Bounding the mass of the graviton using gravitational-wave observations of inspiralling compact binaries

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If gravitation is propagated by a massive field, then the velocity of gravitational waves (gravitons) will depend upon their frequency as $(v_g/c)^2 = 1-(c/f\lambda_g)^2$, and the effective Newtonian potential will have a Yukawa form $\propto r^{-1}\exp(-r/\lambda_g)$, where $\lambda_g = h/m_g c$ is the graviton Compton wavelength. In the case of inspiralling compact binaries, gravitational waves emitted at low frequency early in the inspiral will travel slightly slower than those emitted at high frequency later, resulting in an offset in the relative arrival times at a detector. This modifies the phase evolution of the observed inspiral gravitational waveform, similar to that caused by post-Newtonian corrections to quadrupole phasing. Matched filtering of the waveforms could bound such frequency-dependent variations in propagation speed, and thereby bound the graviton mass. The bound depends on the mass of the source and on noise characteristics of the detector, but is independent of the distance to the source, except for weak cosmological redshift effects. For observations of stellar-mass compact inspiral using ground-based interferometers of the LIGO-VIRGO type, the bound on λ_g could be of the order of 6×10^{12} km, about double that from solar-system tests of Yukawa modifications of Newtonian gravity. For observations of massive black hole binary inspiral at cosmological distances using the proposed Laser Interferometer Space Antenna (LISA), the bound could be as large as 6×10^{16} km. This is three orders of magnitude weaker than model-dependent bounds from galactic cluster dynamics. [S0556-2821(98)05104-2]

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I. INTRODUCTION

The detection of gravitational radiation by either laser interferometers or resonant cryogenic bars will, it is widely stated, usher in a new era of gravitational-wave astronomy [1]. Furthermore, according to conventional wisdom, it will yield new and interesting tests of general relativity (GR) in its radiative regime. These tests are generally based on three aspects of gravitational radiation: its back-reaction on the source, its polarization, and its speed.

(i) Gravitational back-reaction. This plays an important role only in the inspiral of compact objects. The equations of motion of inspiral include the nonradiative, nonlinear post-Newtonian corrections of Newtonian motion, as well as radiation back-reaction and its nonlinear post-Newtonian corrections. The evolution of the orbit is imprinted on the phasing of the inspiral waveform, to which broadband laser interferometers are especially sensitive through the use of matched filtering of the data against theoretical templates derived from GR. A number of tests of GR using matched filtering of binary inspiral have been proposed, including putting a bound on scalar-tensor gravity [2], measuring the nonlinear ‘‘tail term’’ in gravitational radiation damping [3], and testing the GR ‘‘no hair’’ theorems by mapping spacetime outside black holes [4,5]. A concrete test of gravitational back-reaction, albeit at the lowest order of approximation, has already been provided by the binary pulsar PSR 1913+16, where the tracer of the orbital phase was the radio emission from a pulsar rather than matched filtering of gravitational waves [6].

(ii) Polarization of gravitational waves. In GR, gravitational waves come in at most two polarization states, independently of the source, while in alternative theories of gravity, there are as many as six polarizations [7,8]. Using a sufficiently large number of gravitational antennas suitably oriented, it is possible to determine or limit the polarization content of an incident wave, and thereby to test theories. For example, should an incident wave be shown definitively to have three polarizations, the result would be devastating for GR. Although some of the details of implementing such polarization observations have been worked out for arrays of resonant cylindrical, disk-shaped, and spherical detectors [7,9], rather little has been done to assess whether the ground-based laser-interferometers [Laser Interometric Gravitational Wave Observatory (LIGO), VIRGO, GEO600, TAMA] could perform interesting polarization measurements. The results depend sensitively on the relative orientation of the detectors' arms, which are now cast (literally) in concrete.

(iii) Speed of gravitational waves. According to GR, in the limit in which the wavelength of gravitational waves is small compared to the radius of curvature of the background spacetime, the waves propagate along null geodesics of the background spacetime, i.e., they have the same speed, c, as light. In other theories, the speed could differ from c because of coupling of gravitation to ‘‘background’’ gravitational fields. For example, in the Rosen bimetric theory [10] with a flat background metric $\boldsymbol{\eta}$, gravitational waves follow null geodesics of $\boldsymbol{\eta}$, while light follows null geodesics of $\mathbf{g}$ [11,8].

Another way in which the speed of gravitational waves could differ from one is if gravitation were propagated by a massive field (a massive graviton), in which case, v_g would be given by, in a local inertial frame,

$$\frac{v_g^2}{c^2} = 1 - \frac{m_g^2 c^4}{E^2}, \tag{1.1}$$

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where m_g and E are the graviton rest mass and energy, respectively.

The most obvious way to test this is to compare the arrival times of a gravitational wave and an electromagnetic wave from the same event, e.g., a supernova. For a source at a distance D, the resulting value of the difference $1-v_g/c$ is

$$1-\frac{v_g}{c}=5\times10^{-17}\left(\frac{200\ \mathrm{Mpc}}{D}\right)\left(\frac{\Delta t}{1\ \mathrm{s}}\right), \tag{1.2}$$

where Δt is the "time difference," given by

$$\Delta t\equiv\Delta t_a-(1+Z)\Delta t_e, \tag{1.3}$$

where Δt_a and Δt_e are the differences in arrival time and emission time, respectively, of the two signals, and $Z \simeq DH_0/c$ is the redshift of the source, with H_0 the Hubble parameter. In many cases, Δt_e is unknown, so that the best one can do is employ an upper bound on Δt_e based on observation or modeling. The result will then be a bound on $1-v_g/c$.

