The standard model from a new phase transition on the lattice

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Several years ago it was conjectured in the so-called Roma approach that gauge fixing is an essential ingredient in the lattice formulation of chiral gauge theories. In this paper we discuss in detail how the gauge-fixing approach may be realized. As in the usual (gauge-invariant) lattice formulation, the continuum limit corresponds to a Gaussian fixed point that now controls both the transversal and the longitudinal modes of the gauge field. A key role is played by a new phase transition separating a conventional Higgs or Higgs-confinement phase from a phase with broken rotational invariance. In the continuum limit we expect to find a scaling region where the lattice correlators reproduce the Euclidean correlation functions of the target (chiral) gauge theory, in the corresponding continuum gauge. [S0556-2821(98)03901-0]

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I. INTRODUCTION

The great difficulty in constructing chiral gauge theories, such as the standard model, using lattice regularization is related to the *doubling problem* [2–5]. In lattice QCD, species doubling occurs when the discretized fermion action has an unwanted symmetry that should be anomalous in the continuum. When one uses Wilson fermions, the Wilson term eliminates the doublers at the price of breaking all the axial symmetries explicitly. In the continuum limit, one expects to recover the axial symmetries, except the anomalous U(1), by tuning the fermion hopping parameter to a critical value.

In the lattice discretization of a chiral gauge theory, one has to account for the fact that a Weyl fermion in a complex representation contributes to the gauge anomaly. This means that a lattice action for a single chiral fermion cannot be gauge invariant. We will assume below that, as in the continuum, the lattice fermion action involves a sum over different complex representations, whose total gauge anomaly is zero. The question is to what extent the violations of gauge invariance, coming from the individual representations, cancel each other.

Consider the *regularized* effective action obtained by integrating out an anomaly free set of chiral fermions. In the continuum, one can use dimensional regularization to define the effective action for smooth gauge fields that vanish rapidly at infinity. Using the freedom to add local counter-terms, the violations of gauge invariance are proportional to the dimensionless parameter $\epsilon = d - 4$, and so they vanish in the limit $\epsilon \to 0$. This extends to topologically non-trivial background fields, using for example the ζ-function regularization [6].

On the lattice one encounters a fundamentally different situation. The lattice spacing is a dimensionful parameter that plays a dual role. First, it provides a UV cutoff, by replacing the infinite range of momentum integrals with an integration over the periodic Brillouin zone. In addition, the lattice spacing enters the (multi-valued) mapping from the compact link variables $U_{x,\mu}$, to the Lie-algebra valued A_μ field. These differences in the global structure imply that a generic *lattice* gauge transformation, considered as a mapping that acts on the Fourier space of the lattice A_μ-field, is qualitatively different from the corresponding mapping defined by a *continuum* gauge transformation. The result is that the lattice effective action suffers from generic violations of gauge invariance which are not controlled by any small parameter. We know of no method that ensures the smallness of these violations on the entire lattice gauge orbit, at the price of tuning any finite number of parameters (see Sec. IV C or the review article [7] for more details).

In the so-called Roma Approach [1], it was conjectured that gauge fixing is a crucial ingredient in the lattice formulation of chiral gauge theories. A gauge fixing action should assign a bigger Boltzmann weight to a smooth gauge field, relative to a rough field that belongs to the same gauge orbit. This should reduce lattice-artefact violations of gauge invariance, because the latter are associated with the roughness of the lattice gauge field.

In spite of this promising picture, the gauge-fixing approach has remained elusive. A naive discretization of the Lorentz gauge-fixing action leads to a lattice action that has a dense set of *lattice* Gribov copies with no continuum counterparts. These lattice artifact Gribov copies exist even for the classical vacuum. (Remarkably, the proliferation of Gribov copies on the lattice resembles the fermion doubling problem in a number of ways.) As a result, the Boltzmann weight of too many rough lattice configurations is not suppressed.

In this paper we construct a lattice gauge-fixing action that accommodates this problem. (See Ref. [8] for a preliminary version of this work.) The gauge-fixing action is associated with a new generic phase diagram. We argue that, in this phase diagram, there is a Gaussian critical point that belongs to the universality class of a gauge-fixed continuum theory. In comparison with the gauge-invariant lattice definition of QCD, the weak coupling limit here controls not only the transversal modes of the gauge field, but also the longitudinal ones.

As discussed above, gauge invariance of the target continuum theory, as well as the residual Becchi-Rouet-Stora-Tyutin (BRST) invariance, are both explicitly broken on the lattice. By tuning a finite number of counter-terms, one

hopes to recover BRST invariance in the continuum limit, provided the fermion spectrum is anomaly free [1]. The BRST identity that requires the vanishing of the renormalized gauge boson mass, $m_r^2 = 0$, plays a key role. Since the regularization is not gauge invariant, a mass counter-term has to be introduced, and its parameter needs to be tuned, in order to enforce this BRST identity. Usually, a negative renormalized mass-squared indicates spontaneous symmetry breaking. Here we encounter a new feature, namely, the *gauge field* condenses if its mass-squared parameter becomes too negative. This implies that the new critical point is located on the boundary between a conventional phase, which is invariant under lattice rotations, and a new phase where the lattice rotation symmetry is broken spontaneously by the vacuum expectation value (VEV) of the gauge field.

The construction of the gauge-fixing action is presented in Sec. II. The main results are (a) the gauge-fixing action has a unique absolute minimum, $U_{x,\mu} = I$, and (b) perturbation theory around this minimum is manifestly renormalizable. In Sec. III we discuss a simple chiral fermion action. The validity of perturbation theory implies the onset of a scaling behavior in the weak-coupling limit. Up to the regularization-dependent counter-terms, the continuum Lagrangian that controls the scaling behavior can be read off from the marginal and relevant terms of the lattice action. The scaling region should therefore faithfully reproduce the correlation functions of the target chiral gauge theory, in the corresponding gauge.

Without the new gauge-fixing action, the fermion action of Sec. III *does not* lead to a chiral gauge theory in the continuum limit. The longitudinal modes fluctuate strongly, and their non-perturbative dynamics ultimately renders the fermion spectrum vector-like. (This applies to many other chiral fermion proposals, see Refs. [7,9].) In the second part of this paper, we examine the dynamics of the lattice longitudinal modes from a broader point of view. We explain the problems created by this dynamics, and how they may be solved within the present approach.

In Sec. IV we discuss the lattice effective action, and the role of lattice artefact Gribov copies. In Sec. V we discuss the complete phase diagram in the limit of a vanishing gauge coupling. In Sec. VI we explain how our approach evades the no-go theorems. Several open questions are discussed in Sec. VII, and our conclusions are offered in Sec. VIII.

II. CONSTRUCTING THE LATTICE GAUGE-FIXING ACTION

A. The phase transition associated with a critical vector boson

This section is devoted to a step-by-step construction of the gauge-fixing action. As discussed in the Introduction, the lattice-regularized theory has no symmetry that protects the masslessness of the vector bosons. Therefore, in the relevant part of the phase diagram, the lattice vector field is generically not critical. Now, according to the standard lore, the correlation length should diverge close to a continuous phase transition associated with the condensation a Bose field. In this paper we wish to apply this to the lattice vector field.

As a preliminary requirement for a continuous transition, one needs a higher-power term, that stabilizes the classical potential $V_{cl}(A_\mu)$ when the coefficient of the quadratic term

changes sign. In a weakly coupled theory, the actual location of the transition should be close to its tree-level value, and near the transition one expects the onset of a scaling behavior governed by renormalized (continuum) perturbation theory.

Our aim is to achieve criticality of the lattice vector field, very much like the way this is done in the familiar Φ^4 theory. However, going from a spin-0 to a spin-1 field presents new difficulties. The lattice theory is formulated in terms of the link variables $U_{x,\mu}$, which are group-valued parallel transporters. On the other hand, renormalized perturbation theory, that governs the scaling region, is more naturally formulated in terms of the Lie-algebra valued A_μ field. Thus, it takes some trial and error to find the U_μ-dependent action that best suits our purpose.

Another complication arises because not every renormalizable vector theory is unitary. A unitary, physical Hilbert space exists if and only if the vector theory is actually an anomaly free gauge theory in a gauge fixed form. This requires us to choose the lattice action, such that the marginal gauge symmetry breaking terms in the tree-level vector Lagrangian have the form

$$\frac{1}{2\xi_0}(\text{gauge condition})^2. \qquad (2.1)$$

An appropriate Faddeev-Popov ghost action will be necessary too.

B. A higher-derivative Higgs action

Our starting point is the lattice action

$$S = S_G(U) + S_H(\phi, U). \qquad (2.2)$$

Here $S_G(U)$ is the usual plaquette action. The Higgs action is

$$S_H = \text{tr} \sum \ (-\kappa \phi^\dagger \Box(U)\phi + \widetilde{\kappa} \phi^\dagger \Box^2(U)\phi), \qquad (2.3)$$

where

$$\Box_{xy}(U) = \sum_\mu \ (\delta_{x+\hat{\mu},y} U_{x,\mu} + \delta_{x-\hat{\mu},y} U_{y,\mu}^\dagger) - 8\delta_{x,y}, \qquad (2.4)$$

is the standard nearest-neighbor covariant Laplacian. The lattice spacing a is equal to one. Both $U_{x,\mu}$ and ϕ_x take values in a Lie group G. The first term on the right-hand side (RHS) of Eq. (2.3) is a conventional lattice Higgs action, whereas the second term is a higher derivative (HD) action.

(HD actions were recently discussed by Jansen, Kuti and Liu [10]. Here we are interested in a different critical point from the one studied in Ref. [10]. At the technical level, this allows us to introduce only a Laplacian-squared HD term, whereas for the purpose of Ref. [10] it was crucial to introduce also a Laplacian-cubed one.)

The action Eq. (2.2) is gauge invariant, where the lattice gauge transformation is given by $U_{x,\mu} \to g_x U_{x,\mu} g_{x+\hat{\mu}}^\dagger$ and $\phi_x \to g_x \phi_x$ for $g_x \in G$. Now, since $\phi_x \in G$ too, we may use

the lattice gauge invariance to eliminate completely the ϕ_x field. Note that this operation affects only S_H. We introduce the notation

$$S_V(U) = S_H(\phi, U)|_{\phi_x = I}. \tag{2.5}$$

The subscript of S_V stands for ''vector.'' S_H can be recovered from S_V by making the substitution $U_{x,\mu} \rightarrow \phi_x^\dagger U_{x,\mu} \phi_{x+\hat\mu}$. (The significance of the ϕ_x field, which is associated with the longitudinal degrees of freedom, is discussed in Sec. V and Sec. VI.)

