Five-brane instantons and R^2 couplings in $N=4$ string theory

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We compute the gravitational coupling F_1 for type IIA string theory on $K3\times T^2$ and use string-string duality to deduce the corresponding term for heterotic string on T^6. The latter is an infinite sum of gravitational instanton effects which we associate with the effects of Euclidean five-branes wrapped on T^6. These five-branes are the neutral five-branes or zero-size instantons of heterotic string theory.
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I. INTRODUCTION

The principle of second quantized mirror symmetry [1] allows one to map world-sheet instanton effects in compactifications of type IIA string theory to spacetime instanton effects in dual heterotic string theories. For the most part this has been studied in $N=2$ dual pairs [2,1] and the nonperturbative effects deduced in the heterotic string in this way can be attributed to Yang-Mills instanton effects, suitably dressed up by string theory.

In this paper we will study this phenomenon in the much simpler context of the $N=4$ dual pair consisting of the type IIA string on $K3\times T^2$ and the heterotic string on T^6 [3]. By studying purely gravitational couplings we will be able to map genus-1 world-sheet instanton effects on the type IIA side to gravitational instanton effects on the heterotic side. These instantons are the neutral five-branes or zero-size-gauge instantons of heterotic string theory [4–6]. These configurations have many dual descriptions. For example, in M theory this five-brane can be viewed as the zero-size instanton of M theory which sits at the intersection of the Coulomb and Higgs branches of the M theory five-brane moduli space. We call it a gravitational instanton since the gauge fields vanish in the corresponding solution of the low-energy field theory and the fermion zero modes involve the gravitino and dilatino but not the gaugino fields.

II. CURVATURE-SQUARED COUPLINGS FOR THE TYPE IIA STRING ON $K3\times T^2$

Much effort has been devoted to the study of special higher-derivative F terms in string theory with $N=2$ spacetime supersymmetry. As shown in [7] these F terms are related to the topological amplitudes F_g studied in [8].

While the F_g have been much studied in $N=2$ compactifications, in fact, they are not completely trivial in $N=4$ compactifications. In this paper we study the first of these quantities, F_1, in the simpler context of the $N=4$ dual pair of type IIA string theory on $K3\times T^2$ and the heterotic string on T^6. From a mathematical point of view the computation is rather trivial. From a physical point of view, it is not. In a companion paper we will consider a closely related $N=2$ dual pair for which the nonperturbative F_1 can be written exactly [9].

Returning to $N=4$ theory, we first consider the type IIA side. The T^2 has moduli (T,U) which are the complexified Kahler modulus and complex structure modulus of T^2, respectively. The global moduli space on $K3\times T^2$ takes the form

$$(\mathrm{O}(22,6;\mathbb{Z})\backslash \mathrm{O}(22,6;\mathbb{R})/[\mathrm{O}(22)\times \mathrm{O}(6)]) \times (\mathrm{Sl}(2,\mathbb{Z})\backslash \mathrm{Sl}(2;\mathbb{R})/\mathrm{U}(1)), \tag{2.1}$$

where the final factor is associated with the Kähler modulus T.[1] The moduli parametrizing the first factor are the $K3$ σ-model moduli, the type IIA dilaton, the complex structure modulus U of T^2, and the Wilson lines on T^2 of the Ramond-Ramond (RR) gauge fields.

A. Computation of F_1

The quantity of F_1 in type IIA string theory is defined as a fundamental domain integral [8]:

$$F_1\equiv\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\,[\mathrm{Tr}_{R,R}(-1)^{J_L}J_L(-1)^{J_R}J_R q^H\overline{q}^{\widetilde{H}}-\mathrm{const}], \tag{2.2}$$

where the trace is over the Ramond-Ramond sector of the internal superconformal algebra (SCA). The constant term is determined by the massless spectrum and ensures that the integral is convergent. As in [10] one can analyze the states that contribute to F_1 by decomposing them under the left- and right-moving superconformal algebras. Only the RR

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[1] In this section we use automorphic conventions with Im $T>0$.