If the frequency of the gravitational waves is such that $hf \gg m_g c^2$, where h is Planck's constant, then $v_g/c\approx 1-\frac{1}{2}(c/\lambda_g f)^2$, where $\lambda_g=h/m_g c$ is the graviton Compton wavelength, and the bound on $1-v_g/c$ can be converted to a bound on λ_g, given by

$$\lambda_g>3\times10^{12}\ \mathrm{km}\left(\frac{D}{200\ \mathrm{Mpc}}\frac{100\ \mathrm{Hz}}{f}\right)^{1/2}\left(\frac{1}{f\Delta t}\right)^{1/2}. \tag{1.4}$$

The foregoing discussion assumes that the source emits *both* gravitational and electromagnetic radiation in detectable amounts, and that the relative time of emission can be established (by one means or another) to sufficient accuracy, or can be shown to be sufficiently small.

However, there is a situation in which a bound on the graviton mass could be set using future observations of gravitational radiation alone. That is the case of the inspiralling compact binary. Because the frequency of the gravitational radiation sweeps from low frequency at the initial moment of observation to higher frequency at the final moment, the speed of the gravitons emitted will vary, from lower speeds initially to higher speeds (closer to c) at the end. This will cause a distortion of the observed phasing of the waves and result in a shorter than expected overall time Δt_a of passage of a given number of cycles. Furthermore, through the technique of matched filtering, the parameters of the compact binary can be measured accurately [12], and thereby the emission time Δt_e can be determined accurately. Roughly speaking, the "phase interval" $f\Delta t$ in Eq. (1.4) can be measured to an accuracy $1/\rho$, where ρ is the signal-to-noise ratio.

Thus we can estimate the bounds on λ_g achievable in principle for various compact inspiral systems, and for various detectors. For stellar-mass inspiral (neutron stars or black holes) observed by the LIGO-VIRGO class of ground-based interferometers, we have $D\approx 200$ Mpc, $f\approx 100$ Hz, and $f\Delta t\sim\rho^{-1}\approx 1/10$ [13]. The result is $\lambda_g>10^{13}$ km. For massive binary black holes (10^4 to $10^7 M_\odot$) observed by the proposed Laser Interferometer Space Antenna (LISA), it is expected that $D\approx 3$ Gpc, $f\approx 10^{-3}$ Hz, and $f\Delta t\sim\rho^{-1}\approx 1/1000$ [14]. The result is $\lambda_g>10^{17}$ km.

TABLE I. Bounds on λ_g from gravitational-wave observations of inspiralling compact binaries, using ground-based (LIGO-VIRGO) and space-based (LISA) observatories. Masses are in $M_\odot$.

m_1	m_2	Distance (Mpc)	Bound on λ_g (km)
Ground-based (LIGO-VIRGO)			
1.4	1.4	300	4.6×10^{12}
1.4	10	630	5.4×10^{12}
10	10	1500	6.0×10^{12}
Space-based (LISA)			
10^7	10^7	3000	6.9×10^{16}
10^6	10^6	3000	5.4×10^{16}
10^5	10^5	3000	2.3×10^{16}
10^4	10^4	3000	0.7×10^{16}

We have refined these crude estimates by explicit calculations using matched filtering (Table I). We first calculate the effect of the frequency-dependent massive graviton velocity on the observed gravitational-wave phasing. We assume that the evolution of the system, driven by gravitational back-reaction, is given correctly by general relativity, apart from corrections of fractional order $(r/\lambda_g)^2$, where r is the size of the binary system; these corrections can be shown to be negligible for the cases of interest. Including GR post-Newtonian (PN) and tail terms (1.5PN) in the phasing, and assuming circular orbits and nonspinning bodies, we determine the accuracy with which the parameters of the system can be measured ("chirp" mass of the system, reduced mass, fiducial phase, and fiducial time), and simultaneously find the accuracy with which the effect of a graviton mass can be bounded (effectively, we find an upper bound on λ_g^{-1}). We use noise curves appropriate for the advanced LIGO detectors, and for the proposed LISA observatory. It is interesting to note that, despite the apparent distance dependence in Eq. (1.4), the bound for a given system is independent of its distance, because the signal-to-noise ratio, which determines the accuracy of $f\Delta t$, is inversely proportional to distance. As a result, the bound on λ_g depends only on the measured masses of the objects and on detector characteristics. The only effect of distance is a weak Z dependence arising from cosmological effects. The results that could come from the two kinds of detectors for various sources are given in Table I. These correspond to bounds on the graviton rest mass of order 2.5×10^{-22} eV for ground-based, and 2.5×10^{-26} eV for space-based observations.

