Cosmological solutions, *p*-branes, and the Wheeler-DeWitt equation

H. Lü,[1] J. Maharana,[2,3] S. Mukherji,[4] and C. N. Pope[4,5]
[1]*Laboratoire de Physique Théorique de l'École Normale Supérieure, Paris CEDEX 05, France*
[2]*Institute of Physics, Bhubaneswar, India*
[3]*Centre Physique Théorique, Ecole Polytechnique, 91128 Palaiseau, France*
[4]*Physics Department, Texas A & M University, College Station, Texas 77843*
[5]*SISSA, Via Beirut No. 2-4, 34013 Trieste, Italy*
(Received 24 July 1997; published 27 January 1998)

The low energy effective actions which arise from string theory or M-theory are considered in the cosmological context, where the graviton, dilaton and antisymmetric tensor field strengths depend only on time. We show that previous results can be extended to include cosmological solutions that are related to the E_N Toda equations. The solutions of the Wheeler-DeWitt equation in minisuperspace are obtained for some of the simpler cosmological models by introducing intertwining operators that generate canonical transformations which map the theories into free theories. We study the cosmological properties of these solutions, and also briefly discuss generalized Brans-Dicke models in our framework. The cosmological models are closely related to p-brane solitons, which we discuss in the context of the E_N Toda equations. We give the explicit solutions for extremal multi-charge $(D-3)$-branes in the truncated system described by the $D_4=O(4,4)$ Toda equations. [S0556-2821(98)01806-2]

PACS number(s): 98.80.Hw, 11.25.Mj, 11.27.+d

I. INTRODUCTION

There has been considerable attention given to the investigation of the cosmological consequences of string theory. It is hoped that string theory will provide answers to deep questions in quantum gravity and therefore, it is natural that the problem of the evolution of the Universe at early epochs be addressed in the string theory framework [1–4]. One of the important and intriguing problems for cosmology is to explain the mechanism of inflation. When one tries to understand inflation from the perspective of string theory, it is hoped that it should arise naturally from the theory itself. It has been recognized that the dilaton field might play an important rôle in the explanation of inflation. However, the dilaton also determines the coupling constant in string theory, and therefore it must decouple at late times so that the well-known results of late-time cosmology are not affected by dilaton interactions, in view of the fact that it can affect masses and coupling constants and other parameters at late times. This has motivated a search for mechanisms that can account for the dilaton decoupling.

A mechanism has been proposed in the pre-big-bang scenario [5] which exploits symmetries that are particular to string theory. The starting point is the tree level string effective action for the dilaton and graviton. There exist solutions that describe an expanding Universe with decceleration. This solution can be related, by means of stringy symmetries, scale factor duality and a time reversal transformation, to a solution which describes a Universe that is expanding and accelerating. An attractive scenario emerges from these two symmetry-related solutions in which the Universe begins with rapid expansion, i.e. there is a pole driven inflation, with a deccelarating expansion at later times, eventually connecting smoothly to a Friedmann-Robertson-Walker (FRW) Universe.

Recently, some attention has been focused on the investigation of the cosmological aspects of the p-branes that arise as classical solutions of string effective actions or those of supergravity theories [6–14]. It is found that these solutions can be classified into two broad catagories, depending on whether the solution is supported by field strengths carrying electric charges or magnetic charges. In some cases, dualities can relate the two kinds of solution. Although the field equations appear to be quite complicated, even in the cosmological context where the fields depend only on time, nevertheless wide classes of exact classical solutions can be obtained. In fact the equations of motion can be cast into the form of one-dimensional Liouville or Toda equations [9–11]. In particular, in certain cases one encounters the $SL(N+1,\mathbb{R})$ Toda equations [7,11]. Later, we shall show that this can be extended to the E_N Toda equations.

Since one would like to understand the evolution of the Universe at very early times, it is natural to consider string cosmology in a quantum framework. One approach is to solve the Wheeler-DeWitt (WDW) equation in a minisuperspace, and to examine the properties of the solutions [15–17]. One of the interesting applications of quantum string cosmology is to provide a resolution for the graceful exit problem, since no-go theorems have established that it cannot be resolved in classical string cosmology in the pre-big-bang scenario [18].

The purpose of this article is to explore further the cosmological solutions of string effective actions in the presence of generalized gauge potentials, and to examine their properties. First we review the classical solutions and then we study them at the quantum level. We shall show that formal solutions to the WDW equation can derived in an elegant manner in the minisuperspace model. We shall present explicit solutions in some simple cases as illustrative examples.

In perturbative string theory, there are global symmetries, including the extensively-studied $O(d,d)$ T-duality symmetry. This symmetry makes it possible to solve the WDW equation in many cases, allowing the wave function to be completely classified by the $O(d,d)$ quantum numbers [19]. We shall analyze the WDW equation for a wide class of cosmological models that arise in the low-energy effective string theory. The organizing symmetry that naturally appears in these cases is $SL(N+1,\mathbb{R})$ or E_N; the $O(d,d)$ symmetry is not manifest here since we use Poincare duality to write down different form fields that appear in the models. In what follows, we shall discuss how the $SL(N+1,\mathbb{R})$ symmetry helps us to solve the WDW equation in minisuperspace.

The paper is organized as follows. In Sec. II, we discuss the different cosmological models that arise naturally in string theories or M-theory in their low energy limit. In particular, in Secs. II A, II B, and II C we review the single-charge, multi-charge and $SL(3,\mathbb{R})$ Toda models that have been analyzed in detail in [11]. In Sec. II D, we present a new class of E_N Toda models. In Sec. II, we also introduce the notion of canonical transformations which map sets of interacting classical equations into sets of free equations. This, in turn, allows one to solve the equations of motion in a very simple manner. The quantum version of this canonical transformation is introduced in Sec. III with the help of an intertwining operator closely following the work of [20]. This operator maps an interacting quantum theory to a free theory through a set of *non-unitary* transformations. This property of the operator is then exploited to solve the WDW equations of some of the models introduced in Sec. II. We conclude the paper with a discussion of Brans-Dicke theory extended to include form fields. In particular, we show that solutions of the classical equations of motion and also the solutions of the WDW equation can be obtained from the models discussed in Secs. II and III by simple rescalings and redefinitions of fields. In the Appendix, we consider classes of p-brane solutions that are closely related to the cosmological E_N Toda models. In particular, we show how to solve the E_N Toda equations for extremal $(D-3)$-branes, taking a simplifying truncation to the $D_4=O(4,4)$ Toda system as an explicit example that has not previously been presented in the literature.

II. COSMOLOGICAL MODELS WITH NEVEU-SCHWARZ–NEVEU-SCHWARZ (NS)-(NS) OR RAMOND-RAMOND (R-R) FORM FIELDS

The effective low-energy limits of string theory or M-theory compactified on the tori give rise to maximal supergravities in lower dimensions. In [11,12], many cosmological solutions were obtained and analyzed. These included single-charge, multi-charge and $SL(N+1,\mathbb{R})$ Toda models. It was found that the classical equations of motion could be reduced to a set of Liouville or Toda equations. In Secs. II A, II B, and II C we review these models and also, in Sec. II D we introduce a new class of models which are E_N Toda cosmological models. We also introduce a set of canonical transformations which, at the classical level, maps the Liouville or Toda theories to theories governed by free Hamiltonians.