Bogomol'nyi-Prasad-Sommerfield (BPS) states in short (but not medium) representations of the spacetime $N=4$ supersymmetry algebra contribute.

The integral (2.2) is easily evaluated for $K3\times T^2$ compactifications. Both the left and right SCA's decompose as

$$\widetilde{\mathcal{A}}\,^{N=2}_{\tilde{c}=3}\oplus\widetilde{\mathcal{A}}\,^{N=4}_{\tilde{c}=6}. \tag{2.3}$$

Correspondingly, $J=J^{(1)}+J^{(2)}$ where $J^{(2)}=2J^3$ from the $c=6$ $\mathcal{N}=(4,4)$ superconformal algebra with J^3 the Cartan generator of an SU(2) current algebra and hence $\mathrm{Tr}J^{(2)}(-1)^{J^{(2)}}=0$. Therefore, only the term with $\mathrm{Tr}(-1)^{J^{(2)}}=\chi(\mathrm{K3})=24$ contributes and we can write Eq. (2.2) as

$$\begin{aligned}F_1&=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\,[\mathrm{Tr}^{(1)}_{R,R}J^{(1)}_LJ^{(1)}_R(-1)^{J^{(1)}_L+J^{(1)}_R}q^H\overline{q}^{\widetilde{H}}\ \mathrm{Tr}^{(2)}_{R,R}\\&\quad\times(-1)^{J^{(2)}_L+J^{(2)}_R}q^H\overline{q}^{\widetilde{H}}-24]\\&=-\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\,(24Z_{\Gamma^{2,2}}(T,U)-24)\\&=24\left(\ln\|\eta^2(T)\|^2+\ln\|\eta^2(U)\|^2-\ln\left[\frac{8\pi e^{1-\gamma_E}}{\sqrt{27}}\right]\right),\end{aligned} \tag{2.4}$$

where $\|\eta^2(T)\|^2\equiv\mathrm{Im}\,T|\eta^2(T)|^2$ is the invariant norm squared, and in the last line we have used the result of [11].

Note that Eq. (2.4) is invariant under the $\mathrm{Sl}(2,\mathbb{Z})$ group acting on T. However, the expression is not O(22,6) invariant since the complex structure modulus U mixes into other moduli in the O(22,6) coset. Of course, the equations of motion of the low-energy effective theory must be U-duality invariant.

B. Relation to the effective action

It was shown in [12,8] that for a Calabi-Yau σ model, quite generally, F_1 splits as a sum,

$$F_1=F_1^{\mathrm{complex}}+F_1^{\mathrm{K\ddot{a}hler}}, \tag{2.5}$$

which depends only on complex and Kähler moduli, respectively, and are exchanged by mirror symmetry. Which of these two functions couples to $\mathrm{tr}R\wedge R$ depends on whether we discuss the type IIA or IIB theory. In type IIA theory only the term $F_1^{\mathrm{K\ddot{a}hler}}$ in Eq. (2.5) appears in the low-energy effective theory and this term is invariant under both $\mathrm{Sl}(2,\mathbb{Z})$ and O(22,6). In the type IIB theory we would keep the complex structure term in Eq. (2.5) but the moduli space (2.1) is also changed by the interchange of T and U.

Supersymmetry constrains the local Wilsonian couplings of R^2 to the Kähler moduli to be holomorphic.[2] Of course, $F_1^{\mathrm{K\ddot{a}hler}}$ extracted from Eq. (2.4) is not holomorphic. This amplitude is related to an effective coupling. Nevertheless, we may extract from it the holomorphic Wilsonian coupling to R^2. (The relation of the nonholomorphy of effective couplings and the holomorphy constraints of Wilsonian couplings is subtle and is discussed at length in [15–18,14].) The bosonic terms in the Wilsonian action for the T modulus, including leading couplings to gravity fields, are (in Minkowski space)

$$\begin{aligned}I&=\frac{1}{2\kappa_4^2}\int\sqrt{-g}\,\frac{\partial_\mu T\partial^\mu\overline{T}}{(\mathrm{Im}\,T)^2}+I^{\mathrm{gauge\ \ fields}}\\&\quad+\frac{1}{16\pi}\,\mathrm{Re}\left[\int\ \frac{\ln[\eta(T)]^{24}}{2\pi i}\,\mathrm{tr}(R-iR^*)^2\right].\end{aligned} \tag{2.6}$$

