Nonperturbative three-point vertex in massless quenched QED and perturbation theory constraints

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Dong, Munczek, and Roberts have shown how the full 3-point vertex that appears in the Schwinger-Dyson equation for the fermion propagator can be expressed in terms of a constrained function W_1 in massless quenched QED. However, this analysis involved two key assumptions: that the fermion anomalous dimension vanishes in the Landau gauge and that the transverse vertex has a simplified dependence on momenta. Here we remove these assumptions and find the general form for a new constrained function U_1 that ensures the multiplicative renormalizability of the fermion propagator nonperturbatively. We then study the restriction imposed on U_1 by recent perturbative calculations of the vertex and compute its leading logarithmic expansion. Since U_1 should reduce to this expansion in the weak coupling regime, this should serve as a guide to its nonperturbative construction. We comment on the perturbative realization of the constraints on U_1.
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I. INTRODUCTION

The behavior of the fermion propagator in any gauge theory is determined by the fermion–gauge-boson vertex. While in perturbation theory a bare vertex is sufficient, in the strong coupling regime it is well known that this simple ansatz can lead to unacceptable results, such as an unrenormalizable fermion propagator and gauge-dependent chiral symmetry breaking [1]. The Ward-Green-Takahashi identity for the vertex determines what is often called its longitudinal part [2]. The remaining transverse part has long been known to play a crucial role in ensuring the multiplicative renormalizability of the fermion propagator [3,4,5]. However, it is only very recently that a general form for the transverse vertex involving an odd number of gamma matrices has been written down for quenched QED [1,6]. With simplifying assumptions, this ansatz ensures that the fermion propagator is multiplicatively renormalizable and that if a dynamical mass is generated then this phase transition occurs at a gauge-independent value of the critical coupling [6]. This ansatz involves two unknown functions $W_1(x)$ and $W_2(x)$ of a dimensionless ratio x of momenta, each satisfying an integral and a derivative constraint. The integral condition on W_1 guarantees that the fermion propagator is multiplicatively renormalizable, whereas that on W_2 ensures that the critical coupling is a gauge-independent quantity. The derivative conditions are consequences of the transverse vertex being free of kinematic singularities. In the case of massless fermions, W_2 drops out and only W_1 dictates what the transverse vertex is. However, this construction involves the assumption that the transverse vertex vanishes in the Landau gauge and has no dependence on the angle between the fermion momenta. Here we remove these assumptions and introduce a new constrained function $U_1(x)$. In terms of this, we present the most general nonperturbative construction of the transverse vertex required by the multiplicative renormalizability of the fermion propagator.

In this paper, we go on to discuss how perturbation theory can provide additional constraints on U_1. Physically meaningful solutions of the Schwinger-Dyson equations must agree with perturbation theory in the weak coupling limit. Its importance in dictating the nonperturbative structure of the vertex has been appreciated in earlier work [4–7]. We obtain the perturbative expansion of $U_1(x)$ to $\mathcal{O}(\alpha)$, in the limit when $x \to 0$, to which every nonperturbative construction of U_1 must reduce. This is made possible by the recent perturbative calculation of the transverse vertex by Kizilersü *et al.* [7] U_1, being related to a Green's function beyond lowest order, is renormalization scheme dependent. In this paper we have used the cutoff regularization scheme as is natural when discussing multiplicative renormalizability, whereas the calculation of the transverse vertex by Kizilersü *et al.* [7] was performed in the dimensional regularization scheme most useful in perturbation theory, but which does not distinguish between ultraviolet and infrared behaviors. In order to retain consistency, the perturbative evaluation of U_1 has been re-

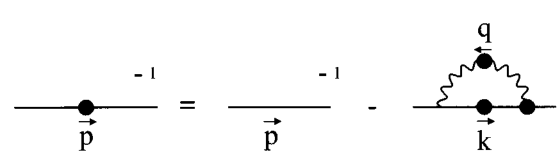

FIG. 1. Schwinger-Dyson equation for the fermion propagator. The straight lines represent fermions and the wavy line the photon. The solid dots indicate full, as opposed to bare, quantities.

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stricted to leading logarithms alone, as these are scheme independent. This fact, however, prevents us from checking explicitly that the integral condition on U_1 is preserved in perturbation theory, though consistency requires that it is.

We check the validity of the derivative constraint on U_1 in perturbation theory. This condition holds in the limit when $x \to 1$. Analytical calculation of U_1 and its derivative in this region is a prohibitively difficult task. However, numerical evaluation is possible. We find that the numerical results are in excellent agreement with the proposed condition.