A. Single-charge cosmological models

The simplest cosmological model in D dimension involves the metric, a dilaton and an n-rank field strength F_n [11]. The action is given by

$$S=\int d^Dx\sqrt{-g}\left[R-\frac{1}{2}(\partial\phi)^2-\frac{1}{2n!}e^{a\phi}F_n^2\right], \tag{1}$$

where the constant a can be parametrized as [27]

$$a^2=\Delta-\frac{2(n-1)(D-n-1)}{D-2}. \tag{2}$$

We will assume that all the fields depend only on time. The background metric is assumed to have the form

$$ds^2=-e^{2U}dt^2+e^{2A}d\overline{s}^2+e^{2B}dy^m dy^m, \tag{3}$$

where $d\overline{s}^2$ represents the p-dimensional metric on the spatial section of a d-dimensional spacetime, with $d=p+1$. We shall consider spatial metrics of the maximally symmetric form

$$d\overline{s}^2=\frac{dr^2}{1-kr^2}+r^2d\Omega^2, \tag{4}$$

where $d\Omega^2$ is the metric on a unit $(p-1)$-sphere. Without loss of generality, the constant k may be taken to be equal to 0,1 or -1, in which case the metric $d\overline{s}^2$ describes flat, spherical, or hyperboloidal spatial sections respectively. In Eq. (3), m runs over q dimensions so that $D=1+p+q$.

In the gauge $U=pA+qB$, the action (1) reduces to

$$S=\int dt\Bigg[(\dot{\Phi})^2+\frac{2q(D-2)a^2}{p-1}\dot{Y}^2-\frac{2p\Delta}{p-1}\dot{X}^2-\Delta\lambda^2e^{\Phi}$$
$$+2kp\Delta(p-1)e^{2X}\Bigg]. \tag{5}$$

In writing down this action, one can use either of two ansätze for the field strength F_n that are compatible with the symmetries of the metric (3), giving rise to electric or magnetic cosmological solutions. In the electric solutions, the ansatz for the antisymmetric tensor is given in terms of its potential, and in a coordinate frame takes the form

$$A_{m_1m_2\ldots m_q}=f\epsilon_{m_1m_2\ldots m_q}, \tag{6}$$

and hence

$$F_{0m_1m_2\ldots m_q}=\dot{f}\epsilon_{m_1m_2\ldots m_q}, \tag{7}$$

where f depends on t only. For the electric solutions, we have $p=D-n, q=n-1$. For the magnetic cosmological solutions, the ansatz for the tangent-space components for the antisymmetric tensor is

$$F_{a_1a_2\ldots a_p}=\lambda e^{-pA}\epsilon_{a_1a_2\ldots a_p}, \tag{8}$$

where λ is a constant. Thus we have $p=n, q=D-n-1$. In the action (5), X,Y and Φ are related to A,B and ϕ in the following way:

$$X\equiv qB+(p-1)A,$$

$$Y\equiv B+\frac{p-1}{\epsilon a(D-2)}\phi,$$

$$\Phi \equiv -\epsilon a\phi + 2qB. \tag{9}$$

Here $\epsilon=1$ is for electric case and $\epsilon=-1$ for the magnetic case. Note that in the electric case, the constant λ arises as the integration constant for the function f in Eq. (7).

The equations of motion for X, Φ, and Y are

$$\ddot{X}+k(p-1)^2e^{2X}=0,$$
$$\ddot{\Phi}+\frac{1}{2}\Delta\lambda^2e^{\Phi}=0, \tag{10}$$
$$\ddot{Y}=0.$$

The variation of the action (1) with respect to the lapse function $\sqrt{g_{00}}$ provides the canonical constraint:

$$\dot{\Phi}^2+\Delta\lambda^2e^{\Phi}+\frac{2q(D-2)a^2}{p-1}\dot{Y}^2 = \frac{2p\Delta}{p-1}\dot{X}^2+2kp\Delta(p-1)e^{2X}. \tag{11}$$

Since X and Φ both satisfy Liouville equations, it is straightforward to solve these equations directly:

$$e^{-X}=\begin{cases}\dfrac{p-1}{\tilde{P}_X}\cosh(\tilde{P}_X t) & \text{if } k=1;\\ \dfrac{p-1}{\tilde{P}_X}\sinh(\tilde{P}_X t) & \text{if } k=-1;\end{cases}$$
$$X=-\tilde{P}_X t, \quad \text{if } k=0, \tag{12}$$

where $\tilde{P}_X$ is an arbitrary constant. Similarly the solution for Φ is

$$e^{-(1/2)\Phi}=\frac{\lambda\sqrt{\Delta}}{2\tilde{P}_\Phi}\cosh(\tilde{P}_\Phi t), \tag{13}$$

where $\tilde{P}_\Phi$ is again constant. The solution for Y may be taken to be simply

$$Y=-\tilde{P}_Y t. \tag{14}$$

The constraint (11) therefore implies that

$$\tilde{P}_\Phi{}^2=\frac{p\Delta\tilde{P}_X{}^2-q(D-2)a^2\tilde{P}_Y{}^2}{2(p-1)}. \tag{15}$$

The Hamiltonians for the fields X, Φ, and Y are given by

$$H_X=\frac{1}{2}P_X^2+\frac{1}{2}k(p-1)^2e^{2X},$$
$$H_\Phi=\frac{1}{2}P_\Phi^2+\frac{1}{2}\Delta\lambda^2e^{\Phi},$$
$$H_Y=\frac{1}{2}P_Y^2, \tag{16}$$

where P_X, P_Φ, and P_Y correspond to momenta conjugate to X, Φ, and Y coordinates. Notice that the solutions for X, Φ, and Y can be cast in a different form in terms of their phase-space variables. For example, when $k=1$, the solution for X can be written as

$$e^{-X}=\frac{p-1}{\tilde{P}_X}\cosh\tilde{X},$$
$$P_X=-\tilde{P}_X\tanh\tilde{X}, \tag{17}$$

where $\tilde{X}=\tilde{P}_X t$. In fact, these equations can be viewed as a canonical transformation from the interacting Liouville system, with phase-space coordinates (X,P_X), to a free system with phase-space coordinates $\tilde{X},\tilde{P}_X$ with the Hamiltonian $\tilde{H}_X=\frac{1}{2}\tilde{P}_X^2$, by re-writing (17) as

$$P_X=(p-1)e^X\sinh\tilde{X},$$
$$\tilde{P}_X=(p-1)e^X\cosh\tilde{X}. \tag{18}$$

The generating function $F(X,\tilde{X})$ has the following form:

$$F(X,\tilde{X})=(p-1)e^X\sinh\tilde{X}, \tag{19}$$

such that

$$P_X=\frac{\partial F}{\partial X}, \quad \tilde{P}_X=\frac{\partial F}{\partial\tilde{X}}. \tag{20}$$

These are the same equations as in Eq. (18). Obviously, since H_Φ has also the same structure, a similar set of canonical transformations will also bring it to a free Hamiltonian form. Thus by solving a set of free systems and using the canonical mapping (17), we can generate the solutions of the interacting theory given by the action (1). As we shall discuss in Sec. III, these transformations can be implemented in the quantum version of the model. This, in turn, will allow us to solve the corresponding WDW equations in a straightforward manner.