Can bounds be placed on λ_g using other observations or experiments? If the graviton is massive, then one expects that, in the nonradiative near zone of a body like the Sun, the gravitational potential will be modified from GM/r to the Yukawa form

$$V(r)=\frac{GM}{r}\exp(-r/\lambda_g). \tag{1.5}$$

Strictly speaking, such a conclusion would require a complete gravitational theory of a massive graviton, capable of making predictions both in the radiative and nonradiative regimes, and which otherwise agrees with observation. However, as several authors have pointed out [15–18], construction of such a theory is a nontrivial question. Thus, in the absence of a well-defined theoretical foundation, we shall make the phenomenological assumption that, if the graviton is massive in the propagation of gravitational waves, the Newtonian potential takes the form of Eq. (1.5), with the same value of λ_g.

With this assumption, one can place bounds on λ_g using solar-system dynamics. Essentially, the orbits of the inner planets agree with standard Newtonian gravity (including its post-Newtonian GR corrections) to an accuracy of order 10^{-8}. Since the observed corrections to Newtonian gravity in the limit $\lambda_g \gg r$ go as $(r/\lambda_g)^2$ (it is the acceleration, not the potential that is important), this implies a rough bound $\lambda_g > 10^4$ astronomical units, or 10^{12} km. Talmadge *et al.* [19] surveyed solar system data in the context of bounding the range and strength of a ''fifth force,'' a Yukawa term *added* to Newtonian gravity. The best bound comes from observations that verify Kepler's third law for the inner planets: from observations of Mars, we find $\lambda_g > 2.8\times 10^{12}$ km. Bounds from other planets are summarized in Table IV. Apart from the Yukawa potential assumption, this bound is solid and model independent.

Thus the bound inferred from gravitational radiation observations of stellar mass compact binary inspiral could be twice as large as the solar-system bound, while that from massive binary inspiral as observed by LISA could be 2×10^4 times larger.

Some have argued for a larger bound on λ_g from galactic and cluster dynamics [20,16,17], noting that the evidence of bound clusters and of clear tidal interactions between galaxies argues for a range λ_g at least as large as a few megaparsecs (6×10^{19} km). Indeed this is the value quoted by the Particle Data Group [21]. However, in view of the uncertainties related to the amount of dark matter in the universe, and the absence of a theory that can encompass a massive graviton and cosmology, these bounds should be viewed with caution.

The remainder of this paper provides the details underlying these results. In Sec. II, we study the propagation of a massive graviton in a cosmological background, to find the relation between emission interval and arrival interval. In Sec. III, using the standard ''restricted PN approximation,'' in which the gravitational waveform is expressed as an amplitude accurate to the lowest, quadrupole approximation, and a phase accurate through 1.5PN order $[O(v/c)^3]$ beyond the quadrupole approximation, we determine the effect of graviton propagation time on the Fourier transform of the waveform, which is the central ingredient in matched filtering. In Sec. IV, we calculate the Fisher information matrix and determine the accuracy with which the compact binary's parameters can be measured, including a bound on the effect of graviton mass. This approach is a reasonable approximation to real matched filtering for Gaussian noise and large signal-to-noise ratio. We apply the results to specific noise curves and binary systems appropriate for ground-based (LIGO-VIRGO) and space-based (LISA) detectors. Section V discusses bounds on the graviton mass using solar-system dynamics. Henceforth, we use units in which $G=c=1$.

II. PROPAGATION OF A MASSIVE GRAVITON

Because some of the detectable compact binaries could be at cosmological distances, we study the propagation of a massive graviton in a background Friedmann-Robertson-Walker (FRW) homogeneous and isotropic spacetime. We take the line element to have the form [22]

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2.1}$$

where $a(t)$ is the scale factor of the universe and $\Sigma(\chi)$ is equal to χ, $\sin\chi$ or $\sinh\chi$ if the universe is spatially flat, closed or open, respectively. For a graviton moving radially from emitter $\chi=\chi_e$ to detector $\chi=0$, it is straightforward to show that the component of 4-momentum $p_\chi =$ const. Using the fact that $m_g^2 = -p^\alpha p^\beta g_{\alpha\beta} = E^2 - a^{-2}p_\chi^2$, where $E=p^0$, together with $p^\chi/E = d\chi/dt$, we obtain

$$\frac{d\chi}{dt} = -\frac{1}{a}\left(1 + \frac{m_g^2 a^2}{p_\chi^2}\right)^{-1/2}, \tag{2.2}$$

where $p_\chi^2 = a^2(t_e)(E_e^2 - m_g^2)$. Assuming that $E_e \gg m_g$, expanding Eq. (2.2) to first order in $(m_g/E_e)^2$, and integrating, we obtain

$$\chi_e = \int_{t_e}^{t_a} \frac{dt}{a(t)} - \frac{1}{2}\frac{m_g^2}{a^2(t_e)E_e^2}\int_{t_e}^{t_a} a(t)dt. \tag{2.3}$$