II. WAVE FUNCTION RENORMALIZATION $F(p^2,\Lambda^2)$

The Schwinger-Dyson equation for the fermion propagator $S_F(p)$ in QED with a bare coupling e is displayed in Fig. 1 and is given by

$$iS_F^{-1}(p) = iS_F^{0^{-1}}(p) - e^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu S_F(k) \Gamma^\nu(k,p) \Delta_{\mu\nu}(g), \tag{1}$$

where $q=k-p$. For massless fermions, $S_F(p)$ can be expressed in terms of a single Lorentz scalar function $F(p^2,\Lambda^2)$, called the wave function renormalization, so that

$$S_F(p) = \frac{F(p^2,\Lambda^2)}{\not p},$$

where Λ is the ultraviolet cutoff used to indicate that the integrals involved are divergent and need to be regularized. The bare propagator $S_F^0(p)=1/\not p$. The photon propagator remains unrenormalized in quenched QED:

$$\Delta_{\mu\nu}(q) = \frac{1}{q^2}\left(g_{\mu\nu} + (\xi-1)\,\frac{q_\mu q_\nu}{q^2}\right) \equiv \Delta^T_{\mu\nu}(q) + \xi\,\frac{q_\mu q_\nu}{q^4},$$

where $\Delta^T_{\mu\nu}(q)$, called the transverse part of the propagator, is defined by the above equation and ξ is the standard covariant gauge parameter. $\Gamma^\mu(k,p)$ is the full fermion-boson vertex, for which we must make an ansatz in order to solve Eq. (1). Keeping in mind that the vertex satisfies the Ward-Green-Takahashi identity

$$q^\mu \Gamma_\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p), \tag{2}$$

Ball and Chiu [2] considered the vertex as a sum of longitudinal and transverse components:

$$\Gamma^\mu(k,p) = \Gamma_L^\mu(k,p) + \Gamma_T^\mu(k,p), \tag{3}$$

where $\Gamma_T^\mu(k,p)$ is defined by

$$q_\mu \Gamma_T^\mu(k,p) = 0. \tag{4}$$

To satisfy Eq. (1) in a manner free of kinematic singularities, which in turn ensures that the Ward identity is satisfied, we have (following Ball and Chiu)

$$\Gamma_L^\mu(k,p) = a(k^2,p^2)\gamma^\mu + b(k^2,p^2)(\not k + \not p)(k+p)^\mu, \tag{5}$$

where

$$a(k^2,p^2) = \frac{1}{2}\left(\frac{1}{F(k^2,\Lambda^2)} + \frac{1}{F(p^2,\Lambda^2)}\right),$$

$$b(k^2,p^2) = \frac{1}{2}\left(\frac{1}{F(k^2,\Lambda^2)} - \frac{1}{F(p^2,\Lambda^2)}\right)\frac{1}{k^2-p^2}, \tag{6}$$

and

$$\Gamma_T^\mu(p,p) = 0. \tag{7}$$

Ball and Chiu [2] demonstrated that a set of 8 vectors $T_i^\mu(k,p)$ formed a general basis for the transverse part, so that

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau_i(k^2,p^2,q^2) T_i^\mu(k,p). \tag{8}$$

Equations (4), (7) are then satisfied provided that in the limit $k \to p$, the $\tau_i(p^2,p^2,0)$ are finite. As shown by Kızılersü *et al.* [7] a modification of the original Ball-Chiu basis is required to achieve this in an arbitrary covariant gauge in perturbation theory. One can then define the Minkowski space basis to be [7]

$$T_1^\mu(k,p) = p^\mu(k\cdot q) - k^\mu(p\cdot q),$$
$$T_2^\mu(k,p) = T_1^\mu(\not k + \not p),$$
$$T_3^\mu(k,p) = q^2\gamma^\mu - q^\mu \not q,$$
$$T_4^\mu(k,p) = q^2[\gamma^\mu(\not k + \not p) - k^\mu - p^\mu] - 2(k-p)^\mu \sigma_{\lambda\nu}k^\lambda p^\nu,$$
$$T_5^\mu(k,p) = -\sigma^{\mu\nu}q_\nu,$$
$$T_6^\mu(k,p) = \gamma^\mu(k^2-p^2) - (k+p)^\mu(\not k - \not p),$$
$$T_7^\mu(k,p) = -\tfrac{1}{2}(k^2-p^2)[\gamma^\mu(\not k + \not p) - k^\mu - p^\mu] + (k+p)^\mu \sigma_{\lambda\nu}k^\lambda p^\nu,$$
$$T_8^\mu(k,p) = \gamma^\mu \sigma_{\lambda\nu}k^\lambda p^\nu - k^\mu \not p + p^\mu \not k. \tag{9}$$