B. Multi-charge cosmological models

Multi-charge solutions in D-dimensional maximal supergravity can be described by the truncated action

$$S=\int d^Dx\sqrt{-g}\left[R-\frac{1}{2}(\partial\vec{\phi})^2-\frac{1}{2n!}\sum_{\alpha=1}^{N}e^{\vec{c}_\alpha\cdot\vec{\phi}}F_\alpha^2\right], \tag{21}$$

when the dilaton vectors for the set of N field strengths F_α of rank $n\geqslant 2$ satisfy the dot products [27]

$$M_{\alpha\beta}=\vec{c}_\alpha\cdot\vec{c}_\beta=4\delta_{\alpha\beta}-\frac{2(n-1)(D-n-1)}{D-2}. \tag{22}$$

The maximum value $N_{\max}$ for N depends on the rank of the field strengths, and on the dimension D. For example for two-form field strengths, $N_{\max}=2$ for $6\leqslant D\leqslant 9$; $N_{\max}=3$ in $D=5$; and $N_{\max}=4$ in $3\leqslant D\leqslant 4$ [27]. As before, we define fields

$$X\equiv qB+(p-1)A,$$

$$Y\equiv B+\frac{p-1}{\epsilon a(D-2)}\sum_{\alpha,\beta}(M^{-1})_{\alpha\beta}\varphi_\beta,$$

$$\Phi_\alpha\equiv-\epsilon\varphi_\alpha+2qB, \tag{23}$$

where $\varphi_\alpha=\vec{c}_\alpha\cdot\vec{\phi}$. The solutions of the equations of motion that follows from the action (21) in terms of these fields are (see [11] for details)

$$e^{-X}=\begin{cases}\dfrac{p-1}{\widetilde{P}_X}\cosh(\widetilde{P}_X t), & \text{if } k=1;\\[2mm] \dfrac{p-1}{\widetilde{P}_X}\sinh(\widetilde{P}_X t), & \text{if } k=-1;\end{cases}$$

$$X=-\widetilde{P}_X t,\quad \text{if } k=0,$$

$$e^{-1/2\Phi_\alpha}=\frac{\lambda_\alpha\sqrt{\Delta}}{2\widetilde{P}_{\Phi_\alpha}}\cosh(\widetilde{P}_{\Phi_\alpha}t),$$

$$Y=-\widetilde{P}_Y t, \tag{24}$$

where λ_α is the charge associated with the form field F_α and $\widetilde{P}_X$, $\widetilde{P}_{\Phi_\alpha}$, $\widetilde{P}_Y$ are constants satisfying the following constraint:

$$\sum_\alpha \widetilde{P}^2_{\Phi_\alpha}=\frac{2p\Delta\widetilde{P}_X^2-2q(D-2)a^2\widetilde{P}_Y^2}{\Delta(p-1)}. \tag{25}$$

Here $\Delta=4/N$ and a^2 is given in Eq. (2). The Hamiltonian that follows from Eq. (21) is given by

$$H=\sum_\alpha H_{\Phi_\alpha}+\frac{8q(D-2)a^2}{(p-1)\Delta}H_Y-\frac{8p}{p-1}H_X, \tag{26}$$

where

$$H_{\Phi_\alpha}=\frac{1}{2}P^2_{\Phi_\alpha}+2\lambda_\alpha^2e^{\Phi_\alpha}. \tag{27}$$

As in the previous subsection the Hamiltonian (26) can be brought to a free Hamiltonian, by means of a set of $N+1$ canonical transformations which act on the phase space variables $(\Phi_\alpha,P_{\Phi_\alpha})$ and (X,P_X).

C. $SL(3,R)$ Toda cosmological models

As discussed in [11], when the space-time dimension is $D=2n$, the n-rank field strength can carry both electric (7) and magnetic (8) charges. In this case, $p=n$ and $q=n-1$. Making the same gauge choice as before, the equations of motion that follow from the action (1) reduce to

$$\ddot{X}+k(n-1)^2e^{2X}=0,$$

$$\ddot{q}_1=-e^{\alpha q_1+(1-\alpha)q_2},$$

$$\ddot{q}_2=-e^{(1-\alpha)q_1+\alpha q_2}, \tag{28}$$

where

$$X=(n-1)(A+B),$$

$$B=\frac{1}{4(n-1)}[q_2+q_1-2\log((n-1)\lambda_1\lambda_2)], \tag{29}$$

$$\phi=\frac{a}{2(n-1)}(q_2-q_1)+\frac{1}{a}\log\frac{\lambda_1}{\lambda_2},$$

and the constant α is given by

$$\alpha=\frac{1}{2}+\frac{a^2}{2(n-1)}=\frac{\Delta}{2(n-1)}. \tag{30}$$

The first-order constraint in this case reduces to

$$\tfrac{1}{2}\alpha(\dot{q}_1^2+\dot{q}_2^2)+(1-\alpha)\dot{q}_1\dot{q}_2+e^{\alpha q_1+(1-\alpha)q_2}+e^{\alpha q_2+(1-\alpha)q_1}$$
$$=2n[\dot{X}^2+k(n-1)^2e^{2X}]. \tag{31}$$

In Eq. (28), λ_1 and λ_2 correspond to electric and magnetic charges. In particular, the choice $\lambda_1=\lambda_2$ will correspond to a self-dual cosmological model. Although for a generic value of α the equations seem not to be integrable, when $\alpha=2$ Eq. (28) reduces to the $SL(3,R)$ Toda equations, which can be exactly solved. This value of α can arise for a two-form field strength in $D=4$, with $\Delta=4$. The Hamiltonian of the (q_1,q_2) system can be written as

$$H_{(q_1,q_2)}=\frac{1}{3}(P_1^2+P_2^2+P_1P_2)+e^{2q_1-q_2}+e^{2q_2-q_1}, \tag{32}$$

where P_1,P_2 are the momenta given by

$$P_1=2\dot{q}_1-\dot{q}_2,\quad P_2=2\dot{q}_2-q_1. \tag{33}$$

As in the Liouville case, there exists a set of canonical transformations which maps the above Toda Hamiltonian to a free Hamiltonian. The mapping is given in [20]:

$$e^{-q_1}=\frac{e^{\widetilde{q}_1}}{\widetilde{P}_1(\widetilde{P}_1-\widetilde{P}_2)}+\frac{e^{\widetilde{q}_2}}{\widetilde{P}_2(\widetilde{P}_1-\widetilde{P}_2)}+\frac{e^{(-\widetilde{q}_1-\widetilde{q}_2)}}{\widetilde{P}_1\widetilde{P}_2},$$

$$e^{-q_2}=\frac{e^{-\widetilde{q}_1}}{\widetilde{P}_1(\widetilde{P}_1-\widetilde{P}_2)}+\frac{e^{-\widetilde{q}_2}}{\widetilde{P}_2(\widetilde{P}_1-\widetilde{P}_2)}+\frac{e^{(-\widetilde{q}_1-\widetilde{q}_2)}}{\widetilde{P}_1\widetilde{P}_2}, \tag{34}$$

and

TABLE I. The dilaton vectors $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$ generate the E_n Dynkin diagram.

