Gauss law operator algebra and double commutators in chiral gauge theories

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We calculate within an algebraic Bjorken-Johnson-Low method anomalous Schwinger terms of fermionic currents and the Gauss law operator in chiral gauge theories. The current algebra is known to violate the Jacobi identity in an iterative computation. Our method takes the subtleties of the equal-time limit into account and leads to an algebra that satisfies the Jacobi identity. The noniterative terms appearing in the double commutators can be traced back directly to the projective representation of the gauge group. [S0556-2821(97)02222-4]

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I. INTRODUCTION

Chiral gauge theories suffer after fermionic quantization from an anomalous breaking of gauge invariance due to the chiral anomaly [1]. The chiral anomaly is directly related (via cohomological descent equations [1]) to the anomalous Schwinger term in the algebra of the Gauss law operator G in these theories [2]. The Gauss law operator consists of two parts $G=G_A+G_\psi$, where G_A generates (time-independent) gauge transformations on the gauge field and $G_\psi=J^0$ acts on fermions. Here J^0 is the zero component of the (consistent) fermionic current. The algebra of G has been studied in many ways [2–16], since it is connected to the question of consistency of quantized chiral gauge theories. Whereas the cohomological prediction has been verified by the results, the algebras of $G_{A/\psi}$ do not coincide in general. Moreover, it is well known that an iterative calculation of double commutators containing $G_{A/\psi}$ leads to a violation of the Jacobi identity. Double commutators of fermionic currents obtained within an iterative computation do not satisfy a consistency condition, which relates them to the anomaly [11]. If one quantizes the gauge field, the gauge field part G_A is formally given by the covariant derivative of the chromo-electric field E^a [5,10,16]. In these calculations an anomalous Schwinger term occurs in the commutator $[E^a,E^b]$. The double commutator $[E^a,[E^b,E^c]]$ obtained in an iterative computation violates the Jacobi identity. Therefore it is not clear whether the identification $G_A=-D\cdot E$ is correct. An explanation of these facts would give some insight to the structure of chiral gauge theories.

Since a violation of the Jacobi identity also takes place in the case of free fermionic axial and vector currents, one can study the free theory as a toy model. The Schwinger terms in the algebra of free currents [17,18] have been shown to be operator valued. Considerations concerning the commutator algebra of composite operators in chiral gauge theories [19] indicate that this is also true for the Schwinger terms in the algebra of $G_{A/\psi}$.

In [11,20] a Bjorken-Johnson-Low- (BJL-)type prescription [21] was presented for the calculation of double commutators, which respects the Jacobi identity and satisfies the consistency condition mentioned before. Nevertheless it was not clear whether this prescription properly defines a double commutator, since in the free theory the prescription fails to give the correct result [18].

In the present paper we give an algebraic BJL-type derivation of commutators and double commutators of $G_{A/\psi}$ based only on consistency. This is related to the definition of a suitable regularization and renormalization of the time evolution operator U of the theory, and shows up in a nontrivial renormalization factor of U (e.g. [3,4]). The derivation gives a proper definition of commutators and double commutators in chiral gauge theories. Since little is known about a consistent quantization of the gauge field in anomalous gauge theories, we treat the gauge field as an external field with $A(t=-\infty)=0$. This defines an adiabatic solution of the theory [16]. We show that the Schwinger terms of commutators can be derived algebraically from the anomaly without further assumptions and without calculating diagrams. The present approach also provides a simple explanation for the discrepancies in the algebra of $G_{A/\psi}$ in the literature. The same method is applied to the calculation of double commutators. As in the case of commutators we can derive all Schwinger terms from the anomaly only. The Jacobi identity is fulfilled nontrivially for the algebra of $G_{A/\psi}$. The result confirms the validity of the BJL-type prescription given in [11] for the double commutators of $G_{A/\psi}$. It also indicates, that this scheme should be valid in general.