On multiplying Eq. (1) by $\not p$, taking the trace, and making use of Eqs. (2), (3), (5), (6), (8), (9) we have, on Wick rotating to Euclidean space,

$$\frac{1}{F(p^2,\Lambda^2)} = 1 - \frac{\alpha}{4\pi^3}\frac{1}{p^2}\int d^4k\,\frac{F(k^2,\Lambda^2)}{k^2q^2}\Bigg\{a(k^2,p^2)\,\frac{1}{q^2}[-2\Delta^2 - 3q^2k\cdot p] + b(k^2,p^2)\,\frac{1}{q^2}[-2\Delta^2(k^2+p^2)]$$
$$- \frac{\xi}{F(p^2,\Lambda^2)}\frac{p^2}{q^2}(k^2 - k\cdot p) + \tau_2(k^2,p^2,q^2)[-\Delta^2(k^2+p^2)] + \tau_3(k^2,p^2,q^2)[2\Delta^2 + 3q^2k\cdot p]$$
$$+ \tau_6(k^2,p^2,q^2)[3(k^2-p^2)k\cdot p] + \tau_8(k^2,p^2,q^2)[2\Delta^2]\Bigg\}. \tag{10}$$

where $\Delta^2=(k\cdot p)^2-k^2p^2$. Note that only those T_i^μ with odd numbers of gamma matrices contribute in the case of massless fermions—incidentally, these are then the same as in the basis proposed in [2]. At this stage, it appears impossible to proceed any further without demanding that the τ_i be independent of the angle between the fermion momentum vectors k and p, i.e., independent of q^2. This assumption allows us to carry out the angular integration in Eq. (10). We shall show later in this paper that this assumption is not a necessary requirement for solving the above Schwinger-Dyson equation and this can readily be undone. In order to distinguish the transverse components which are assumed to be independent of q^2 from the real ones which explicitly depend on q^2 [7], we denote the former by $\tau_i^{\rm eff}$, suggesting that these are only effective τ_i. Now carrying out the angular integration,

$$\frac{1}{F(p^2,\Lambda^2)}=1-\frac{\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2}\,F(k^2,\Lambda^2)\Bigg[\frac{k^4}{p^4}\Bigg\{b(k^2,p^2)\left[\frac{3}{2}\,(k^2+p^2)\right]+\tau_2^{\rm eff}(k^2,p^2)\left[-\frac{1}{4}\,(k^2+p^2)(k^2-3p^2)\right]$$
$$+\tau_3^{\rm eff}(k^2,p^2)\left[\frac{1}{2}\,(k^2-3p^2)\right]+\tau_6^{\rm eff}(k^2,p^2)\left[\frac{3}{2}\,(k^2-p^2)\right]+\tau_8^{\rm eff}(k^2,p^2)\left[\frac{1}{2}\,(k^2-3p^2)\right]\Bigg\}\theta(p^2-k^2)$$
$$+\Bigg\{b(k^2,p^2)\left[\frac{3}{2}\,(k^2-p^2)\right]-\frac{\xi}{F(p^2)}+\tau_2^{\rm eff}(k^2,p^2)\left[-\frac{1}{4}\,(k^2+p^2)(p^2-3k^2)\right]+\tau_3^{\rm eff}(k^2,p^2)\left[\frac{1}{2}\,(p^2-3k^2)\right]$$
$$+\tau_6^{\rm eff}(k^2,p^2)\left[\frac{3}{2}\,(k^2-p^2)\right]+\tau_8^{\rm eff}(k^2,p^2)\left[\frac{1}{2}\,(p^2-3k^2)\right]\Bigg\}\theta(k^2-p^2)\Bigg]. \qquad (11)$$

Following Dong, Munczek, and Roberts [1], Bashir and Pennington [6] have proposed an ansatz for the transverse vertex. They require it to be chosen such that the fermion propagator is multiplicatively renormalizable, and in the case of massive fermions, the chiral symmetry-breaking phase transition takes place at a gauge-invariant value of the coupling. They show that the transverse vertex can be written in terms of two unknown functions W_1 and W_2, each obeying an integral and a derivative condition. In the case of a chirally symmetric solution, the transverse vertex reduces to being a function of W_1 alone. However, this construction involves the additional assumption that the transverse vertex is zero in the Landau gauge. In general, the solution of Eq. (11) imposed by multiplicative renormalizability in quenched QED is

$$F(p^2,\Lambda^2)=A(p^2/\Lambda^2)^\gamma \qquad (12)$$

in any covariant gauge with A a constant. The assumption of a vanishing transverse vertex in the Landau gauge means that the anomalous dimension γ is equal to $\nu\equiv\alpha\xi/4\pi$. Crucially this is not the general solution; nor is it even in agreement with perturbation theory [8]. The anomalous dimension γ is not zero in the Landau gauge. Consequently, we fix the effective transverse vertex quite generally in terms of a function $U_1(x)$ through a series of steps analogous to those followed in Refs. [1,6].

