Staggered fermion matrix elements using smeared operators

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We investigate the use of two kinds of staggered fermion operators, smeared and unsmeared. The smeared operators extend over a 4^4 hypercube, and tend to have smaller perturbative corrections than the corresponding unsmeared operators. We use these operators to calculate kaon weak matrix elements on quenched ensembles at $\beta=6.0$, 6.2, and 6.4. Extrapolating to the continuum limit, we find $B_K(\mathrm{NDR,2\ GeV})=0.62\pm0.02(\mathrm{stat})\pm0.02(\mathrm{syst})$. The systematic error is dominated by the uncertainty in the matching between lattice and continuum operators due to the truncation of perturbation theory at one loop. We do not include any estimate of the errors due to quenching or to the use of degenerate s and d quarks. For the $\Delta I=3/2$ electromagnetic penguin operators we find $B_7^{(3/2)}=0.62\pm0.03\pm0.06$ and $B_8^{(3/2)}=0.77\pm0.04\pm0.04$. We also use the ratio of unsmeared to smeared operators to make a partially nonperturbative estimate of the renormalization of the quark mass for staggered fermions. We find that tadpole improved perturbation theory works well if the coupling is chosen to be $\alpha_{\overline{\mathrm{MS}}}(q^*=1/a)$. [S0556-2821(98)05201-1]

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I. INTRODUCTION

An important goal of lattice QCD is to provide reliable calculations of electroweak matrix elements. The major sources of error in present calculations are the use of finite lattice spacing, the use of one-loop perturbation theory to match continuum and lattice operators, and the use of the quenched approximation [1]. In this paper we address the first two errors for calculations using staggered fermions. In particular, we test the efficacy of ''smeared'' operators [2]. These extend over a 4^4 hypercube, and thus are larger than the usual (''unsmeared'') operators which are confined to a 2^4 hypercube. Nevertheless, in many cases they are closer to the continuum operators in the sense that the one-loop matching coefficients are closer to unity.

We apply these operators to the study of three quantities. The first is the kaon B parameter B_K which we study using both smeared and unsmeared operators. The initial motivation for introducing smeared operators was the discovery of large discretization errors in the results for unsmeared operators [3,4]. At the time, it was unclear whether the discretization errors were proportional to the lattice spacing a or to a^2 (up to logarithmic corrections). The smeared operators were designed to reduce possible $O(a)$ errors—they match onto continuum operators with no errors of $O(a)$ at tree level. It was subsequently realized that the $O(a)$ parts of the lattice operators do not contribute to B_K, and that the errors are automatically of $O(a^2)$ for both the unsmeared and smeared operators [5–7]. This theoretical argument has since been tested numerically [7,8]. Thus the interest in using the smeared operators is that they provide an estimate of the error in the matching of continuum and lattice operators. The results from different operators should differ at finite a, but agree upon extrapolation to $a=0$, up to higher order perturbative corrections.

Preliminary results from this study of B_K were presented in Ref. [5], and one of our purposes here is to present final results. Although, by present standards, these come from a small statistical sample, the errors are nevertheless small enough to assess the impact of smeared operators. We have also improved our estimates of the error due to the truncation of the perturbative matching factors, using the method introduced in Ref. [9].

Our second application is the calculation of $B_7^{3/2}$ and $B_8^{3/2}$. These B parameters, and in particular $B_8^{3/2}$, determine the size of the electromagnetic penguin contribution to ϵ'. In contrast to B_K, the use of staggered fermions for these quantities offers no clear advantage over Wilson fermions. However, since the systematic errors in the results with the two types of fermion are different, an important check of the reliability of the lattice calculations is to show that the two formulations give consistent results in the continuum limit. To this end we compare the staggered data with the recent results obtained using Wilson fermions at $\beta=6/g^2=6$ in the quenched approximation [9].

The calculation of $B_7^{3/2}$ and $B_8^{3/2}$ demonstrates the importance of using several discretizations of continuum operators. It turns out that one cannot use the unsmeared operators because the one-loop correction to the matching coefficients approaches 100% [10]. On the other hand, the one-loop corrections are much smaller for the smeared operators

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($\sim 25\%$), and we can use them to calculate $B_7^{3/2}$ and $B_8^{3/2}$. The only way in which one could use the unsmeared operators would be to develop a non-perturbative method of determining the matching coefficients (which is possible in principle using external quark states [11]).

Our final application concerns the calculation of the matching relation between the lattice and continuum regularization schemes, particularly in cases where the reliability of perturbative estimates is questionable. The ratio of such matching factors for two different discretizations of an operator can be estimated nonperturbatively by taking ratios of appropriate matrix elements. The nonperturbative results so obtained can be used to test the reliability of the one-loop perturbative estimates. In particular, one can use the results to fix the scale q^* at which to evaluate the coupling constant entering into the perturbative expressions. For the pseudoscalar density we find that $q^* \approx 1/a$. Our conclusions are, however, preliminary, since we do not have results at enough values of lattice spacing to check the extrapolations we use to remove discretization errors.

We use this method to assess the reliability of the matching factor Z_m. Z_m relates the bare lattice mass to the continuum mass in, say, the modified minimal subtraction ($\overline{\rm MS}$) scheme, and is a crucial ingredient in the calculation of continuum light quark masses from the lattice. Recent work has suggested that continuum quark masses are smaller than previously thought, but this is based on trusting one-loop perturbation theory for Z_m [12]. For staggered fermions, the one-loop contribution to Z_m is large, roughly a 60% correction at $\beta=6$, even after tadpole improvement [2]. This casts doubt on the reliability of the perturbative Z_m, and therefore also on the extracted value of quark masses. We find, however, that our partly nonperturbative estimate suggests that the one-loop evaluation is close to the correct answer.

The organization of this paper is as follows. In the following section we describe the method we use to match continuum operators to lattice operators composed of staggered fermions. In Sec. III we give a short description of the numerical methods and data sample. The three subsequent sections contain our analysis and results for B_K, $B_{7,8}^{3/2}$, and the nonperturbative ratios of matching factors, respectively. We close with some conclusions.

II. THEORETICAL REVIEW

In this section we explain our method of calculating B parameters using staggered fermions. This requires combining a variety of results already in the literature, and we focus here only on the essential details. For a more extensive description of the method see Ref. [13].

The continuum operators of interest are

$$\mathcal{Q}_K = \overline{s}_a \gamma_\mu^L d_a \overline{s}_b \gamma_\mu^L d_b , \tag{1}$$

$$\mathcal{Q}_7^{3/2} = \overline{s}_a \gamma_\mu^L d_a [\overline{u}_b \gamma_\mu^R u_b - \overline{d}_b \gamma_\mu^R d_b] + \overline{s}_a \gamma_\mu^L u_a \overline{u}_b \gamma_\mu{}^R d_b , \tag{2}$$

$$\mathcal{Q}_8^{3/2} = \overline{s}_a \gamma_\mu^L d_b [\overline{u}_b \gamma_\mu^R u_a - \overline{d}_b \gamma_\mu^R d_a] + \overline{s}_a \gamma_\mu^L u_b \overline{u}_b \gamma_\mu{}^R d_a . \tag{3}$$

where a,b are color indices and $\gamma_\mu^{R,L} = \gamma_\mu(1 \pm \gamma_5)$. All operators are in Euclidean space, and we use Hermitian γ matrices. The superscripts on $\mathcal{Q}_{7,8}$ indicate that these are the $I=3/2$ parts of the operators $\mathcal{Q}_{7,8}$, i.e., the $I=1/2$ component has been removed. We make this restriction because the calculation of the matrix elements of the $I=3/2$ parts is much simpler. In the limit of exact flavor SU(3), which is the limit we work in here, the $I=3/2$ parts give rise only to ''eight'' diagrams, i.e., those in which the quark fields in the operator are contracted with fields in the external mesons. These are the same type of diagrams which contribute to the matrix element of $\mathcal{Q}_K$. The $I=1/2$ parts, by contrast, give rise also to ''penguin'' or ''eye'' diagrams, which are much more difficult to calculate. The restriction to the $I=3/2$ parts of $\mathcal{Q}_{7,8}$ does not, however, diminish the phenomenological interest in the results, because $\mathcal{Q}_{7,8}^{3/2}$ are the only operators which give an imaginary part to the $K^+ \to \pi^+ \pi^0$ amplitude.