Consider gravitons emitted at two different times t_e and t_e', with energies E_e and E_e', and received at corresponding arrival times (χ_e is the same for both). Assuming that $\Delta t_e \equiv t_e - t_e' \ll a/\dot{a}$, and noting that $m_g/E_e = (\lambda_g f_e)^{-1}$, where f_e is the emitted frequency, we obtain, after eliminating χ_e,

$$\Delta t_a = (1+Z)\left[\Delta t_e + \frac{D}{2\lambda_g^2}\left(\frac{1}{f_e^2} - \frac{1}{f_e'^2}\right)\right], \tag{2.4}$$

where $Z \equiv a_0/a(t_e) - 1$ is the cosmological redshift, and

$$D \equiv \frac{(1+Z)}{a_0}\int_{t_e}^{t_a} a(t)dt, \tag{2.5}$$

where $a_0 = a(t_a)$ is the present value of the scale factor. Note that D is not a conventional cosmological distance measure, like the luminosity distance $D_L \equiv a_0\Sigma(\chi_e)(1+Z)$, or the proper distance $D_P \equiv a_0\chi_e$. For $Z \ll 1$, it is given by the standard formula $D = Z/H_0$; for a matter dominated, spatially flat universe, D and D_L are given by

$$D = (2/5H_0)(1+Z)(1-(1+Z)^{-5/2}), \tag{2.6a}$$

$$D_L = (2/H_0)(1+Z)(1-(1+Z)^{-1/2}). \tag{2.6b}$$

The ratio D/D_L will play a role in our analysis of the bound on λ_g. It has the following representative behavior:

$$\frac{D}{D_L} = \begin{cases} 1 - Z + O(Z^2), & Z \ll 1, \text{ all } \Omega_0 \\ \dfrac{1 + (2+Z)(1+Z+\sqrt{1+Z})}{5(1+Z)^2}, & \Omega_0 = 1, \text{ all } Z \end{cases} \tag{2.7}$$

where Ω_0 is the density parameter. At $Z=1$, the factor D/D_L varies from 0.5 for $\Omega_0=0.01$ to 0.6 for $\Omega_0=2$. For simplicity, we shall henceforth assume that $\Omega_0\equiv 1$.

III. MASSIVE GRAVITON PROPAGATION AND THE PHASING OF GRAVITATIONAL WAVES

We shall treat the problem of a binary system of compact bodies of locally measured masses m_1 and m_2 in a quasicircular orbit, that is an orbit which is circular apart from an adiabatic inspiral induced by gravitational radiation reaction within GR. We ignore tidal interactions and spin effects. For matched filtering of gravitational waves using LIGO-VIRGO or LISA type detectors, it is sufficient for our purpose to write the gravitational waveform $h(t)$ in the ''restricted post-Newtonian form'' [23,24,12], in terms of an amplitude $A(t)$ expressed to the lowest, quadrupole approximation, and a phase $\Phi(t)$, expressed as a post-Newtonian expansion several orders beyond the quadrupole approximation,

$$h(t)\equiv A(t)e^{-i\Phi(t)}, \tag{3.1a}$$

$$\Phi(t)\equiv\Phi_c+2\pi\int_{t_c}^{t} f(t)dt, \tag{3.1b}$$

where $f(t)$ is the observed frequency of the waves, and Φ_c and t_c are ''fiducial'' phase and time respectively. The amplitude A is given by

$$A(t)=\frac{2\mu}{a_0\Sigma(\chi_e)}\frac{m}{r(t)}F(i,\theta,\phi,\psi), \tag{3.2}$$

where $m\equiv m_1+m_2$ and $\mu\equiv m_1m_2/m$ are the total and reduced mass of the system (we also define the reduced mass parameter $\eta\equiv\mu/m$), $r(t)$ is the orbital separation, and F is an angular function related to the orientation of the orbit (angles i, ψ) and the direction of the source relative to the antenna (angles θ, ϕ), given by

$$F^2(i,\theta,\phi,\psi)=\frac{1}{4}(1+\cos^2 i)^2F_+^2+\cos^2 iF_\times^2, \tag{3.3}$$

where $F_+(\theta,\phi,\psi)$ and $F_\times(\theta,\phi,\psi)$ are beam pattern factors quoted, for example in Eqs. (104) of [1]. For simplicity, we shall average over all four angles, and use the fact that $\langle F^2\rangle=4/25$.

We next compute the Fourier transform of $h(t)$. Expanding $h(t)$ about the time $\tilde{t}$ at which the observed frequency is $\tilde{f}$, i.e., $f(\tilde{t})\equiv\tilde{f}$, and using the stationary-phase approximation, we obtain

$$\tilde{h}(\tilde{f})=\frac{A(\tilde{t})}{\sqrt{\dot{f}(\tilde{t})}}e^{i\Psi(\tilde{f})}, \tag{3.4}$$

where

$$A(\tilde{t})=\frac{4}{5}\frac{\mathcal{M}_e}{a_0\Sigma(\chi_e)}(\pi\mathcal{M}_e\tilde{f}_e)^{2/3}, \tag{3.5a}$$

$$\Psi(\tilde{f})=2\pi\int_{f_c}^{\tilde{f}}(t-t_c)df+2\pi\tilde{f}t_c-\Phi_c-\pi/4, \tag{3.5b}$$

where $\mathcal{M}_e=\eta^{3/5}m$ is the ''chirp'' mass of the emitter, and where we have used the Newtonian relation $m/r(\tilde{t})=(\pi m\tilde{f}_e)^{2/3}$. The subscript ''e'' denotes ''at the emitter.'' We next substitute Eq. (2.4) into (3.5b) to relate the time at the detector to that at the emitter, noting that, because of the cosmological redshift, $f_e=(1+Z)f$. The result is