We will denote the first formulation of the theory [Eq. (2.2)] as the *Higgs picture*. The alternative formulation [Eq. (2.5)] where only $U_{x,\mu}$ (but not ϕ_x) is present, is called the *vector picture*. The equality of the partition functions in the two pictures extends to observables. Any observable in the vector picture is mapped to a gauge-invariant observable in Higgs picture, and vice versa. Thus, we are dealing with two mathematically equivalent formulations of the same theory [11].

In this section we assume $\widetilde\kappa \gg 1$. The physics in this parameter range is more easily accounted for in the vector picture, which is used below to study the classical potential, and to set up the weak-coupling expansion. In the vector picture, the gauge non-invariance of the action resides in S_V. Therefore, we will ultimately demand that the marginal terms in S_V have the form of a gauge-fixing action, *cf.* Eq. (2.1).

C. The classical potential

For simplicity we consider the classical potential in the U(1) case. The essential features generalize to the non-Abelian case. Making use of the standard weak-coupling expansion

$$U_{x,\mu} = \exp(ig_0 A_{x+\hat\mu/2,\mu}), \tag{2.6}$$

and considering a constant A_μ field, the action S_V leads to the following classical potential:

$$V_{cl} = \kappa \mathcal{F}(g_0 A_\mu) + \widetilde\kappa \mathcal{F}^2(g_0 A_\mu), \tag{2.7}$$

$$\mathcal{F}(g_0 A_\mu) = 2\sum_\mu [1 - \cos(g_0 A_\mu)]. \tag{2.8}$$

Note that S_G is zero for a constant Abelian field.

For $\kappa > 0$, the absolute minimum of the classical potential is $A_\mu = 0$ (mod $2\pi/g_0$). Since the quadratic term in V_{cl} comes only from the κ-term, a non-zero vector condensate arises for $\kappa < 0$. The classical features of the transition can be determined by keeping only the leading term in the expansion of $\mathcal{F}(g_0 A_\mu)$, separately for the κ- and $\widetilde\kappa$-terms. (This approximation is consistent for $|\kappa| \ll \widetilde\kappa$.) The result is the quartic potential

$$V_{cl} \approx \kappa g_0^2 \sum_\mu A_\mu^2 + \widetilde\kappa g_0^4 \left(\sum_\mu A_\mu^2 \right)^2. \tag{2.9}$$

Equation (2.9) closely resembles the potential of a Φ^4 theory. For small negative κ, the minimum is

$$\|\langle A_\mu \rangle\| = \frac{1}{g_0} \left(\frac{|\kappa|}{2\widetilde\kappa} \right)^{1/2}, \quad \kappa < 0. \tag{2.10}$$

Equation (2.10) exhibits the mean-field critical exponent 1/2. It is easy to check that this is the absolute minimum of the classical potential.

We note that Eq. (2.10) is invariant under arbitrary SO(4) rotations, reflecting the symmetry of the approximate potential Eq. (2.9). When higher-order corrections are taken into account, the rotational symmetry of the potential is reduced to the lattice hypercubic symmetry.

Below, the phase with a non-zero vector condensate will be denoted as the FMD phase. We will speak about the FMD transition, referring to the transition from the rotationally invariant phase to the FMD phase in the large-$\widetilde\kappa$ region. FMD stands for *ferromagnetic directional*. The preferred spacetime direction of the FMD phase is defined by the vectorial VEV. For $g_0 \neq 0$, there are no Goldstone bosons in the FMD phase, because the lattice rotation group is discrete. The limiting $g_0 = 0$ theory is discussed in Sec. V, and in particular we explain there in what sense the FMD phase is ferromagnetic.

D. The weak-coupling expansion

We now want to study fluctuations around the classical vacuum $U_{x,\mu} = I$ (equivalently $A_\mu = 0$) in the rotationally invariant phase, close to the FMD transition where the theory defined by Eqs. (2.2)–(2.5) is expected to be critical. The FMD transition is given by $\kappa = 0$ in the classical approximation. As mentioned earlier, we are assuming $\widetilde\kappa \gg 1$. We, therefore, focus on the HD term in its vector picture form. Relaxing the assumption of a constant A_μ field, we find

$$\widetilde\kappa \phi^\dagger \Box^2(U) \phi|_{\phi_x = 1} = \widetilde\kappa g_0^2 \left(\left(\sum_\mu \Delta_\mu^- A_\mu \right)^2 \right.$$
$$\left. + g_0^2 \left(\sum_\mu A_\mu^2 \right)^2 + \cdots \right), \tag{2.11}$$

where the dots stand for irrelevant operators. Δ_μ^- is the backward lattice derivative, defined as $\Delta_\mu^- f_x = f_x - f_{x-\hat\mu}$ for any function f_x.

Equation (2.11) contains a longitudinal kinetic term. We define

$$\frac{1}{2\xi_0} \equiv \widetilde\kappa g_0^2, \tag{2.12}$$

and we will assume that ξ_0 is an $O(1)$ parameter. This means that the longitudinal kinetic term belongs to the tree-level Lagrangian. Remember that a transversal kinetic term is provided by the gauge-invariant plaquette action S_G. Finally, we assume that the tree-level vector boson mass is zero. Under these assumptions, the tree-level vector propagator is

$$G_{\mu\nu}(p) = \frac{\Pi_{\mu\nu}^\perp(\hat p) + \xi_0 \Pi_{\mu\nu}^\parallel(\hat p)}{\hat p^2}, \tag{2.13}$$

where

$$\Pi^{\perp}_{\mu\nu}(\hat{p}) = \delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu / \hat{p}^2, \qquad (2.14)$$

$$\Pi^{\parallel}_{\mu\nu}(\hat{p}) = \hat{p}_\mu \hat{p}_\nu / \hat{p}^2, \qquad (2.15)$$

and $\hat{p}_\mu = 2 \sin(p_\mu/2)$.

Massless weak-coupling perturbation theory is defined by the vector propagator Eq. (2.13), and by a set of vertices which can be read off from the lattice action using Eq. (2.6) in the usual way.

E. The gauge-fixing action

In view of the presence of kinetic terms for all polarizations, lattice perturbation theory is manifestly *renormalizable*. According to the standard lore, renormalizability implies a Lorentz invariant *scaling* behavior in the vicinity of the Gaussian critical point $g_0 = 1/\widetilde{\kappa} = 0$. The scaling behavior is achieved by tuning a finite number of counter-terms, that correspond to the relevant and the marginal operators.

At this stage, the marginal gauge symmetry breaking terms in the tree-level vector action [see Eq. (2.11)] do not have the form of a gauge-fixing action, *cf.* Eq. (2.1). The way to remedy this is to add another term to the HD action. There are two options. The new term can be chosen to cancel the quartic term in Eq. (2.11). The remaining marginal term—the longitudinal kinetic term—has the form of a gauge-fixing action for the linear Lorentz gauge $\partial \cdot A = 0$. Alternatively, the new HD term can lead to a mixed marginal term proportional to $(\partial \cdot A)A^2$. In this case one recovers the non-linear gauge $\partial \cdot A + gA^2 = 0$.

The linear gauge $\partial \cdot A = 0$ is more familiar, and less complicated to implement in perturbation theory. Moreover, the above non-linear gauge is consistent only for U(1) or SU(N)×U(1), whereas the linear gauge is consistent for any gauge group. The linear gauge has, however, one technical disadvantage. The quartic term in Eq. (2.11) is the stabilizing term of the classical potential [see Eq. (2.9)]. In its absence, one has to reanalyze the classical potential, and make sure that it is stabilized by a higher-power term (in practice this is an A^6 term). This task is done in Ref. [12], which is henceforth referred to as *II*.

Here we will consider only the non-linear gauge. Since the necessary mixed term contains a derivative, one can modify the HD action while leaving the classical potential intact. This simplifies our task, as the large-$\widetilde{\kappa}$ study of the phase diagram in Sec. II C remains valid. The new HD action is

$$S^{\text{n.l.}}_{\text{HD}} = \frac{1}{2\xi_0 g_0^2} \operatorname{tr} \sum \left(\phi^\dagger \Box^2(U)\phi + 2B \sum_\mu \Delta^-_\mu V_\mu \right), \qquad (2.16)$$

$$B_x = \sum_\mu \left(\frac{V_{x-\hat{\mu},\mu} + V_{x,\mu}}{2} \right)^2, \qquad (2.17)$$

$$V_{x,\mu} = \frac{1}{2i} (\phi_x^\dagger U_{x,\mu} \phi_{x+\hat{\mu}} - \text{H.c.}). \qquad (2.18)$$

Going to the vector picture and applying the weak-coupling expansion, we have

$$V_\mu|_{\phi_x=I} = g_0 A_\mu - \tfrac{1}{6}(g_0 A_\mu)^3 + \cdots . \qquad (2.19)$$

It is easy to check that the desired mixed term is now present in the tree-level vector action. We comment that, in the Higgs picture, V_μ is a gauge-invariant local vector field, whose expectation value serves as an order parameter for the FMD phase. (The corresponding order parameter in the vector picture is the expectation value of $V_\mu|_{\phi_x=I}$.) At the classical level, $\langle V_\mu \rangle = g_0 \langle A_\mu \rangle$, where the latter is given by Eq. (2.10).

For the Laplacian-squared HD action [see Eq. (2.3)], it is evident that $U_{x,\mu} = I$ is the unique absolute minimum for *all* configurations, and not only for the constant ones considered in the classical potential (we assume $\phi_x = I$). This property, which is necessary to validate the weak-coupling expansion, applies to the new HD action (2.16) as well. The proof is given in *II*. The symmetric combination used in the definition of B_x [Eq. (2.17)], which does not affect the marginal term contained in $B\Sigma_\mu \Delta^-_\mu V_\mu$, is essential for the proof.