In the second section we discuss the properties of the BJL limit and give a definition of the Gauss law operator G at arbitrary times as the time evolution of $G(t=-\infty)$. The projective representation of the gauge group (on the Fock space) shows up in the properties of the time evolution operator. The following two sections are dedicated to the derivation of equal-time commutators and double commutators only using the anomaly equation. The last section summarizes the results.

II. THE GAUSS LAW OPERATOR

In the following the gauge field is treated as an external field. The fermionic action of a chiral gauge theory is

$$S[\bar\psi,\psi,A]=i\int d^4x\,\bar\psi\left(\not\partial+\frac{1-\gamma_5}{2}\not A\right)\psi, j_a^\mu=i\bar\psi\frac{1-\gamma_5}{2}\gamma^\mu t^a\psi \tag{2.1}$$

with the notation

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$$A=A^a t^a, \quad \mathrm{Tr} t^a t^b=-\frac{1}{2}\delta^{ab}, \quad [t^a,t^b]=f^{abc}t^c,$$

$$\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \epsilon^{0123}=\epsilon^{123}=1. \tag{2.2}$$

The Gauss law operator G in gauge theories is the generator of time-independent gauge transformations and thus the generator of gauge transformations on the Fock space. The appropriate gauge in this framework is the Weyl gauge $A^0=0$. However, we cannot neglect terms dependent on A^0 at intermediate steps, since we will use functional derivatives $(\delta/\delta A_0)A^0$ which contribute. Only the variation $(\delta/\delta A_0)|_{A_0=0}A_0$ vanishes. Commutators of G (defined on the Fock space) are given by the equal-time commutators of

$$G^a(x)=-i\int dx_0 D_i^{ab}(x)\frac{\delta}{\delta A_i^b(x)}+j_a^0(x),$$

$$D_\mu^{ab}=\partial_\mu\delta^{ab}+f^{acb}A_\mu^c, \tag{2.3}$$

where the first part of G generates time-independent gauge transformations on the (external) gauge field and the second part acts on fermions. However, for an anomalous gauge theory we have to be careful with this identification. Therefore we start at $t=-\infty$ with vanishing gauge field $A=0$, where the identification is justified. The time evolution of $G(t=-\infty,\boldsymbol{x})$ defines G at later times. Since we work with an external gauge field, the time evolution operator U is given by

$$U(-\infty,x_0)\simeq T^*\exp\left\{i\int d^3x\int_{-\infty}^{x_0}dtA\cdot j(x)\right\}, \tag{2.4}$$

$$U(y_0,x_0)=U(-\infty,x_0)U(y_0,-\infty),$$

where T^* is the Lorentz covariantized time ordered product. U has to be regularized and renormalized (e.g., [4] and references therein). If the representation of the gauge group is projective (on the Fock space), $U(x_0,x_0)=\mathbb{1}$ cannot be maintained in general. For our purpose it is sufficient to discuss the properties of

$$\begin{aligned} iW[A]&=\ln\langle 0|T^*\exp\left\{i\int d^4xA\cdot j(x)\right\}|0\rangle \\ &=\ln\langle 0|U(-\infty,\infty)|0\rangle. \end{aligned} \tag{2.5}$$

Integrability (consistency) of $W[A]$ is crucial for the following calculations. Using a gauge covariant regularization of the current j [22], the integrability of $W[A]$ shows up in the (consistent) anomaly equation

$$\begin{aligned} \mathcal{A}^a(x)&=D_\mu^{ac}(x)\frac{\delta}{\delta A_\mu^c(x)}W[A] \\ &=\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\,\mathrm{Tr}\left[t^a\left(A_\nu\partial_\rho A_\sigma+\frac{1}{2}A_\nu A_\rho A_\sigma\right)\right]. \end{aligned} \tag{2.6}$$

Equation (2.6) also indicates the existence of a nontrivial renormalization factor in Eq. (2.4). In the following we will only use the definition of $W[A]$ with a regularized current in Eq. (2.5) and the anomaly Eq. (2.6) as an input to calculate commutators and double commutators.

