribution function [29,28].

More importantly, because the smearing is over a cell size very much greater than $\hbar$, an effect essentially the same as decoherence of the *small quantum system* is produced. To be precise, suppose the initial state of the small quantum system consisted of a superposition of localized wavepackets, e.g., coherent states. Then, in the Wigner function, the interference terms between these wavepackets would appear as terms which rapidly oscillate in phase space on a scale the size of $\hbar$. It is well known that smearing the Wigner function

over a region much large in size than $\hbar$ strongly suppresses these terms (see Refs. [30, 31], for example). Therefore, for all practical purposes we may replace the initial Wigner function with a generally mixed state Wigner function in which the interference terms between wavepackets has been thrown away.

Effectively what is happening here is that the small quantum system alternatives are approximately decoherent because they are approximately correlated with the decohered large system alternatives. A similar phenomenon, in the context of quantum measurement theory, was noted by Hartle [24,32].

It could be the case, of course, that the interesting parameters for the model in a particular application are such that Eq. (2.9) is not in fact positive. Then the weight function can no longer be interpreted as a probability distribution on $\overline{q(t)}$. However, what one is ultimately interested in is the probability distribution for histories of the large particle, Eq. (2.4), is this is positive by construction, for all choices of parameters.

In this simple linear model, a further simplification may be obtained by inserting Eq. (2.9) into Eq. (2.5), and carrying out the integration over $\overline{q(t)}$, with the result

$$p[\overline{X(t)}]=\int dp_0 dq_0 W_0^B(p_0,q_0)\int \mathcal{D}Q\ W_0^A(M\dot{Q}_0,Q_0)\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}\right)$$
$$\times\exp\left[-\frac{1}{4\hbar^2\widetilde{D}}\int dt\left(M\ddot{Q}+\lambda^2\int dt' G(t,t')Q(t')+\lambda\left[q_0\cos\omega t+\frac{p_0}{m\omega}\sin\omega t\right]\right)^2\right]. \tag{2.11}$$

Loosely speaking, this equation tells us to solve the classical equations of motion for the small particle in terms of their initial data p_0 and q_0, and then regard these as stochastic variables with a probability distribution given by a smeared Wigner function.

Another convenient way of writing the weight function (2.5) is

$$w_Q[\overline{q(t)}]=\int \mathcal{D}q\ \mathcal{W}[q,Q]\exp\left(-\frac{\lambda^2}{4\hbar^2\widetilde{D}\eta}\int dt(q-\overline{q})^2\right), \tag{2.12}$$

where

$$\mathcal{W}[q,Q]=\int \mathcal{D}\xi_2\exp\left(-\frac{i\lambda}{\hbar}\int dt Q\xi_2\right)$$
$$\times\exp\left(\frac{i}{\hbar}S_B\left[q+\frac{1}{2}\xi_2\right]-\frac{i}{\hbar}S_B\left[q-\frac{1}{2}\xi_2\right]\right)\rho_0^B(x_0,y_0), \tag{2.13}$$

where, recall, $q=\frac{1}{2}(x+y)$ and $\xi_2=x-y$, and $S_B[x]$ is the free action of the small particle. The quantity $\mathcal{W}[q,Q]$ is the Wigner functional, introduced by Gell-Mann and Hartle [10], and is defined, in Eq. (2.13), by a functional Wigner transform of the fine-grained decoherence functional for the small particle position histories. It is analogous to the ordinary Wigner function in relation to the density matrix [33]. Hence, in Eq. (2.12), we see that the weight function is a smeared Wigner functional. Like the Wigner function, the Wigner functional is not positive in general, but in this case at least, the smeared Wigner functional (2.12) is.