We define the lattice gauge-fixing action to be

$$S^{\text{n.l.}}_{\text{gf}}(U) \equiv S^{\text{n.l.}}_{\text{HD}}(\phi, U)|_{\phi_x=I}. \qquad (2.20)$$

As expected, $S^{\text{n.l.}}_{\text{gf}}$ has the classical continuum limit $(1/2\xi_0)(\partial \cdot A + gA^2)^2$. Because of the irrelevant terms it contains, one cannot write $S^{\text{n.l.}}_{\text{gf}}$ as the (sum over x of the) square of a local function of the U_μ-s. Consequently, the gauge-fixed lattice action is not invariant under BRST transformations.

It is interesting that the breaking of (gauge and) BRST invariance is a common feature of the chiral fermion action and the gauge-fixing action. In the case of the gauge-fixing action, it has to be so because of a theorem by Neuberger [13], which asserts that any lattice BRST-invariant (gauge-fixed) partition function must vanish due to lattice artefact Gribov copies. We return to the role of lattice Gribov copies in Sec. IV.

Before we introduce fermions, the complete lattice action (in the vector picture) is therefore

$$S^{\text{n.l.}}_V = S_G + S^{\text{n.l.}}_{\text{gf}} + S^{\text{n.l.}}_{\text{fp}} + S_{\text{ct}}. \qquad (2.21)$$

For the non-linear gauge, the continuum Faddeev-Popov action involves the operator (we suppress the group structure constants) $\partial^2 + igA \cdot \partial + g\{A, \partial + igA\}$. The last term is absent in the case of the linear gauge. For the discretization of ∂^2 we take the standard (free) lattice Laplacian. We are discretizing a second-order operator, and our choice avoids the appearance of any FP doublers. For the interaction terms, any lattice operator with the correct classical continuum limit should do. For the discretization of $gA \cdot \partial$, for example, one can take $\Sigma_\mu V_\mu \Delta_\mu$ where Δ_μ is the antisymmetric difference operator. (Since BRST symmetry is broken anyway by the gauge-fixing action, we make no attempt to preserve any exact relation between the discretized versions of ∂^2 and $A \cdot \partial$.) We note that the ghost fields contribute to the effective potential only through loops, and so they do not modify the tree-level considerations.

S_{ct} is the counter-term action. The role of S_{ct} is to enforce BRST invariance in the low momentum limit of lattice perturbation theory [1]. The BRST symmetry is violated in particular by (marginal) SO(4)-breaking lattice operators. Therefore, enforcing BRST invariance should also restore full SO(4)-invariance in the continuum limit. The counter-term action is more naturally written in terms of A_μ. We define S_{ct} as a local functional of the U_μ-s by trading A_μ with V_μ using Eq. (2.19). (The second term in the expansion of V_μ, which breaks SO(4) invariance, is needed only for the dimension-two mass counter-term. In all other cases one simply replaces $g_0 A_\mu$ with V_μ.)

The BRST identity $m_r^2 = 0$, which says that the (renormalized) vector boson mass must vanish to all orders in perturbation theory, is consistent with taking the continuum limit at the FMD transition. As a mass counter-term one can take the κ-term in Eq. (2.3). This means that κ is tuned to $\kappa_{c.l.}(g_0, \xi_0)$, where in perturbation theory $\kappa_{c.l.} = \sum_{n \geqslant 1} c_n(\xi_0) g_0^{2(n-1)}$. Note that the coefficient of the mass term in Eq. (2.9) is κg_0^2. The absence of an $O(1/g_0^2)$ term in the expansion of $\kappa_{c.l.}$ is in agreement with the vanishing of the tree-level vector boson mass.

In this paper we have simplified things by considering only the most important counter-term, namely, the dimension-two mass term. In the case of the non-linear gauge, the next most important counter-term is the dimension-four SO(4)-breaking term $\sum_\mu A_\mu^4$. As for the linear gauge, the classical potential is stabilized by an A^6 term, and $(\sum_\mu A_\mu^2)^2$ too occurs only as a counter-term. In this case, the effect of the dimension-four non-derivative counter-terms is discussed in *II*. One finds that the conventional (hyper-cubic invariant) phase and the FMD phase both extend into the higher-dimensional phase diagram. Also, the FMD transition remains continuous when the dimension-four counter-terms are tuned to their critical values. The crucial features leading to these conclusions are (a) it is justified to expand U_μ up to a finite order in A_μ (equal to the dimension of the stabilizing term) when looking for the absolute minimum of the potential; (b) the coefficients of the counter-terms are $O(1)$, whereas the coefficient of the gauge-fixing action is $O(1/g_0^2)$. Since these features are true in the case of the non-linear gauge as well, we expect a similar robustness against the inclusion of additional counter-terms.

In this section we have discussed the phase diagram only in the large-$\tilde{\kappa}$ limit. The phase diagram for arbitrary κ and $\tilde{\kappa}$ is studied in Sec. V A. This study, as well as additional arguments presented in Sec. VI A, further clarify why the continuum limit of the gauge-fixing approach should be defined at the FMD transition.

III. CHIRAL FERMIONS

In a gauge-invariant lattice theory, the minimum of the plaquette action is unique up to a gauge transformation, and the transversal kinetic term is sufficient to define a valid weak-coupling expansion. In the *absence* of gauge invariance, there exists a valid weak-coupling expansion provided the gauge-fixing action of Sec. II E is added to the plaquette action. This applies also to the gauge-fixing action presented in *II* for the linear gauge $\partial \cdot A = 0$. Using either of these gauge-fixing actions, there is a lot of freedom in the choice of the chiral fermion action. We consider here (in the vector picture) an action which is the most economic in the number of fermionic degrees of freedom [14]. (For related work see Refs. [15,16].) Other fermion actions have certain advantages over the one presented here, and in particular they can reduce the required fine-tuning.

According to Ref. [14], one introduces a two-component lattice fermion field χ_x, to account for a single Weyl fermion in the target continuum theory. The fermion action is (suppressing coordinates summations)

$$S_F = \sum_\mu \bar{\chi} \sigma_\mu D_\mu(U) \chi - \frac{w}{4}(\chi \square \chi + \text{H.c.}), \qquad (3.1)$$

$$D_{xy,\mu}(U) = \tfrac{1}{2}(\delta_{x+\hat{\mu},y} U_{x,\mu} - \delta_{x-\hat{\mu},y} U_{y,\mu}^\dagger). \qquad (3.2)$$

Here $\square_{xy}$ is the free lattice Laplacian [Eq. (2.4) for $U_{x,\mu} = I$], and $\chi_x \chi_y \equiv \epsilon_{\alpha\beta} \chi_{x,\alpha} \chi_{y,\beta} = \chi_x^T \epsilon \chi_y$ where ϵ is the antisymmetric two-by-two matrix. We assume $w = O(1)$. The first term in Eq. (3.1) is the naive lattice discretization of the continuum Weyl action. The second term is a Majorana-Wilson (MW) term, that breaks explicitly gauge invariance as well as the fermion number symmetry. The latter is unwanted, because fermion number is not conserved in the continuum theory.

In order to understand the properties of the lattice fermion path integral, it is convenient to recast the fermion action Eq. (3.1) in terms of four components fields ψ_M and $\bar{\psi}_M$. By definition, $P_L \psi_M = \chi$ and $P_R \psi_M = \epsilon \bar{\chi}^T$, where $P_{R,L} = \tfrac{1}{2}(1 \pm \gamma_5)$ denote chirality projectors. $\bar{\psi}_M$ is not an independent field, and is given by

$$\bar{\psi}_M \equiv \psi_M^T C. \qquad (3.3)$$

Here C is the antisymmetric four-by-four charge conjugation matrix, obeying $C^2 = -1$, $\gamma_\mu^T C = -C\gamma_\mu$, and $\gamma_5 C = C\gamma_5$. In terms of these four component fields, the fermion action takes the form

$$S_F = \tfrac{1}{2} \sum_\mu \bar{\psi}_M[\gamma_\mu D_\mu(U) P_L + \gamma_\mu D_\mu(U^*) P_R]\psi_M$$

$$- \frac{w}{4} \bar{\psi}_M \square \psi_M. \qquad (3.4)$$

Let us first examine Eq. (3.4) in perturbation theory. The tree-level fermion propagator is the (massless) Wilson propagator [14]. Because of the Majorana-like condition Eq. (3.3), the symmetry factors in Feynman graphs are the same as for Majorana fermions. Now, if we go to the small momentum limit, we find that the two chiralities of ψ_M couple to the gauge field according to *complex conjugate* representations of the gauge group. One sees that the role of Eq. (3.3) is to maintain consistently the identification $P_L \psi_M \leftrightarrow \chi$, $P_R \psi_M \leftrightarrow \bar{\chi}$, at the level of Feynman diagrams. That Eq. (3.4) correctly describes a single left-handed Weyl fermion, can be verified by calculating the non-analytic part of one-fermion-loop diagrams, that should agree with the continuum result in the limit of a vanishing external momentum (the role of

counter-terms is discussed below). As usual, the non-analytic contribution comes from an infinitesimal neighborhood of the origin in the Brillouin zone. In this neighborhood one can neglect the Wilson term in both numerators and denominators. The left-handed and right-handed components of ψ_M are no longer coupled, and one can reexpress the Feynman integrand in terms of the continuum propagator for a single Weyl fermion. [A left-handed fermion loop is equal to a right-handed fermion loop in the complex conjugate representation. The ''double-counting'' is compensated by a one-half symmetry factor for each closed fermion loop, which arises from the Majorana-like condition Eq. (3.3).]

We next discuss the rigorous definition of the fermionic path integral. We introduce the $2N \times 2N$ fermion matrix $Q_{\alpha\beta}$ by writing the action in the generic form $S_F = \frac{1}{2} \Sigma_{\alpha,\beta} \psi_M^\alpha Q_{\alpha\beta} \psi_M^\beta$. (The charge conjugation matrix C is absorbed into the definition of Q.) It is easy to check that Q is antisymmetric. The fermion path integral takes the following form [17]

$$\int \prod_\alpha d\psi_M^\alpha \exp\left(\frac{1}{2} \sum_{\alpha,\beta} \psi_M^\alpha Q_{\alpha\beta} \psi_M^\beta \right)$$

$$= \frac{1}{2^N N!} \epsilon_{\alpha_1,\beta_1,....,\alpha_N,\beta_N} Q_{\alpha_1,\beta_1} \cdots Q_{\alpha_N,\beta_N}$$

$$\equiv \mathrm{pf}\left(\frac{Q}{2} \right). \tag{3.5}$$

There is no integration over the (dependent) variables $\overline{\psi}_M$. According to Eq. (3.5), the fermionic path integral is a *Pfaffian*. In general, $\mathrm{pf}(Q/2)$ is complex, as expected from the Euclidean path integral for a single Weyl fermion.