At $t=-\infty$ we deal with a free theory. $G(-\infty,\boldsymbol{x})$ [see Eq. (2.3)] is given by

$$G(-\infty,\boldsymbol{x})=G_A(-\infty,\boldsymbol{x})+G_\psi(-\infty,\boldsymbol{x}), \tag{2.7}$$

where the fermionic part is the zero component of the consistent current J:

$$G_\psi(-\infty,\boldsymbol{x})=J^0(-\infty,\boldsymbol{x})$$

with

$$J(-\infty,\boldsymbol{x})=\frac{\delta}{\delta A(-\infty,\boldsymbol{x})}\int A\cdot j. \tag{2.8}$$

Since j in Eq. (2.5) is gauge field dependent due to the regularization, J differs from j for arbitrary time by a term proportional to $\int A_\mu(\delta/\delta A)j^\mu$. G_A is the generator of time-independent gauge transformations on the gauge field and is given by

$$G_A(-\infty,\boldsymbol{x})=-i\int dtD_i^{ab}(t,\boldsymbol{x})\frac{\delta}{\delta A_i^b(t,\boldsymbol{x})}. \tag{2.9}$$

The equal-time commutators of $G_{A/\psi}(-\infty,\boldsymbol{x})$ are the canonical commutators, since the gauge field vanishes $[A(-\infty,\boldsymbol{x})=0]$. Now we define the Gauss law operator $G(x)$ for arbitrary times x_0 as the time evolution of $G(-\infty,\boldsymbol{x})$. The fermionic part $G_\psi(x)$ of G is given by the zero component of the consistent current J:

$$G_\psi(x)=J^0(x)$$

with

$$J(x)=U(x_0,-\infty)J(-\infty,\boldsymbol{x})U(-\infty,x_0). \tag{2.10}$$

The time evolution of G_A is given by

$$\begin{aligned} G_A(x)&=U(x_0,-\infty)G_A(-\infty,\boldsymbol{x})U(-\infty,x_0) \\ &=-i\int dtD_i^{ab}(t,\boldsymbol{x})\boldsymbol{\delta}^i{}_b(t,x), \end{aligned} \tag{2.11}$$

where $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta}^i{}_b(t,x)=U(x_0,-\infty)U(-\infty,x_0)\frac{\delta}{\delta A_i^b(t,\boldsymbol{x})}+\Gamma^i_b(t,x) \tag{2.12}$$

with

$$\Gamma^i_b(t,x)=i\,\theta(x_0-t)J^i_b(t,\boldsymbol{x}).$$

Γ^i can be seen as a nontrivial connection for the derivative $\delta/\delta A$ as in the two-dimensional theory [16]. However, in contrast to [16] it is not possible to construct Γ^i explicitly in four dimensions. We relate expectation values of G_A to derivatives of $W[A]$ with respect to the gauge field. The expectation value of G_A is connected to $W[A]$ via

$$\langle G_A(x)\rangle=\int dt\,\theta(x_0-t)D^{ab}_i(t,\boldsymbol{x})\frac{\delta}{\delta A^b_i(t,\boldsymbol{x})}W[A], \tag{2.13}$$

where the expectation value $\langle\cdots\rangle$ refers to the background defined by $W[A]$.

In the following we will present equal-time commutators and double commutators as derivatives of $W[A]$ with respect to the gauge field. The components $G_{A/\psi}$ of G are connected to the following A derivatives:

$$\widetilde{G}^a_A(x)=\int dt\,\theta(x_0-t)D^{ab}_i(t,\boldsymbol{x})\frac{\delta}{\delta A^b_i(t,\boldsymbol{x})},$$

$$\widetilde{G}^a_\psi(x)=\int dt\,\theta(x_0-t)\partial_0\frac{\delta}{\delta A^a_0(t,\boldsymbol{x})}=\frac{\delta}{\delta A^a_0(x)}. \tag{2.14}$$