Here $\kappa_4^2=1/M^2_{\mathrm{Planck}}$. The curvature tensor is regarded as a two-form with values in the Lie algebra of SO(3,1), $R=\frac{1}{2}R^a_{b\mu\nu}dx^\mu dx^\nu$, the dual on R is taken on the tangent space indicesand the trace is over these indices.

We recall the coupling to the gauge fields which follows from the general constraints of $d=4$, $N=4$ supergravity [19]. The scalar geometry is fixed to be an $\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(6,n)$ coset. Following [20] we may write the action in the present case by introducing U(1) gauge field strengths F^I (considered as two-forms), $I=1,...,28$, a quadratic form $\langle v,w\rangle=v^IL_{IJ}w^J$ defining O(22,6), and M_{IJ}, a matrix of scalar moduli for the O(22,6) coset such that $M^T=M$, $(ML)^2=1$. We define the projection operators $\Pi_\pm=\frac{1}{2}(1\pm ML)$ onto the graviphotons and vector multiplet field strengths, respectively, and also define $\mathcal{F}^\eta_\epsilon\equiv\Pi_\epsilon(F+\eta i*F)$ with $\epsilon=\pm$, $\eta=\pm$.

Under $\mathrm{SL}(2,\mathbb{R})$, $\mathcal{F}^\eta_\epsilon$ transforms as a modular form of weight (0,1) when $\epsilon\eta=1$ and of weight (1,0) when $\epsilon\eta=-1$:

$$\begin{aligned}\mathcal{F}^+_+&\to(c\overline{T}+d)\Pi_+(F+i*F),\\\mathcal{F}^+_-&\to(cT+d)\Pi_-(F+i*F).\end{aligned} \tag{2.7}$$

Moreover, $\mathcal{F}^\eta_\epsilon\to\Omega\mathcal{F}^\eta_\epsilon$ under O(22,6) transformations Ω. The coupling to gauge fields is

$$I^{\mathrm{gauge\ \ fields}}=-\frac{1}{16\pi}\,\mathrm{Re}\left\{\int_{\mathbb{R}^{1,3}}\overline{T}[\langle\mathcal{F}^+_+,\mathcal{F}^+_+\rangle+\langle\mathcal{F}^-_-,\mathcal{F}^-_-\rangle]\right\}. \tag{2.8}$$

Finally, let us discuss the invariances of the action (2.6). As emphasized in [21], Eq. (2.8) is not manifestly invariant. This is not surprising since the gauge fields undergo duality rotations under $\mathrm{SL}(2,\mathbb{R})$. On the other hand, the Einstein metric is $\mathrm{SL}(2,\mathbb{R})$ invariant, and hence the coupling of T to it must be invariant. This is the key difference between the gravity coupling in Eq. (2.6) and the gauge coupling in Eq. (2.8). Actually, Eq. (2.6) is not exactly invariant because $\ln[\eta(T)]^{24}$ suffers a shift under $\mathrm{SL}(2,\mathbb{R})$. As explained in [15–18,14] this is closely connected with σ-model duality anomalies. Indeed, the gravitinos, dilatinos, and gauginos are chiral under $\mathrm{SL}(2,\mathbb{R})$. Since all the fields are neutral under the 28 gauge fields, the anomalous variation will have an imaginary part proportional only to $\mathrm{tr}R\wedge R$. The anomalous

[2]Supersymmetric completions of terms of the form $\int z\,\mathrm{tr}R\wedge R$ have been discussed extensively in [13,14]. When comparing with these expressions it is important to bear in mind that string amplitudes are only computed on shell.

variation of the fermion determinant cancels the shift of $\ln[\eta(T)]^{24}$. It is worth emphasizing that the nonholomorphic terms in F_1 are nonzero in this example, even though the ''gravitational β function'' of [22] is zero.[3]

It would be very interesting to extend the above discussion to the higher F_g terms.