(As a further check that our fermion path integral describes a Weyl fermion, we can consider a ''two-generation'' model, where each complex representation occurs twice in the fermion spectrum. Using the identity $\mathrm{pf}^2(Q/2) = \det(Q)$, that holds for a general antisymmeric matrix, this two-generation model can be defined by an action similar to Eq. (3.4), where we now drop an overall one-half factor, substitute $\psi_M \to \psi_D$, $\overline{\psi}_M \to \overline{\psi}_D$, and regard ψ_D and $\overline{\psi}_D$ as *independent* Dirac-like variables. The counting of degrees of freedom is now straightforward. Since the two chiralities of ψ_D belong to complex conjugate representations, this action actually describes two left-handed Weyl fermions in the *same* complex representation.)

Until now we have implicitly discussed the fermions in the background of a fixed external gauge field. The main result of the previous section is that, with the gauge-fixing action, perturbation theory is valid for a dynamical gauge field as well. Therefore, with appropriate counter-terms, *the continuum fields describing the scaling behavior are in one-to-one correspondence with the massless poles of the various tree-level propagators*. If we choose an anomaly free fermion spectrum, the scaling region should be governed by a continuum chiral gauge theory, in the relevant gauge. We note that if one chooses an *anomalous* fermion spectrum, the scaling region will still be governed by a renormalizable Lagrangian, but BRST invariance and, hence, unitarity will be violated.

Finally, let us discuss the fermion mass counter-terms. As with ordinary Wilson fermions, a mass counter-term $m_0 \overline{\psi}_M \psi_M$ is necessary to maintain the masslessness of each chiral fermion. (A different fermion action that does not require mass counter-terms will be discussed elsewhere [36].) The renormalized Majorana-like mass is proportional to $(m_0 - m_c)$, where m_c is (minus) the fluctuations-induced mass. If $(m_0 - m_c)$ is small but non-zero, BRST invariance will be explicitly broken in the scaling region. The scaling behavior is then governed by a renormalizable continuum theory which is not gauge invariant (hence also non-unitary). By tuning m_0 to m_c, assuming all other counter-terms already have their critical values, we recover BRST invariance simultaneously with the masslessness of the chiral fermions. (The situation on the lattice is similar to what one would encounter in the continuum, if a gauge non-invariant regularization is employed for a chiral gauge theory. As on the lattice, Majorana-like mass counter-terms may be needed, alongside with other gauge non-invariant counter-terms, to cancel the breaking of gauge invariance induced by the regularization, and to ensure that the renormalized amplitudes are gauge invariant.)

An important question in the literature on lattice chiral gauge theories, is how to reproduce correctly fermion number violation in the continuum limit. Different solutions have been proposed to the problem [18–21]. We hope that the present approach can shed new light on it.

For definiteness, we adopt the strategy of Ref. [14]. Namely, we demand that the lattice fermion action should have no symmetry which is not present in the target continuum theory. Now, while the action Eq. (3.1) is not invariant under global U(1) transformations with an arbitrary phase, it is still invariant under the residual *discrete* symmetry $\chi \to -\chi$, $\overline{\chi} \to -\overline{\chi}$. This symmetry implies a (mod 2) conservation law for each fermion species, which still causes a problem. Consider for definiteness an SU(5) grand unified theory (GUT), with one generation that contains a $\overline{\mathbf{5}}$ and a $\mathbf{10}$. In an instanton background, the numbers of zero modes for these representations are, respectively, one and three. This is in conflict with the above (mod 2) conservation laws. Thus, on top of the MW terms present in Eq. (3.1) for each representation, one has to introduce an additional gauge-non-invariant MW term that couples the $\overline{\mathbf{5}}$ and the $\mathbf{10}$. (As a result, a Majorana-like mass counter-term that mixes the $\overline{\mathbf{5}}$ and the $\mathbf{10}$ will be necessary too.) With this new MW term, the remaining discrete symmetry leads only to (mod 2) conservation of the total fermion number for each generation.

IV. WHY GAUGE FIXING

We now return to the lattice effective action, and consider some of its properties in more detail. We keep the discussion at an informal level. Our approach has been presented in detail in the previous sections, and the aim here is to clarify the nature of the *problems* that it is meant to solve.

In this section we assume that the lattice chiral fermion action S_F is bilinear in the fermion fields, and that it depends in addition only on the link variables $U_{x,\mu}$. (This corresponds to the vector picture.) It is also assumed that S_F is

(mildly) local. The lattice spacing a will be shown explicitly in this section.

The difficulties encountered in the construction of lattice chiral gauge theories can be addressed at a more rigorous level [5,7]. This complementary discussion, which focuses on the robustness of the Nielsen-Ninomiya theorem, is given in Sec. VI.

A. Rough lattice gauge transformations and the need for gauge fixing

The lattice effective action is defined by integrating out the fermions

$$S_{\text{eff}}(U) = -\log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_F(U,\psi,\bar{\psi})}. \qquad (4.1)$$

Clearly, the well-defined object is $\exp(-S_{\text{eff}})$, rather than S_{eff} itself. For our purpose it will be sufficient to consider the *perturbative* effective action, and so we will ignore the problems associated with the global definition of S_{eff}.

The variation of S_{eff}, in response to an infinitesimal lattice gauge transformation at the point x_0, has the following general form

$$\delta_{x_0} S_{\text{eff}} \approx c_0 \mathcal{O}_{x_0}^{con} + \sum_{n \geq 1} a^n \sum_i c_{n,i} \mathcal{O}_{x_0}^{n,i}. \qquad (4.2)$$

The $\approx$ sign indicates that the right-hand side (RHS) is computed perturbatively. Note that the gauge field is external, and so the gauge-field action is not needed at this stage. $\mathcal{O}_x^{con}$ and $\mathcal{O}_x^{n,i}$ are local lattice operators that depend on A_μ, *cf.* Eq. (2.6). The dimension of $\mathcal{O}_x^{n,i}$ is $4+n$, and the i-summation is over linearly independent operators of this dimension. $\mathcal{O}_x^{con}$ is some discretized version of the consistent anomaly. We assume that all operators of dimension less than or equal to four, other than $\mathcal{O}_x^{con}$, have been cancelled by counter-terms. If, moreover, we choose a set of complex representations that satisfies the anomaly cancellation condition, then $c_0 = 0$.

The infinite sum on the RHS of Eq. (4.2) accounts for lattice artifact violations of gauge invariance. The precise form of these violations is model dependent, but their existence is generic. As can be easily seen by going to momentum space, which is the usual setting for a perturbative computation, this sum represents a double expansion in $|g_0 a A_\mu|$ and $|a p_\mu|$.

Let us first consider a smooth gauge field A_μ^0 which is characterized by some physical scale $\Lambda_{\text{phys}} \ll a^{-1}$, and the corresponding configuration of link variables $U_{x,\mu}^0$ defined via Eq. (2.6). Since the dimensionful quantities A_μ and p_μ are $O(\Lambda_{\text{phys}})$, both of the above expansion parameters are small. Equation (4.2) is the gradient of S_{eff} with respect to a motion inside the lattice gauge orbit. Therefore, S_{eff} is (approximately) constant on the orbit in the vicinity $U_{x,\mu}^0$. The constancy of S_{eff} extends to that portion of the orbit which is reachable from $U_{x,\mu}^0$ by a smooth gauge transformation.

The problem is that, on the lattice, smooth gauge transformations represent a tiny part of the local gauge group. Let $U_{x,\mu}^{(g)} = g_x U_{x,\mu}^0 g_{x+\hat{\mu}}^\dagger$ be a *generic* configuration in the orbit of

$U_{x,\mu}^0$, and let $A_\mu^{(g)}$ be related to $U_{x,\mu}^{(g)}$ via Eq. (2.6). Since the operators that occur on the RHS of Eq. (4.2) are *not* gauge invariant, they are sensitive to the value of g_x for x in the vicinity of x_0. Now, the g_x-s on different sites a *uncorrelated*. As a result, the expansion parameters $|g_0 a A_\mu^{(g)}|$ and $|a \Delta_\nu A_\mu^{(g)}/A_\mu^{(g)}|$ are $O(1)$ for a generic lattice gauge transformation. We conclude that S_{eff} fails to be (approximately) constant on most of the lattice gauge orbit. This is true for *any* orbit, including orbits that have a smooth representative.

B. Proliferation of lattice Gribov copies

The above problem stems from the *roughness* of generic lattice gauge transformations. Following Ref. [1] we make no attempt to reduce the violations of gauge invariance at the level of the effective action. Instead, our aim is to suppress the Boltzmann weight of rough gauge field configurations (relative to smooth configurations that belong to the same orbit) consistently with the gauge invariance of the physical Hilbert space, namely, via *gauge fixing*.

In lattice QCD, gauge fixing is a matter of choice, since it has no effect on the gauge-invariant observables. Here, the fermion action is not gauge invariant. As a result, the gauge-fixing method is an *integral part* of the definition of the theory. Different gauge-fixing methods may in general give rise to different phase diagrams with different critical points. There is no guarantee that every gauge-fixing method will lead to a non-trivial continuum limit, let alone to a chiral gauge theory.

Still, in order to make progress, one has to choose *some* gauge-fixing method. Vink [22] proposed to use the Laplacian gauge, where a maximally smooth representative is chosen on each gauge orbit by global minimization. The Laplacian gauge is highly non-local, and this creates difficulties both in the analytic and in the numerical study of this method.

As discussed in detail in Sec. II, we build a local gauge-fixing lattice action, that (a) has the unique absolute minimum $U_{x,\mu} = I$, and (b) reduces to a covariant gauge-fixing action in the classical continuum limit. Our gauge-fixing actions [Eqs. (2.16)–(2.18) for the non-linear gauge, see *II* for the linear gauge] are clearly not the most naive discretizations of the corresponding continuum actions. Focusing for simplicity on the linear gauge $\partial \cdot A = 0$, let us examine what goes wrong with a naively discretized gauge-fixing action. We thus consider the following action:

$$S_{\text{gf}}^{\text{naive}} = \frac{1}{2\xi_0 g_0^2} \text{tr} \sum_x \mathcal{G}_x^2, \qquad (4.3)$$

$$\mathcal{G}_x = \sum_\mu \Delta_\mu^- V_{x,\mu} = \frac{1}{2i} \sum_\mu (\Delta_\mu^- U_{x,\mu} - \text{H.c.}). \qquad (4.4)$$

Note that $g_0^{-1} \Sigma_\mu \Delta_\mu^- V_{x,\mu}$ reduces in the classical continuum limit to $\partial \cdot A$, as it should.