The result is

$$\overline{\tau_{\rm eff}}(k^2,p^2)=\frac{1}{4}\,\frac{1}{k^2-p^2}\,\frac{1}{s_1(k^2,p^2)}\left[U_1\!\left(\frac{k^2}{p^2}\right)-U_1\!\left(\frac{p^2}{k^2}\right)\right]-\frac{2\pi}{\alpha}\,\frac{\gamma-\nu}{k^2-p^2}\left(\frac{1}{F(k^2,\Lambda^2)}-\frac{1}{F(p^2,\Lambda^2)}\right), \qquad (13)$$

$$\tau_6^{\rm eff}(k^2,p^2)=-\frac{1}{2}\,\frac{k^2+p^2}{(k^2-p^2)^2}\left(\frac{1}{F(k^2,\Lambda^2)}-\frac{1}{F(p^2,\Lambda^2)}\right)+\frac{1}{3}\,\frac{k^2+p^2}{k^2-p^2}\,\overline{\tau_{\rm eff}}(k^2,p^2)+\frac{1}{6}\,\frac{1}{k^2-p^2}\,\frac{1}{s_1(k^2,p^2)}\left[U_1\!\left(\frac{k^2}{p^2}\right)+U_1\!\left(\frac{p^2}{k^2}\right)\right]-\frac{4\pi}{3\alpha}\,\frac{\gamma-\nu}{k^2-p^2}\left(\frac{1}{F(k^2,\Lambda^2)}+\frac{1}{F(p^2,\Lambda^2)}\right), \qquad (14)$$

where

$$s_1(k^2,p^2)=\frac{k^2}{p^2}\,F(k^2,\Lambda^2)+\frac{p^2}{k^2}\,F(p^2,\Lambda^2)$$

and

$$\overline{\tau_{\rm eff}}(k^2,p^2)=\tau_3^{\rm eff}(k^2,p^2)+\tau_8^{\rm eff}(k^2,p^2)-\tfrac{1}{2}(k^2+p^2)\,\tau_2^{\rm eff}(k^2,p^2). \qquad (15)$$

To see how these forms arise uniquely, let us substitute Eqs. (13), (14) into Eq. (11) to obtain

$$\frac{1}{F(p^2,\Lambda^2)}=1+\nu\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2,\Lambda^2)}{F(p^2,\Lambda^2)}-\frac{\alpha}{8\pi}\int_0^{p^2}\frac{dk^2k^2}{p^4}\frac{U_1(k^2/p^2)F(k^2,\Lambda^2)}{(k^2/p^2)F(k^2,\Lambda^2)+(p^2/k^2)F(p^2,\Lambda^2)}$$
$$-\frac{\alpha}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{U_1(p^2/k^2)F(k^2,\Lambda^2)}{(k^2/p^2)F(k^2,\Lambda^2)+(p^2/k^2)F(p^2,\Lambda^2)}+(\gamma-\nu)\int_0^{p^2}\frac{dk^2k^2}{p^4}+(\gamma-\nu)\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2,\Lambda^2)}{F(p^2,\Lambda^2)}, \tag{16}$$

where we recall $\nu\equiv\alpha\xi/(4\pi)$. Multiplicative renormalizability requires that the renormalized F_R be related to the unrenormalized F through a multiplicative factor Z by

$$F(p^2,\Lambda^2)=Z(\mu^2/\Lambda^2)F_R(p^2,\mu^2), \tag{17}$$

so that the solution of this equation is

$$\frac{F_R(k^2,\mu^2)}{F_R(p^2,\mu^2)}=\frac{F(k^2,\Lambda^2)}{F(p^2,\Lambda^2)}=\left(\frac{k^2}{p^2}\right)^{\gamma}. \tag{18}$$

Now this power behavior is the solution of

$$\frac{1}{F(p^2,\Lambda^2)}=1+\gamma\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2,\Lambda^2)}{F(p^2,\Lambda^2)}. \tag{19}$$

Consequently, from Eq. (16), this imposes the following restriction on the transverse vertex and hence the function $U_1(k^2/p^2)$:

$$\frac{\alpha}{8\pi}\int_0^{p^2}\frac{dk^2k^2}{p^4}\frac{U_1(k^2/p^2)[F(k^2,\Lambda^2)/F(p^2,\Lambda^2)]}{(k^2/p^2)[F(k^2,\Lambda^2)/F(p^2,\Lambda^2)]+(p^2/k^2)}+\frac{\alpha}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{U_1(p^2/k^2)}{(k^2/p^2)+(p^2/k^2)[F(p^2,\Lambda^2)/F(k^2,\Lambda^2)]}$$
$$-\frac{1}{2}(\gamma-\nu)=0. \tag{20}$$

Introducing the variable x, where

$$x=k^2/p^2\quad\forall 0\leqslant k^2<p^2,$$
$$x=p^2/k^2\quad\forall p^2\leqslant k^2<\Lambda^2, \tag{21}$$

in the first two terms of the above equation, Eq. (20) becomes simply

$$\int_0^{-1}dx\,\frac{U_1(x)x^{1+\gamma}}{x^{-1}+x^{1+\gamma}}+\int_{p^2/\Lambda^2}^{1}dx\,\frac{U_1(x)x^{-1}}{x^{-1}+x^{1+\gamma}}=\frac{4\pi}{\alpha}(\gamma-\nu). \tag{22}$$

We can now let $\Lambda^2\to\infty$, and so we simply have

$$\int_0^1 dx\;U_1(x)=\frac{4\pi}{\alpha}(\gamma-\nu). \tag{23}$$

Note that the previous construction [1,6] explicitly assumed that $\gamma=\nu=\alpha\xi/(4\pi)$ and then $U_1(x)\to W_1(x)$ and $\int_0^1 dx W_1(x)=0$. Moreover, the simplified vertex of [4] corresponds to setting $W_1(x)=0$.