$\vec{b}_{12}$		$\vec{b}_{23}$		$\vec{b}_{34}$		$\vec{b}_{45}$		$\vec{b}_{56}$		$\vec{b}_{67}$		$\vec{b}_{78}$
o	—	o	—	o	—	o	—	o	—	o	—	o
				$\vert$								
				o								
				$\vec{a}_{123}$								

$$(2P_1+P_2)e^{-q_1}=-\frac{(2\tilde{P}_1-\tilde{P}_2)e^{\tilde{q}_1}}{\tilde{P}_1(\tilde{P}_1-\tilde{P}_2)}-\frac{(2\tilde{P}_2-\tilde{P}_1)e^{\tilde{q}_2}}{\tilde{P}_2(\tilde{P}_1-\tilde{P}_2)}+\frac{(\tilde{P}_1+\tilde{P}_2)e^{(-\tilde{q}_1-\tilde{q}_2)}}{\tilde{P}_1\tilde{P}_2},$$

$$(2P_2+P_1)e^{-q_2}=\frac{(2\tilde{P}_1-\tilde{P}_2)e^{-\tilde{q}_1}}{\tilde{P}_1(\tilde{P}_1-\tilde{P}_2)}+\frac{(2\tilde{P}_2-\tilde{P}_1)e^{-\tilde{q}_2}}{\tilde{P}_2(\tilde{P}_1-\tilde{P}_2)}-\frac{(\tilde{P}_1+\tilde{P}_2)e^{(\tilde{q}_1+\tilde{q}_2)}}{\tilde{P}_1\tilde{P}_2}. \qquad (35)$$

With this transformation, in terms of the new variables, the Toda Hamiltonian reduces to a free Hamiltonian of the form

$$H_{(\tilde{q}_1,\tilde{q}_2)}=\frac{1}{3}(\tilde{P}_1^2+\tilde{P}_2^2-\tilde{P}_1\tilde{P}_2). \qquad (36)$$

D. E_N Toda cosmological models

Maximal supergravities in D dimensions coming from the toroidal compactification of 11-dimensional supergravity have E_{11-D} global symmetries. It is natural therefore to expect that there might exist p-brane or cosmological solutions that arise as solutions of the E_N Toda equations. It has been observed that the dilaton vectors for all the axions are precisely in one-to-one correspondence with the positive roots of the E_{11-D} algebra. In particular, the simple roots can be taken to be $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$ [21]. Thus in all dimensions we may summarize the information about the dot products of the dilaton vectors for the full sets of axions by the Dynkin diagram (see Table I).

In each dimension D, the diagram is truncated to the part that survives when only the simple roots with indices $i\leqslant 11-D$ are retained.

It is straightforward to verify that when the axions take the standard electric or magnetic ansätze, the full Lagrangian can be consistently truncated to one of the form (21) with the N field strengths $F_\alpha=(F_0^{(123)},\mathcal{F}_0^{(12)},\mathcal{F}_0^{(23)},\ldots)$, and associated dilaton vectors given by $\vec{c}_\alpha=(\vec{a}_{123},\vec{b}_{12},\vec{b}_{23},\ldots)$. Now the dilaton dot produts $M_{\alpha\beta}$ are no longer given by Eq. (22); instead $M_{\alpha\beta}$ is precisely the Cartan matrix for E_N. We shall now show that this has the consequence that the equations of motion of the system can be cast into the form of the one-dimensional E_N Toda equations. To do this, we first consistently truncate the Lagrangian further to

$$e^{-1}\mathcal{L}=R-\frac{1}{2}\sum_{\alpha,\beta=1}^{N}(M^{-1})_{\alpha\beta}\partial_M\varphi_\alpha\partial^M\varphi_\beta-\frac{1}{2}\sum_{\alpha=1}^{N}e^{\varphi_\alpha}(\partial\chi_\alpha)^2, \qquad (37)$$

where $\varphi_\alpha=\vec{c}_\alpha\cdot\vec{\phi}$. We shall discuss the electric solutions in detail, for which $p=D-1$ and $q=0$. (The discussion for the magnetic solutions is analogous.) The metric *Ansatz* in this case is thus given by

$$ds^2=-e^{2U}dt^2+e^{2A}d\overline{s^2}, \qquad (38)$$

where $d\overline{s^2}$ is again the metric on the spatial sections, typically taking the form (4). It is convenient to make the gauge choice $U=(D-1)A$, which implies that the equations of motion can be written as

$$\ddot{A}+k(D-2)e^{2(D-2)A}=0, \qquad (39)$$

$$\ddot{\Phi}_\alpha=-\lambda_\alpha^2\exp\left(\frac{1}{2}\sum_\beta M_{\alpha\beta}\Phi_\beta\right), \qquad (40)$$

where $\Phi_\alpha=-2\Sigma_\beta(M^{-1})_{\alpha\beta}\varphi_\beta$, together with a first-order equation

$$\frac{1}{4}\sum_{\alpha\beta}M_{\alpha\beta}\dot{\Phi}_\alpha\dot{\Phi}_\beta+\sum_a\lambda_\alpha^2\exp\left(\frac{1}{2}\sum_\beta M_{\alpha\beta}\Phi_\beta\right)=2(D-2)(D-1)(\dot{A}^2+ke^{2(D-2)A}). \qquad (41)$$

Defining $\Phi_\alpha=q_\alpha-4\Sigma_\beta(M^{-1})_{\alpha\beta}\log(\lambda_\beta)$ to remove the charges, the equations for Φ_α then become

$$\ddot{q}_1=-e^{2q_1-q_4},\qquad \ddot{q}_2=-e^{2q_2-q_3},$$

$$\ddot{q}_3=-e^{-q_2+2q_3-q_4},\qquad \ddot{q}_4=-e^{-q_1-q_3+2q_4-q_5},$$

$$\ddot{q}_5=-e^{-q_4+2q_5-q_6},\qquad \ddot{q}_6=-e^{-q_5+2q_6-q_7},$$

$$\ddot{q}_7=-e^{-q_6+2q_7-q_8},\qquad \ddot{q}_8=-e^{-q_7+2q_8}. \qquad (42)$$

These are precisely the E_8 Toda equations. Here we present only the E_8 case, since the lower cases for E_N with $N\leqslant 7$ are obtained by straightforward truncation. (In fact an alternative truncation can instead be made that reduces the E_8 Toda equations to the $SL(N+1,\mathbb{R})$ equations that were discussed previously [7].) The left-hand side of the first-order equation (41) is the Hamiltonian for the Toda equations (42), given by

$$\mathcal{H}=\frac{1}{4}\sum_{\alpha\beta}M_{\alpha\beta}\dot{q}_a\dot{q}_\beta+\sum_\alpha\exp\left(\frac{1}{2}\sum_\beta M_{\alpha\beta}q_\beta\right). \qquad (43)$$

The right-hand side of the equation (41) is a constant, given by $2(D-1)P_X^2/(D-2)$, since the function A, satisfying Eq. (39), is given by Eq. (12) with $X=2(D-2)A$. Thus the first-order equation (41) means no more than that the Hamiltonian is a conserved quantity, given by

$$\mathcal{H}=\frac{2(D-1)P_X^2}{D-2}. \tag{44}$$

III. THE WHEELER-DEWITT EQUATION

Recently, there have been attempts to solve the WDW equation in string cosmology and study its implications. Let us now construct the WDW equations for the string cosmological models that we are considering in this paper. We recall that the classical equations of motion, which correspond to interacting Liouville or Toda systems, can be reduced to free field equations after implementing the canonical transformations discussed in the previous section. In fact it has been shown that these classical transformations can be extended to the quantum level. This is achieved by introducing intertwining operators, which implement the canonical transformations on the quantum mechanical operators and wave functions [20]. In what follows, we explicitly construct the intertwining operators for the cosmological models that have been discussed in Secs. II A and II B, and we use these to obtain solutions of the corresponding WDW equations. We end this section with the analysis of some of the solutions of the WDW equations, by imposing proper boundary conditions on the wave functions.

A. Intertwining operators and the solutions of WDW equation

The canonical transformation between the classical Liouville and free theories that have been discussed in the previous section can be implemented at the quantum level. This is done by introducing intertwining operators [20] which generate canonical transformations on the quantum operators and on the wave functions. In order to construct such operators we first focus on the simplest of all the models that have been analyzed in Sec. II, namely, the cosmological model with a single charge.