To form B parameters we also need the matrix elements of the axial and pseudoscalar densities:

$$A_\mu = \overline{s}_a \gamma_\mu \gamma_5 d_a , \quad P = \overline{s}_a \gamma_5 d_a . \tag{4}$$

We can then define

$$B_K = \frac{\langle K^0 | \mathcal{Q}_K | \overline{K^0} \rangle}{(8/3)\langle K^0 | A_4 | 0 \rangle \langle 0 | A_4 | \overline{K^0} \rangle}, \tag{5}$$

$$B_7^{3/2} = \frac{\langle \pi^+ | \mathcal{Q}_7^{3/2} | K^+ \rangle}{(2/3)\langle K^0 | P | 0 \rangle \langle 0 | P | \overline{K^0} \rangle - \langle K^0 | A_4 | 0 \rangle \langle 0 | A_4 | \overline{K^0} \rangle}, \tag{6}$$

$$B_8^{3/2} = \frac{\langle \pi^+ | \mathcal{Q}_8^{3/2} | K^+ \rangle}{2\langle K^0 | P | 0 \rangle \langle 0 | P | \overline{K^0} \rangle - (1/3)\langle K^0 | A_4 | 0 \rangle \langle 0 | A_4 | \overline{K^0} \rangle}. \tag{7}$$

All external particles have been assumed to be at rest. For brevity, we have used SU(3) flavor symmetry to rewrite all the denominators in terms of kaon matrix elements. Note that, in general, both numerators and denominators of these ratios depend upon the renormalization scale μ and the scheme used to define the operators. When quoting physical values we use the naive dimensional regularization (NDR) scheme, i.e., $\overline{\rm MS}$ renormalization combined with a particular set of rules for treating γ_5 away from four dimensions, and choose the renormalization scale to be $\mu = 2$ GeV.

The extraction of the above matrix elements using staggered fermions is complicated by the mixing between the spin and flavor degrees of freedom. As explained in Refs. [10,13], we proceed in two stages. We first match the continuum matrix elements onto those in an ''enlarged'' continuum theory, and then match from that theory onto the lattice. The enlarged theory differs from QCD by having eight copies of each physical quark. The eight copies of the strange quark spinor are collected into two 4×4 matrices $S_{\beta,b}$ and $S'_{\beta,b}$, where β is a spinor index, and $b=1-4$ is a ''staggered-flavor'' index. Similar fields are constructed for the up and down quarks. The correspondence between matrix elements in the continuum and in the enlarged theory is

$$\langle 0|A_4|\overline{K^0}\rangle = \sqrt{\frac{1}{N_f}}\,\langle 0|\,\overline{S(}\,\gamma_4\gamma_5\otimes\xi_5)D|\overline{K^0_G}\rangle, \tag{8}$$

$$\langle K^0|A_4|0\rangle = \sqrt{\frac{1}{N_f}}\,\langle K'^0_G|\,\overline{S'}\,(\gamma_4\gamma_5\otimes\xi_5)D'|0\rangle, \tag{9}$$

$$\langle 0|P|\overline{K^0}\rangle = \sqrt{\frac{1}{N_f}}\,\langle 0|\,\overline{S(}\,\gamma_5\otimes\xi_5)D|\overline{K^0_G}\rangle, \tag{10}$$

$$\langle K^0|P|0\rangle = \sqrt{\frac{1}{N_f}}\,\langle K'^0_G|\,\overline{S'}\,(\gamma_5\otimes\xi_5)D'|0\rangle, \tag{11}$$

$$\langle K^0|\mathcal{Q}_K|\overline{K^0}\rangle = \frac{2}{N_f}\,\langle K'^0_G|[(V-A)\times P]_I + [(V-A)\times P]_{II}|\overline{K^0_G}\rangle, \tag{12}$$

$$\langle \pi^+|\mathcal{Q}_7^{3/2}|K^+\rangle = \frac{1}{N_f}\,\langle K'^0_G|2[(P-S)\times P]_I + [(V+A)\times P]_{II}|\overline{K^0_G}\rangle, \tag{13}$$

$$\langle \pi^+|\mathcal{Q}_8^{3/2}|K^+\rangle = \frac{1}{N_f}\,\langle K'^0_G|2[(P-S)\times P]_{II} + [(V+A)\times P]_I|\overline{K^0_G}\rangle. \tag{14}$$

We have used flavor symmetry to rewrite all matrix elements in terms of those between external kaon states. The notation for matrix elements in the enlarged theory is that of Refs. [2, 10], and we give only a brief summary. The matrices appearing in bilinears are tensor products of spin and staggered-flavor matrices. For example, $(\gamma_5\otimes\xi_5)$ indicates a pseudoscalar density with staggered-flavor matrix γ_5. The states $|\overline{K^0_G}\rangle$ and $|\overline{K'^0_G}\rangle$ are those created from the vacuum by $\overline{D(}\,\gamma_5\times\xi_5)S$ and $\overline{D'}\,(\gamma_5\times\xi_5)S'$, respectively. The states are normalized, which leads to the factors of $N_f=4$, the usual multiplicity factor for staggered fermions. The subscript G indicates that these states are the pseudo Goldstone bosons corresponding to the axial U(1) symmetry which is unbroken when one discretizes the enlarged theory using staggered fermions and takes the chiral limit. The notation for four-fermion operators is from Ref. [10]. For example, $[(V-A)\times P]_I$ represents the one color loop contraction of the four-fermion operator with spin structure $\gamma_\mu\cdot\gamma_\mu-\gamma_\mu\gamma_5\cdot\gamma_5\gamma_\mu$, and in which both bilinears have staggered flavor γ_5. Finally, the factor of 2 on the right-hand side (RHS) of Eq. (12) arises from the difference in the number of Wick contractions in QCD and the enlarged theory due to the use of ''primed'' quarks in the latter theory.

All these equalities hold identically between the quenched versions of the two theories. In the presence of internal fermion loops, they hold if, in the enlarged theory, each loop is multiplied by 1/8, i.e., the fermion determinant is taken to the power 1/8.

The next step is to relate the operators in the enlarged continuum theory to those in the corresponding lattice theory. The enlarged theory has been chosen so that it is the continuum limit of a discretized theory in which one uses a single staggered species for each 4×4 matrix field, e.g., χ_S for S and $\chi_{S'}$ for S'. This means that the relation between operators is simple at tree level. For example, using the notation of Ref. [2],

$$\overline{S(}\,\gamma_5\otimes\xi_5)D = \overline{\chi_S(\gamma_5\otimes\xi_5)}\,\chi_D[1+O(a)+O(\alpha_s)]. \tag{15}$$

Here $\overline{(\gamma_5\otimes\xi_5)}$ is that matrix in the space of possible positions in a 2^4 hypercube which corresponds to the continuum spin-flavor matrix $(\gamma_5\otimes\xi_5)$. Similar relations hold for the four fermion operators [10].

In this paper we use one-loop perturbation theory to match operators in the enlarged continuum theory to those on the lattice. In other words, we include the terms of $O(\alpha_s)$ in equations such as Eq. (15). More precisely, we use the ''horizontal matching'' procedure discussed in detail in Ref. [9]. This consists of two steps. We first match the lattice and continuum operators at an intermediate scale $q^*\sim1/a$ using

$$\mathcal{O}_i^{\rm cont}(q^*) = \mathcal{O}_i^{\rm lat} + \frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}\sum_j\,[-\gamma_{ij}^{(0)}\,\ln(q^*a)+c_{ij}]\mathcal{O}_j^{\rm lat} + O(\alpha^2)+O(a). \tag{16}$$

Here $\mathcal{O}_i^{\rm lat}$ are bare lattice operators, $\gamma^{(0)}$ is the one-loop anomalous dimension matrix, and the c_{ij} are the one-loop matching coefficients. We give, below, numerical values for the c_{ij} of interest. The second step is to evolve the result from q^* to the final scale $\mu=2$ GeV using the continuum two-loop anomalous dimension matrix. The two-loop anomalous dimensions for the continuum operators of interest are collected in Ref. [9], and we do not repeat them here.