The transverse vertex has no kinematic singularities. Motivated by the perturbative calculation of Ball and Chiu [2] in the Feynman gauge and later by Kizilersü *et al.* [7] in arbitrary covariant gauges, it is a plausible assumption that even nonperturbatively this is achieved by the individual τ_i's being free of kinematic singularities. The antisymmetry of $\tau_6^{\rm eff}(k^2,p^2)$ under $k^2\leftrightarrow p^2$ interchange then requires that

$$\lim_{k^2\to p^2}(k^2-p^2)\tau_6^{\rm eff}(k^2,p^2)=0. \tag{24}$$

This imposes another constraint on $U_1(x)$:

$$U_1(1)+U_1'(1)=-6\gamma+\frac{8\pi}{\alpha}(\gamma-\nu)(2-\gamma). \tag{25}$$

For later, let us note that Eqs. (13) and (14) can be inverted to write U_1 in terms of the $\tau_i^{\rm eff}$:

$$U_1\left(\frac{k^2}{p^2}\right)=s_1(k^2,p^2)\Bigg[(k^2-3p^2)\overline{\tau_{\rm eff}}(k^2,p^2)+\frac{3}{2}\frac{k^2+p^2}{k^2-p^2}\left(\frac{1}{F(k^2,\Lambda^2)}-\frac{1}{F(p^2,\Lambda^2)}\right)+3(k^2-p^2)\tau_6^{\rm eff}(k^2,p^2)+\frac{8\pi}{\alpha}(\gamma-\nu)\frac{1}{F(k^2,\Lambda^2)}\Bigg]. \tag{26}$$

We now set about using recent perturbative calculations of the structure of the vertex to determine the weak coupling limit of this U_1.

III. REAL VERTEX AND EFFECTIVE VERTEX

Calculation of $U_1(x)$ is nontrivial. This is because to be able to solve the Schwinger-Dyson equation for the fermion propagator requires assumptions to be made about the way the fermion-boson vertex $\Gamma^\mu(k,p,q)$ depends upon q^2. Indeed, it seems impossible to proceed analytically without assuming that the vertex is independent of the photon momentum q; otherwise, we cannot carry out the integration over the angular variable. A motivation for this simplifying assumption comes from the large momentum behavior of the vertex in perturbation theory, where it does, indeed, only depend on the variables k^2 and p^2, and *not on* q^2 [4]:

$$\Gamma_T^\mu(k,p) \simeq -\frac{\alpha\xi}{8\pi}\ln\frac{k^2}{p^2}\left[\gamma^\mu - \frac{k^\mu \not k}{k^2}\right]. \tag{27}$$

However, it is clear from the perturbative calculation of Kizilersü, Reenders, and Pennington [7] that the same does not hold true for all the ranges of k^2 and p^2. Instead, the q^2 dependence occurs in almost every term of each of the τ_i. We should, therefore, keep in mind that whenever we are neglecting the q^2 dependence, we are not talking about the exact, but only the *effective* vertex. In order to find a connection between the two, we compare Eqs. (10) and (11), which yields the following exact relation between the real and the effective τ_i:

$$\tau_2^{\rm eff}(k^2,p^2) = \frac{1}{f(k^2,p^2)}\int_0^\pi d\theta\,\frac{\sin^2\theta}{q^2}\,\tau_2(k^2,p^2,q^2)\Delta^2,$$

$$\tau_3^{\rm eff}(k^2,p^2) = \frac{1}{f(k^2,p^2)}\int_0^\pi d\theta\,\frac{\sin^2\theta}{q^2}\,\tau_3(k^2,p^2,q^2)\times\left(\Delta^2 + \frac{3}{2}q^2 k\cdot p\right),$$

$$\tau_6^{\rm eff}(k^2,p^2) = \frac{1}{f_6(k^2,p^2)}\int_0^\pi d\theta\,\frac{\sin^2\theta}{q^2}\,\tau_6(k^2,p^2,q^2)k\cdot p,$$

$$\tau_8^{\rm eff}(k^2,p^2) = \frac{1}{f(k^2,p^2)}\int_0^\pi d\theta\,\frac{\sin^2\theta}{q^2}\,\tau_8(k^2,p^2,q^2)\Delta^2, \tag{28}$$

where

$$f(k^2,p^2) = \frac{\pi}{8}\left[\frac{k^2}{p^2}(k^2-3p^2)\theta(p^2-k^2) + \frac{p^2}{k^2}(p^2-3k^2)\theta(k^2-p^2)\right],$$

$$f_6(k^2,p^2) = \frac{\pi}{4}\left[\frac{k^2}{p^2}\theta(p^2-k^2) + \frac{p^2}{k^2}\theta(k^2-p^2)\right].$$