With Eqs. (2.13) and (2.14) we derive the well-known relation [23] between the time derivative of $G=G_A+G_\psi$ and the anomaly (using $A^0=0$)

$$\partial_{x_0}\langle G^a(x_0,\boldsymbol{x})\rangle=\partial_{x_0}\widetilde{G}(x)W[A]=D^{ab}_\mu\frac{\delta}{\delta A^b_\mu}W[A]=\mathcal{A}^a. \tag{2.15}$$

The equal-time commutator of $G_A(x_0,\boldsymbol{x})$ with an operator

$$O(y)=U(y_0,-\infty)O(-\infty,\boldsymbol{y})U(-\infty,y_0)$$

with

$$[A,O]=0 \tag{2.16}$$

is given by

$$[U(x_0,-\infty)G_A(-\infty,\boldsymbol{x})U(-\infty,x_0),U(x_0,-\infty)O(-\infty,\boldsymbol{y})U(-\infty,x_0)]. \tag{2.17}$$

With $D^i(\delta/\delta A^i)U(x_0,x_0)=0$ we would conclude

$$[G_A(x_0,\boldsymbol{x}),O(x_0,\boldsymbol{y})]=U(x_0,-\infty)[G_A(-\infty,\boldsymbol{y}),O(-\infty,\boldsymbol{y})]U(-\infty,x_0). \tag{2.18}$$

Since we expect a projective representation of the gauge group, $D^i(\delta/\delta A^i)U(x_0,x_0)=0$ cannot be assumed. Indeed we have $\widetilde{G}_A(-\infty,\boldsymbol{x})U(x_0,x_0)\neq 0$. This can be taken into account by carefully calculating the equal-time limit

$$\lim_{t\to x_0}G_A(-\infty,\boldsymbol{x})U(-\infty,t)U(x_0,-\infty)=\lim_{t\to x_0}G_A(-\infty,\boldsymbol{x})U(x_0,t)=-i\lim_{p_0\to\infty}\int dt e^{ip_0(t-x_0)}D^{ab}_i(t,\boldsymbol{x})\frac{\delta}{\delta A^b_i(t,\boldsymbol{x})}U(-\infty,\infty). \tag{2.19}$$

The limit $p_0\to\infty$ in the last line projects on the terms with $t=x_0$ in $D(\delta/\delta A)U(-\infty,\infty)$. Thus we conclude

$$\langle[G_A(x),O(y)]_{\mathrm{ET}}\rangle=-i\lim_{p_0\to\infty}\int dx_0 e^{ip_0(x_0-y_0)}D^{ab}_i(x)\frac{\delta}{\delta A^b_i(x)}\langle O(y)\rangle. \tag{2.20}$$

For operators O containing $(\delta/\delta A)$ we have additional terms proportional to $(\delta/\delta A)[G_A]$. Equation (2.20) has the form of a BJL limit [21], which connects the time-ordered product of two (bosonic) operators A,B with their equal-time commutator. We have formally

$$\int dx_0 e^{ip_0x_0}\partial_{x_0}TA(x)B(0)=\int dx_0 e^{ip_0x_0}\partial_{x_0}[\theta(x_0)A(x)B(0)+\theta(-x_0)B(0)A(x)]$$

$$=[A(x),B(0)]_{x_0=0}+\int dx_0 e^{ip_0x_0}T\partial_{x_0}A(x)B(0). \tag{2.21}$$