The point of this procedure is to account for the fact that there are two scales in the problem: the matching scale $q^*\sim1/a$, and the final scale μ. These can differ substantially—indeed the ratio q^*/μ can be as large as 10 in our calculation. The coupling constant can thus be quite different at the two scales, and it is important to make sure that the appropriate coupling is used at each stage. Furthermore, it becomes necessary to sum up the leading logarithms of q^*/μ. Both of these requirements are accomplished by horizontal matching. The scale in the coupling in the first step should be q^*, and the renormalization group evolution in the second step sums up the leading logarithms. Of course, since we truncate perturbation theory, there are errors in both steps. In the matching relation (16) the truncation errors are of $O[\alpha_s(q^*)^2]$, while the errors in the continuum evolution are proportional to both $\alpha_s(q^*)^2$ and $\alpha_s(\mu)^2$. We note, however, that the errors from the continuum evolution are the same for any choice of lattice discretization of the continuum operator. This point will be important in our discussion of results for B_K using different lattice operators.[1]

To use horizontal matching we need to choose the intermediate matching scale q^*. If one worked to all orders in perturbation theory, the final result would be independent of q^*. This is not true if one truncates perturbation theory:

[1]For further discussion of horizontal matching, and in particular of its relation to the exact matching formula of Ji [15], see Ref. [9].

different choices lead to results differing by $O[\alpha(q^*)^2]$. One should choose q^* to be, roughly speaking, the average momentum flowing through the quark-gluon vertices in the matching calculation [16]. It is difficult, however, to make this into a precise prescription for matching calculations involving operators with nonzero anomalous dimensions. The only method we know of was proposed in Ref. [9], and involves applying the Brodsky-Lepage-Mackenzie (BLM) prescription of Ref. [17] to the calculation of the two-loop lattice anomalous dimension matrix. This calculation has not been done, and so we have chosen to use a range of values for q^*. Since the lattice and continuum operators are constructed to have the same matrix elements at long distances, the dominant contributions to the matching calculation are from momenta near the lattice cutoff (as long as the continuum renormalization point is also of this size). Thus we take $q^*=K/a$ with K a constant. Since we use tadpole improved operators we expect that $K\approx 1$ rather than $K\approx\pi$, as explained in Ref. [16]. So we have chosen $q^*=1/a$ for our central values, and used $q^*=\pi/a$ in order to estimate the uncertainty due to the truncation of perturbation theory.[2] This is certainly a crude estimate, but it is the best that we can do.

In our preliminary analysis of B_K we did not use horizontal matching, but rather used the one-loop matching relation Eq. (16) connect directly from the lattice to the final scale μ [5]. To estimate the truncation error we took the difference between the results of using $\alpha_s(\mu)$ and $\alpha_s(q^*)$ in the matching equation. Our present methods both of matching and of estimating the truncation error are more reliable, although, as we will see below, the final answer is little changed.

The $\overline{\rm MS}$ coupling constant appearing in Eq. (16) is determined using the method of Ref. [16], which incorporates tadpole improvement. We first solve

$$-\ln\Box=\frac{4\pi}{3}\alpha_V(3.41/a)[1-1.185\alpha_V(3.41/a)] \quad (17)$$

for α_V, where $\Box$ is the expectation value of the plaquette normalized to be unity in the continuum limit. We then convert to the $\overline{\rm MS}$ scheme using

$$\alpha_{\overline{\rm MS}}(3.41/a)=\alpha_V(e^{5/6}3.41/a)(1+2\alpha_V/\pi). \quad (18)$$

This coupling is then run to other scales using the two-loop β function.

The lattice operators we use are of two types: the standard operators, which are contained in 2^4 hypercubes (and which we refer to as ''unsmeared'' operators); and the ''smeared'' operators introduced in Ref. [2], which live on 4^4 hypercubes. The smeared operators have the same form as the unsmeared, except for the replacement

$$\chi(y+A)\rightarrow\frac{1}{4}\sum_{\mu=1}^{4}\chi(y+A+2\hat{\mu}[1-2A_\mu]), \quad (19)$$

where y is the position of the origin of the 2^4 hypercube, and A is the position within the hypercube. This definition makes the average position of the quark field lie at the center of the hypercube, which is why there are no $O(a)$ corrections when matching to the continuum operator at tree level. Both types of operator are made gauge invariant by fixing to lattice Landau gauge. Both are also tadpole improved, using the mean link determined from the plaquette $u_0^4=\Box$.

Although the discretization errors in the matching equation are, in general, of $O(a)$, it turns out that for the matrix elements of interest the corrections are actually of $O(a^2)$. This is because the $O(a)$ parts have the wrong staggered flavor [5,14], and by using staggered flavor ξ_5 for the incoming and outgoing states we project against them up to errors of $O(a^2)$. For the same reason, we need only include in the matching equations those operators which have the correct staggered flavor. This is fortunate, since typically many operators of the wrong flavor are needed to match the continuum operators.

The matching matrices c_{ij} for all continuum operators of interest, and for both smeared and unsmeared lattice operators, can be determined from the results of Ref. [10]. When doing this one must be careful to account for the following three issues. First, the numerical results for the c_{ij} are quoted in Ref. [10] for continuum operators defined in the DR$\overline{\rm EZ}'$, rather than the NDR, scheme. Thus one must convert between these schemes, and a recipe for doing so is given in [10]. Second, tadpole improvement in [10] is based on the average trace of the link in Landau gauge, rather than on the average plaquette which we use here. This causes a small change in the coefficients

$$c_{ij}=c_{ij}(\text{Ref. [10]})+N_B(4/3)(\pi^2-9.17479)\delta_{ij}, \quad (20)$$

where $N_B=1$ for bilinears and $N_B=2$ for four fermion operators. Third, the coefficient of the logarithm in the matching equation (16) has been changed from q^*a/π to q^*a. This shifts the c_{ij} of Ref. [10] by $\gamma_{ij}^{(0)}\ln\pi$. This change allows one to see the size of the matching corrections for our standard value $q^*a=1$ directly from the size of the c_{ij}.

As the procedure for calculating the matching coefficients is rather involved, we collect here the relevant parts of the final results. For the bilinears, the matching involves no mixing, only multiplicative renormalization. The results are

$$c_A^{\rm UN}=0.9264, \quad c_A^{\rm SM}=-1.0120, \quad (21)$$

$$c_P^{\rm UN}=-39.1414, \quad c_P^{\rm SM}=-8.8882. \quad (22)$$

Here c_A is a shorthand for the diagonal coefficient for the axial density, etc., and the superscript refers to whether or not the quark fields are smeared. These coefficients are multiplied by $\alpha_{\overline{\rm MS}}(q^*)/4\pi$, which, in our calculation, varies in the range 0.01–0.015. Thus these corrections are numerically small for the axial currents ($\sim 1\%$) and of moderate size for

[2]Reference [16] advocates the use of a different coupling constant α_V, rather than $\alpha_{\overline{\rm MS}}$. α_V is defined in terms of the quark-antiquark potential. We prefer to use $\alpha_{\overline{\rm MS}}$ because this is the scheme used in continuum calculations of coefficient functions, and the matching formulae are simpler if one uses the same coupling in the continuum and on the lattice. The two couplings are quite similar—our choice of $q^*=1/a$ in the $\overline{\rm MS}$ scheme corresponds to using $\alpha_V(1.6/a)$. This is indeed a typical value for q^* for tadpole improved quantities.

TABLE I. Parameters of numerical simulations used in this analysis.

β	6.0	6.2	6.4
Lattice size	$24^3\times 40$	$32^3\times 48$	$32^3\times 48$
Number of lattices	13	23	24
Samples per lattice	1	2	2
Quark masses	0.01, 0.02, 0.03	0.005, 0.01, 0.015	0.005, 0.01, 0.015
$1/a$ in GeV from m_ρ	1.9	2.6	3.45
Minimum M_π in GeV	0.454	0.376	0.414
Average plaquette	0.5937	0.6137	0.6306
$\alpha_{\overline{\rm MS}}(\pi/a)$	0.134	0.125	0.117
$\alpha_{\overline{\rm MS}}(1/a)$	0.192	0.173	0.158
$\alpha_{\overline{\rm MS}}(2\ {\rm GeV})$	0.188	0.190	0.191
$1/a$ for $\alpha_{\overline{\rm MS}}(2\ {\rm GeV})=0.190$	1.94	2.6	3.4

the smeared pseudoscalar density ($\sim 10\%$). For the unsmeared pseudoscalar density, however, they are large ($\sim 50\%$), suggesting that two-loop terms will be important. This is an example of the advantage of using smeared operators.