What is common to our gauge-fixing action(s) and to the naive gauge-fixing action Eq. (4.3), is that they contain a longitudinal kinetic term. The trouble with $S_{\text{gf}}^{\text{naive}}$ is that it supports a *dense set* of Gribov copies for the identity field $U_{x,\mu} = I$. Each of these Gribov copies is a classical vacuum

of $S_G + S_{\text{gf}}^{\text{naive}}$. The superposition of contributions coming from all these classical vacua, which *cannot* be calculated perturbatively, may ultimately render the fermion spectrum vector-like. Even without fermions, stability of the classical potential is lost, and it is unclear whether the FM-FMD transition (associated with a divergent vector-field correlation length) can be maintained in the weak-coupling limit. (If the gauge-fixing action in Eq. (2.21) is replaced by $S_{\text{gf}}^{\text{naive}}$, the resulting classical potential is identically zero. If we assume $\kappa = O(1/g_0^2)$, which implies the presence of a tree-level mass term, one has $V_{\text{cl}}^{\text{naive}} = \kappa \mathcal{F}(g_0 A_\mu)$ [compare Eq. (2.7)]. The minimum of $V_{\text{cl}}^{\text{naive}}$ is $A_\mu = 0$ for $\kappa > 0$, and $A_\mu = \pi/g_0$ for $\kappa < 0$.)

We now demonstrate the existence of a dense set of lattice Gribov copies for the identity field. Consider first the U(1) case. The condition $\mathcal{G}_x = 0$ is satisfied if the imaginary part of $U_{x,\mu}$ is zero everywhere. The latter is true if we consider only lattice gauge transformations where $g_x = \pm 1$. In other words, in spite of the presence of the gauge-fixing action $S_{\text{gf}}^{\text{naive}}$, the Gribov copies of the identity field still exhibit a local Z_2 symmetry. [This is also true for the non-linear gauge, if one replaces $\mathcal{G}_x$ in Eq. (4.3) by $\mathcal{G}_x^{\text{n.l.}} = \Sigma_\mu (\Delta_\mu^- V_{x,\mu} + V_{x,\mu}^2)$.] An "elementary" Gribov copy is created if we choose $g_x = -1$ for $x = x_0$, and $g_x = 1$ elsewhere. This clearly shows that the Gribov copies are *local lattice artefacts*.

The above example generalizes to non-Abelian groups. In the case of SU(2N) and SO(2N) groups, simply replace ± 1 by $\pm I$. Moreover, for any SU(N) group, one can choose an SU(2) subgroup (which for simplicity we assume to lie at the top left corner) and repeat the above construction with $g_x = \text{diag}(\pm 1, \pm 1, 1, 1, \ldots)$. The discrete local symmetry of the Gribov copies is therefore larger than Z_2 in the general case. For a more detailed discussion of lattice Gribov copies see Ref. [23].

A number of remarkable similarities draw us to say that the *proliferation of Gribov copies* is the spin-1 counter-part of the *fermion doubling* problem. In both cases, one deals with the discretization of a first-order differential operator: in the spin-$\frac{1}{2}$ case, this is the Dirac (or Weyl) equation; in the spin-1 case, this is a covariant gauge condition (see Sec. II E). In both cases the problem arises when a non-compact continuum variable is replaced with a compact lattice variable: fermion doubling arises because, unlike in the continuum, the lattice momentum is periodic; in the spin-1 case, also the non-compact continuum gauge field is replaced with compact group variables. In both cases, there are theorems that establish an impasse under certain mild-looking conditions: the Karsten-Smit [3] and Nielsen-Ninomiya [4] theorems which predict fermion doubling, and Neuberger's theorem [13] which asserts that any BRST-invariant partition function must vanish identically. Finally, in both cases the solution is to reduce the symmetry of the lattice theory, by adding *irrelevant* terms to the naively discretized action. In the case of Wilson fermions, this is the role of the Wilson term. As for our gauge-fixing action(s), one can show (see *II*) that it is equal to the square of a discretized gauge condition, plus irrelevant terms. Thus, again, the irrelevant terms reduce the symmetry, this time by breaking explicitly BRST invariance.

C. Other approaches

A different approach to the dynamical problems created by rough lattice gauge transformations is to adopt a more sophisticated definition for the effective action. The prominent representatives of this approach are the interpolation method [24,25] and the overlap formalism [26].

In the interpolation method, one constructs a continuum interpolating field $A_\mu^{\text{cont}} = A_\mu^{\text{cont}}(y; U_{x,\mu})$, $y \in R^4$, for each configuration of the lattice gauge field. The determinant of the Weyl operator, $\sigma \cdot (\partial + igA^{\text{cont}})$, is then defined using a separate regulator. Associated with the fermion regularization is a new cutoff parameter, denoted generically Λ_{f}, which must be sent to infinity before the lattice spacing a is sent to zero. [A concrete method [25] is to discretize the Weyl action on a finer lattice with a lattice spacing $a_{\text{f}} \ll a$, using *e.g.* the fermion action of Ref. [14] [Eq. (3.1)]. In this case $\Lambda_{\text{f}} = a_{\text{f}}^{-1}$.]

Consistent regularizations of the Weyl determinant break gauge invariance for finite values of the cutoff, even when the fermion spectrum is anomaly free. Therefore, gauge-non-invariant counter-terms are needed in the interpolation method too. (It has been proposed that gauge-non-invariant counter-terms may be avoided, if the real part of the effective action is regulated separately from the imaginary part [27]. In spite of attempts in this direction [25], it remains unclear whether this procedure can be implemented beyond perturbation theory without violating locality, and, eventually, unitarity.)

For given $U_{x,\mu}$-s, the interpolating field assigns a *local winding number* to each hypercube (in the non-Abelian case), or to each plaquette (in the Abelian case). In the non-Abelian case, this is the winding number of the continuum gauge transformation defined on the faces of the hypercube, that brings the interpolating field to a prescribed axial gauge; in the Abelian case, the continuum gauge transformation is defined on the perimeter of each plaquette. Now, a fundamental requirement is that the fermion determinant should be gauge invariant in the limit $\Lambda_{\text{f}} \to \infty$. Gauge invariance can be established only if $A_\mu^{\text{cont}}(y)$ is globally bounded [24,25]. Gauge invariance is therefore recovered in the limit $\Lambda_{\text{f}} \to \infty$ only on that portion of the lattice gauge orbit, where all the local winding numbers are zero [7]. The solution is to apply a gauge transformation that sets all local winding numbers to zero *before* computing the fermion determinant (for simplicity we consider a trivial global topology). We note that the smoothing gauge transformation is non-local, and so a careful study of potential problems associated with the infinite-volume limit is required.

In the overlap approach, while the real part of the effective action is gauge invariant by construction, the imaginary part is not. Again, we expect that gauge-non-invariant counter-terms will be needed, starting at some finite loop order. Potentially severe problems with the overlap approach were pointed out in Ref. [28]. According to our judgement, subsequent works (including in particular Ref. [29]) fail to address the issues raised in Ref. [28]. Numerical evidence for the lack of gauge invariance (in the non-Abelian case) has been found in Ref. [30] (see Fig. 1 therein).

V. THE REDUCED MODEL

Returning to our approach, we consider in this section the limit of a vanishing gauge coupling. Since $1/g_0^2$ is the coef-
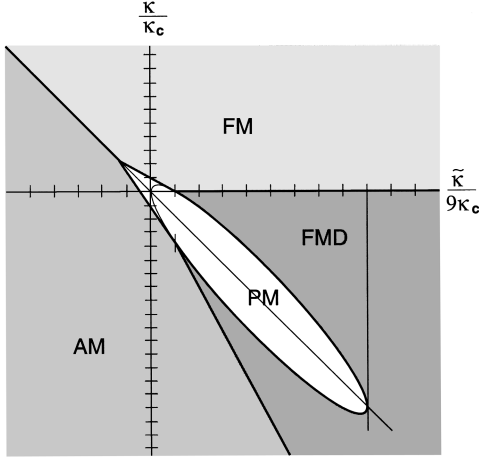
FIG. 1. Mean-field phase diagram. See Table I for the definition of the various phases.

ficient of the plaquette action, the $g_0 = 0$ limit constrains the lattice gauge field to the trivial orbit.

The theory defined by $g_0 = 0$ limit is called the *reduced model*. If we use the *vector picture*, the reduced model is obtained by substituting $U_{x,\mu} \to \phi_x^\dagger \phi_{x+\hat\mu}$ in the lattice action. The lattice gauge field measure $\Pi \int dU_{x,\mu}$ is replaced by $\Pi \int d\phi_x$. [Alternatively, starting from the *Higgs picture* that already involves both $U_{x,\mu}$ and ϕ_x (see Sec. II B), one obtains the reduced model by simply setting $U_{x,\mu} = I$.]

In the weak gauge-coupling limit, the transversal modes are perturbative at the lattice scale. Many important features, including the fermion spectrum, are determined by the dynamics of the longitudinal modes. The utility of the reduced model is that it allows us to study the longitudinal dynamics in isolation, without making any *a priori* assumption. The reduced model accounts for dynamical situations ranging from a divergent longitudinal correlation length, as in our approach, down to a very short correlation length. In this section we study a prototype reduced model. The entire dynamical range is realized in different regions of its phase diagram. In Sec. VI we discuss the effects of the longitudinal dynamics on the fermion spectrum, first in general terms and then in our approach.

A. The phase diagram

In this subsection (and the next one) we discuss the reduced model related to the lattice action of Sec. II B, for $\phi_x \in U(1)$. Since Eq. (2.3) is written in the Higgs picture, the reduced model is obtained by setting $U_{x,\mu} = 1$. This leads to the action

$$S'_H = \sum \ (-\kappa \phi^\dagger \Box \phi + \widetilde\kappa \phi^\dagger \Box^2 \phi). \tag{5.1}$$

Our first task is to derive the mean-field phase diagram in the $(\widetilde\kappa, \kappa)$-plane. This phase diagram is in fact generic, and pertains also to the more relevant theories defined in Sec. II E and in *II*.