In order to proceed, let us first concentrate on H_X given in Eq. (16). It is known that there exists an operator C_X which transforms the Liouville Hamiltonian to a free one [20]. In particular,

$$C_X H_X C_X^{-1}=\widetilde{H}_X. \tag{45}$$

As a result, the wave functions ψ_X and $\widetilde{\psi}_X$ of the Liouville and free theories are related by $\psi=C_X^{-1}\widetilde{\psi}$. The operator C_X has been constructed in [20], and takes the following form:

$$C_X=\mathcal{P}_{(p-1)\sinh X}P_X^{-1}\mathcal{I}\mathcal{P}_{\ln X}, \tag{46}$$

where each of the constituent pieces has the following action:

$$\begin{aligned}
&\mathcal{P}_{\ln X}: \quad X\to\ln X, \quad P_X\to XP_X,\\
&\mathcal{I}_X: \quad X\to P_X, \quad P_X\to -X,\\
&P_X^{-1}: \quad X\to P_X^{-1}XP_X, \quad P_X\to P_X,\\
&\mathcal{P}_{(p-1)\sinh X}: \quad X\to(p-1)\sinh X, \quad P_X\to\frac{P_X\cosh^{-1}X}{p-1}.
\end{aligned} \tag{47}$$

Taking into account the commutation relation $[P_X,X]=-i$, it is immediate that the combined action of Eq. (47) is to map the Liouville Hamiltonian H_X to the free Hamiltonian $\widetilde{H}_X=\frac{1}{2}\widetilde{P}_X^2$. Similarly, the operator C_X has the following action on the wave function [20]:

$$C_X^{-1}: \quad e^{ikX}\to N_k K_{ik}(e^X), \tag{48}$$

where K_{ik} is a modified Bessel function. Owing to the fact that the canonical transformation described by C_X is nonunitary (as it must be, since the Liouville theory is not simply equivalent to the free theory), the normalization of the transformed wave function is not just the same as the normalization of the free wave function. It can be determined by calculating the effect of the transformation on the Hilbert-space inner product, leading to the result [20]

$$N_k=\frac{1}{\pi}\sqrt{2k\sinh(\pi k)}. \tag{49}$$

Now consider the WDW equation, which is simply

$$H\Psi(X,\Phi,Y)=0. \tag{50}$$

Here the total Hamiltonian of the system is given by

$$H=H_\Phi+\frac{2q(D-2)a^2}{p-1}H_Y-\frac{2p\Delta}{p-1}H_X. \tag{51}$$

It is clear now from the structure of the Hamiltonian that the wave function $\Psi(X,\Phi,Y)$ will have the following form:

$$\Psi(X,\Phi,Y)=\Psi_X\Psi_\Phi e^{iP_YY}, \tag{52}$$

where Ψ_X and Ψ_Φ depend on X and Φ, respectively. Following our previous discussion, there is an intertwining operator which will convert the interacting Hamiltonian H to a sum of free Hamiltonians. It is given by

$$C=\mathcal{P}_{(p-1)\sinh X}P_X^{-1}\mathcal{I}_X\mathcal{P}_{\ln X}\mathcal{P}_{\sqrt{\Delta}\lambda\sinh\Phi}P_\Phi^{-1}\mathcal{I}_\Phi\mathcal{P}_{\ln\Phi}. \tag{53}$$

Its action on the Hamiltonian is

$$CHC^{-1}=\widetilde{H}_\Phi+\frac{2q(D-2)a^2}{p-1}H_Y-\frac{2p\Delta}{p-1}\widetilde{H}_X. \tag{54}$$

It is now easy to read off the action of C on the wave functions:

$$\Psi(X,Y,\Phi)=\frac{1}{\sqrt{2\pi}}N_{k_X}N_{k_\Phi}K_{i(p-1)k_X}(e^X)K_{i\sqrt{\Delta}\lambda k_\Phi}(e^\Phi)e^{ik_YY}, \tag{55}$$

where N_{k_X} and N_{k_Φ} are momentum-dependent normalization constants which can be determined from Eq. (49).

So far, we have been discussing the case $k=1$. Following similar arguments, we can also study the WDW wave function for an open universe, for which $k=-1$. In this case, the analog of Eq. (47) is

$$\mathcal{P}_{\ln X}: \quad X\to \ln X, \quad P_X\to XP_X,$$

$$\mathcal{I}_X: \quad X\to P_X, \quad P_X\to -X,$$

$$P_X^{-1}: X\to P_X^{-1}XP_X, \quad P_X\to P_X,$$

$$\mathcal{P}_{(p-1)\cosh X}: \quad X\to (p-1)\cosh X, \tag{56}$$

$$P_X\to \frac{P_X \sinh^{-1} X}{p-1}.$$

The operator C is now

$$C_X=\mathcal{P}_{(p-1)\cosh X}P_X^{-1}\mathcal{I}\mathcal{P}_{\ln X}, \tag{57}$$

whose action on the wave functions can be evaluated using similar methods to those decsribed in [20], which we used above in the $k=1$ case.

We shall not discuss the $k=0$ case in detail. Following the above discussion, the structure of the wave function is also easily obtained in this case.

Consider now the multi-charge cosmological models discussed in Sec. II B. As we saw, by proper choice of variables the Hamiltonian can be brought to the form of a sum of $N+1$ Liouville equations, together with a free part, as given in Eq. (27). Thus, following the above discussion, one can immediately construct the quantum intertwining operator C for this case. For $k=1$, it is

$$C=\mathcal{P}_{(p-1)\sinh X}P_X^{-1}\mathcal{I}_X\mathcal{P}_{\ln X}\prod_\alpha \mathcal{P}_{\sqrt{\Delta\lambda}\sinh \Phi_\alpha}P_{\Phi_\alpha}^{-1}\mathcal{I}_{\Phi_\alpha}\mathcal{P}_{\ln \Phi_\alpha}. \tag{58}$$

The action of C on the wave function Ψ is again easily read off:

$$\Psi(X,Y,\Phi)=\frac{1}{\sqrt{2\pi}}N_{k_X}K_{i(p-1)k_X}(e^X)e^{ik_YY} \times\prod_\alpha N_{k_{\Phi_\alpha}}K_{i\sqrt{\Delta}\lambda k_{\Phi_\alpha}}(e^{\Phi_\alpha}), \tag{59}$$

and the normalization constants N_{k_X} and $N_{k_{\Phi_\alpha}}$ can be determined from Eq. (49).

We shall not discuss the $k=-1$ and $k=0$ cases seperately here, since the wave functions can be obtained easily by following the previous discussion. We should like to mention here that for the case of the $SL(3,R)$ Toda model, the intertwining operator can also be constructed, by generalizing the transformation of Eq. (34) at the operator level [20]. The corresponding wave function can also be computed.

B. Analysis of WDW wave functions

Here we analyze some of the solutions of the WDW equation discussed in the previous subsection, by imposing proper boundary conditions on the wave functions.

In order to study the solutions of the WDW equation obtained above, we first note that it is necassary to specify the intial boundary conditions. When we look at the classical cosmological solutions given in Sec. II, we see from Eqs. (12)–(14), that one has to specify p, $\tilde{P}_X$, Δ, a, $\tilde{P}_\Phi$ and $\tilde{P}_Y$. Furthermore, we have to choose the value $k=-1$, 0, or 1. We shall present two specific cases to illustrate how we can obtain explicit solutions, and then discuss their properties.