For the four-fermion operators of interest here, we can decompose the matrices of matching coefficients into two parts. The first is a square matrix with indices running over operators

$$i,j=([(V-A)\times P]_{\rm I},[(V-A)\times P]_{\rm II},[(V+A)\times P]_{\rm I},[(V+A)\times P]_{\rm II}), \tag{23}$$

for which we find

$$c_{ij}^{\rm UN}=\begin{pmatrix} -4.3079 & 0.4611 & 0.2638 & -0.7913 \\ -1.0002 & 0.0761 & 0 & -2.1102 \\ 0.2638 & -0.7913 & -8.9740 & -1.5387 \\ 0 & -2.1102 & -4.9998 & 1.4093 \end{pmatrix}, \tag{24}$$

$$c_{ij}^{\rm SM}=\begin{pmatrix} -2.1376 & -5.4068 & 0.0850 & -0.2551 \\ -5.8701 & -0.7474 & 0 & -0.6802 \\ 0.0850 & -0.2551 & 4.5595 & -2.5366 \\ 0 & -0.6802 & -0.1299 & -2.6608 \end{pmatrix}. \tag{25}$$

These coefficients are needed for all three B parameters, and the corrections are of moderate size for both smeared and unsmeared operators. The second matrix we need is rectangular, having indices[3]

$$i=\{[(P-S)\times P]_{\rm I},[(P-S)\times P]_{\rm II}\}, \tag{26}$$

$$j=\{[(P-S)\times P]_{\rm I},[(P-S)\times P]_{\rm II},[(P+S)\times P]_{\rm I},[(P+S)\times P]_{\rm II}\}. \tag{27}$$

For this we find

$$c_{ij}^{\rm UN}=\begin{pmatrix} 0.7562 & -14.7309 & 5.3559 & -16.0674 \\ -3 & -34.4364 & 0 & -42.8464 \end{pmatrix}, \tag{28}$$

$$c_{ij}^{\rm SM}=\begin{pmatrix} -2.8862 & -3.1611 & 1.6759 & -5.0277 \\ -3 & -3.3692 & 0 & -13.4072 \end{pmatrix}. \tag{29}$$

These results are needed for $B_7^{3/2}$ and $B_8^{3/2}$, and it is here that the use of smeared operators is most important. In particular, the numbers in the second row in Eq. (29) are much smaller than in the corresponding row in Eq. (28). Much of this reduction can be traced back to the similarly large difference in the coefficients c_P shown in Eq. (22). For the range of couplings we study, it turns out that the one-loop matching correction for the operators appearing in the numerator of $B_7^{3/2}$ and $B_8^{3/2}$ actually exceeds 100% if one uses the unsmeared operators with $q^*\approx 1/a$. Our results for these quantities are, therefore, obtained exclusively with smeared operators.

III. NUMERICAL DETAILS

Our results are based on ensembles of lattices at three different lattice spacings. A summary of the important parameters is given in Table I. The lattices at $\beta=6$ are part of a sample previously used to study the hadron spectrum, and a complete description of how we generated the lattices and calculated quark propagators can be found in Ref. [18]. The lattices at $\beta=6.2$ and 6.4 have been discussed previously in Refs. [3, 19]. They were generated using overrelaxed and Metropolis sweeps in a 4:1 ratio, and separated by 1000 sweeps. They were divided roughly equally into two independent streams at $\beta=6.2$, and three at $\beta=6.4$. Quark propagators were calculated using the conjugate gradient algorithm.

We consider here only pions composed of two degenerate quarks. For the quark masses that we use, the masses of the lattice pions bracket that of the physical kaon at each β. To illustrate this we include in the table the mass of the lightest pion, converted to physical units using $1/a$ determined from m_ρ. These scales, also listed in the table, have been previously reported by one of us from a fit to the hadron spectrum [19]. We refer to these below as the ''m_ρ scales.''

[3]Note that, at one-loop, tensor operators do not appear.

We also give the values of $\alpha_{\overline{\rm MS}}$ needed in the one-loop matching between lattice and continuum operators. These are obtained using Eqs. (17) and (18) from the plaquette values listed in the table. Since in the end we run our results to 2 GeV, a consistency check on the applicability of tadpole improved perturbation theory is that the values of $\alpha_{\overline{\rm MS}}(2\ {\rm GeV})$ agree for different β. The table shows that they are in reasonable agreement, although there is a small systematic increase as a decreases.

This systematic dependence on a suggests using a different set of scales, defined so that $\alpha_{\overline{\rm MS}}(2\ {\rm GeV})$ is independent of a. We choose $\alpha_{\overline{\rm MS}}(2\ {\rm GeV})=0.190$, and the resulting scales are listed in the last row of the table. We refer to these below as the ''α scales.'' We use these scales to determine our central values for matrix elements, and use the m_ρ scales to estimate the error introduced by the uncertainty in determining a.

Our method for calculating B parameters was developed for calculating B_K with staggered fermions [20]. It has also been used for Wilson fermions [21]. A detailed description of the latter application, and the extension to other operators is given in Ref. [9]. Here we confine ourselves to a brief description of the features which are special to the present work.

The essential feature of the method is the use of wall sources (i.e., sources confined to a single time-slice and extended over the entire timeslice), which are designed to create particles with specific quantum numbers. In the current applications we want to create only the lattice pseudo Goldstone pion at rest, and this can be accomplished using a linear combination of two wall sources, as explained in Ref. [18]. The only other states created are rho mesons and excited pions (π'). Both of these are, however, considerably more massive than the pseudo Goldstone pion, and their contribution can be largely removed by moving far enough away from the source in Euclidean time.

The basic method is then to calculate ratios such as

$$B_K(t)=\frac{3V_y\langle W(t_1)\Sigma_{\vec{y}}\mathcal{Q}_K(\vec{y},t)W(t_2)\rangle}{8\langle W(t_1)\Sigma_{\vec{y}'}A_4(\vec{y}',t)\rangle\langle\Sigma_{\vec{y}''}A_4(\vec{y}'',t)W(t_2)\rangle}. \tag{30}$$

Here the operators are the lattice versions of the continuum operators obtained after the matching explained in the previous section has been carried out. The wall sources are denoted by W and are located at times t_1 and t_2. The operators are placed at an intermediate time t satisfying $t_1\ll t\ll t_2$, and are summed over the V_y spatial hypercubes. Finally, the expectation values in Eq. (30) are averages over quenched configurations.

The expression in Eq. (30) is designed so that for large enough $t-t_1$ and t_2-t it is independent of t, and gives B_K directly. The point is that the exponential factors from Euclidean time evolution cancel if only a single state, here the pseudo Goldstone pion, contributes. This has to be true separately for the numerator and denominator, i.e., both must exhibit a ''plateau'' over intermediate times in which they are independent of t. Thus the signal can be improved by averaging the numerator and denominator over the time slices in this plateau. This is what we do in practice, using the same range of times for both numerator and denominator.

This method has several positive features. First, it involves no fitting. Second, statistical errors are reduced by directly calculating B parameters rather than the matrix elements themselves. Third, the ability to average over all the spatial points and over a number of time slices improves the statistics. This is possible because we know the propagator from the wall sources to all points in the lattice. For the same reason, we can calculate matrix elements of nonlocal operators without additional quark propagators. And, finally, we can use the same set of quark propagators to calculate matrix elements with different choices of lattice operator. The latter two features are particularly important for staggered fermions where most operators are nonlocal. If we placed the source of the propagators at the position of the operators we would need 2^4 sources for unsmeared operators (4^4 for smeared operators).

To calculate statistical errors, we use single elimination jackknife in the following manner. On each jackknife sample, we first match at the scale q^*, then linearly interpolate the results to physical kaon mass, and then evolve to the final scale $\mu=2$ GeV, and, finally, take the ratio which defines the B parameter. The error is then obtained from the variation between samples.