Let us first consider the ordinary VEV, v, as an order parameter (or the staggered VEV v_{AM}). An additional order parameter will be introduced shortly. By definition

$$v = \langle \phi_x \rangle, \tag{5.2}$$

$$v_{AM} = \langle \epsilon_x \phi_x \rangle, \tag{5.3}$$

where $\epsilon_x = (-1)^{\Sigma_\mu x_\mu}$. For $\widetilde\kappa = 0$, we recover the familiar non-linear sigma model. On the κ-axis there is a symmetric (PM) phase for $|\kappa| < \kappa_c$, a ferromagnetic (FM) phase for $\kappa > \kappa_c$, and an antiferromagnetic (AM) phase for $\kappa < -\kappa_c$. The field redefinition $\phi_x \to \epsilon_x \phi_x$ maps κ to $-\kappa$, thus implying a symmetry of the κ-axis.

We now extend the discussion to the full $(\widetilde\kappa, \kappa)$-plane. Mean-field approximation in d-dimensions yields the following equation for the FM-PM line

$$\kappa + (2d+1)\widetilde\kappa = \kappa_c, \quad \text{FM-PM.} \tag{5.4}$$

The equation for the AM-PM line is

$$\kappa + (6d-1)\widetilde\kappa = -\kappa_c, \quad \text{AM-PM.} \tag{5.5}$$

The FM-PM and AM-PM transitions are continuous. The symmetry of the κ-axis extends to $\widetilde\kappa \neq 0$. Under the field redefinition $\phi_x \to \epsilon_x \phi_x$, the point $(\widetilde\kappa, \kappa)$ is mapped in four dimensions to $(\widetilde\kappa, -\kappa - 32\widetilde\kappa)$. This implies that the linear equation

$$\kappa + 16\widetilde\kappa = 0, \tag{5.6}$$

defines a symmetry line of the phase diagram. The FM-PM and AM-PM lines meet in the second quadrant, at the point $(-\kappa_c/7, 16\kappa_c/7)$ on the symmetry line. It can be shown that, beyond this point, the symmetry line is a first-order transition line separating the FM and AM phases [31].

In condensed matter physics, it is well known that spin models with competing interactions tend to develop a ground state that breaks translation and rotation invariance. If a small antiferromagnetic interaction is added to a dominant ferromagnetic one, the spin orientation of the ground state will *rotate* slowly with a wave vector $q_\mu \neq 0$ (see Ref. [32] for a recent review).

In the reduced model defined by Eq. (5.1), competing interactions occur when κ and $\widetilde\kappa$ have opposite signs. In order to look for a similar phenomenon, we introduce the mean-field ansatz

$$\langle \phi_x \rangle = v e^{iq_\mu x_\mu}. \tag{5.7}$$

We assume $0 \leq q_\mu < 2\pi$. A non-zero q_μ signals the spontaneous breaking of translation and rotation invariance. More precisely, translation invariance is broken in the direction defined by q_μ, but it remains unbroken in the transversal directions. In condensed matter, phases with a non-zero q_μ are known as *helicoidal-ferromagnetic* ones. Here, the helicoidal-ferromagnetic phase of the reduced model is the $g_0 = 0$ boundary of the FMD phase of the full theory (see Sec. II C), and the name FMD will be used both in the full theory and in the reduced model.

A simple mean-field method is based on a *factorized probability measure* (see *e.g.* Ref. [33]). For the FM-PM transition, one can use the following factorized probability measure

$$\mathcal{P}_0(\theta_x) = \frac{1 + 2v \, \cos(\theta_x)}{2\pi}. \tag{5.8}$$

In order to accommodate a non-zero q_μ, we generalize this to

$$\mathcal{P}(\theta_x) = \frac{1 + 2v \, \cos(\theta_x - q_\mu x_\mu)}{2\pi}. \tag{5.9}$$

One has $\langle 1 \rangle_\mathcal{P} = 1$ and $\langle e^{i\theta_x} \rangle_\mathcal{P} = v e^{iq_\mu x_\mu}$ [in agreement with Eq. (5.7)]. Because of its factorized nature, the q_μ-dependence of $\mathcal{P}(\theta_x)$ affects only the *internal energy*, but not the *entropy*. Introducing the notation $S'_H = \Sigma_x \mathcal{H}_x$, we find

$$\langle \mathcal{H} \rangle_\mathcal{P} = (1 - v^2)(8\kappa + 72\widetilde{\kappa}) + v^2[\kappa\mathcal{F}(q_\mu) + \widetilde{\kappa}\mathcal{F}^2(q_\mu)], \tag{5.10}$$

where the function $\mathcal{F}$ is defined in Eq. (2.8). While the choice of the factorized probability measures Eqs. (5.8) and (5.9) is somewhat arbitrary, the internal energy (5.10) is a universal feature of any mean-field approximation for S'_H.

If we consider a point in the phase diagram well to the right of the FM-PM line, the value of v is finite in lattice units. Making the self-consistent assumption that q_μ is small, the location of the FM-FMD transition can be determined by minimizing the internal energy with respect to q_μ. Remarkably, the q_μ-dependent part of the internal energy coincides with the classical potential (2.7), if we make the identification $g_0 A_\mu \leftrightarrow q_\mu$. Consequently, there is complete agreement between the mean-field properties of the FM-FMD transition in the reduced model, and the classical properties of the FMD transition in the $g_0 \neq 0$ theory. The mean-field location of the FM-FMD transition is $\kappa = 0$. For $\kappa > 0$ one is in the FM phase, whereas for $\kappa < 0$ one is in the FMD phase. Close to the FM-FMD line, q_μ is given by Eq. (2.10) where $g_0 A_\mu$ is replaced by q_μ (after this replacement g_0 drops out, and one has $q^2 = -\kappa/2\widetilde{\kappa}$ for a small negative κ). The FM-FMD line ends when it hits the FM-PM line. The multi-critical point where the PM, FM and FMD phases meet is known as a Lifshitz point [34]. Its mean-field value is $(\kappa_c/9,0)$. (Lifshitz points exhibit rich critical behavior. This was discussed recently in a field theoretic context by J. Kuti [35].)

The remaining features of the mean-field phase diagram are as follows (see Fig. 1). The transformation $\phi_x \to \epsilon_x \phi_x$ maps q_μ to $q_\mu + \pi$. This implies the existence of a second FMD region (and another Lifshitz point) below the symmetry line in the fourth quadrant. The PM-FMD line, separating the paramagnetic phase from the FMD phase, can be determined by first minimizing the (internal) energy with respect to q_μ, and then the (free) energy with respect to v. The PM phase occupies a bounded region in the phase diagram. The two FMD regions above and below the symmetry line belong to a single FMD phase. The PM-FMD line lies on an ellipse, with the FM-PM and AM-PM lines tangent to it at the two Lifshitz points. A more detailed mean-field calculation will be presented elsewhere [31].

The mean-field ansatz (5.7) implies the simultaneous breaking of the internal U(1) symmetry, as well as of rotation and translation invariance. We believe that one cannot break translation invariance without at the same time breaking an internal symmetry. However, one can conceive of a phase

TABLE I. Phases of the reduced model. The entries indicate which order parameters were non-zero in each phase.

Phase	v	v_{AM}	v_H	$V_\mu^\parallel$
PM	no	no	no	no
FM	yes	no	yes	no
AM	no	yes	yes	no
FMD	no	no	yes	yes
PMD(?)	no	no	no	yes

(denoted PMD) where only rotation symmetry is broken, while the internal symmetries as well as translations are unbroken. The order parameter for a PMD phase of the reduced model is the expectation value of the composite vector field [compare Eq. (2.18)]

$$V_{x,\mu}^\parallel = \frac{1}{2i}(\phi_x^\dagger \phi_{x+\hat\mu} - \text{H.c.}). \tag{5.11}$$

At the moment, however, we have no evidence for a PMD phase.

The phases of the reduced model are depicted in Table I. The order parameter v_H is defined as $v_H = \langle \phi_x e^{-iq_\mu x_\mu} \rangle$. Note that in the special case $q_\mu = (0,0,0,0)$ $(q_\mu = (\pi,\pi,\pi,\pi))$, v_H coincides with v (v_{AM}). In this paper we usually do not distinguish between v and v_H, since the correct meaning can be understood from the context. However, this distinction is important in numerical simulation. The value of q_μ, to be used in the measurement of v_H, can be determined for example by measuring $\langle \phi_x^\dagger \phi_{x+\hat\mu} \rangle$ and extracting its phase.

The relation between the $(\widetilde{\kappa},\kappa)$-phase diagram of the reduced model and the $(\widetilde{\kappa},\kappa,g_0)$-phase diagram of the full theory is the following. In the U(1) case, the symmetric (PM) phase is the boundary of a Coulomb phase, and the broken (FM or AM) phase is the boundary of a Higgs phase. In the non-Abelian case, the PM, FM and AM phases correspond to the boundary of a single Higgs-confinement phase. Finally, the helicoidal-ferromagnetic (FMD) phase of the reduced model is the boundary of the FMD phase of the full theory.

In the $g_0 \to 0$ limit of the full theory, we find (approximately) massless gauge bosons close to the FM-FMD line, as well as close to the PM-FM line, and in the entire PM phase. An interesting observation is that, even without fermions, if we want to study a Lorentz gauge-fixed Yang-Mills theory on the lattice, then the appropriate critical line is the FM-FMD line. The reason is that, in order to keep the longitudinal kinetic term in the tree-level action, $\widetilde{\kappa}$ must scale like $1/g_0^2$ (see Sec. II). In the large-$\widetilde{\kappa}$ limit, a PM phase does not exist, and criticality can only be achieved by approaching the FM-FMD line.

B. The weak-coupling expansion in the reduced model

In view of the properties of the weak-coupling expansion in the full theory (Sec. II), and in particular Eq. (2.12), one expects that $1/\widetilde{\kappa}$ will play the role of a coupling constant in the reduced model. We will now demonstrate this explicitly. We do not carry out here any detailed calculations that re-

quire the full lattice Feynman rules. Therefore, we work in the continuum approximation, *i.e.* we extract from the lattice action the marginal and relevant terms, that control the critical behavior in the vicinity of the Gaussian critical point $1/\widetilde{\kappa}=0$.