The perturbative evaluation of $\tau_i^{\rm eff}$ using Eq. (28) is made possible by the calculation of Kizilersü *et al.* [7] for the real τ_i:

$$\tau_2(k^2,p^2,q^2) = \frac{\alpha}{8\pi\Delta^2}\left\{J_0\left[\frac{1}{2}(\xi-2)\left(\frac{3}{2\Delta^2}q^2k^2p^2 + (k^2+p^2)\right) + k\cdot p\right] - \ln\frac{k^2}{p^2}\left[(\xi-2)\frac{3}{4\Delta^2}(k^2-p^2)k\cdot p + \frac{\xi}{2}\frac{(k+p)^2}{k^2-p^2}\right] + \ln\frac{q^4}{k^2p^2}\left[(\xi-2)\frac{3}{4\Delta^2}q^2k\cdot p + \xi - 1\right] + (\xi-2)\right\}, \tag{29}$$

$$\tau_3(k^2,p^2,q^2) = \frac{\alpha}{8\pi\Delta^2}\left\{J_0\left[\frac{(\xi-2)}{8}\left(\frac{3}{\Delta^2}(k^2-p^2)^2(k\cdot p)^2 + (k^2+p^2)^2\right) - \Delta^2\right] + \ln\frac{k^2}{p^2}\left[(\xi-2)\frac{k^2-p^2}{4}\left(1 - \frac{3}{2\Delta^2}(k+p)^2k\cdot p\right)\right] + \ln\frac{q^4}{k^2p^2}\left[(\xi-2)\frac{k\cdot p}{2}\left(\frac{3}{4\Delta^2}(k^2-p^2) - 1\right)\right] + \frac{1}{2}(\xi-2)(k+p)^2\right\}, \tag{30}$$

$$\frac{\tau_6(k^2,p^2,q^2)}{k^2-p^2} = \frac{\alpha(\xi-2)}{32\pi\Delta^2}\left\{J_0\left[\frac{q^2}{2}\left(1 - \frac{3}{\Delta^2}(k\cdot p)^2\right) + \Delta^2\right] + \ln\frac{k^2}{p^2}\left[\frac{3}{2\Delta^2}k\cdot p(k^2-p^2) - \frac{(k+p)^2}{k^2-p^2}\right] + \ln\frac{q^4}{k^2p^2}\left[\frac{-3}{2\Delta^2}q^2k\cdot p\right] - 2\right\}, \tag{31}$$

$$\tau_8(k^2,p^2,q^2) = \frac{\alpha}{8\pi\Delta^2}\left\{q^2\left[k\cdot pJ_0 + \ln\frac{q^4}{k^2p^2}\right] - (k^2-p^2)\ln\frac{k^2}{p^2}\right\}, \tag{32}$$

where

$$J_0=\frac{2}{i\pi^2}\int d^4\omega\,\frac{1}{\omega^2(\omega-p)^2(\omega-k)^2},$$

$$=\frac{2}{\Delta}\left[f\left(\frac{k\cdot p-\Delta}{p^2}\right)-f\left(\frac{k\cdot p+\Delta}{p^2}\right)+\frac{1}{2}\ln\frac{q^2}{p^2}\ln\left(\frac{k\cdot p-\Delta}{k\cdot p+\Delta}\right)\right], \tag{33}$$

and

$$f(x)=\mathrm{Sp}(1-x)=-\int_x^1 dy\,\frac{\ln y}{1-y}. \tag{34}$$

Although the Eqs. (29)–(32) appear a little complicated, the nice thing is that all the τ_i are expressed in terms of elementary functions and a single scalar integral J_0. Kizilersü *et al.* have carried out the calculation in the dimensional regularization scheme, whereas here, so as to be able to identify the ultraviolet behavior readily, we use the cutoff method. Consequently, we restrict our discussion to leading logarithms, which are independent of the choice of the regularization scheme. In the asymptotic limit $k^2 \gg p^2$, the integrals can be evaluated analytically and the separation between the leading and the next to leading terms becomes apparent.