Providing a suitable regularization for the operators A and B, the second term vanishes in the limit $p_0\to\infty$. The extension to double commutators is obvious [20]:

$$[A(x),[B(y),C(0)]]_{\mathrm{ET}}=\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int dx_0dy_0e^{ip_0x_0}e^{iq_0y_0}\partial_{x_0}\partial_{y_0}TA(x)B(y)C(0), \tag{2.22}$$

where the subscript ET denotes equal time. In Eq. (2.22) we first have to perform the $q_0 \to \infty$ limit. Moreover this only provides a proper definition of the double commutator, if the regularization of the time ordered product $TA(x)B(y)C(0)$ does not affect the θ functions. However, performing the BJL limit perturbatively in Feynman integrals, this condition is violated by the exchange of integration and limit procedure. The diagrammatic calculation of double commutators with formally equivalent BJL limits does not give the same result in general (e.g., fermionic currents [11,15,20]), which indicates the failure of the iterative BJL limit. This is the reason for the violation of the Jacobi identity within the iterative BJL procedure [15,18].

Commutators containing G_A can be expressed as the limit of time-ordered products with the BJL method. With use of Eqs. (2.11) and (2.20) we get (O does not contain $(\delta/\delta A)$)

$$\langle [G_A(x), O(0)]_{\mathrm{ET}} \rangle = \lim_{p_0 \to \infty} \int dx_0 e^{ip_0 x_0} \partial_{x_0} \langle T^* G_A(x) O(0) \rangle = -i \lim_{p_0 \to \infty} \int dt e^{ip_0 t} D_i^{ab}(t, \boldsymbol{x}) \frac{\delta}{\delta A_i^b(t, \boldsymbol{x})} \langle O(0) \rangle. \tag{2.23}$$

The covariantized time ordering T^* appears naturally in the definition of the BJL limit, if G_A is involved. However, if

$$\langle 0 | T^* O(y) \exp\left(i \int A \cdot j \right) | 0 \rangle = \langle 0 | T O(y) \exp\left(i \int A \cdot j \right) | 0 \rangle, \tag{2.24}$$

the results do not depend on the use of the usual time ordering T or T^*. The operators O mentioned here have the property Eq. (2.24).

Equation (2.23) can be extended to arbitrary commutators and double commutators of G_A, if we take into account derivatives of G_A with respect to the gauge field.

We want to emphasize that Eq. (2.11) and the BJL limit Eq. (2.23) coincide with the BJL formulas for $-D \cdot E$ in a chiral theory with quantized gauge field, where only fermionic loops are taken into account [6].

III. THE ALGEBRA OF COMPONENTS OF G

In the following we use the properties of $G_{A/\psi}, W[A]$ and the BJL method for the calculation of the equal-time commutators of $G_{A/\psi}$. The results are derived only from the consistent anomaly. It is well known that the Schwinger terms of the different commutators are related by functional derivatives of the anomaly (e.g., [7]). Given these relations we only have to calculate one Schwinger term from the (consistent) anomaly $\mathcal{A}$. First we derive the relations between Schwinger terms within the formalism introduced in Sec. II. It follows by Eq. (2.23):

$$\langle [G^a(x), G_\psi^b(y)]_{\mathrm{ET}} \rangle = -i \lim_{p_0 \to \infty} \int dx_0 e^{ip_0(x_0 - y_0)} \partial_0^x \widetilde{G}^a(x) \langle J_b^0(y) \rangle = -i \lim_{p_0 \to \infty} \int dx_0 e^{ip_0(x_0 - y_0)} \frac{\delta}{\delta A_0^b(y)} \partial_0^x \widetilde{G}_A^a(x) W[A], \tag{3.1}$$

where the integrability of $W[A]$ was used by commuting the derivatives with respect to A. With Eq. (2.15) we conclude

$$\begin{aligned} \langle [G^a(x), G_\psi^b(y)]_{\mathrm{ET}} \rangle &= -i \lim_{p_0 \to \infty} \int dx_0 e^{ip_0(x_0 - y_0)} \frac{\delta}{\delta A_0^b(y)} \mathcal{A}^a(x) + i \lim_{p_0 \to \infty} \int dx_0 e^{ip_0(x_0 - y_0)} \frac{\delta}{\delta A_0^b(y)} f^{acd} A_0^c(x) \frac{\delta}{\delta A_0^d(x)} W[A] \\ &= i \langle G_\psi^{[a,b]}(x) \rangle \delta(\boldsymbol{x} - \boldsymbol{y}) - i \int dx_0 \frac{\delta}{\delta A_0^b(y)} \mathcal{A}^a(x). \end{aligned} \tag{3.2}$$