$$\Psi(\tilde{f})=2\pi\int_{\tilde{f}_{ec}}^{\tilde{f}_e}(t_e-t_{ec})df_e-\frac{\pi D}{f_e\lambda_g^2}+2\pi\tilde{f}\,\overline{t_c}-\overline{\Phi_c}-\frac{\pi}{4}, \tag{3.6}$$

where $\overline{t_c}=t_c-D/[2(1+Z)\lambda_g^2f_c^2]$, and $\overline{\Phi_c}=\Phi_c-2\pi D/[(1+Z)\lambda_g^2f_c]$. To find t_e-t_{ec} as a function of f_e, we integrate the equation for radiation reaction between t_{ec} and t_e:

$$\frac{df_e}{dt_e}=\frac{96}{5\pi\mathcal{M}_e^2}(\pi\mathcal{M}_ef_e)^{11/3}\times\left[1-\left(\frac{743}{336}+\frac{11}{4}\eta\right)(\pi mf_e)^{2/3}+4\pi(\pi mf_e)\right], \tag{3.7}$$

where we have included the first post-Newtonian (PN) term and the 1.5PN ''tail'' term in the radiation-reaction equation (see, e.g., [23]). After absorbing further constants of integration into $\overline{t_c}$ and $\overline{\Phi_c}$, dropping the bars on those two quantities, and reexpressing everything in terms of the *measured* frequency $\tilde{f}$ [note that $(\dot{f})^{1/2}=(df_e/dt_e)^{1/2}/(1+Z)$], we obtain

$$\tilde{h}(\tilde{f})=\begin{cases}\tilde{A}(\tilde{f})e^{i\Psi(\tilde{f})}, & 0<\tilde{f}<\tilde{f}_{\rm max}\\ 0, & \tilde{f}>\tilde{f}_{\rm max}\end{cases} \tag{3.8a}$$

$$A(\tilde{f})\equiv\mathcal{A}\tilde{f}^{-7/6}=\sqrt{\frac{\pi}{30}}\frac{\mathcal{M}^2}{D_L}u^{-7/6}, \tag{3.8b}$$

$$\Psi(\tilde{f})=2\pi\tilde{f}t_c-\Phi_c-\pi/4+\frac{3}{128}u^{-5/3}-\beta u^{-1}+\frac{5}{96}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\eta^{-2/5}u^{-1}-\frac{3\pi}{8}\eta^{-3/5}u^{-2/3}, \tag{3.8c}$$

where $u\equiv\pi\mathcal{M}\tilde{f}$, and $\mathcal{M}$ is the ''measured chirp mass,'' related to the source chirp mass by a redshift: $\mathcal{M}=(1+Z)\mathcal{M}_e$. The parameter β is given by

$$\beta\equiv\frac{\pi^2D\mathcal{M}}{\lambda_g^2(1+Z)}. \tag{3.9}$$

The frequency $\tilde{f}_{\rm max}$ represents an upper cutoff frequency where the PN approximation fails. Equations (3.8a)–(3.8c) are the basis for an analysis of parameter estimation using matched filtering.

Before turning to matched filtering, we must address our approximation of the motion and gravitational radiation damping as being general relativistic up to corrections of order $(r/\lambda_g)^2$. In the radiation-reaction formula Eq. (3.7), we included corrections to the quadrupole formula at 1.5PN order, corresponding to corrections of order v^3. Thus our neglect of massive graviton effects amounts to assuming that $r^2\lambda_g^{-2}v^{-3} \ll 1$ for all systems of interest. Because $v^2 \simeq m/r$ for circular orbits, we can rewrite this condition as $(m/\lambda_g)v^{-5/2} \ll 1$. Since typically $v > 10^{-2}$ for all systems of interest, and $\lambda_g > 10^{12}$ km from solar-system bounds, this condition is easily satisfied.

IV. BOUNDS ON THE GRAVITON MASS USING MATCHED FILTERING

A. Matched-filter analysis

To obtain a more reliable estimate of the bound that can be placed on the graviton mass, we carry out a full matched-filter analysis following the method outlined for compact binary inspiral by Cutler and Flanagan [23] and Finn and Chernoff [24]. The details here parallel those of [25].

With a given noise spectrum $S_n(f)$, one defines the inner product of signals h_1 and h_2 by

$$(h_1|h_2) \equiv 2\int_0^\infty \frac{\tilde{h}_1^* \tilde{h}_2 + \tilde{h}_2^* \tilde{h}_1}{S_n(f)} df, \tag{4.1}$$

where $\tilde{h}_a$ is the Fourier transform of the waveform defined in Eqs. (3.8a)–(3.8c) (henceforth, we drop the tilde on frequencies). The signal-to-noise ratio for a given signal h is given by

$$\rho[h] \equiv S/N[h] = (h|h)^{1/2}. \tag{4.2}$$

If the signal depends on a set of parameters θ^a which are to be estimated by the matched filter, then the rms error in θ^a in the limit of large S/N is given by

$$\Delta\theta^a \equiv \sqrt{\langle(\theta^a - \langle\theta^a\rangle)^2\rangle} = \sqrt{\Sigma^{aa}}, \tag{4.3}$$

where Σ^{aa} is the corresponding component of the inverse of the covariance matrix or Fisher information matrix Γ_{ab} defined by

$$\Gamma_{ab} \equiv \left(\frac{\partial h}{\partial \theta^a} \middle| \frac{\partial h}{\partial \theta^b} \right). \tag{4.4}$$