C. Recovery of the naive semiclassical approximation

It is also of interest to examine Eq. (2.4) or (2.11) in the limit of very weak coupling, i.e., small λ. One readily finds that, to lowest order,

$$p[\overline{X(t)}]=\int \mathcal{D}Q\ W_0^A(M\dot{Q}_0,Q_0)\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}\right)$$
$$\times\exp\left(-\frac{1}{4\hbar^2\widetilde{D}}\int dt[M\ddot{Q}+\lambda\langle q(t)\rangle]^2\right), \tag{2.14}$$

where

$$\langle q(t)\rangle=\langle q_0\rangle\cos\omega t+\frac{\langle p_0\rangle}{m\omega}\sin\omega t+\lambda\int dt' G(t,t')Q(t'). \tag{2.15}$$

The probability distribution is peaked about the naive semiclassical equation (1.5), and here we see it emerging in the limit of very weak coupling. This is not surprising, since weak coupling corresponds to a very imprecise measurement, and the first thing a very broad measurement sees is the average value.

Although the naive semiclassical approach is very limited, the above considerations show how it might still be useful if used with discretion for certain types of initial states. Note first of all that the expression (2.14), the naive semiclassical result, is not only valid for small λ. It will also be a valid approximation to Eq. (2.11) for initial states whose Wigner function is strongly peaked about the mean values of p and q, as is the case for a coherent state.

Now consider the key case of an initial state that consists of a superposition of two well-separated phase space localized states, one localized around p_1 and q_1, the other localized around p_2 and q_2:

$$|\Psi\rangle = \alpha_1|\Psi_{p_1q_1}\rangle + \alpha_2|\Psi_{p_2q_2}\rangle. \tag{2.16}$$

Then the Wigner function for this initial state has the form

$$W = |\alpha_1|^2 W_{p_1q_1} + |\alpha_2|^2 W_{p_2q_2} + \text{interference terms}, \tag{2.17}$$

where $W_{p_1q_1}$ denotes the Wigner function for the state $|\Psi_{p_1q_1}\rangle$, and thus is concentrated around p_1 and q_1, and similarly for p_2 and q_2. Now the point is that, as we have argued above, the interference terms in the initial Wigner function become highly suppressed as a result of the smearing in Eq. (2.9). Therefore the probability distribution (2.11) has the form

$$p[\overline{X(t)}] \approx |\alpha_1|^2 p_1[\overline{X(t)}] + |\alpha_2|^2 p_2[\overline{X(t)}], \tag{2.18}$$

where $p_1[\overline{X(t)}]$ denotes the probability distribution (2.4), but with initial Wigner function $W_{p_1q_1}$ (for the small system) and similarly for $p_2[\overline{X(t)}]$. But these Wigner functions are strongly peaked about their mean values, and hence the naive semiclassical expression (2.14) is valid for $p_1[\overline{X(t)}]$ and $p_2[\overline{X(t)}]$ seperately. The effective description of the coupled classical and quantum system is therefore that the classical system follows the equations of motion

$$M\ddot{Q} + \lambda\langle q(t)\rangle_1 = 0, \tag{2.19}$$

with probability $|\alpha_1|^2$, and follows

$$M\ddot{Q} + \lambda\langle q(t)\rangle_2 = 0, \tag{2.20}$$

with probability $|\alpha_2|^2$, where

$$\langle q(t)\rangle_1 = \langle \Psi_{p_1q_1}|q(t)|\Psi_{p_1q_1}\rangle \tag{2.21}$$

and similarly for $\langle q(t)\rangle_2$. This is clearly the intuitively sensible result. Applied undiscerningly, the mean field equations (1.5) would not give this result, although here we see that, given the small amount of insight provided by decoherence, the mean field equations can be used to good effect.

III. CONTINUOUS MEASUREMENT THEORY

The weight function $w_Q[\overline{q(t)}]$ turns out to be closely related to continuous measurement theory [13–17]. As we have noted, the interaction of the large particle with the small one constitutes a continuous but imprecise measurement of the position of the small particle. In the quantum theory of continuous measurements, the probability for measuring a trajectory $\overline{q(t)}$ up to an imprecision σ_1 is given by the path integral expression

$$\begin{aligned} p[\overline{q(t)}] = \int \mathcal{D}x\ \mathcal{D}y\ \rho_0^B(x_0,y_0) & \\ \times \exp\left(-\int dt\ \frac{(x-\overline{q})^2}{2\sigma_1^2} - \int dt\ \frac{(y-\overline{q})^2}{2\sigma_1^2}\right) & \\ \times \exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 - \lambda Qx\right)\right] & \\ \times \exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2y^2 - \lambda Qy\right)\right]. & \end{aligned} \tag{3.1}$$