We end this section with some details specific to our particular lattices.

At $\beta=6$, we use wall sources in Coulomb gauge, while on the other lattices the sources are in Landau gauge. The choice of gauge should have no impact on the final result (as long as the nonlocality introduced by the gauge fixing does not extend from the source to the operator [18]) but might affect the statistical errors.

At $\beta=6$, we use periodic boundary conditions (PBCs) in space, and Dirichlet bounday conditions in time. We place the wall sources next to the boundary, i.e., at $t_1=0$ and $t_2=39$. The merits of this procedure have been discussed in Ref. [18]. We only note here that we use a plateau region of $t=10-29$.

The lattices at $\beta=6.2$ and 6.4 have too short an extent in time to follow the method adopted at $\beta=6$. Instead we use PBCs in all four directions, having first periodically doubled the lattice in time. We place wall sources at $t_1=0$ and $t_2=72$, and use the plateau region $t=32-40$. Note that the propagator with source at $t_1=0$ ($t_2=72$) is, due to the PBCs and doubling of the lattice, the same as if the source was at $t_1=48$ ($t_2=24$). Because of this, and the fact that each source produces pions propagating in both forward and backward directions, we can make a second measurement with the plateau region at times $t=8-16$. We treat these two results as independent in our jackknife error analysis.

To study possible sources of systematic errors due to contamination from excited states or ''off-shell'' matrix elements we have also made the following measurements. Taking the same two wall sources to lie at $t_1=0$ and $t_2=24$ we use the region $t=32-40$ to obtain an estimate of the ''off-shell'' matrix elements, i.e., those in which both of the pions approach the operator from the same side. Placing the sources at $t_1=0$ and $t_2=48$, and considering the plateau region at $t=23-24$, we get an estimate of the on-shell matrix elements having a greater contamination from excited states, but a smaller contribution from off-shell matrix elements. Further discussion of these constructions and the as-

TABLE II. Results for B_K(NDR,2 GeV) at the physical kaon mass, using the α scales. UN and SM refer to unsmeared and smeared operators, respectively.

Operator	q^*	$\beta=6.0$	$\beta=6.2$	$\beta=6.4$	$a=0$
UN	$1/a$	0.728(9)	0.677(9)	0.685(24)	0.628(20)
UN	π/a	0.725(9)	0.669(9)	0.677(24)	0.614(20)
SM	$1/a$	0.681(6)	0.647(6)		0.604(15)
SM	π/a	0.694(6)	0.655(6)		0.606(15)
Ratio UN/SM	$1/a$	1.069(11)	1.046(13)		1.018(32)
Ratio UN/SM	π/a	1.044(11)	1.021(14)		0.991(34)

sociated systematic errors is given in Ref. [3]. In practice, we find that the various methods yield very similar results for the B parameters we consider, and that these sources of error are considerably smaller than others as discussed later.[4] Thus we give no further details.

IV. RESULTS FOR B_K

Using the methods described in the previous two sections we extract the continuum B_K at $\mu=2$ GeV for each lattice spacing. The only feature of the analysis not discussed above is the interpolation to the physical kaon mass. This we do by fitting B_K itself (for our three mass points) to a linear function of the squared lattice kaon mass $m^2_{K,\mathrm{lat}}$. This is reasonable for staggered fermions because chiral symmetry constrains B_K to have the same form as in the continuum and in particular to be finite in the chiral limit [13,22]. For degenerate quarks, the explicit form is

$$B_K = B[1 - 3y\ \ln y + by + O(y^2)], \tag{31}$$

where $y=m_K^2/(4\pi f_\pi)^2$ is the usual chiral expansion parameter. For our range of $m^2_{K,\mathrm{lat}}$ the $y\ln y$ contribution is well represented by a linear function. We note in passing that for Wilson fermions there is, in general, an additional term proportional to a/y in Eq. (31) because of the explicit breaking of chiral symmetry.

We present results only for the α scales, i.e., the lattice spacings determined by requiring that $\alpha_{\overline{\mathrm{MS}}}(2\ \mathrm{GeV})=0.190$. These results are collected in Table II, and displayed in Fig. 1. We include the results of an extrapolation to the continuum limit assuming quadratic dependence on a. Note that we do not have results for smeared operators at $\beta=6.4$. Our results are not extensive enough either to test whether the dependence on a is indeed quadratic, or whether terms of higher order than quadratic are needed for our range of lattice spacings. The best confirmation of the validity of the quadratic dependence comes from the work of the JLQCD Collaboration, who have more extensive results than ours for both the unsmeared operator and its gauge-invariant version [7].

[4]The same is not true for the auxiliary parameters B_A and B_V defined in Ref. [22]. For these we have no useful results at $\beta=6.2$ and 6.4.

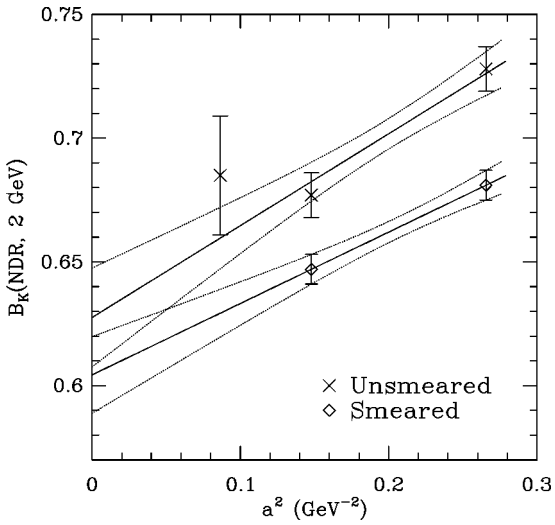

FIG. 1. Results for B_K including extrapolation to the continuum limit.

We can use our results to estimate the systematic errors arising from the choice of lattice spacings, the truncation of perturbation theory in the matching factors, and the contamination from excited states and off-shell matrix elements. We begin with the dependence on the choice of lattice spacings. This we estimate by comparing the results using the α scales (listed in Table II) to those obtained using the scales determined from m_ρ. The values of B_K at each lattice spacing change by no more than 0.002, and the maximum change in the extrapolated value is 0.004. Thus we take ± 0.004 for our estimate of this systematic error. This is much smaller than the statistical errors.

The error due to the truncation of perturbation theory in the matching factors can be estimated in two ways. The first, discussed in Sec. II, uses the q^* dependence of the result. We take the error to be the difference between the extrapolated results with $q^*a=1$ and $q^*a=\pi$. The table shows that this is comparable to the statistical errors. The largest difference is that for the unsmeared operators, and we take this for our estimate, yielding ± 0.014. This 3% error is a reasonable estimate for a two-loop correction given that the one-loop matching corrections for unsmeared and smeared operators are $\sim 5-15\%$. Our estimate of this error turns out to be the same as that quoted in our preliminary result [5], although the method we use here is more reliable, as explained in Sec. II.

The second way of estimating the perturbative error is to compare the results using the two types of operator. If the matching factors were correct then they should yield the same result in the continuum limit. We estimate the error as half the difference between the smeared and unsmeared results at $q^*a=1$, and thus obtain ± 0.012. In fact, this is likely to be an overestimate of the difference in the results from the two operators. This is because there is an extra data point at $\beta=6.4$ for the unsmeared operators, and, as can be seen from Fig. 1, this shifts the extrapolated result away from that of the smeared operator. Perhaps a better way of estimating the error is to take the ratio of the results using the two operators, and extrapolate this to the continuum limit. This removes the data point at $\beta=6.4$ (since we have no smeared result there), and also accounts for the correlations

between the results for the two operators. The results for this ratio are given in Table II, and show that although the results from the two operators are significantly different at finite a, they are consistent in the continuum limit. Because of this, we take our estimate of the perturbative error from the dependence on q^*.