Let us consider the string effective action in $D=10$, such that $p=3$, $q=6$, $a^2=1$ and $\Delta=4$. Furthermore, we look at the magnetic sector of this NS-NS case, and so $\epsilon=-1$; see Eqs. (1)–(9) for definitions of the parameters we specified above. We shall take $k=0$ from now on. The solutions correspond to

$$X=-\tilde{P}_Xt, \quad Y=-\tilde{P}_Yt, \quad e^{-2\phi}=\frac{\lambda}{\tilde{P}_\Phi}\cosh(\tilde{P}_\Phi t+\gamma), \tag{60}$$

where ϕ is the dilaton. Note that for the case at hand, the coupling constant of the theory is identified to be $g_s=e^{-\phi}$. We recall that the integration constants satisfy the constraint

$$\tilde{P}_\Phi^2=3\tilde{P}_X^2-12\tilde{P}_Y^2 \tag{61}$$

as is evident from Eq. (15). Instead of examining the two-parameter problem, (note that we can set $\gamma=0$ without any loss of generality and keep λ as an arbitrary parameter) let us look at two cases separately with specific choices for the value of the parameters.

CASE I:

Let us first consider the case when $\tilde{P}_Y=0$ and $\tilde{P}_X<0$. Then it follows that $\tilde{P}_X=-(1/\sqrt{3})\tilde{P}_\Phi$ and $U=3A+6B$, where

$$e^{(8/3)A}=\frac{\lambda}{\tilde{P}_\Phi}\cosh(\tilde{P}_\Phi t)e^{(4/3)X},$$

$$e^{-8B}=\frac{\lambda}{\tilde{P}_\Phi}\cosh(\tilde{P}_\Phi t)e^{(4/3)X}. \tag{62}$$

Now, let us examine the behavior of e^U, e^A and e^B, for $t\to\pm\infty$,

$$e^U\to e^{(3/8)\tilde{P}_\Phi|t|+(1/\sqrt{3})t}. \tag{63}$$

We see that as $t\to+\infty$, $e^U\to+\infty$ and as $t\to-\infty$, $e^U\to 0$. We can define a comoving time as follows: $d\tau=e^Udt$. Therefore, $0\leqslant\tau\leqslant\infty$ since $-\infty\leqslant t\leqslant\infty$. We can define two scale factors, R_A and R_B, respectively, as $R_A=e^A$ and $R_B=e^B$. Notice that for large τ $R\to\tau^\alpha,0<\alpha<0$. We also note that for $\tau\to 0$ and $\tau\to\infty$ this scale factor tends to ∞. The other scale factor R_B tends to zero in these two limits. Note that $R_A'\geqslant 0$ and $R_B'\leqslant 0$ in this case. Since $e^{-\phi}$ is the coupling constant for this magnetically-charged case, we see that for $\tau=0$ and $\tau=\infty$, we end up in the strong coupling phase.

Let us look at the wave function obtained from the solution of the WDW equation. The Bessel functions of our choice are

$$\Psi(X,Y,\Phi)=N_X N_Y N_\Phi e^{iP_X X} K_{2i\lambda P_\Phi}(e^\Phi), \qquad (64)$$

where the normalization constants are determined from Eq. (49) as usual. We recall that for $t\to\pm\infty$, $\Phi\to 0$; and from the relation between Φ and dilaton ϕ, we also know that the coupling constant $e^{-\phi}$ diverges in this limit. The wave function

$$\Psi(X,Y,\Phi)=N_X N_Y N_\Phi \Gamma(2i\lambda P_\Phi) e^{2i\lambda P_\Phi \ln 2} e^{P_X X} e^{-2i\lambda P_\Phi \Phi} \qquad (65)$$

is obtained in the limit when the scale factor A tends to large values and Φ tends to zero.

CASE II:

Let us consider another interesting case when P_Y is nonzero and $P_\Phi=6P_Y$. The constraint equation implies that $P_X=\pm 4P_Y$, for which we shall choose the plus sign. Note that for this choice we have

$$e^A\to e^{5P_Y t}, \quad e^B\to e^{-P_Y t}, \quad e^U\to e^{(21/2)\,t}, \qquad (66)$$

as $t\to\infty$. In the limit of $t\to-\infty$, we have

$$e^A\to e^{(1/2)\,P_Y t}, \quad e^B\to e^{(1/2)\,P_Y t}, \qquad (67)$$

and it is easy to see that R'_A is positive and R''_A is negative. When we consider the wavefunction for this case, namely $\Psi(X,Y,\Psi)$, in the limit when $\Phi\to 0$, it has the form

$$\Psi(X,Y,\Phi)=N_X N_Y N_\Phi e^{4iP_Y X} e^{iP_Y} e^{12i\lambda P_Y \ln 2} \times\Gamma(12i\lambda P_Y) e^{-12i\lambda P_Y \Phi}. \qquad (68)$$

CASE III:

Now we consider the situation when there is non-zero spatial curvature, corresponding to $k=-1$. For the sake of simplicity, let us choose $P_Y=0$ and therefore, from the constraint equation, we have $P_X=\pm(1/\sqrt{3})\,P_\Phi$. We shall again choose the plus sign. It follows from the solutions of the equations of motion that e^X has a singularly at $t=0$. If we extract the two scale parameters: $R_A=e^A$ and $R_B=e^B$ we find that R_A is singular at $t=0$ and R_B is not. Furthermore, $\Phi\to$constant as $t\to 0$. Now let us look at the $t\to\infty$ limit. We find in this case that $R_A\to\infty$, $R_B\to 0$ and $\Phi\to 0$. We can extract the wave function in these two limits from the behavior of the Bessel functions.

When $t\to 0$, the wave function takes the form

$$\Psi(X,Y,\Phi)=N_X N_Y N_\Phi \sqrt{\frac{\pi}{2Z}}\, e^{-Z} K_{i2\lambda P_\Phi}(e^\Phi), \qquad (69)$$

where $Z=e^X$. Note that in this limit the argument of the Bessel function takes a finite value. In the other limit, i.e. $t\to 0$, we find that e^X and Φ tend to zero, and so we must take limits in both of the Bessel functions. Thus the wave function is

$$\Psi(X,Y,\Phi)=N_X N_Y N_\Phi e^{2i\,(1/\sqrt{3})\,P_\Phi \ln 2},$$
$$\Gamma\left(i\frac{2}{\sqrt{3}}P_\Phi\right) e^{2i\lambda P_\Phi \ln 2},$$
$$\Gamma(2i\lambda P_\Phi) e^{2i\lambda P_\Phi \Phi}. \qquad (70)$$

IV. GENERALIZED BRANS-DICKE MODEL

In the cosmological context, one of the extensively discussed models is the Brans-Dicke (BD) model [22]. Although at late times it reduces to Einstein's gravity, at early times its behavior is very different. This is essentially because of the presence of a scalar which couples to the metric in a non-trivial way. Here, we are interested in a similar model, but extended to include different form fields. A subclass of such models has previously been analyzed, for example in [23–25]. The action for the theory is given by

$$S=\int d^D x\sqrt{\tilde{g}}\, e^{-\tilde{\phi}}\left[\tilde{R}-\omega(\partial\tilde{\phi})^2+\frac{1}{2n!}e^{c\tilde{\phi}}\tilde{F}_n^2\right], \qquad (71)$$

where the constant ω is known as the BD parameter.

We would like to show here that starting from the action (1), and then properly rescaling the metric and redefining various fields, we end up with an action of the form (71). Thus the solutions of the classical equations of motion of the generalized BD theories can simply be obtained from our analysis in Sec. II. Furthermore, the solutions of the WDW equation in the mini-superspace will follow simply from those in Sec. III, after making the necessary field redefinitions.