As above, introduce the variables $q = \frac{1}{2}(x+y)$, $\xi = x - y$, and hence

$$\begin{aligned} p[\overline{q(t)}] = \int \mathcal{D}q\ \mathcal{D}\xi\ \rho_0^B\left(q_0 + \frac{1}{2}\xi_0, q_0 - \frac{1}{2}\xi_0\right) & \\ \times \exp\left(-\int dt\ \frac{(q-\overline{q})^2}{\sigma_1^2} - \int dt\ \frac{\xi^2}{4\sigma_1^2}\right) & \\ \times \exp\left(-\frac{i}{\hbar}\int dt\,\xi(m\ddot{q} + m\omega^2 q + \lambda Q)\right. & \\ \left. -\frac{i}{\hbar}m\dot{q}(0)\xi(0)\right). & \end{aligned} \tag{3.2}$$

Comparing with the expression for the weight function (2.5), we see that it is very similar, if σ_1, which is so far arbitrary, is taken to be

$$\sigma_1^2 = \frac{4\hbar^2\widetilde{D}\eta}{\lambda^2} \sim \frac{M\gamma kT}{\lambda^2}, \tag{3.3}$$

although note that Eqs. (3.1) and (3.2) differ from Eq. (2.5) by the absence of the term

$$\exp\left(-\int dt\ \frac{\xi^2}{4\sigma_1^2}\right). \tag{3.4}$$

in Eq. (2.5). Hence Eq. (2.5) is not *exactly* the same as the continuous measurement formula (3.1). However, we will see that they are very close.

Carrying out the ξ integration we obtain

$$p[\overline{q(t)}]=\int \mathcal{D}q\ W_0^B(m\dot{q}_0,q_0)\exp\left(-\int dt\ \frac{(q-\overline{q})^2}{\sigma_1^2}-\frac{\sigma_1^2}{\hbar^2}\int dt(m\ddot{q}+m\omega^2 q+\lambda Q)^2\right). \tag{3.5}$$

With the above choice of σ_1,

$$\frac{\sigma_1^2}{\hbar^2}\sim\frac{M\gamma kT}{\hbar^2\lambda^2}. \tag{3.6}$$

For macroscopic values of M, γ, and T, and assuming λ^2 is not unusually large, the factor of $\hbar^2$ in the denominator ensures that $\sigma_1^2/\hbar^2$ is very large. The second exponential in Eq. (3.5) is therefore very close to a δ function, and Eq. (3.5) is therefore very close in form to the alternative expression for the weight function (2.7).

Note that the width σ_1 of the effective ''measurement'' of q depends on two things. First of all, it depends on the coupling λ and is smaller the larger λ is, corresponding to the notion that stronger interactions produce more precise measurements. Second, it depends on the combination $M\gamma kT$, which is a measure of the thermal fluctuations endured by the large particle as a result of its interaction with the environment and, hence, is a measure of the precision to within which the trajectory of the large particle is defined. σ_1 increases with increasing $M\gamma kT$, which is to be expected, since the precision with which the large particle can measure the small particle depends on the precision with which the large particles's properties are themselves defined.

Given the close resemblance to continuous quantum measurement theory, the question remains, why did we not get *exactly* the formula for continuous quantum measurements? After all, in the decoherent histories approach it is possible to derive standard quantum measurement theory (i.e., measurements represented by exact projection operators at discrete moments of time), under certain idealized conditions [24,32]. The answer to this is probably to be found in the nature of the simple model we are considering and, in particular, the couplings between the subsystems. For example, in the model considered here, we coupled x to just a single degree of freedom X (in turn coupled to the environment). Coupling to a large number of degrees of freedom may lead to the missing factor, Eq. (3.4), in the same way that the coupling of X to a large environment produces the factor involving $\int dt(X-Y)^2$ in Eq. (2.1). Hence it is quite possible that a closer connection between decoherent histories and continuous quantum measurements might be found by exploring more general types of models couplings. This is tangential to the main theme of this paper, and so will be explored elsewhere.