III. CURVATURE-SQUARED COUPLINGS FOR THE HETEROTIC STRING ON T^6

Under six-dimensional string-string duality the T modulus of the type IIA theory on $T^2\times K3$ is exchanged with the dilaton-axion multiplet or axiodil $\tau_S \equiv 4\pi i S$ of the heterotic string on T^6. Thus to obtain the R^2 couplings in the heterotic theory we may simply replace $T\rightarrow\tau_S$ everywhere in the previous section.

It is also easy to argue for this result directly in the heterotic string by insisting on S duality. At tree level the Bianchi identity for H, which follows from implementing the Green Schwarz mechanism, requires a term in the Minkowskian action:

$$\frac{1}{8\pi}\int \mathrm{Re}(\tau_S)[\mathrm{tr}\, R\wedge R-\mathrm{tr}\, F\wedge F], \qquad (3.1)$$

where as usual the gauge trace is in the fundamental of SO(32) or 1/30 times the trace in the adjoint for $E_8\times E_8$. The coefficient should be exactly as given, since otherwise instantons would not break the continuous SL(2, $\mathbb{R}$) duality group to SL(2, $\mathbb{Z}$). Supersymmetry then requires the coupling of S to R^2 to be

$$\frac{1}{8\pi}\int \mathrm{Re}(\tau_S)\mathrm{tr}(R\wedge R)-\mathrm{Im}(\tau_S)\mathrm{tr}(R\wedge R^*). \qquad (3.2)$$

From S duality itself, we know that the S-dual completion must take

$$\tau_S\rightarrow\frac{24}{2\pi i}\ln\,\eta(\tau_S), \qquad (3.3)$$

which leads to the R^2 couplings

$$\frac{1}{16\pi}\mathrm{Re}\left[\int\frac{\ln\,\eta^{24}(\tau_S)}{2\pi i}\mathrm{tr}(R-iR^*)^2\right], \qquad (3.4)$$

which reproduces Eq. (2.6) after exchanging T and τ_S.

In terms of effective couplings we may state the result in terms of the equations of motion for the dilaton multiplet:

$$\frac{1}{2\kappa_4^2}\left[\frac{\nabla^2\tau_S}{(\mathrm{Im}\,\tau_S)^2}+i\,\frac{\nabla^\mu\tau_S\nabla_\mu\tau_S}{(\mathrm{Im}\,\tau_S)^3}\right]+*\,\frac{1}{16\pi}[\langle\mathcal{F}_+^+,\mathcal{F}_+^+\rangle + \langle\mathcal{F}_-^-,\mathcal{F}_-^-\rangle-\widehat{E_2}(\overline{\tau_S})\mathrm{Tr}(R_{\mu\nu}+iR^*_{\mu\nu})^2]=0. \qquad (3.5)$$

The first two terms in Eq. (3.5) transform under SL(2, $\mathbb{R}$) transformations covariantly with weight 2. The last term breaks the invariance to SL(2, $\mathbb{Z}$). The Eisenstein series E_2 transforms with a shift while $\widehat{E_2}=E_2-3/\pi\mathrm{Im}\tau$ transforms covariantly. Equations (3.5) are the R^2 corrections to the S-duality-invariant equations of [21].