We begin with the action (1), which is written in the Einstein frame. After rescaling the dilaton according to

$$\phi=\sqrt{2\left(\frac{D-1}{D-2}+\omega\right)}\,\tilde{\phi}, \qquad (72)$$

we get

$$S=\int d^D x\sqrt{g}\left[R-\left(\frac{D-1}{D-2}+\omega\right)(\partial\tilde{\phi})^2+\frac{1}{2n!}e^{b\tilde{\phi}}F_n^2\right]. \qquad (73)$$

Here ω is a constant and $b=a\sqrt{[(D-1)/(D-2)+\omega]}$. Now, after the further metric rescaling

$$\tilde{g}_{\mu\nu}=e^{2\tilde{\phi}/(D-2)}g_{\mu\nu}, \qquad (74)$$

we get Eq. (71), with $c=b+2(n-1)/(D-2)$.

As mentioned earlier, the action (71) has been analyzed for $D=4$ and $n=3$ in [23–25], and its cosmological properties have been studied. However, following our previous discussion, it is immediate that cosmological solutions of Eq. (71) can be obtained by simply taking the solutions of Sec. II and scaling the fields as in Eqs. (72) and (74). As the procedure is straightforward, we will not carry it out here. However, we should like to make the following comments. In most of the solutions, the dilaton field ϕ becomes singular at some value of the proper time. Moreover, since the field scalings in Eq. (74) involve the dilaton, the cosmological properties of the Brans-Dicke (BD) metric will be considerably different from those seen in Sec. II.

V. CONCLUSIONS

In this paper, we have analyzed some of the cosmological models arising as solutions of the low-energy effective actions of string theories or M-theory. These solutions involve time-dependent metric tensor, dilaton, and antisymmetric tensor fields. The *Ansatz* and conventions of Ref. [12] were used in constructing general solutions. As is well known, the maximal supergravities in D dimensions, which arise from the toroidal compactification of 11-dimensional supergravity, have E_{11-D} global symmetries. We showed that in certain cases the equations of motion can be cast in the form of the one-dimensional E_N Toda equations. This was demonstrated explicitly for the maximal E_8 case.

We then studied some of the simpler cosmological models (which are related to the Liouville equation) at the quantum level, by obtaining the solutions of the Wheeler-DeWitt equation in a minisuperspace approximation. The WDW solutions were obtained using the techniques described in [20]. This involves constructing a quantum-mechanical canonical transformation that maps the Liouville theory to a free theory. The Liouville wave functions are then obtained from the free wave functions by means of intertwining operators. We presented solutions of the WDW equations in several cases, where we discussed the initial boundary conditions and studied the evolution of the wave functions. We showed how the wave functions relate different domains of the theory, and we studied their asymptotic behavior at large times. We also briefly discussed generalized Brans-Dicke theories including higher-degree field strengths, and showed how they can be related to the our framework. Finally, in the appendix, we studied the E_N Toda equations for $(D-3)$-brane solitions, which are closely related to the E_N cosmological models, and showed how they may be solved in the extremal limit. The case of the $D_4=O(4,4)$ Toda equations was presented explicitly.

ACKNOWLEDGMENTS

The work of C.N.P. is supported in part by DOE grant DE-FG03-95ER40917 and S.M is supported by NSF grant PHY-9411543. J.M. would like to thank Professors C. Bachas and A. Chakrabarti for their gracious hospitality and he would like to acknowledge Jawaharlal Nehru Memorial Fund for financial support.

APPENDIX: EXTREMAL E_N TODA p-BRANES

In Sec. II, we showed that we can obtain cosmological solutions supported by one-form field strengths, whose equations of motion can be cast into E_N Toda equations. One-form field strengths can also support electric instanton or magnetic $(D-3)$-branes. In this appendix, we shall discuss extremal E_N Toda instantons or $(D-3)$-branes using the same set of field strengths discussed in Sec. II. (Note that the non-extremal p-branes are equivalent to cosmological solutions that we discussed in Sec. II, after certain Wick rotations are performed [12], and we shall not consider these here.) The Lagrangian is given by Eq. (37). We shall first consider magnetic $(D-3)$ branes, with the standard metric and field strength *Ansätze*

$$ds^2=\eta_{\mu\nu}dx^\mu dx^\nu+2^{B(r)}(dr^2+r^2d\theta^2),$$

$$\chi_\alpha=4Q_\alpha\theta. \tag{A1}$$

The extremal condition implies that $B=-1/2\Sigma_\alpha\Phi_\alpha$, with $\Phi_\alpha=-2\Sigma_\beta(M^{-1})_{\alpha\beta}\varphi_\beta$. The equations of motion can be then written as

$$q''_\alpha=\exp\left(\frac{1}{2}\sum_{\alpha,\beta}M_{\alpha\beta}q_\beta\right), \tag{A2}$$

with extremality implying that its Hamiltonian

$$\mathcal{H}=4\sum_{\alpha,\beta}(M^{-1})_{\alpha\beta}p_\alpha p_\beta-\sum_\alpha\exp\left(\frac{1}{2}\sum_\beta M_{\alpha\beta}q_\beta\right) \tag{A3}$$

vanishes. Here a prime denotes a derivative with respect to $\rho=\log r$.

Note that the positive sign on the right-hand side of Eq. (A2) is the opposite of that in the cosmological equations that we discussed previously. Correspondingly, there is a minus sign in the second term in the Hamiltonian (A3), whereas in the cosmological case, the Hamiltonian (43) is positive definite. This means that unlike in the cosmological case, here we can find simple but non-trivial solutions for which the Hamiltonian vanishes [7]. These are in fact extremal p-branes, for which the functions e^{-q_α} can be expanded in terms of polynomials in ρ:

$$e^{-q_\alpha}=\sum_{m=0}^{n_\alpha}a_{\alpha m}\rho^m. \tag{A4}$$

The integrability of the Toda equations implies that the above series have only finite degrees n_α, which we shall now determine. To do this, we first consider the following simple *Ansatz* for a particular, special solution:

$$e^{-q_\alpha}=c_\alpha H^{n_\alpha}, \tag{A5}$$

where the c_α are constants and $H=1+c\rho$ is a single ''harmonic'' function. Thus we have $q''_\alpha=n_\alpha c^2H^{-2}$, and so by substituting into the Toda equations (A2), we find that they are all satisfied provided that the exponents n_α in Eq. (A5) are chosen to be

$$n_\alpha=4\sum_\beta(M^{-1})_{\alpha\beta}, \tag{A6}$$

and that the constants c_α are chosen appropriately. Thus in this special case we see that the highest powers n_α of ρ in the polynomials (A5) are given by Eq. (A6). In fact this special solution corresponds to choosing the charges Q_α to occur in a certain fixed ratio for which the solution reduces to a single-scalar one [26, 27]. More generally, if we relax this latter condition, we get solutions of the form (A4) for which the number of free parameters is equal to the number of charges [7]. The degrees n_α of the polynomials continue to be given by Eq. (A6). The charges Q_α are related to the values $q_\alpha(0)$ of the variables q_α at $\rho=0$ (i.e. at spatial infinity) by [7]

$$e^{q_\alpha(0)}=\prod_\beta (4Q_\beta)^{4(M^{-1})_{\alpha\beta}}. \tag{A7}$$

In the case of E_8, the Toda equations (A2) are given by

$$q_1''=e^{2q_1-q_4}, \qquad q_2''=e^{2q_2-q_3},$$
$$q_3''=e^{-q_2+2q_3-q_4}, \quad q_4''=e^{-q_1-q_3+2q_4-q_5},$$
$$q_5''=e^{-q_4+2q_5-q_6}, \quad q_6''=e^{-q_5+2q_6-q_7}, \tag{A8}$$
$$q_7''=e^{-q_6+2q_7-q_8}, \quad q_8''=e^{-q_7+2q_8}.$$