It is perhaps of interest to note that the above result on continuous measurement can be reexpressed in terms of evolution equations, and this in fact casts our results in a form very close to the original mean field equations (1.5) [18]. For a pure initial state, the probability formula for continuous measurements Eq. (3.1), has the form $\langle\Psi_{\overline{q}}|\Psi_{\overline{q}}\rangle$, for a wave function $\Psi_{\overline{q}}$, whose path integral representation is given by ''half'' of Eq. (3.1). From this one can define a normalized state $|\psi\rangle$ whose time evolution is given by the nonlinear stochastic equation

$$\frac{d}{dt}|\psi\rangle=\left(-\frac{i}{\hbar}H-\frac{1}{\sigma_1^2}(\hat{q}-\langle q\rangle)^2\right)|\psi\rangle+\frac{1}{\sigma_1}(\hat{q}-\langle q\rangle)|\psi\rangle\,\eta(t). \tag{3.7}$$

Here $\eta(t)$ is Gaussian white noise whose linear and quadratic means are

$$\langle\eta(t)\rangle_S=0,\quad \langle\eta(t)\,\eta(t')\rangle_S=\delta(t-t'), \tag{3.8}$$

where $\langle\ \rangle_S$ denotes stochastic averaging. Here η is related to the measured variable $\overline{q}$ by

$$\overline{q}=\langle\psi|\hat{q}|\psi\rangle+\tfrac{1}{2}\sigma_1\eta(t). \tag{3.9}$$

H is the Hamiltonian for the small system [in this case a harmonic oscillator, with $X(t)$ as an external source].

Hence, to the extent that the semiclassical equations we have derived are equivalent to continuous quantum measurement, the new equations that replace the mean field equations (1.5) are

$$M\ddot{Q}+\lambda\langle\psi|q|\psi\rangle+\tfrac{1}{2}\lambda\sigma_1\eta(t)=0, \tag{3.10}$$

where $|\psi\rangle$ evolves according to the stochastic nonlinear equation (3.7). Note that with the value of σ_1 is given by Eq. (3.3), the noise term $\lambda\sigma_1\eta(t)$ is independent of λ and so remains as $\lambda\to 0$ and describes the thermal fluctuations of the large particle.

The noise term describes fluctuations about the mean. This sort of modification, in the context of Eq. (1.1), has been considered before [5,34]. More significant is the fact the state evolves according to Eq. (3.7), and it is the properties of this equation that correspond to the separation of initial superposition states described in Sec. II C [18].

The above scheme was put forward in Ref. [18] as a phenomenological model for the coupling of classical and quantum variables, and the value of σ_1 proposed there on general physical grounds agrees with the one derived here.

IV. NONLINEAR COUPLINGS

Section II concentrated entirely on the case of a free particle linearly coupled to a harmonic oscillator. Now we show how these considerations can be extended to more complicated cases. First, we consider the case of a particle in a potential $V(X)$ coupled to a harmonic oscillator via a coupling of the form $g(X)x$. The decoherence functional (2.1) is therefore replaced by

$$D[\overline{X,Y}]=\int \mathcal{D}X\ \mathcal{D}Y\ \mathcal{D}x\ \mathcal{D}y\ \rho_0^A(X_0,Y_0)\rho_0^B(x_0,y_0)\exp\left(-\int dt\ \frac{(X-\overline{X})^2}{2\sigma^2}-\int dt\ \frac{(Y-\overline{Y})^2}{2\sigma^2}\right)$$
$$\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}M\dot{X}^2-V(X)-\frac{1}{2}M\dot{Y}^2+V(Y)\right)-D\int dt(X-Y)^2\right]\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-g(X)x\right)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-g(Y)y\right)\right]. \tag{4.1}$$

This case is actually handled quite simply using the fact that the integration over X and Y is strongly concentrated around $X=Y$. As before, write $X=Q+\frac{1}{2}\xi_1$ and $Y=Q-\frac{1}{2}\xi_1$. Then we have

$$V(X)-V(Y)=\xi_1V'(Q)+O(\xi_1^3),$$
$$g(X)x-g(Y)y=g(Q)(x-y)+\tfrac{1}{2}\xi_1g'(Q)(x+y)+O(\xi_1^2). \tag{4.2}$$