It is worth noting that in the low-energy field theory limit where we fix the dilaton S to be constant and work on a general Euclidean four-manifold, Eq. (3.4) contributes

$$\exp\left[-\left(\chi+\frac{3}{2}\sigma\right)\ln\,\eta^{12}-\left(\chi-\frac{3}{2}\sigma\right)\ln\,\overline{\eta}^{12}\right] \qquad (3.6)$$

to the Euclidean path integral. Here χ is the Euler character and σ is the signature. We presume that this gravitational S-duality anomaly is related to the S-duality anomaly in the gauge partition function studied in [24]. Note in particular that on a four-dimensional hyper-Kähler manifold where the physical and twisted $N=4$ theories should agree the curvature is automatically anti-self-dual and as a result $\chi=-3\sigma/2$ so that the term discussed here there contributes $(\overline{\eta})^{-24\chi}$ to the Euclidean path integral. It would be interesting to make the connection to the result of [24] more precise.

IV. PHYSICAL INTERPRETATION

As mentioned earlier, we expect that world-sheet instantons effects in the type IIA string should be exchanged with spacetime instanton effects in the heterotic string. The formula for F_1 is, as explained in [8], a sum over genus-1 type-IIA world-sheet instantons. In this section we will identify the spacetime instanton in the heterotic string which leads to the R^2 corrections (3.4). Since the heterotic string and the type IIA string can each be viewed as wrapped five-branes in the dual theory [25,26,6], we expect that the instantons can be viewed as heterotic five-branes wrapped on T^6.

A. Instanton expansion

In order to make the instanton expansion manifest we use the ''string conventions'' with axion-dilaton chiral superfield S with $\mathrm{Re}(S)>0$ and

$$q_S=e^{-8\pi^2 S}=e^{2\pi i\tau_S}.$$

We normalize the four-dimensional gauge action to be $(1/2g^2)\int\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$ with tr the trace in the fundamental representation of U(n) so that a charge one instanton has action $8\pi^2/g^2$. Then $S=1/g^2+i\theta/8\pi^2$, and we can expand:

$$\ln[\eta(\tau_S)]^{24}=-8\pi^2 S-24\left[q_S+\frac{3q_S^2}{2}+\frac{4q_S^3}{3}+\dots\right]. \qquad (4.1)$$

The first term is the tree level coupling, as discussed above. The higher-order terms have the form of instanton corrections to the R^2 couplings where the instanton action is given by $8\pi^2\mathrm{Re}(S)=8\pi^2/g^2$.

B. Wrapped five-branes: Macroscopic analysis

The result (3.4) appears to sum up an infinite set of instanton contributions. To confirm this we would like to identify the instanton configurations in the heterotic string which lead to these corrections to R^2 couplings.

[3]The reader should compare with the discussion in [23].

We now argue that the relevant instanton is the neutral five-brane wrapped on T^6. We will proceed in two steps, first analyzing the instanton using the low-energy analysis of [5] and then discussing the nonperturbative modifications found in [6].

As in the one-instanton contribution to the $N=2$ prepotential [27] the easiest quantity to calculate is not the purely bosonic term in the action but rather the term with the maximal number of fermion fields which is related to the bosonic term by extended supersymmetry. In $N=4$ supergravity a coupling of the form $F(S)\mathrm{tr}R^2$ is paired with eight fermion terms involving the dilatino and gravitino. To see this we note that such eight fermion terms are present in $N=1$, $d=10$ supergravity and are paired with the tree level $S\mathrm{tr}R^2$ coupling by supersymmetry [28]. They must thus be present in the dimensional reduction to the $N=4$ theory in $d=4$. There are of course additional terms with fewer fermion fields: these must be generated by supersymmetric instanton perturbation theory as has been checked in detail for $N=2$ gauge theory [29]. We are thus looking for an instanton in heterotic string theory which has action $e^{-8\pi^2 S}$ and eight fermion zero modes constructed out of the gravitino and dilatino but independent of the gauginos.

In [5] a number of five-brane solutions to heterotic string theory were discussed, the neutral five-brane, gauge five-brane, and symmetric five-brane.[4] The latter two involve finite size instantons of an unbroken non-Abelian gauge group. For simplicity we will restrict our analysis to a generic point in the Narain moduli space where the gauge group is $\mathrm{U}(1)^{28}$ and there will be no finite size gauge instantons. Thus only the neutral five-brane can be relevant to our analysis.