In order to have a perturbative expansion for U_1, we have to go up to $\mathcal{O}(1/k^4)$ in $\tau_3^{\rm eff}$, $\tau_6^{\rm eff}$, and $\tau_8^{\rm eff}$, and $\mathcal{O}(1/k^6)$ in $\tau_2^{\rm eff}$, instead of just keeping the terms of order $\mathcal{O}(1/k^2)$ and $\mathcal{O}(1/k^4)$, respectively. Consequently, in an arbitrary gauge, we have to go up to $\mathcal{O}(1/k^7)$ in evaluating J_0 for k^2 large. The expansion of J_0, keeping only the logarithms, to the required order in the limit when $k^2 \gg p^2$ is

$$J_0=\frac{2}{k^2}\left[1+\frac{k\cdot p}{k^2}-\frac{1}{3}\frac{p^2}{k^2}+\frac{4}{3}\frac{(k\cdot p)^2}{k^4}-\frac{p^2 k\cdot p}{k^4}+2\frac{(k\cdot p)^3}{k^6}+\frac{1}{5}\frac{p^4}{k^4}-\frac{12}{5}\frac{p^2(k\cdot p)^2}{k^6}+\frac{16}{5}\frac{(k\cdot p)^4}{k^8}+\frac{p^4 k\cdot p}{k^6}-\frac{16}{3}\frac{p^2(k\cdot p)^3}{k^8}+\frac{16}{3}\frac{(k\cdot p)^5}{k^{10}}\right]\ln\frac{k^2}{p^2}, \tag{35}$$

Now the perturbative expansion of the real τ_i can be written as

$$\tau_2(k^2,p^2,q^2)$$

$$=-\frac{\alpha}{12\pi k^4}\left\{1+2\frac{k\cdot p}{k^2}+\frac{1}{5k^4}[18(k\cdot p)^2-k^2p^2]-\xi\left[2+3\frac{k\cdot p}{k^2}+\frac{1}{5k^4}[24(k\cdot p)^2+7k^2p^2]\right]\right\}\ln\frac{k^2}{p^2},$$

$$\tau_3(k^2,p^2,q^2)=\frac{\alpha}{12\pi k^2}\left\{1+0\,\frac{k\cdot p}{k^2}-\frac{1}{5k^4}[4(k\cdot p)^2+7k^2p^2]-\frac{k\cdot p}{k^6}[2(k\cdot p)^2+3k^2p^2]+\xi\left[1+\frac{3}{2}\frac{k\cdot p}{k^2}+\frac{1}{5k^4}[12(k\cdot p)^2+k^2p^2]+4\frac{(k\cdot p)^3}{k^6}\right]\right\}\ln\frac{k^2}{p^2},$$

$$\tau_6(k^2,p^2,q^2)=\frac{\alpha(\xi-2)}{24\pi k^2}\left\{1+\frac{k\cdot p}{k^2}+\frac{3}{5k^4}[2(k\cdot p)^2+k^2p^2]+\frac{4k\cdot p}{5k^6}[2(k\cdot p)^2+k^2p^2]\right\}\ln\frac{k^2}{p^2},$$

$$\tau_8(k^2,p^2,q^2)=-\frac{\alpha}{4\pi k^2}\left\{1+\frac{2}{3}\frac{k\cdot p}{k^2}+\frac{2}{3k^4}(k\cdot p)^2\right\}\ln\frac{k^2}{p^2}. \tag{36}$$

We learn the following points from the above calculation.

To the lowest order in $1/k^2$, all four τ_i are independent of the angle between the momenta k and p.

Substituting these τ_i in the expression for the full transverse vertex, we retrieve the perturbative result for the transverse vertex derived by Curtis and Pennington [4], Eq. (27). This serves as one of the checks of the calculation.

Comparing the equations for the real τ_i, Eqs. (29)–(32), with their large k^2 limit, Eq. (36), one can see that all the Δ^2 factors have disappeared from the denominator. Hence, for large k^2, the τ_i are explicitly finite for all values of the angular variable.

We can now use Eq. (28) to find out the large k^2 expansion of the effective τ_i. This yields

$$\tau_2^{\rm eff}(k^2,p^2)=-\frac{\alpha}{12\pi k^4}\left\{1-2\xi+\frac{16}{5}\left(\frac{1}{3}-\xi\right)\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2},$$

$$\tau_3^{\rm eff}(k^2,p^2)=+\frac{\alpha}{12\pi k^2}\left\{1+\frac{1}{4}\xi+\frac{1}{5}\left(\frac{7}{3}-\frac{3}{4}\xi\right)\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2},$$

$$\tau_6^{\rm eff}(k^2,p^2)=\frac{\alpha(\xi-2)}{16\pi k^2}\left\{1+\frac{5}{3}\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2},$$

$$\tau_8^{\rm eff}(k^2,p^2)=-\frac{\alpha}{4\pi k^2}\left\{1+\frac{1}{3}\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2},$$

$$\overline{\tau_{\rm eff}}(k^2,p^2)=-\frac{\alpha}{8\pi k^2}\left\{1+\frac{1}{2}\xi-\frac{1}{3}\left(1-\frac{11}{2}\xi\right)\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2}. \tag{37}$$

Using the definition, Eq. (26), for $U_1(x)$, we then deduce its leading logarithmic form to be simply

$$U_1(x)\overset{x\to 0}{=}\frac{\alpha}{2\pi}\ln x. \tag{38}$$

The above equation is the scheme-independent perturbative expression for $U_1(x)$ for $x \to 0$, to which every nonperturbative construction must reduce in the weak coupling regime. This is the main and remarkably simple result of this section.