The ξ_1 integration is readily done, and we obtain, for the probabilities,

$$p[\overline{X(t)}]=\int \mathcal{D}Q\ \mathcal{D}x\ \mathcal{D}y\ W_0^A(M\dot{Q}_0,Q_0)\rho^B(x_0,y_0)\exp\left[-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}}\right.$$
$$\left.\times\int dt\left(M\ddot{Q}+V'(Q)+\frac{1}{2}g'(Q)(x+y)\right)^2\right]\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-g(Q)x\right)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-g(Q)y\right)\right]. \tag{4.3}$$

This may be written

$$p[\overline{X(t)}]=\int \mathcal{D}Q\ \mathcal{D}\overline{q}\ W_0^A(M\dot{Q}_0,Q_0)w_Q[\overline{q(t)}]\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}(1-\eta)}\int dt[M\ddot{Q}+V'(Q)+g'(Q)\overline{q}]^2\right), \tag{4.4}$$

where

$$w_Q[\overline{q(t)}]=\int \mathcal{D}x\ \mathcal{D}y\ \rho_0^B(x_0,y_0)\exp\left[-\frac{1}{4\hbar^2\widetilde{D}\eta}\int dt[g'(Q)]^2\left(\frac{(x+y)}{2}-\overline{q}\right)^2\right]$$
$$\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-g(Q)x\right)\right]\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-g(Q)y\right)\right]. \tag{4.5}$$

Introducing $q=\frac{1}{2}(x+y)$ and $\xi_2=x-y$, the ξ_2 integral may be done with the result

$$w_Q[\overline{q(t)}]=\int \mathcal{D}q\ W_0^B(m\dot{q}_0,q_0)\exp\left(-\frac{1}{4\hbar^2\widetilde{D}\eta}\int dt[g'(Q)]^2(q-\overline{q})^2\right)\delta[m\ddot{q}+m\omega^2q+g(Q)]. \tag{4.6}$$

This is very similar to the continuous measurement formulas (3.1) and (3.2) if we allow the imprecision parameter σ_1 to depend on the external field.

The next more complicated case we consider is that in which the coupling between the particles is of the form $g(X)f(x)$. It is straightforward to show that the probability is then

$$p[\overline{X(t)}]=\int \mathcal{D}Q\ \mathcal{D}x\ \mathcal{D}y\ W_0^A(M\dot{Q}_0,Q_0)\rho^B(x_0,y_0)\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}}\right.$$
$$\left.\times\int dt\left[M\ddot{Q}+V'(Q)+\frac{1}{2}g'(Q)[f(x)+f(y)]\right]^2\right)\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-g(Q)f(x)\right)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-g(Q)f(y)\right)\right]. \tag{4.7}$$

Again, this may be cast in the form

$$p[\overline{X(t)}]=\int \mathcal{D}Q\ \mathcal{D}\overline{f}\ W_0^A(M\dot{Q}_0,Q_0)w_Q[\overline{f(t)}]\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}(1-\eta)}\int dt[(M\ddot{Q})+V'(Q)+g'(Q)\overline{f}]^2\right), \tag{4.8}$$

where

$$w_Q[\overline{f(t)}]=\int \mathcal{D}x\ \mathcal{D}y\ \rho_0^B(x_0,y_0)\exp\left[-\frac{1}{4\hbar^2\widetilde{D}\eta}\int dt[g'(Q)]^2\left(\frac{(f(x)+f(y))}{2}-\overline{f}\right)^2\right]$$
$$\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-g(Q)f(x)\right)\right]\exp\left[-\frac{i}{\hbar}\int dt\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2-g(Q)f(y)\right)\right]. \tag{4.9}$$

Again, it is similar to the continuous measurement formula, but now the variable being measure is not position x, but $f(x)$, as one would expect, since it is this that couples to the large particle. It is not possible to evaluate w_Q any further in this case.