Finally, we discuss the errors due to contamination in the matrix elements from excited states and off-shell contributions. The only excited state which is allowed to contribute by the symmetries is the π', i.e., the radially excited pion. There are no contributions from ρ mesons (a point not realized in Ref. [5]). As mentioned in Sec. III, the size of the first effect can be estimated by comparing the results with the two sets of sources ($t_1=0$, $t_2=72$ versus $t_1=0$, $t_2=48$), while off-shell contamination can be estimated by studying $B_K(t)$ in regions where the off-shell contributions dominate. We find that the combined shift in B_K, after averaging the results from the two operators, and extrapolating to the continuum limit, is ≈ -0.003. Since this estimate is approximate and small, we do not include this shift in our final result, but instead include it as part of the above overall systematic error.

Putting this all together, we can now quote our final result. For the central value we use $q^*=1/a$, and take the average of the results from the two operators. We use the larger of the statistical errors (that for the unsmeared operators). And we estimate the overall systematic error by combining linearly those from the choice of lattice scales (0.004), from the truncation of perturbation theory (0.014), and from the contaminations (0.003). Thus we quote

$$B_K(\mathrm{NDR},2\ \mathrm{GeV})=0.62\pm0.02(\mathrm{stat})\pm0.02(\mathrm{syst}). \quad (32)$$

This result is consistent with our preliminary number quoted in Ref. [5] ($0.616\pm0.020\pm0.017$), although the precise agreement is somewhat fortuitous given that our method of matching has been improved.

Our results for the unsmeared operator can be checked by comparing them to those from the JLQCD Collaboration [7]. They have results at $\beta=6.0$, 6.2, and 6.4, on lattices of the same spatial sizes as ours, but with statistical errors two or three times smaller. They also have results on larger lattices at $\beta=6$ and 6.4, and at smaller values of β. A direct comparison is possible because they use a method of matching and determining $\alpha_{\overline{\mathrm{MS}}}$ which is very close to ours if we set $q^*=1/a$. At $\beta=6$ and 6.2, our results are larger by about 0.025, a two standard deviation difference. At $\beta=6.4$ our number is consistent with theirs from a 32^3 spatial lattice. They find, however, that B_K decreases on larger lattices, suggesting that our number at $\beta=6.4$ may be afflicted by finite size errors. Nevertheless, our extrapolated value is only 1.5 standard deviations (i.e., 0.03) above theirs. This comparison gives us confidence that our results are correct within the quoted errors, and in particular that our procedure of doubling the lattice in the time direction has not introduced additional systematic errors.

The JLQCD Collaboration has also used the gauge-invariant version of the unsmeared operator. Averaging this with the Landau gauge operator, they quote a preliminary result $B_K(\mathrm{NDR},2\ \mathrm{GeV})=0.587\pm0.007(\mathrm{stat})\pm0.017(\mathrm{syst})$, or adding errors in quadrature: $B_K=0.59\pm.02$. The agreement with our result of $B_K=0.62\pm.03$ is gratifying, since the operators we use have entirely different perturbative and power corrections.[5]

We note that both calculations are systematics limited, and that of the systematic errors, the most important one quoted by JLQCD is estimated by comparing results for different operators. In their preliminary report [23], JLQCD ascribes such differences to terms of order $\alpha_{\overline{\mathrm{MS}}}(\mu)^2$, where $\mu=2$ GeV is the conventional scale at which the answer is quoted. On this point we disagree in principle. We certainly agree that errors of order $\alpha_{\overline{\mathrm{MS}}}(\mu)^n$ are introduced when one uses n-loop evolution to the final scale, but these corrections are universal and should not appear as differences between lattice operators. If, for example, we compare our one-loop corrected smeared and unsmeared operators, the connection between the two is

$$\mathcal{Q}_K^{\mathrm{UN}}(\mu)=\mathcal{Q}_K^{\mathrm{SM}}(\mu)\{1+O[\alpha_{\overline{\mathrm{MS}}}(q^*)^2]+O(a^2)\}, \quad (33)$$

where we have taken the same $q^*=K/a$ in renormalizing both operators.[6] Since $\alpha_{\overline{\mathrm{MS}}}(K/a)^2$ vanishes as $a\to0$, we conclude that when correctly extrapolated, the operators should give the same result at $a=0$. Of course one would need rather precise data to make a fit including the $\alpha_{\overline{\mathrm{MS}}}(K/a)^2$ term, but in principle it could be done if more precision were required. If one uses a simple a^2 extrapolation, however, the α^2 term in Eq. (33) will not extrapolate to zero, but instead to an artifact of size $\alpha_{\overline{\mathrm{MS}}}(q^*)^2$.

V. RESULTS FOR $B_7^{3/2}$ AND $B_8^{3/2}$

We have evaluated $B_7^{3/2}$ and $B_8^{3/2}$ using almost the same method as for B_K. The only difference concerns the extrapolation to the physical kaon mass. This we have done separately for matrix elements appearing in the numerator and denominator of the definitions Eqs. (6) and (7), prior to the evolution from q^* to $\mu=2$ GeV.

Our results are summarized in Table III. The most striking feature is the very strong q^* dependence of the results for unsmeared operators. Indeed, as one goes from $q^*=\pi/a$ to $1/a$, which causes $\alpha_{\overline{\mathrm{MS}}}(q^*)$ to increase by roughly 40%, $B_{7,8}^{3/2}$ change sign because the negative one-loop matching contribution exceeds the tree-level contribution. Clearly, we cannot use one-loop matching for the unsmeared operators. We stress that the large perturbative corrections for unsmeared operators do not invalidate the results for smeared operators. There will always be choices of discretization procedure for which the perturbative series is poorly convergent at the lattice spacing one is working.

For completeness we mention that we expect a similar problem to render the gauge-invariant unsmeared operators unsuitable for a calculation of $B_{7,8}^{3/2}$. The dominant contribu-

[5]We can improve the agreement by dropping our unsmeared data point at $\beta=6.4$, which, as noted above, might be afflicted with finite size errors.

[6]Using a different choice of q^* for the two operators would lead to an additional factor coming from the evolution between the two scales. However, since this factor tends to unity in the continuum limit, our conclusion is unaltered.

TABLE III. Results for $B_7^{3/2}$(NDR,2 GeV) and $B_8^{3/2}$(NDR,2 GeV), at the physical kaon mass, using α scales.

Operator	q^*	$\beta=6.0$	$\beta=6.2$	$a=0$
		$B_7^{3/2}$		
Unsmeared	$1/a$	$-1.951(09)$	$-1.097(15)$	
Unsmeared	π/a	0.606(06)	0.645(06)	
Smeared	$1/a$	0.989(05)	0.823(16)	0.615(30)
Smeared	π/a	1.085(06)	0.903(14)	0.674(32)
		$B_8^{3/2}$		
Unsmeared	$1/a$	$-1.486(11)$	$-0.689(15)$	
Unsmeared	π/a	0.822(05)	0.864(05)	
Smeared	$1/a$	1.240(06)	1.030(16)	0.766(37)
Smeared	π/a	1.288(06)	1.076(17)	0.810(39)

tion to the matrix elements comes from the PP part of the operator. Since this part of the operator is local, and does not require gauge links, it is the same as the corresponding part of the unsmeared operator. But it is this part of the unsmeared operator which leads to the bulk of the large one-loop matching corrections seen in Sec. II. Although in principle it is possible these large corrections will be canceled by the as yet uncalculated contributions from the other parts of the gauge-invariant operator, this seems unlikely in practice.

The situation is much improved for the smeared operators—the q^* dependence, while more significant than for B_K, is only at the 10% level. This is a much larger uncertainty than the statistical errors or that due to the choice of lattice spacings, or that from the contamination by excited states or off-shell matrix elements. Thus we do not give details concerning these other uncertainties. For our final results, we quote

$$B_7^{3/2}(\text{NDR,2 GeV}) = 0.62 \pm 0.03(\text{stat}) \pm 0.06(\text{syst}), \quad (34)$$

$$B_8^{3/2}(\text{NDR,2 GeV}) = 0.77 \pm 0.04(\text{stat}) \pm 0.04(\text{syst}), \quad (35)$$

where the systematic error is our estimate (based on the q^* variation) of the uncertainty due to the truncation of perturbation theory.