The weak-coupling expansion is facilitated by expanding around a broken symmetry vacuum. We first introduce the Goldstone boson (GB) field θ_x via $\phi_x = e^{i\theta_x}$. [This classical expansion is consistent with the mean-field ansatz (5.7), because $v \rightarrow 1$ for $\widetilde{\kappa} \rightarrow \infty$.] Rescaling $\theta \rightarrow (1/\sqrt{2\widetilde{\kappa}})\theta$ we find, in the continuum approximation, the following GB Lagrangian

$$\mathcal{L}_{GB}=\frac{1}{2}\int d^4x\left((\Box\theta)^2+\frac{\kappa}{\widetilde{\kappa}}\partial_\mu\theta\partial_\mu\theta+\frac{1}{2\widetilde{\kappa}}(\partial_\mu\theta\partial_\mu\theta)^2\right).\tag{5.12}$$

Equation (5.12) is valid on the FM side of the transition line. On the FMD side, one first looks for the classical vacuum by assuming $\theta=q_\mu x_\mu$ and minimizing for q_μ. The result is the same as in the mean-field approximation. The weak-coupling expansion on the FMD side is then defined via $\theta \rightarrow q_\mu x_\mu + (1/\sqrt{2\widetilde{\kappa}})\theta$.

We will consider here only the FM side. Taking the Fourier transform of the bilinear part of the Lagrangian, we find the following GB propagator

$$G_0^{-1}(p)=(p^2)^2+m_0^2 p^2,\tag{5.13}$$

where $m_0^2=\kappa/\widetilde{\kappa}$. Analytical continuation to Minkowski space shows the existence of a positive-residue pole at $p_M^2=0$, and a negative-residue (ghost) pole at $p_M^2=m_0^2$. These poles merge into a quartic singularity in the limit $m_0^2 \rightarrow 0$. The continuum limit of the reduced model is therefore not unitary. (In Sec. VI B we discuss the interaction of the GB field with fermions. The crucial requirement is that the non-unitary GB sector, which accounts for the two unphysical polarizations of the gauge bosons, will decouple from the fermions in the continuum limit. This decoupling is discussed in detail in Ref. [36].)

Because of the quartic kinetic term in the GB Lagrangian, the canonical dimension of the GB field $\theta(x)$ is zero. The GB Lagrangian is invariant under the shift symmetry $\theta(x) \rightarrow \theta(x)+$const, which forbids the appearance of non-derivative terms under renormalization. In addition, the GB Lagrangian is invariant under the discrete symmetry $\theta(x) \rightarrow -\theta(x)$. Equation (5.12) is the most general renormalizable Lagrangian allowed by these symmetries.

What marks the Feynman rules of the GB Lagrangian, is that one derivative acts on every line attached to a vertex. The derivatives acting on the two ends of each internal line effectively cancel one factor of $1/p^2$ in the propagator. The result is that the UV power counting of the GB model is the same as in an ordinary $\lambda\Phi^4$ theory. If we ignore the vector index carried by the partial derivatives, one can match each term in the GB Lagrangian with a corresponding term in the $\lambda\Phi^4$ Lagrangian according to the rule $\partial\theta \rightarrow \Phi$.

In the $\lambda\Phi^4$ theory, at the one loop level only the mass term is renormalized, but not the kinetic term. By analogy, in the GB Lagrangian only $(\partial_\mu\theta)^2$, but not $(\Box\theta)^2$, is renor-

malized at the one-loop level. The induced one-loop $(\partial_\mu\theta)^2$-term has a positive sign. This implies

$$\kappa_{c.l.}(\widetilde{\kappa})=-c+O(\widetilde{\kappa}^{-1}).\tag{5.14}$$

[Note that the coefficient of $(\partial_\mu\theta)^2$ in Eq. (5.12) is $\kappa/\widetilde{\kappa}$.] The dimension of the positive constant c is two. Its numerical value, which is $O(1/a^2)$, has to be determined by a lattice calculation. Finally, as in the $\lambda\Phi^4$ theory, the one-loop beta-function is determined by the vertex renormalization, and is found to be positive. Explicitly

$$\beta(\widetilde{\kappa}^{-1})\equiv\Lambda\frac{\partial}{\partial\Lambda}\widetilde{\kappa}^{-1}=\frac{5}{16\pi^2}\widetilde{\kappa}^{-2},\tag{5.15}$$

where Λ is the UV cutoff (the inverse lattice spacing in a lattice calculation).

C. Infra-red divergences of the critical theory

If we tune κ to $\kappa_{c.l.}(\widetilde{\kappa})$, the quadratic kinetic term in Eq. (5.13) vanishes, and the renormalized GB propagator reads

$$G_r^{-1}(p)=Z(p^2)^2,\tag{5.16}$$

where Z accounts for the wave-function renormalization. This quartic propagator leads to IR divergences in four dimensions, like massless bosons with an ordinary kinetic term do in two dimensions.

The IR divergences of massless Goldstone bosons lead to the restoration of continuous symmetries in two dimensions [37]. Only symmetric observables, which are invariant under all the continuous symmetries, can have a non-zero value. There are theorems [38–40] that guarantee the IR-finiteness of the symmetric observables.

Here the quartic propagator (5.16) does not characterize a whole phase, but only the FM-FMD line itself. The order parameter v_H (or v) dips close to the FM-FMD line, and vanishes on that line in several (maybe in all) interesting cases [31,36]. The theorems on the finiteness of symmetric observables, in particular Ref. [40], generalize to four dimensions. Also, as in two dimensions, the predictions of the weak-coupling expansion are often valid, if interpreted carefully [41]. This will be important in Sec. VI B.

D. Realistic reduced models

The key features of the simplified reduced model studied in this section extend to the realistic reduced models, defined from the (gauge-fixing and ghost) action of Sec. II E for the non-linear gauge, or the action of *II* for the linear gauge. This includes the qualitative structure of the phase diagram and in particular the FM and FMD phases, the Gaussian critical point at $\widetilde{\kappa}=\infty$, and the IR divergences on the FM-FMD line.

Particularly interesting are the critical FM-FMD theories in the reduced models that correspond to the linear gauge $\partial\cdot A=0$. In the Abelian case, the linear-gauge reduced model leads to a free theory with a $1/(p^2)^2$ propagator. The properties of this critical theory are analogous to the spin-wave phase of a two-dimensional Abelian theory. This will be discussed in detail elsewhere [31,36]. The critical theory for a

non-Abelian gauge group was investigated by Hata [42] in the continuum approximation. His main result is that, like non-Abelian sigma models in two dimensions, these four-dimensional non-linear models are asymptotically free. It will be interesting to investigate the significance of this result for the construction of gauge-fixed non-Abelian lattice theories via our approach.

VI. FERMIONS IN THE REDUCED MODEL

In a manifestly gauge invariant theory like QCD, the fermion spectrum can be read off from the lattice action by going to the free field limit $g_0 = 0$. Here, the fermion action is not gauge invariant (in the vector picture), and the limit $g_0 = 0$ gives rise to an interacting theory, namely, to the reduced model. We identify the elementary fermions of a general lattice gauge theory with the independent *fermionic massless poles* of the associated reduced model. (If the fermion action is gauge invariant, any ϕ_x dependence of its reduced-model form can be eliminated by a field redefinition.) It is justified to determine the matter spectrum by setting $g_0 = 0$, since, in a scaling region, the transversal degrees of freedom are perturbative at the lattice scale.

A. The robustness of the no-go theorems

Let the gauge field belong to a Lie group G. By construction, the associated reduced model has a global G-symmetry, denoted G_L, that acts on ϕ_x by left multiplication. (The reduced model is obtained from the vector picture via $U_{x,\mu} \rightarrow \phi_x^\dagger \phi_{x+\hat\mu}$, and the product $\phi_x^\dagger \phi_{x+\hat\mu}$ is invariant under left multiplication. Notice also that the gauge-invariant Higgs picture can be obtained by gauging the G_L symmetry of the reduced model.) Now, we demand the existence of massless vector bosons in the scaling region, which can be identified with the gauge bosons of the target continuum theory. These vector bosons couple to the Noether current associated with the G_L symmetry. Thus, assigning the fermions to representations of G_L determines whether the continuum limit is chiral or vector-like.

In previous chiral fermion proposals, it was usually attempted to take the continuum limit in a symmetric phase, where the G_L symmetry is *not* broken spontaneously. (Physical gauge invariance is restored dynamically in a symmetric phase, when we consider the full $g_0 \neq 0$ theory. This means that there are no light unphysical states, whose decoupling in the continuum limit requires fine-tuning. Since the VEV of the ϕ_x field is zero, the physics in a symmetric phase is more easily accounted for in the gauge-invariant Higgs picture.) In a symmetric phase, the fluctuations of the ϕ_x field are usually not controlled by any small parameter. As a result, nonperturbative methods had to be invoked in order to determine the fermion spectrum. Where available, it was always found that the true fermion spectrum is vector-like (see Refs. [7,9,43] for details).

We have discussed this phenomenon in Refs. [5,7], and argued that it has a simple physical explanation. Here we can only outline the key considerations leading to this conclusion, and we refer the reader to Refs. [5,7] for the details. One starts with the observation that, in a symmetric phase of the reduced model, there are generically no massless scalars. Therefore, the only massless particles (if any) are fermions.

(It could be [43] that no massless fermions are present unless a mass term if fine-tuned. Since we are in symmetric phase, a massless fermion obtained by fine-tuning is necessarily a Dirac fermion.) Now, in four dimensions, there are no renormalizable interactions involving only fermion fields. The continuum limit defined by a generic point inside a symmetric phase is therefore a theory of free massless fermions (if it is not empty). One can then construct an *effective lattice Hamiltonian* for the fermions, that satisfies all the assumptions of the *Nielsen-Ninomiya theorem*. (The effective Hamiltonian is defined as the $p_0 = 0$ limit of the inverse of a suitable two-point function.) We refer here in particular to the analytic structure near the zeros of the effective Hamiltonian, and to the existence of a smooth interpolation throughout the rest of the Brillouin zone. This leads to the conclusion that the fermion spectrum is vector-like in a symmetric phase, provided the underlying theory is local. (In the case of a non-local theory one, expects violations of unitarity and/or Lorentz invariance, see Ref. [7] for references to the original literature.)