Overall, therefore, although the evaluation of the path integrals is less explicit in these more complicated cases, the general pattern is the same as in the linear case described in Sec. II: The large particle follows near-deterministic equations of motion, but with a stochastic forcing term due the small particle, whose probability distribution bears a close resemblance to the formula of continuous quantum measurement theory.

V. COUPLING TO ENERGY

Another case of particular interest, especially in connection with the semiclassical Einstein equations (1.1), is the case in which the large particle couples to the energy of the small particle. The considerations of the previous sections apply to this case very easily. In fact, this case turns out to be somewhat simpler.

Let the Hamiltonian of the total closed system, including large system (A), small system (B), and environment ($\mathcal{E}$), be

$$H=H_A+H_{AB}+H_{\mathcal{E}}+H_{A\mathcal{E}}, \tag{5.1}$$

where

$$H_{AB}=\lambda g(X)h \tag{5.2}$$

and h is a harmonic oscillator Hamiltonian. Here $g(X)$ is an arbitrary function of X, and H_A, $H_{\mathcal{E}}$, and $H_{A\mathcal{E}}$ are as before. Let the initial state of the small system be written in terms of energy eigenfunctions,

$$\rho^B=\sum_{EE'}\rho_{EE'}|E\rangle\langle E'|, \tag{5.3}$$

where $h|E\rangle=E|E\rangle$. Then, because h commutes with everything, whenever H operates on $|E\rangle$, h is replaced by the eigenfunction E.

It is then straightforward to see that the probabilities for histories of the large particle are

$$p[\overline{X(t)}]=\sum_E \rho_{EE}\int \mathcal{D}X\ \mathcal{D}Y\ \rho_0^A(X_0,Y_0)$$
$$\times\exp\left(-\int dt\ \frac{(X-\overline{X})^2}{2\sigma^2}-\int dt\ \frac{(Y-\overline{X})^2}{2\sigma^2}\right)$$
$$\times\exp\left[\frac{i}{\hbar}\int dt\left(\frac{1}{2}M\dot{X}^2-\frac{1}{2}M\dot{Y}^2\right)\right.$$
$$\left.-D\int dt(X-Y)^2\right]$$
$$\times\exp\left(-\frac{i}{\hbar}\int dt[g(X)-g(Y)]E\right). \tag{5.4}$$

The summation over E and E' is diagonal because h commutes with everything else so that the states $|E\rangle$ are preserved under evolution by the total Hamiltonian, and the trace in the decoherence functional then contains the term $\langle E|E'\rangle$.

Introducing Q and ξ as before, this becomes

$$p[\overline{X(t)}]=\sum_E \rho_{EE}\int \mathcal{D}Q\ W_0^A(M\dot{Q}_0,Q_0)$$
$$\times\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}}\right.$$
$$\left.\times\int dt[M\ddot{Q}+g'(Q)E]^2\right). \tag{5.5}$$

It is straightforward to then rewrite this in terms of continuous imprecise measurement of energy. For simplicity, take $g(Q)=\lambda Q$; then,

$$p[\overline{X(t)}]=\int d\overline{E}\int \mathcal{D}Q\ W_0^A(M\dot{Q}_0,Q_0)w(\overline{E})$$
$$\times\exp\left(-\int dt\ \frac{(Q-\overline{X})^2}{\sigma^2}-\frac{1}{4\hbar^2\widetilde{D}(1-\eta)}\right.$$
$$\left.\times\int dt(M\ddot{Q}+\lambda\overline{E})^2\right), \tag{5.6}$$

where

$$w(\overline{E})=\sum_{E}\ \rho_{EE}\ \exp\left(-\frac{\lambda^2\tau}{4\hbar^2\widetilde{D}\eta}(E-\overline{E})\right), \qquad (5.7)$$

where τ is the time duration of the histories. Equation (5.7) is the formula for the continuous imprecise measurement of energy, using Gaussian projectors, to a width of order $\hbar^2\widetilde{D}/(\lambda^2\tau)$. Here the connection with continuous measurements is precise, although this is clearly due to the simplicity of this particular case.