The correlation coefficient between two parameters θ^a and θ^b is

$$c^{ab} \equiv \Sigma^{ab}/\sqrt{\Sigma^{aa}\Sigma^{bb}}. \tag{4.5}$$

We estimate the following six parameters, $\ln\mathcal{A}$, Φ_c, $f_0 t_c$, $\ln\mathcal{M}$, $\ln\eta$, and β, where f_0 is a frequency characteristic of the detector, typically a ''knee'' frequency, or a frequency at which $S_n(f)$ is a minimum. The corresponding partial derivatives of $\tilde{h}(f)$ are

$$\frac{\partial \tilde{h}(f)}{\partial \ln\mathcal{A}} = \tilde{h}(f), \tag{4.6a}$$

$$\frac{\partial \tilde{h}(f)}{\partial \Phi_c} = -i\tilde{h}(f), \tag{4.6b}$$

$$\frac{\partial \tilde{h}(f)}{\partial f_0 t_c} = 2\pi i (f/f_0)\tilde{h}(f), \tag{4.6c}$$

$$\frac{\partial \tilde{h}(f)}{\partial \ln\mathcal{M}} = -\left(\frac{5i}{128}u^{-5/3} + \frac{5i}{96}\gamma(\eta)u^{-1} - \frac{i\pi}{4}\eta^{-3/5}u^{-2/3} \right)\tilde{h}(f), \tag{4.6d}$$

$$\frac{\partial \tilde{h}(f)}{\partial \ln\eta} = \left(\frac{5i}{96}\eta\gamma'(\eta)u^{-1} + \frac{9i\pi}{40}\eta^{-3/5}u^{-2/3} \right)\tilde{h}(f), \tag{4.6e}$$

$$\frac{\partial \tilde{h}(f)}{\partial \beta} = -iu^{-1}\tilde{h}(f), \tag{4.6f}$$

where $\gamma(\eta) \equiv (743/336 + 11\eta/4)\eta^{-2/5}$, and $\gamma' \equiv d\gamma/d\eta$. Since we plan to derive the error in estimating β about the nominal or *a priori* GR value $\beta = 0$, we have set $\beta = 0$ in all the partial derivatives.

We assume that the detector noise curve can be approximated by an amplitude S_0, which sets the overall scale of the noise, and a function of the ratio $f/f_0 \equiv x$, which may include additional parameters, that is $S_n(f) = S_0 g_\alpha(x)$, where the subscript α denotes a set of parameters. Then from Eqs. (3.8a)–(3.8c) and (4.2) we find that the signal-to-noise ratio is given by

$$\rho = 2\mathcal{A} f_0^{-2/3}(I(7)/S_0)^{1/2} = \sqrt{\frac{2}{15}} \frac{\mathcal{M}^{5/6}}{D_L}(\pi f_0)^{-2/3}\left(\frac{I(7)}{S_0} \right)^{1/2}, \tag{4.7}$$

where we define the integrals

$$I(q) \equiv \int_0^\infty \frac{x^{-q/3}}{g_\alpha(x)} dx. \tag{4.8}$$

Note that any frequency cutoffs are to be incorporated appropriately into the endpoints of the integrals $I(q)$. If we define the coefficients $I_q \equiv I(q)/I(7)$, then all elements of the covariance matrix turn out to be proportional to ρ^2 times linear combinations of terms of the form $u_0^{-n/3} I_q$ for various integers n and q, where $u_0 = \pi\mathcal{M} f_0$. This overall ρ dependence is characteristic of the large S/N limit. As a result, the rms errors $\Delta\theta^a$ are inversely proportional to ρ, while the correlation coefficients are independent of ρ. Defining $\Delta\beta \equiv \Delta^{1/2}/\rho$, viewing $\Delta\beta$ as an upper bound on β, and combining this definition with Eqs. (3.9) and (4.7) we obtain the *lower* bound on λ_g:

$$\lambda_g > \left(\frac{2}{15} \frac{I(7)}{S_0} \right)^{1/4} \left(\frac{D}{(1+Z)D_L} \right)^{1/2} \frac{\pi^{2/3}\mathcal{M}^{11/12}}{f_0^{1/3}\Delta^{1/4}}. \tag{4.9}$$

TABLE II. The rms errors for signal parameters, the corresponding bound on λ_g, in units of 10^{12} km, and the correlation coefficients $c_{\mathcal{M}\eta}$, $c_{\mathcal{M}\beta}$ and $c_{\beta\eta}$. The noise spectrum is that of the advanced LIGO system, given by Eq. (4.10), and a signal-to-noise ratio of 10 is assumed. Masses are in units of $M_\odot$, Δt_c is in msec.

m_1	m_2	$\Delta\phi_c$	Δt_c	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$	λ_g	$c_{\mathcal{M}\eta}$	$c_{\mathcal{M}\beta}$	$c_{\beta\eta}$
1.4	1.4	4.09	1.13	0.034%	7.88%	4.6	−0.971	−0.993	0.992
1.4	10.0	6.24	2.04	0.191%	12.2%	5.4	−0.978	−0.994	0.994
10.0	10.0	9.26	3.53	1.42 %	57.3%	6.0	−0.983	−0.994	0.997

Note that the bound on λ_g depends only weakly on distance, via the Z dependence of the factor $[D/(1+Z)D_L]^{1/2}$, which varies from unity at $Z=0$ to 0.45 at $Z=1.5$. This is because, while the signal strength, and hence the accuracy, falls with distance, the size of the arrival-time effect increases with distance. Otherwise, the bound depends only on the chirp mass and on detector noise characteristics. We now apply this formalism to specific detectors.