According to the low-energy analysis of [5] the neutral five-brane has (1,0) world-brane supersymmetry with a single hypermultiplet of zero modes. The hypermultiplet consists of four real scalar moduli associated with translations in the four dimensions transverse to the brane and a six-dimensional Weyl fermion. The fermion zero modes arise from the action of the eight components of $N=1$ supersymmetry in $d=10$ which are broken by the five-brane background.

The eight fermion zero modes are precisely what we require to get an instanton-induced eight-fermion interaction term. It is important to check that the collective coordinate integral is well defined. We do this as follows. From the formulas of [5] it is easy to write down the fermion zero modes:

$$\delta\lambda = 2e^{-\phi}\Gamma^m\partial_m\phi\epsilon\otimes\eta,$$

$$\delta\psi_\mu = -\delta_{\mu m}\partial_n\phi\Gamma^{mn}\epsilon\otimes\eta, \qquad (4.2)$$

where $\mu,m=1,...,4$, $\epsilon\otimes\eta$ is a constant spinor in the $(2^+,4)$ of $\mathrm{SO}(4)\times\mathrm{SO}(6)$, and ϕ is given by [5]

$$e^{2\phi} = e^{2\phi_0} + \frac{\alpha'}{x^2}. \qquad (4.3)$$

Thus the gauge coupling diverges ''down the throat'' of the neutral five-brane. Nevertheless, the fermion zero modes are normalizable and localized near the throat (at distances scales $\sim\sqrt{\alpha'}$). Moreover, the eight-fermion term inducing the R^2 interactions can be extracted from Eq. (2.11) of [28]. One finds several different tensor structures, which can be denoted schematically as

$$(\overline{\psi}\Gamma^{(1)}\psi)^4, \quad (\overline{\psi}\Gamma^{(1)}\psi)^3(\overline{\psi}\Gamma^{(3)}\psi), \quad (\overline{\psi}\Gamma^{(1)}\psi)^3(\overline{\psi}\Gamma^{(5)}\psi),$$
$$(\overline{\psi}\Gamma^{(1)}\psi)^3(\overline{\psi}\Gamma^{(7)}\psi), \quad (\overline{\psi}\Gamma^{(1)}\psi)^3(\overline{\psi}\Gamma^{(6)}\lambda),$$
$$(\overline{\psi}\Gamma^{(1)}\psi)^3(\overline{\psi}\Gamma^{(4)}\lambda). \qquad (4.4)$$

The notation $\Gamma^{(n)}$ refers to $\Gamma_{\mu_1\cdots\mu_n}$. Indices are contracted in all possible combinations. All these terms scale in the same way as $x^2\to 0$, and the density in the collective coordinate integral behaves like

$$\int d^4x\,\frac{1}{x^2},$$

as $x^2\to 0$, and so there is no divergence. The integral also converges well for $x^2\to\infty$.

We can also argue that the weight of the instanton action is correct. The wrapped neutral five-brane has an action which is T_5V_6 with T_5 the five-brane tension and V_6 the volume of T^6. The five-brane tension saturates a Bogolomol'nyi bound given in [30] and from this bound the action T_5V_6 is equal to the action of a minimal charge gauge instanton and is thus equal to $8\pi^2\,\mathrm{Re}\,S$ with our conventions. We will also check the action later by comparison to M theory.

C. Wrapped five-brane: Microscopic analysis

So far we have ignored the fact that the five-brane solutions of [5] have regions of strong coupling (''down the throat'') which can invalidate a naive low-energy analysis of the zero-mode structure and lead to novel effects [6]. Let us start with the SO(32) heterotic string in $d=10$. The neutral five-brane has (1,0) world-brane supersymmetry and the zero modes discussed in [5] consist of a single neutral hypermultiplet whose scalar fields give the location in R^4 of the five-brane. According to the analysis of [6] there are additional nonperturbative collective coordinates which consist of a SU(2) gauge multiplet with gauge field $\mathcal{A}$. There are also hypermultiplets in the (2,32) of $\mathrm{SU}(2)\times\mathrm{SO}(32)$.