It is interesting to compare our staggered results with those from Wilson fermions. While a continuum extrapolation is not yet available, the results at $\beta=6.0$ [9] are

$$B_7^{3/2}(\text{NDR,2 GeV,Wilson}) = 0.58 \pm 0.02 \pm 0.07, \quad (36)$$

$$B_8^{3/2}(\text{NDR,2 GeV,Wilson}) = 0.81 \pm 0.03 \pm 0.03. \quad (37)$$

The central values and error bars have been determined in the same way as in this paper. Obviously the agreement is already rather good, and indicates indirectly that the $O(a)$ errors in the Wilson case are not particularly large. This is in contrast to the case of B_K, where the explicit breaking of chiral symmetry disrupts the delicate cancellation between the VV and AA matrix elements, and gives dramatically large $O(a)$ errors. For B_7 and B_8 the result is dominated by the PP matrix element, with no particular constraint from chiral symmetry, presumably yielding more moderate $O(a)$ errors.

VI. NONPERTURBATIVE RESULTS FOR MATCHING CONSTANTS

It is clear from the results of the previous two sections that using finite order perturbation theory to match lattice and continuum operators is an important source of uncertainty in results for matrix elements. In this section we investigate the accuracy of perturbative matching factors by calculating some of them nonperturbatively. In particular, we are able to assess the accuracy of the perturbative matching factor for the quark mass Z_m which is an important ingredient in determining continuum quark masses.

The basic idea is simple, and has been applied extensively with Wilson fermions. A given continuum operator can be discretized in different ways, each discretization having an associated matching factor. Only if these matching factors are chosen correctly will the different choices yield the same matrix elements. This allows a nonperturbative determination of the ratio of matching factors. Note that in such ratios the anomalous dimension factors cancel, implying that the ratios are finite functions of the lattice coupling.

In this section, we apply this idea to the pseudoscalar and axial densities. It could, in principle, be applied also to four-fermion operators such as $\mathcal{Q}_K$, but this requires studying a matrix mixing problem, and thus using several external states, and is beyond the scope of the present work. We only consider operators for which the matching is diagonal.

Our notation in this section differs from that used above. We use P^{UN}, for example, to refer to the bare lattice operator constructed of unsmeared fields:

$$P^{\text{UN}} = (1/\sqrt{N_f})\,\overline{\chi}_S^{\text{UN}}(\overline{\gamma_5 \otimes \xi_5})\chi_D^{\text{UN}}. \quad (38)$$

In other words, we do not include the matching factor in the definition of P^{UN}. The matching equation becomes [cf. Eq. (16)]

$$P^{\text{cont}} = Z_P^{\text{UN}} P^{\text{UN}}[1 + O(a^2)], \quad (39)$$

$$Z_P^{\text{UN}} = 1 + \frac{\alpha_{\overline{\text{MS}}}(q^*)}{4\pi}[-\gamma_P^{(0)} \ln(q^* a) + c_P^{\text{UN}}] + O(\alpha^2). \quad (40)$$

Here we have used the fact that for the matrix elements of interest the corrections are quadratic in the lattice spacing. Similar definitions apply to P^{SM}, A_μ^{UN}, and A_μ^{SM}. We also need to introduce the gauge-invariant version of the axial current A_μ^{GI}, which is the unsmeared current with appropriate gauge links included. Note that $P^{\text{GI}} = P^{\text{UN}}$, i.e., the unsmeared pseudoscalar density is already gauge invariant since it is local.

The quantities we determine nonperturbatively are $Z_P^{\text{SM}}/Z_P^{\text{UN}}$, $Z_A^{\text{SM}}/Z_A^{\text{UN}}$, and Z_A^{UN}. The first we obtain starting from the result

$$\langle 0|P^{\rm cont}|\overline{K^0}\rangle = \langle 0|Z_P^{\rm UN}P^{\rm UN}|\overline{K_G^0}\rangle[1+O(a^2)]$$
$$= \langle 0|Z_P^{\rm SM}P^{\rm SM}|\overline{K_G^0}\rangle[1+O(a^2)] \qquad (41)$$

from which we find the nonperturbative estimate

$$\frac{Z_P^{\rm SM}}{Z_P^{\rm UN}} = \frac{\langle 0|P^{\rm UN}|\overline{K_G^0}\rangle}{\langle 0|P^{\rm SM}|\overline{K_G^0}\rangle}[1+O(a^2)]. \qquad (42)$$

The equations for $Z_A^{\rm SM}/Z_A^{\rm UN}$ are identical except that P is replaced by A_4. The one-loop perturbative results for the relevant ratios are

$$\frac{Z_P^{\rm SM}}{Z_P^{\rm UN}} = \left[1+\frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}(c_P^{\rm SM}-c_P^{\rm UN})\right]$$
$$= 1+\frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}\times 30.2532, \qquad (43)$$

$$\frac{Z_A^{\rm SM}}{Z_A^{\rm UN}} = \left[1+\frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}(c_A^{\rm SM}-c_A^{\rm UN})\right] = 1-\frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}\times 1.9384. \qquad (44)$$

The determination of $Z_A^{\rm UN}$ proceeds slightly differently. We note that the gauge-invariant axial current is partially conserved on the lattice, implying that $Z_A^{\rm GI}=1$. Thus we could determine $Z_A^{\rm UN}$ by taking ratios of matrix elements of $A_\mu^{\rm GI}$ to those of $A_\mu^{\rm UN}$. While we have not calculated matrix elements using $A_\mu^{\rm GI}$, we do have available the matrix elements of its divergence $\partial_\mu A_\mu^{\rm GI}=2mP^{\rm GI}=2mP^{\rm UN}$. This is sufficient as long as the matrix element involves nonzero momentum transfer. Note that the lattice partial conservation equation is exact as long as we use the appropriate lattice derivative [24]. Putting this all together we arrive at

$$\frac{1}{Z_A^{\rm UN}} = \left(\frac{-\sinh(m_K)\langle 0|A_4^{\rm UN}|\overline{K_G^0}\rangle}{2(m_q/u_0)\langle 0|P^{\rm UN}|\overline{K_G^0}\rangle}\right)[1+O(a^2)], \qquad (45)$$

where m_q/u_0 is tadpole improved quark mass, and m_K is the mass of the K_G^0. All quantities on the RHS of this equation are in lattice units. We have used $\sinh(m_K)$ on the RHS, rather than m_K itself, because if we replace $A_4^{\rm UN}$ by $A_4^{\rm GI}$ then the ratio on the RHS is exactly equal to unity. In other words, $\sinh(m_K)$ is the appropriate kinematical factor for the exactly conserved current. We choose to keep it for the unsmeared current in the hope that it will reduce the size of the $O(a^2)$ terms. The perturbative expression to which Eq. (45) should be compared is

$$\frac{1}{Z_A^{\rm UN}} = \left[1+\frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}\frac{4}{3}(\pi^2-9.17479)\right]. \qquad (46)$$

Note that if we had used the average link in Landau gauge to determine u_0, rather than the average plaquette, then $Z_A^{\rm UN}=1$ at one-loop order.

We determine the required ratios of matrix elements using the quantities previously used to determine the vacuum saturation approximants appearing in the B parameters. For example, consider the ratio

TABLE IV. Nonperturbative and perturbative results for ratios of matching constants. Quadratic extrapolations to $a=0$ use α scales.

Quantity	Method	$\beta=6.0$	$\beta=6.2$	$a=0$
$Z_A^{\rm SM}/Z_A^{\rm UN}$	Nonpert	1.131(5)	1.060(1)	0.972(6)
	Pert($q^*=1/a$)	0.970	0.973	0.977
	Pert($q^*=\pi/a$)	0.979	0.981	0.983
$Z_P^{\rm SM}/Z_P^{\rm UN}$	Nonpert	2.245(11)	1.861(6)	1.380(19)
	Pert($q^*=1/a$)	1.462	1.416	1.359
	Pert($q^*=\pi/a$)	1.323	1.301	1.272

$$R_P(t) = \frac{\langle W(t_1)\Sigma_{\vec{y}}P^{\rm UN}(\vec{y},t)\rangle\langle\Sigma_{\vec{y}'}P^{\rm UN}(\vec{y}',t)W(t_2)\rangle}{\langle W(t_1)\Sigma_{\vec{y}}P^{\rm SM}(\vec{y},t)\rangle\langle\Sigma_{\vec{y}'}P^{\rm SM}(\vec{y}',t)W(t_2)\rangle}. \qquad (47)$$

For $t_1 \ll t \ll t_2$ this should be independent of t and gives directly $(Z_P^{\rm SM}/Z_P^{\rm UN})^2$ aside from $O(a^2)$ corrections. We average $R_P(t)$ over the same plateau regions as for the B parameters. Similar ratios are used for the other quantities.