Of perhaps greater interest is the situation in which the energy of the small particle couples in a nontrivial way to the large particle, for example, through a small particle Hamiltonian of the form

$$H=f(X)p^2+g(X)x^2. \qquad (5.8)$$

This is a more realistic model of the way in which matter couples to gravity (for example, in cosmology, a single mode of a scalar field coupled to the scale factor). This is much more complicated to deal with and will be treated elsewhere.

VI. DISCUSSION

We have derived the form of the effective equations of motion for some simple systems consisting of a large particle coupled to a small particle and coupled also to a thermal environment in order to produce the decoherence necessary for classicality of the large particle. The resultant effective theory has the form of a classical variable coupled to a stochastic variable $\overline{x(t)}$, where the probability distribution for the stochastic variable is given by a certain weight function [most generally, Eq. (4.9)]. This weight function is closely related (although not exactly the same) as the probability for continuous imprecise measurements of the position of the small particle. In the case of coupling to energy, it is exactly the same as the continuous measurement theory result.

The weight function has the property that it suppresses the interference between localized wave packets for the small particle. Hence one of the more unsatisfactory features of the naive semiclassical approximation is avoided, and the intuitively sensible result that localized wavepacket initial states may be treated separately is restored.

The derived semiclassical theory suggests the form of a possible semiclassical theory even when the quantum theory of the variables that are taken to be classical is not known. It is the following: In the equations of motion for the classical system, which involves a coupling to the quantum system, replace the quantum variables with stochastic variables whose probabilities are given by a weight function of the form (2.5) (or its generalizations). The classical variable Q is regarded as an external classical source in Eq. (2.5) and the path integral is well defined, even if the quantum theory of Q is not known.

The only aspect of Eq. (2.5) that was inherited from the quantum theory of the classical variables is the width of the Gaussian, $\hbar^2\widetilde{D}^2/\lambda^2$. However, we can see from Eq. (2.4) that, physically, the factor $\hbar^2\widetilde{D}$ is a measure of the precision to within which the trajectories of classical variables are defined, and we can imagine that this number could be determined (or at least bounded) by experiment.

A very similar semiclassical scheme (using the continuous measurement formula) was described in Ref. [18], and the results of this paper give partial substantiations of that scheme.

We have used the decoherent histories approach to derive effective field equations, for reasons stated in Sec. I. It is, however, quite possible that other approaches to emergent classicality may be used, such as the density matrix approach [35,31], the quantum state diffusion picture [36,37,22], or the hybrid representation of composite quantum systems [38–40]. A system similar to that considered in this paper has been analyzed in the quantum state diffusion picture by Zoupas [41] and a simple spin system by Yu and Zoupas [42].

Note added in proof

Some of the qualitative features of the approach described here have previously been discussed by A. Anderson, in *Proceedings of the Fourth Drexel Symposium on Quantum Nonintegrability*, edited by D. H. Feng (International Press, 1996). (See also A. Anderson in Ref. [8].) In particular, he noted the importance of coupling quantum variables to variables rendered quasiclassical by decoherence (rather than fundamentally classical variables), and the desirability of having a stochastic equation predicting that the quasiclassical system ''sees'' distinct elements of a superposition with some probability.

E. Calzetta and B. L. Hu [Phys. Rev. D **49**, 6636 (1994)], in the context of system-environment models (such as quantum Brownian motion), have written down stochastic equations describing the stochastic effect of a thermal environment on the system. E. Calzetta and B. L. Hu [e-print hep-th/9501040, IASSNS-HEP/95/2 (1995)] have also discussed the decoherence of ''correlation histories'' in field theories, and have shown that histories specified by values of the energy momentum tensor are approximately decoherent, and thus, may be assigned probabilities. This leads to the possibility that the right-hand side of Eq. (1.1) may be taken to be a stochastic c-number, $\overline{T_{\mu\nu}}$, whose probabilities are given by the expression derived by Calzetta and Hu, thereby generalizing the results discussed here to the full Einstein equations. Some other related works are E. Calzetta and B. Hu, Phys. Rev. D **52**, 6770 (1995), and E. Calzetta, A. Campos, and E. Verdaguer, *ibid.* **56**, 2163 (1997).

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