B. Ground-based detectors of the LIGO-VIRGO type

The proposed advanced version of LIGO is expected to detect compact binary inspiral to distances of 200 Mpc to 1 Gpc. The sensitive frequency band extends from around 10 Hz to several hundreds of Hz. We adopt the benchmark advanced LIGO noise curve, given by

$$S_n(f)=\begin{cases}\infty, & f<10\ \text{Hz}\\ S_0[(f_0/f)^4+2+2(f/f_0)^2]/5, & f>10\ \text{Hz}\end{cases} \tag{4.10}$$

where $S_0=3\times10^{-48}\ \text{Hz}^{-1}$, and $f_0=70$ Hz. The cutoff at 10 Hz corresponds to seismic noise, while the f^{-4} and f^2 dependences denote thermal and photon shot noise, respectively [13]. We choose an upper cutoff frequency, where the PN approximation fails, corresponding to the innermost stable circular orbit. Although this is known rigorously only for test body motion around black holes [26], a conventional estimate is given by $f_{\text{ISCO}}\approx[6^{3/2}\pi(m_1+m_2)]^{-1}$. Converting this to the measured frequency and chirp mass, we have $x_{\max}=[6^{3/2}\pi\eta^{-3/5}\mathcal{M}f_0]^{-1}$. For this case, we thus have $g(x)=(x^{-4}+2+2x^2)/5$, and $I(q)=\int_{1/7}^{x_{\max}}[x^{-q/3}/g(x)]dx$. We then calculate and invert the covariance matrix and evaluate the errors in the five relevant parameters (the parameter $\ln\mathcal{A}$ decouples from the rest and is relevant only for the calculation of ρ), and the correlation coefficients between $\mathcal{M}$, η and β. For various ''canonical'' compact binary systems observable by advanced LIGO, the results are shown in Table II. Note that, in determining the bound on λ_g, we must include the Z dependence embodied in Eq. (4.9). To do so, we take our assumed value for signal-to-noise ratio $\rho=10$, determine the luminosity distance using Eq. (4.7), and convert that to a redshift using Eq. (2.6b), with an assumed value $H_0=50\ \text{km s}^{-1}\text{Mpc}^{-1}$ and $\Omega_0=1$. We then substitute it, along with Eq. (2.6a) into Eq. (4.9).

It is useful to compare these results to those from parameter estimation calculations using pure GR to 1.5PN order *including* spin-orbit effects (see e.g., [23,25]). There, an additional parameter related to the spin-orbit effect (also called β, with a nominal value of zero) was estimated, although it produced a different u-dependent term in the phasing formula ($u^{-2/3}$ instead of u^{-1}). Nevertheless, the errors in the fiducial phase $\Delta\Phi_c$, time Δt_c and chirp mass $\Delta\ln\mathcal{M}$ are virtually identical in both cases, and somewhat larger than if no additional β parameter were estimated (compare Table II with Table I and II of [23] or Table II of [25]). But in our case, the errors in the reduced mass parameter η are, surprisingly smaller, despite the nearly perfect correlation (u^{-1} dependence) between the 1PN term and the β-term in the phasing, Eq. (3.8c). The error grows dramatically with total mass because the smaller number of observed gravitational-wave cycles reduces the ability of the tail term ($\propto u^{-2/3}$) to break the degeneracy.

C. Space-based detectors of the LISA type

The proposed Laser Interferometer Space Antenna (LISA) is expected to be able to detect the inspiral of massive black hole binaries to cosmological distances, with very large signal-to-noise ratio. The sensitive frequency band extends from around 10^{-4} to 10^{-1} Hz, with a typical integration time of the order of one year. We adopt a noise curve described in

TABLE III. Rms errors on signal parameters, the bound on λ_g, in units of 10^{16} km, and the correlation coefficients. The noise spectrum is that of LISA including white-dwarf binary confusion noise, given by Eq. (4.13). Signal-to-noise ratio ρ is shown, corresponding to a luminosity distance of about 3 Gpc. Masses are in units of $M_\odot$, Δt_c is in sec.

m_1	m_2	ρ	$\Delta\phi_c$	Δt_c	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$	λ_g	$c_{\mathcal{M}\eta}$	$c_{\mathcal{M}\beta}$	$c_{\beta\eta}$
10^7	10^7	1600	0.073	20.0	0.0187%	0.562%	6.9	−0.979	−0.992	0.997
10^7	10^6	710	0.145	22.5	0.0119%	0.362%	3.9	−0.984	−0.995	0.997
10^6	10^6	5800	0.017	0.48	0.0021%	0.108%	5.4	−0.954	−0.985	0.991
10^6	10^5	4300	0.026	0.40	0.0015%	0.062%	3.0	−0.970	−0.992	0.991
10^5	10^5	2100	0.017	0.09	0.0008%	0.072%	2.3	−0.946	−0.975	0.992
10^5	10^4	750	0.048	0.18	0.0007%	0.059%	1.2	−0.955	−0.987	0.989
10^4	10^4	320	0.092	0.22	0.0004%	0.141%	0.7	−0.963	−0.992	0.989

TABLE IV. Bounds on λ_g in units of 10^{12} km from Kepler's third law applied to the solar system. Semimajor axes are in astronomical units, and the appropriate one-sided, 2σ bound on η_p from Talmadge *et al.* [19] is shown.