From Eq. (A6), we find that the degrees n_α of the polynomials for e^{-q_α} in this case will be $n_\alpha=\{136,92,182,270,220,168,114,58\}$. Solving for the 1248 coefficients $a_{\alpha m}$ in terms of the eight independent parameters associated with the eight charges Q_α is mechanical, but somewhat involved. Instead, we shall just present the results for a subclass of solutions, where we truncate the system to the $D_4=O(4,4)$ subalgebra with simple roots $\{\vec{a}_{123},\vec{b}_{23},\vec{b}_{34},\vec{b}_{45}\}$ (see Table I). This is a new solution that lies outside the A_N solutions obtained in [7]. For the D_4 truncation, we find

$$e^{-q_1}=c_1c_3-\frac{1}{8}c_3^6-36c_2c_4+6c_3^3c_4-36c_4^2+c_1\rho+\left(3c_2c_3-\frac{1}{8}c_3^4\right)\rho^2+c_2\rho^3+\frac{1}{24}c_3^2\rho^4+\frac{1}{60}c_3\rho^5+\frac{1}{360}\rho^6,$$

$$e^{-q_3}=c_1c_3-36c_2^2+3c_2c_3^3-\frac{1}{8}c_3^6-36c_2c_4+3c_3^3c_4+(c_1-3c_2c_3^2+3c_3^2c_4)\rho$$
$$+\left(3c_3c_4-\frac{1}{8}c_3^4\right)\rho^2+c_4\rho^3+\frac{1}{24}c_3^2\rho^4+\frac{1}{60}c_3\rho^5+\frac{1}{360}\rho^6,$$

$$e^{-q_5}=c_1c_3-6c_2c_3^3+\frac{3}{8}c_3^6+36c_2c_4-3c_3^3c_4+\left(c_1-6c_2c_3^2+\frac{1}{2}c_3^5-3c_3^2c_4\right)\rho$$
$$-\left(3c_2c_3-\frac{3}{8}c_3^4+3c_3c_4\right)\rho^2-\left(c_2+c_4-\frac{1}{6}c_3^3\right)\rho^3+\frac{1}{24}c_3^2\rho^4+\frac{1}{60}c_3\rho^5+\frac{1}{360}\rho^6, \tag{A9}$$

$$e^{-q_4}=c_1^2-6c_1c_2c_3^3+\frac{1}{4}c_1c_3^5+\frac{3}{4}c_2c_3^7-\frac{1}{32}c_3^{10}+216c_2^2c_3c_4-45c_2c_3^4c_4+\frac{3}{2}c_3^7c_4+216c_2c_3c_4^2-9c_3^4c_4^2$$
$$+\left(\frac{3}{4}c_2c_3^6-\frac{1}{4}c_1c_3^4+216c_2^2c_4-36c_2c_3^3c_4+216c_2c_4^2\right)\rho$$
$$+\left(18c_2^2c_3^2-\frac{1}{2}c_1c_3^3-\frac{3}{2}c_2c_3^5+\frac{3}{32}c_3^8+18c_2c_3^2c_4-3c_3^5c_4+18c_3^2c_4^2\right)\rho^2$$
$$+\left(12c_2^2c_3-\frac{1}{2}c_1c_3^2-\frac{1}{2}c_2c_3^4+\frac{1}{24}c_3^7+12c_2c_3c_4-2c_3^4c_4+12c_3c_4^2\right)\rho^3$$
$$+\left(3c_2^2+3c_4^2-\frac{1}{4}c_1c_3+\frac{1}{4}c_2c_3^3+3c_2c_4-\frac{1}{2}c_3^3c_4\right)\rho^4+\left(-\frac{1}{20}c_1+\frac{3}{20}c_2c_3^2+\frac{1}{240}c_3^5\right)\rho^5$$
$$+\frac{7}{720}c_3^4\rho^6+\frac{1}{180}c_3^3\rho^7+\frac{1}{480}c_3^2\rho^8+\frac{1}{2160}c_3\rho^9+\frac{1}{21600}\rho^{10},$$

where c_i $(i=1,2,3,4)$ are arbitrary constants.

The relation between the four arbitrary constants in Eq. (A9) and the four independent charges in the D_4 Toda extremal p-brane solutions is given by Eq. (A7). The mass of the solution is expressible in terms of the charges *via* a polynomial equation whose degree is equal to the dimension of the Weyl group of the Lie algebra characterizing the Toda equations [7]. In the case of the D_4 example above, this means that there will be a polynomial of degree 192 relating the mass and the charges. Obtaining the relation between the mass and the eight charges of the E_8 Toda p-branes would require a more challenging calculation, since the polynomial is of degree 696,729,600.

[1] G. W. Gibbons and P. K. Townsend, Nucl. Phys. **B282**, 610 (1987).
[2] R. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 319 (1988).
[3] B. Campbell, A. Linde, and K. Olive, Nucl. Phys. **B355**, 146 (1991).
[4] A. A. Tseytlin, Int. J. Mod. Phys. D **1**, 111 (1992).
[5] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).
[6] E. J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D **50**, 4868 (1994).
[7] H. Lü and C. N. Pope, Int. J. Mod. Phys. A **12**, 2061 (1997).
[8] K. Behrndt and S. Förste, Nucl. Phys. **B430**, 441 (1994).
[9] A. Lukas, B. A. Ovrut, and D. Waldram, Phys. Lett. B **393**, 65 (1997).
[10] N. Kaloper, Phys. Rev. D **55**, 3394 (1997).
[11] H. Lü, S. Mukherji, C. N. Pope, and K.-W. Xu, Phys. Rev. D **55**, 7926 (1997).
[12] H. Lü, S. Mukherji, and C. N Pope, hep-th/9612224.
[13] R. Poppe and S. Schwager, Phys. Lett. B **393**, 51 (1997).
[14] F. Larsen and F. Wilczek, Phys. Rev. D **55**, 4591 (1997).
[15] A. Lyons and S. W. Hawking, Phys. Rev. D **44**, 3802 (1991).
[16] M. Gasperini, J. Maharana, and G. Veneziano, Nucl. Phys. **B427**, 349 (1996).
[17] J. Maharana, S. Mukherji, and S. Panda, Mod. Phys. Lett. A **12**, 447 (1997).
[18] R. Brustein and G. Veneziano, Phys. Lett. **B329**, 429 (1994).
[19] A. A. Kehagias and A. Lukas, Nucl. Phys. **B477**, 549 (1996).
[20] A. Anderson, B. E. W. Nilsson, C. N. Pope, and K. S. Stelle, Nucl. Phys. **B430**, 107 (1994).
[21] E. Cremmer, B. Julia, H. Lü, and C. N. Pope, hep-th/9710119.
[22] J. Barrow and P. Parsons, Phys. Rev. D **55**, 1906 (1997), and references therein.
[23] S. Mukherji, Mod. Phys. Lett. A **12**, 639 (1997).
[24] J. E. Lidsey, gr-qc/9609063.
[25] E. Copeland, J. Lidsey, and D. Wands, Nucl. Phys. **B506**, 407 (1997).
[26] H. Lü, C. N. Pope, E. Sezgin, and K. S. Stelle, Nucl. Phys. **B456**, 669 (1995).
[27] H. Lü and C. N. Pope, Nucl. Phys. **B465**, 127 (1996).