The resulting data for $Z_P^{\rm SM}/Z_P^{\rm UN}$ and $Z_A^{\rm SM}/Z_A^{\rm UN}$ are well represented by a linear function of m_K^2. This dependence on m_K^2 is an $O(p^2a^2)$ discretization error, because any physical dependence cancels in the ratio of matrix elements, We remove this error by extrapolating to the chiral limit. We do a similar extrapolation for $1/Z_A^{\rm UN}$, although the dependence on m_K^2 is much weaker, presumably because of the $\sinh(m_K)$ factor in Eq. (45).

Our nonperturbative results for the ratio of smeared to unsmeared matching factors after chiral extrapolation are collected in Table IV. What is most striking is the substantial dependence on lattice spacing, particularly for the ratio of Z_P's. This is due to a combination of $O(a^2)$ discretization errors and the variation of the perturbative matching factors which depend on $g^2(a)$. To analyze these results we assume the following form for the ratios

$$\text{Ratio(nonpert)} = \text{Ratio(one loop;}q^*) + a^2\Lambda^2, \qquad (48)$$

where the one-loop results are given above, and Λ is an unknown constant. In other words we ignore completely higher powers of a, and assume that higher powers of α are well represented by the appropriate choice of $q^*=K/a$. The difference ratio(nonpert)$-$ratio(one loop) should then, for the right choice of q^*, extrapolate to zero in the continuum limit. Conversely, one could regard this procedure as providing an approximate nonperturbative definition of q^*. To show the individual variations in the perturbative[7] and nonperturbative results we give, in Table IV, the continuum value for each obtained by linear extrapolation in a^2. We do this for our two standard choices $q^*a=1$ and π. What we find remarkable is that, modulo the simplifying assumption of Eq. (48), the nonperturbative and perturbative predictions

[7]Note that we are here making use of the fact that our extrapolation to the continuum limit does not remove terms which vary logarithmically. This is an example of the problem discussed in Sec. IV.

TABLE V. Results for $1/Z_A^{\rm UN}$.

Method	$\beta=6.0$	$\beta=6.2$	$\beta=6.4$
Nonperturbative	0.978(09)	1.010(09)	0.998(20)
Pert ($q^*=1/a$)	0.986	0.987	0.988
Pert ($q^*=\pi/a$)	0.990	0.991	0.991

agree if we use $q^*\approx 1/a$ but not for $q^*\approx\pi/a$. The discrepancy for $q^*=\pi/a$ is particularly significant for $Z_P^{\rm SM}/Z_P^{\rm UN}$.

The results for $1/Z_A^{\rm UN}$, given in Table V, behave very differently. There is very little dependence on the lattice spacing—presumably because the unsmeared current is very similar to the gauge-invariant current for which all the numbers in the table would be unity independent of lattice spacing. In fact, the errors are such that an extrapolation to $a=0$ is not useful, and so we compare our results to perturbation theory at each lattice spacing. The results are reasonably consistent, but we cannot distinguish between different values of q^* in this case.

An important application of the above results is to estimate the reliability of the one-loop result for the matching factor Z_m. This factor converts the lattice results for quark masses to a continuum scheme such as $\overline{\rm MS}$, as discussed in Ref. [12]. The perturbative result after tadpole improvement is

$$Z_m(\mu=q^*)=\frac{1}{Z_P^{\rm UN}(\mu=q^*)}$$
$$=1-\frac{\alpha_{\overline{\rm MS}}(q^*)}{4\pi}[-\gamma_P^{(0)}\ln(q^*a)+c_P^{\rm UN}], \qquad (49)$$

where for simplicity we have chosen to consider the case where the final scale μ equals the matching scale q^*. The one-loop correction to Z_m is large at typical lattice spacings. For example, at $\beta=6$, and taking $q^*=1/a$, $Z_m=1.598$. This suggests that higher order corrections may be important. We can, however, rewrite Z_m as

$$Z_m=\left(\frac{Z_P^{\rm SM}}{Z_P^{\rm UN}}\right)\left(\frac{1}{Z_P^{\rm SM}}\right). \qquad (50)$$

The results of Table IV show that the bulk of the perturbative correction lies in the first factor $Z_P^{\rm SM}/Z_P^{\rm UN}$. At $\beta=6$ it is 1.462, while the second factor is 1.136 (again for $\mu=q^*=1/a$). Thus, one would expect that the dominant source of higher order terms in Z_m is the first factor, and that the uncertainty which they introduce could be substantially reduced by obtaining a nonperturbative estimate of this factor. We have attempted such an estimate above, with the preliminary conclusion that perturbation theory with $q^*=1/a$ works to within a few percent, aside from discretization errors. If we accept this result then we obtain the partly nonperturbative estimate[8] $Z_m=1.462\times 1.136=1.66$. The point we wish to stress is that this is not very different from the one-loop estimate of 1.60. In particular, the difference is much smaller than the naive estimate of the two-loop contribution $0.6^2=0.36$ based on the assumption of geometric growth. This analysis thus suggests that the one-loop perturbative value of Z_m used in the analysis of quark masses is good to about 5% for $\beta\geqslant 6.0$ provided one uses $\alpha_{\overline{\rm MS}}(q^*\approx 1/a)$.

VII. CONCLUSIONS

In this paper we have presented a variety of results for weak matrix elements using staggered fermions. Our major focus has been on the importance of using a variety of discretizations of continuum operators. There are two reasons for doing so. First, comparing results with different lattice operators gives an estimate of the uncertainty in the matching factors between continuum and lattice operators. For B_K this may be the dominant source of error in future calculations, aside from that due to quenching. Second, for some operators the perturbative matching factors are not convergent at present couplings, and so one must use different discretizations. It turns out that the smeared operators have uniformly moderate perturbative corrections. Using them we are able to obtain the first results for $B_7^{3/2}$ and $B_8^{3/2}$ using staggered fermions.

Our results for the B parameters confirm and extend existing lattice results. In particular, for B_K we find that smeared operators give results consistent with those from unsmeared and gauge-invariant operators. We confirm the low value found in our preliminary study [5], a result which has been improved and extended by the JLQCD Collaboration [7]. For $B_7^{3/2}$ and $B_8^{3/2}$ we find results consistent with those using Wilson fermions. All these numbers are important inputs into analyses attempting to constrain the Cabibbo-Kobayashi-Maskawa (CKM) matrix. It is encouraging that the errors we are considering are at the few percent level. It is important to stress, however, that we are still using the quenched approximation, and also working with a kaon composed of degenerate quarks. For a discussion of the importance of these approximations see Refs. [1,13].

As an offshoot of our study, we have calculated several ratios of matching factors nonperturbatively. These ratios are finite functions of the lattice coupling, and thus allow a test of tadpole improved perturbation theory. We find that one-loop perturbation theory works well if we set the scale in the one-loop coupling $\alpha_{\overline{\rm MS}}(q^*)$ to be $q^*=1/a$, but not for $q^*=\pi/a$. This is consistent with the expectations of Ref. [16]. This conclusion is, however, preliminary because we have results only at two lattice spacings. To convincingly disentangle discretization errors from perturbative corrections will require precise results at several lattice spacings.

We have used the results for ratios of matching factors to make a partly nonperturbative estimate of the size of the

[8]It is not advantageous to directly use the nonperturbative results, e.g., $Z_P^{\rm SM}/Z_P^{\rm UN}=2.245$ at $\beta=6$. Doing so introduces additional $O(a^2)$ errors which one would then have to remove by extrapolation.

matching factor for the quark mass Z_m. Our result is $\sim 5\%$ higher than the one-loop perturbative result. This is a small enough change that it does not alter the essential conclusion of Ref. [12], namely that light quark masses are considerably smaller than previously thought.

Finally, we note that our results show many examples of significant discretization errors. To make progress with simulations of full QCD, where one is restricted to larger lattice spacings, it may be necessary to improve the staggered fermion action.

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