Planet	a_p	Bound on η_p	λ_g
Mercury	0.387	1.4×10^{-8}	0.5
Venus	0.723	1.5×10^{-9}	1.1
Mars	1.523	-6.5×10^{-10}	2.8
Jupiter	5.203	-6×10^{-8}	1.3

the LISA pre-Phase A report [14], augmented by a fit to ''confusion noise'' generated by a population of close white dwarf binaries in our galaxy [27], given by the equations:

$$S_0=4.2\times10^{-41}\ \ \mathrm{Hz}^{-1}, \tag{4.11}$$

$$f_0=10^{-3}\ \ \mathrm{Hz}, \tag{4.12}$$

$$g(x)=\sqrt{10}x^{-14/3}+1+x^2/1000+313.5x^{-(6.398+3.518\log_{10}x)}. \tag{4.13}$$

In order, the four terms in $g(x)$ correspond to: test-mass acceleration noise, photon shot noise, loss of sensitivity when the arm lengths exceed the gravitational wavelength, and a fit to the white-dwarf binary confusion noise. For the maximum frequency, we again adopt that of the innermost stable circular orbit. The minimum frequency is set by the characteristic integration time for LISA, nominally chosen to be one year. We calculate the time T_e remaining until the system reaches the innermost stable orbit by integrating Eq. (3.7) using only the dominant, Newtonian contribution, convert from time at the emitter to observation time T using Eq. (2.4) (ignoring the small correction due to massive graviton propagation), and obtain

$$x_{\rm min}\approx\frac{1}{\pi\mathcal{M}f_0}\left(\frac{5\mathcal{M}}{256T}\right)^{3/8}. \tag{4.14}$$

For massive binaries ranging from 10^4 to $10^7 M_\odot$, and for integration time $T=$ 1 year, we estimate the errors in the five parameters and determine a bound on λ_g. We choose the signal-to-noise ratio ρ for each case such that the luminosity distance to the source $\simeq 3$ Gpc, so that cosmological effects do not become too severe. The results are shown in Table III.

V. SOLAR-SYSTEM BOUNDS ON THE GRAVITON MASS

If the Newtonian gravitational potential is modified by a massive graviton to have the Yukawa form of Eq. (1.5), then the acceleration of a test body takes the form

$$\mathbf{g}=-\frac{\mathbf{n}}{r^2}\mu(r), \tag{5.1}$$

where

$$\mu(r)\equiv M(1+r/\lambda_g)\exp(-r/\lambda_g) = M\left[1-\frac{1}{2}\left(\frac{r}{\lambda_g}\right)^2+O\left(\frac{r}{\lambda_g}\right)^3\right]. \tag{5.2}$$

For a planet with semimajor axis a_p and period P_p, Kepler's third law gives $a_p(2\pi/P_p)^{2/3}=\mu(a_p)^{1/3}$. For a pure inverse-square law, $\mu\equiv$constant, and its value is determined accurately using the orbit of the Earth. Thus, by checking Kepler's third law for other planets, one can test the constancy of μ. For a given planet, we define the parameter η_p by

$$1+\eta_p\equiv\left(\frac{\mu(a_p)}{\mu(a_\oplus)}\right)^{1/3}. \tag{5.3}$$

Combining Eq. (5.3) and (5.2), we obtain a bound on λ_g in terms of η_p

$$\lambda_g>\left(\frac{1-a_p^2}{6\eta_p}\right)^{1/2}, \tag{5.4}$$

where λ_g and a_p are expressed in astronomical units (1.5×10^8 km). Table IV lists the observed bounds on η_p for Mercury, Venus, Mars and Jupiter compiled by Talmadge *et al.* [19], and the resulting bounds on λ_g.

A Yukawa violation of the inverse square law will also produce anomalous perhelion shifts of orbits, of the form $\delta\omega\approx\pi(a_p/\lambda_g)^2$. By comparing measured shifts with the general relativistic prediction for Mercury and Mars (Venus and Jupiter are unsuitable for this purpose) [19], one obtains bounds a factor two weaker than those from Kepler's law.

Systems like the binary pulsar do not yield useful bounds, mainly because they are so compact. Since the deviations from GR go as $(a/\lambda_g)^2$, where a is the size of the system, the bound is roughly $\lambda_g>a/\epsilon^{1/2}$, where ϵ is the accuracy of agreement with observations. Since $a\sim10^6$ km, and $\epsilon\sim10^{-3}$, the bound is far from competitive with that from the solar system.

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