Note first and most importantly, all terms of the type $\ln x/x$ in the equations for the τ_i *neatly* cancel out in the expression for $U_1(x)$. If this had not happened, such terms would have led to nonintegrable contributions. This cancellation is consistent with Eq. (23), which shows that there can be no $\ln x/x$ term to $\mathcal{O}(\alpha)$. Note, second, that the leading logarithmic perturbative expression for $U_1(x)$ turns out to be independent of the gauge parameter. While we could imagine checking that $\int_0^1 dx\ U_1(x) = 4\pi(\gamma - \nu)/\alpha$ numerically by constructing the integrand explicitly from Eqs. (26), (28)–(32) the lack of consistency arising from the use of two different schemes would render such an attempt meaningless (beyond leading logarithms).

Importantly, our results are in agreement with the rules of the Landau-Khalatnikov transformation [9]. These determine the gauge dependence of a Green's function, once one knows its behavior in some covariant gauge. Thus, if in the Landau gauge

$$F(p^2,\Lambda^2) = A_0(p^2/\Lambda^2)^{\gamma_0},$$

then these rules [9,10,11,12] applied to quenched QED require that, in a general covariant gauge,

$$F(p^2,\Lambda^2) = A(p^2/\Lambda^2)^{\gamma},$$

where $\gamma = \gamma_0 + \alpha\xi/4\pi$ and A, A_0 are constants. Thus $\nu = \alpha\xi/(4\pi)$ provides the only gauge dependence to the anomalous dimension [10–12]. Consequently, in Eqs. (13)–(15), (20)–(23), (25), (26), the factor $\gamma - \nu = \gamma_0$ is gauge independent and in perturbation theory of $\mathcal{O}(\alpha^2)$. Thus, $\int_0^1 dx\ U_1(x)$ too must be of $\mathcal{O}(\alpha)$ and generally gauge independent, like its $x \to 0$ limit, Eq. (38).

Because the derivative condition, Eq. (25), is merely a statement of the transverse vertex being free of kinematic singularities, regardless of in what scheme it has been calculated, it can be checked numerically. To $\mathcal{O}(\alpha)$, the derivative condition reads

$$\omega \equiv U_1(1) + U_1'(1) - \frac{16\pi}{\alpha}\ \gamma_0 = -\frac{3\alpha\xi}{2\pi}. \tag{39}$$

Making use of the complete expressions in Eqs. (26), (28)–(32), we plot ω/α versus the gauge parameter ξ in Fig. 2. The numerical and analytical results are in excellent agreement with each other.

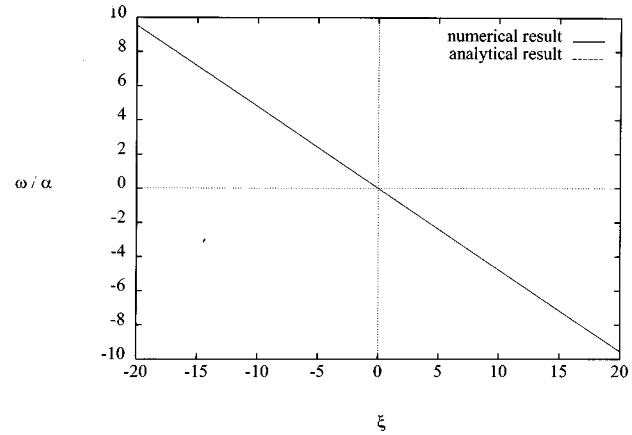

FIG. 2. ω/α of Eq. (39) is plotted as a function of the gauge parameter ξ. The solid line which represents the numerical result lies completely on top of the dashed analytical result, $-3\xi/4\pi$ of Eq. (39), in perfect agreement.

IV. CONCLUSIONS

The nonperturbative study of the fermion propagator through its Schwinger-Dyson equation requires an ansatz for the fermion–gauge-boson vertex. Here we have shown that this vertex (in the case of massless fermions) can be expressed in terms of a single unknown function $U_1(x)$ constrained to ensure the multiplicative renormalizability of the fermion propagator. We have devised a general nonperturbative form for this function and so developed a simple construction for the full fermion-boson vertex. We have then calculated its perturbative expansion and found the remarkably simple result that to $\mathcal{O}(\alpha)$:

$$U_1(x) \overset{x\to 0}{=} \frac{\alpha}{2\pi} \ln x.$$

Any nonperturbative ansatz for $U_1(x)$ should agree with this in the weak coupling limit. This should help in pinning down the only unknown part of the full interaction vertex, Eqs. (5), (6), (8), (13)–(15), and so finally encapsulate the physics encoded in the Schwinger-Dyson equation for the fermion propagator.

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