At a generic point in the Narain moduli space SO(32) Wilson lines will break the four-dimensional gauge group to $\mathrm{U}(1)^{28}$ and give mass to the (2,32) hypermultiplets. The five-brane collective coordinates governing zero-energy deformations of the five-brane will then consist of the neutral hypermultiplet plus the values of the flat SU(2) connections and their fermion partners.

It is convenient to wrap the five-brane on T^6 in two steps by regarding T^6 as $T^4\times\mathcal{T}^2$. From string duality we know that the Kähler modulus of the $\mathcal{T}^2$ in this decomposition is equal to the S modulus of the original type IIA string theory. We first consider the neutral five-brane wrapped on T^4. Then as in [6] the flat SU(2) connections on T^4 are just Wilson

[4]The gauge five-brane was first discussed in [30]. The neutral five-brane is also discussed in [4].

lines around the one-cycles γ_i of T^4

$$U_i = P \exp \int_{\gamma_i} \mathcal{A}. \tag{4.5}$$

If $e^{\pm i\theta_i}$ are the eigenvalues of U_i, then the moduli space of flat SU(2) connections has periodic coordinates θ_i subject to the Weyl group identification $\theta_i \rightarrow -\theta_i$. The moduli space of flat connection is thus $\mathcal{T}^4/Z_2$ where we use $\mathcal{T}$ to denote the torus with coordinates θ_i in order to distinguish it from the compactification torus T^4. Thus the five-brane wrapped on T^4 yields a string in ten dimensions with transverse coordinates propagating on the space $\mathbb{R}^2 \times \mathcal{T}^2 \times \mathcal{T}^4/Z_2$. As predicted by string-string duality, this is precisely the structure of the type IIA string compactified on $K3$ as long as it is correct to view the orbifold $\mathcal{T}^4/Z_2$ as equivalent to $K3$. We henceforth refer to $\mathcal{T}^4/Z_2$ as $\mathcal{K}3$. This soliton description is implicitly in static gauge, but we should be able to consider the soliton type IIA string constructed in this way more abstractly. We now consider the effects of wrapping the five-brane on the full T^6. These instantons can be viewed as world-sheet instantons of the type IIA soliton string in the target space $\mathcal{T}^2 \times \mathcal{K}3$. Summing over these instantons will give precisely the same sum as in the original type IIA string theory, but with the replacement $T \rightarrow \tau_S$.

Thus in this example we have a very direct mapping from second quantized mirror symmetry not only between terms in the Lagrangian but also between explicit instanton configurations. In the original type IIA theory we have world-sheet instantons which are genus-1 holomorphic curves on $K3 \times T^2$. In heterotic theory these map to spacetime instanton effects which can be viewed as world-sheet instantons of the soliton type IIA string given by genus-1 holomorphic curves on $\mathcal{K}3 \times \mathcal{T}^2$. We expect that this point of view will be useful also in $N=2$ dual pairs. In this case we can start with world-sheet instantons of the fundamental type IIA string on a Calabi-Yau space which is a $K3$ fibration. On the heterotic side we have a dual pair consisting of the heterotic string on $K3 \times T^2$ with a specific choice of gauge bundle. Indeed, if the $K3$ surface is elliptically fibered, as in F compactification [31], we may attempt to use the adiabatic argument of [32] and write the ''fibration''

$$\begin{array}{c} \widetilde{T}^2 \times T^2 \rightarrow K3 \times T^2 \\ \downarrow \\ \mathbb{P}^1 \end{array} \tag{4.6}$$

whose generic fiber is a four-torus. Once again we can consider five-brane instantons wrapped on $K3 \times T^2$ in a two-step process. In the first step we wrap the five-brane on a generic fiber $\widetilde{T}^2 \times T^2$ to obtain a soliton type IIA string. This soliton string propagates on the ''fibration''