This impasse extends, by continuity, to the fermion spectrum on any phase transition line that separates a symmetric phase from a broken phase. In particular, even though the gauge boson mass vanishes on the PM-FM line, we do not expect to find a chiral gauge theory by taking the continuum limit at the PM-FM line. The fermion spectrum will be vector-like if the PM-FM line is approached from the PM phase. If we approach the PM-FM line from the FM phase, we can only obtain a mirror fermion model [44], but we cannot decouple the unwanted mirror fermions.

B. Evading the no-go theorems

Let us now investigate what changes when the continuum limit is taken at the FM-FMD line. We will consider the simplest case, namely a U(1) gauge group with the gauge-fixing action pertaining to the linear gauge, *cf. II*. As mentioned in Sec. V D, the properties of the critical FM-FMD theory are analogous to the spin-wave phase of a two-dimensional Abelian theory.

We go from (the vector picture of) the full theory to the reduced model according to the rule $U_{x,\mu} \rightarrow \phi_x^\dagger \phi_{x+\hat\mu}$. The fermion action Eq. (3.1) becomes (we use the two-component notation)

$$S_F' = \sum_\mu (\bar\chi \phi^\dagger)\sigma_\mu \Delta_\mu(\phi\chi) - \frac{w}{4}(\chi\Box\chi + \text{H.c.}). \quad (6.1)$$

The fermion variables in Eq. (6.1) are *neutral* with respect to G_L. If, instead, we use the *charged* variables $\chi_c = \phi\chi$, the fermion action reads

$$S_F' = \sum_\mu \bar\chi_c \sigma_\mu \Delta_\mu \chi_c - \frac{w}{4}((\phi^\dagger\chi_c)\Box(\phi^\dagger\chi_c) + \text{H.c.}).$$
$$(6.2)$$

According to the rules of the weak-coupling expansion (see Sec. V B), the tree-level fermion action is obtained by substituting the classical vacuum $\phi_x = 1$. Using Eq. (6.2) for definiteness, we get

$$S_F^0 = \sum_\mu \bar{\chi}_c \sigma_\mu \Delta_\mu \chi_c - \frac{w}{4}(\chi_c \square \chi_c + \text{H.c.}). \qquad (6.3)$$

In the limit $w = 0$, only the kinetic term $\Sigma_\mu \bar{\chi}_c \sigma_\mu \Delta_\mu \chi_c$ is left. Thus, the $w = 0$ action exhibits the infamous doubling, with sixteen Weyl fermions altogether. (Each fermion is associated with a point in the Brillouin zone, whose lattice momentum components are equal to either 0 or π.) Since we take $w = O(1)$, the MW term eliminates the doublers, and the pole in the tree-level fermion propagator describes a single Weyl field (see Sec. III).

Had we started from the fermion action written in terms of the neutral variables [Eq. (6.1)], the substitution $\phi_x = 1$ would lead to a tree-level action identical to Eq. (6.3), but with the neutral field χ replacing the charged field χ_c. Now, deep in the FM phase this makes no difference, because the G_L symmetry is broken anyway by the ϕ_x-VEV, which is $O(1)$ in lattice units. However, the G_L symmetry is restored right on the FM-FMD line [31,36]. It is therefore a meaningful (and important) question to ask what are the G_L-quantum numbers of the massless fermions.

The fact that one cannot simply read off the quantum numbers of one-fermion states from the tree-level action, is a consequence of the IR-divergent nature of the GB propagator, $1/(p^2)^2$. The way to proceed is to examine a family of fermionic two-point functions $\Gamma_n = \langle(\phi^n \chi)(\bar{\chi}\phi^{\dagger n})\rangle$. Assuming all mass parameters have been tuned to their critical values, Γ_n will in general contain terms proportional to $(\not{p})^{-1} \log^k(p^2)$ for any k. The presence of logarithmic terms (which typically lead to power law corrections when summed over all orders) means that the operator $\phi^n \chi$ *does not create a one-particle state* [41,43]. Only when there are no logarithmic corrections do we have a simple massless pole, and the quantum numbers of the intermediate one-fermion state must coincide with the quantum numbers of the interpolating fermion field.

The fermion spectrum in the reduced model can be studied in detail using the weak-coupling expansion. While the actual calculations require a substantial amount of work, the conclusions are robust, as they really depend on universal properties of the low-energy effective (continuum) Lagrangian. A one-loop calculation, which is also supported by numerical simulations, will be presented elsewhere [36]. Here we will list the key results, as they apply to the MW fermion action.

Logarithmic terms are *absent* only for $n = 1$, namely in the two-point function $\langle \chi_c \bar{\chi}_c \rangle$. Consequently, the massless fermion has the quantum numbers of the χ_c field. The latter is *charged*, which means that the χ_c Weyl fermion will couple to the transversal gauge field, when the latter is turned on.

As can be expected on general grounds (see Sec. III), divergent Majorana-like mass terms are induced at the one-loop level. These must be cancelled by suitable counter-terms, to maintain the masslessness of each chiral fermion.

When the Majorana-like mass counter-terms are tuned to their critical values, the unphysical GB field *decouples* from the χ_c-fermions. The continuum limit is a direct product of (in general) several free theories, one associated with the unphysical GB field, and one associated with every species of chiral χ_c-fermions.

This establishes an agreement between the predictions of the weak-coupling expansion in the full theory and in the reduced model, thus supporting the consistency of our approach. The properties of the reduced model are true for an arbitrary fermion spectrum, and this is consistent with the vanishing of the anomaly in the absence of a transversal gauge field.

In comparison with previous chiral fermion proposals (see Sec. VI A) we note two key differences that allow us to escape from a similar impasse. First, one may worry that the need to tune mass counter-terms may indicate that we got the wrong spectrum (*e.g.* Dirac instead of Weyl fermions). Now, when the (Majorana-like) mass counter-terms are *not* tuned to their critical values, the fermions remain *coupled* to the IR-singular GB field in the low-energy limit. Due to potential IR divergences, it is not at all clear that (massive) one-fermion states could be consistently defined in this case, nor that such states would have well-defined G_L-quantum numbers. The off-critical theory remains to be investigated in the future. However, in view of the above IR subtleties, the general conclusion is that by considering the role of fermion mass perturbations, one does not end up with an argument *against* the existence of a chiral spectrum at the critical point. (See also the discussion of fermion mass counter-terms in Sec. III.)

The other key difference is that the continuum limit is now taken at the phase transition separating *two broken phases* of the reduced model. Off the FM-FMD line (on both sides) the G_L symmetry is broken spontaneously, and all asymptotic states do not have well-defined G_L-quantum numbers. On the FM-FMD line itself, the G_L symmetry is restored, and the question arises whether we do not run into the same old conflict with the No-Go theorems. The answer is contained in the analytic structure discussed above. Thanks to the presence of the highly IR-singular GB field, a zero in the inverse propagator does not necessarily imply the existence of a one-fermion state with the same quantum numbers. As an illustration, consider the four-component unit-charge field ψ_c, whose left-handed component is $P_L\psi_c = \chi_c$, and whose right-handed component is $P_R\psi_c = \phi^2 \bar{\chi}_c$. If we consider the inverse two-point function of ψ_c, we may erroneously conclude that it interpolates a massless Dirac fermion. In reality, only the left-handed channel of this inverse propagator has a simple zero $\sim \not{p}$, implying the existence of a unit-charge left-handed fermion. In the right-handed channel, one finds a $\not{p} \log(p^2)$ correction in the one-loop approximation, which implies the absence of a right-handed fermion with the same charge.

VII. OPEN QUESTIONS

In Sec. II E, the criterion for fixing the counter-terms was to enforce BRST invariance (and, hence, unitarity) order by order. This perturbative prescription is incomplete. Ultimately, the counter-terms should be determined by a non-perturbative method. To define rigorously the continuum limit, one has to specify a trajectory in the Higgs (or Higgs-confinement) phase, that ends at the Gaussian point $g_0 = 1/\tilde{\kappa} = 0$ on the FMD boundary. (See Sec. V A for the phase diagram.) In addition, one has to construct a BRST operator, and prove its nilpotency in the continuum limit.

Enforcing BRST invariance should also lead to the restoration of full SO(4) invariance, because the marginal SO(4)-breaking operators violate the BRST symmetry too.

We comment that similar problems are encountered in lattice QCD with Wilson fermions, where the axial-flavour symmetries are broken on the lattice, in analogy with the BRST symmetry in our gauge-fixing approach. When using Wilson fermions, tuning is required not only at the level of the lattice action, but also in the construction of renormalized operators with well defined axial-flavour transformation properties [45]. This is analogous to the problem of defining BRST-invariant operators in our gauge-fixing approach. (In QCD, the fine-tuning problem can be solved using domain-wall fermions [46–48]. Whether a similar solution exists for the tuning problem in the gauge-fixing approach, is an interesting question.)

Our gauge-fixing formulation can be tested by applying it to asymptotically free gauge theories which are *not* chiral. In particular, in the absence of fermions, one should study whether the confining behavior and the mass gap of Yang-Mills theories are reproduced. One possibility is that the FMD transition becomes weakly first order due to nonperturbative effects. This scenario is favorable, at least from the point of view of numerical simulations. Another possibility is that the correlation length of the vector field strictly diverges at the FMD transition, already for $g_0 \neq 0$. In this case, a consistent continuum limit may exist provided all the massless excitations are unphysical.

VIII. CONCLUSIONS

In a *regularized* chiral gauge theory, the longitudinal modes of the gauge field couple to the fermions. Before the regularization is removed, there are violations of gauge in-variance even if the fermion spectrum is anomaly free. When we use the lattice regularization, the longitudinal modes should decouple in the continuum limit, but it may be too much to expect for exact decoupling when the lattice spacing is still finite.

The gauge-fixing approach aims to decouple the longitudinal modes in the continuum limit. In this paper we have discussed how the gauge-fixing approach may be realized, thus making the first step of a systematic investigation of the gauge fixing approach. We have constructed a lattice gauge-fixing action that has a unique classical vacuum. The gauge-fixing action contains a longitudinal kinetic term, and leads to a renormalizable weak-coupling expansion, which is valid even if the lattice fermion action is not gauge invariant. We have argued that the continuum fields, needed to describe the scaling behavior, are in one-to-one correspondence with the poles of the tree-level lattice propagators. This should accommodate any consistent theory, including anomaly free chiral gauge theories.

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