$$\begin{array}{c} \mathcal{K}3 \rightarrow X_3 \\ \downarrow \\ \mathbb{P}^1 \end{array}, \tag{4.7}$$

giving a Calabi-Yau threefold X_3, where the $\mathcal{K}3$ fiber is constructed as before as the moduli space of flat SU(2) connections on $\widetilde{T}^2 \times T^2$. World-sheet instantons where the soliton type IIA string wraps the $\mathbb{P}^1$ will then give rise to nonperturbative spacetime instanton effects in the heterotic string. The adiabatic argument will need to be corrected, since, for example, the $\mathcal{K}3$ fibers degenerate over $\mathbb{P}^1$ as do the $\widetilde{T}^2$ fibers over $\mathbb{P}^1$. Nevertheless, it should be possible again to map directly the spacetime instantons to the world-sheet instantons of the dual soliton type IIA string on a $\mathcal{K}3$ fibration Calabi-Yau space. It is an interesting problem to determine the relation between these two Calabi-Yau spaces and to figure out how the data of the heterotic gauge bundle are encoded in this description. The answer is provided, in part, by F compactification [31].

Even in the $N=4$ context discussed here there are several aspects of this identification which deserve further investigation. As in [6] we have ignored effects which may be associated with cancellation between SU(2) and SO(32) Wilson lines. However, the general picture seems robust and should continue to hold true whether or not quantum effects modify the orbifold $\mathcal{T}^4/Z_2$ to a smooth $K3$ surface, as discussed in [6,33].

V. COMMENTS ON *M*-THEORY FIVE-BRANES

It is clear that the five-brane instanton effects described above must have a description in M theory since the SO(32) heterotic string is T dual to the $E_8 \times E_8$ theory which can be obtained from M theory on S^1/Z_2 [34].

From an analysis of the fermion zero modes it is clear that in M theory the required five-brane cannot be the bulk five-brane with (2,0) world-brane supersymmetry but must instead be a zero-size-gauge five-brane which lives at the boundary of the Higgs and Coulomb branches of the M-theory five-brane moduli space. This is the tensionless string theory when considered in $\mathbb{R}^{1,5}$ [35–38] but here compactified in Euclidean signature on T^6.

We can perform one small check on the M-theory description by computing the action of the instanton directly in M theory. On general grounds this must give the same answer as before.

Following the conventions of [39] the M-theory five-brane tension is given in terms of the 11-dimensional Planck constant by

$$T_5^M = \frac{\pi^{1/3}}{2^{1/3} \kappa_{11}^{4/3}}. \tag{5.1}$$

On the other hand, it was shown in [40] that the E_8 gauge coupling λ in M theory obeys the relation

$$\kappa_{11}^4 = \frac{\lambda^6}{4(2\pi)^5}. \tag{5.2}$$

Combining this with Eq. (5.1) gives

$$T_5^M V_6 = \frac{4\pi^2 V_6}{\lambda^2} = \frac{8\pi^2}{(g)^2}, \tag{5.3}$$

where the final factor of 2 arises from the fact that the normalization used in [40] for the gauge kinetic term differs by a factor of 2 from the normalization we use in which the instanton action is $8\pi^2/g^2$.

VI. CONCLUSIONS

We have used string-string duality to compute the S-dependent corrections to R^2 couplings in $N=4$ heterotic string theory. These are given by an infinite set of spacetime instanton corrections and we have identified the instanton as the neutral five-brane of heterotic string theory or equivalently the zero-size five-brane of the M theoretic description of heterotic string theory. We have also argued that there is in this example a direct map from the world-sheet instantons of the type IIA string on $K3\times T^2$ to spacetime instantons in the heterotic string consisting of world-sheet instantons of the type IIA soliton string on a dual $K3\times T^2$. This direct map from world-sheet instantons to spacetime instantons viewed as world-sheet instantons of a soliton string is likely to have application to other dual pairs involving only $N=2$ or $N=1$ spacetime supersymmetry.

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