The relations between the other commutators follow similarly. We quote the results

$$\langle [G^a(x), G_A^b(y)]_{\mathrm{ET}} \rangle = i \langle G_A^{[a,b]}(x) \rangle \delta(\boldsymbol{x} - \boldsymbol{y}) + i \int dx_0 D_j^{bd}(y) \frac{\delta}{\delta \partial_0 A_j^d(y)} \mathcal{A}^a(x),$$

$$\langle [G^a(x), G_\psi^b(y)]_{\mathrm{ET}} \rangle = i \langle G_\psi^{[a,b]}(x) \rangle \delta(\boldsymbol{x} - \boldsymbol{y}) - i \int dx_0 \frac{\delta}{\delta A_0^b(y)} \mathcal{A}^a(x), \tag{3.3}$$

$$\langle [G^a(x), G^b(y)]_{\mathrm{ET}} \rangle = i \langle G^{[a,b]}(x) \rangle \delta(\boldsymbol{x} - \boldsymbol{y}) - i \int dx_0 \left[\frac{\delta}{\delta A_0^b(y)} - D_i^{bc}(y) \frac{\delta}{\delta \partial_0 A_i^c(y)} \right] \mathcal{A}^a(x).$$

Now we calculate the commutator $[G_A^a, G_A^b]$ with Eq. (3.3) and considerations concerning symmetry properties. The anomalous Schwinger terms are connected to terms in $W[A]$ containing at least cubic powers of the gauge field. Hence the Schwinger terms contain at least linear powers of the gauge field. The only term with the correct symmetry properties is

$$\langle [G_A^a(x),G_A^b(y)]_{\rm ST}\rangle = q\epsilon^{ijk}D_i^{ac}(x)D_j^{bd}(y)\ {\rm Tr}[\{t^c,t^d\}A_k]\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{3.4}$$

which is connected to a U(1) curvature in field space [5,10,16]. In the present approach it follows by observation that the Schwinger term, Eq. (3.4), is directly related to $D^i(\delta/\delta A^i)U(x_0,x_0)\neq 0$, where $U(x_0,x_0)$ defines a loop with fixed point $t=-\infty$. It remains to determine the constant q. Equation (3.3) establishes the relation between the Schwinger term, Eq. (3.4), and $[G_\psi^a,G_A^a]_{\rm ET}$

$$\langle [G_\psi^a(x),G_A^b(y)]_{\rm ET}\rangle = -\langle [G_A^a(x),G_A^b(y)]_{\rm ST}\rangle + i\int dx_0 D_j^{bd}(y)\frac{\delta}{\delta\partial_0 A_j^d(y)}\mathcal{A}^a(x). \tag{3.5}$$

It follows from Eqs. (3.4) and (3.5) that

$$\langle [G_\psi^a(x),G_A^b(y)]_{\rm ET}\rangle \sim -q\epsilon^{ijk}D_j^{bd}(y)\ {\rm Tr}[\{t^a,t^d\}A_k(x)]\partial_i^x\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{3.6}$$

We only have to evaluate these terms contributing to $[G_\psi^a,G_A^a]_{\rm ST}$ to determine q. For this purpose we introduce the covariant current $\tilde{J}$, which differs from the consistent current by the Bardeen-Zumino polynomial ΔJ [24]:

$$\langle J_a^\mu\rangle = \langle \tilde{J}_a^\mu\rangle - \Delta J_a^\mu$$

with

$$\Delta J_a^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,{\rm Tr}\left[t^a\{A_\nu,\partial_\rho A_\sigma\}+\frac{3}{2}A_\nu A_\rho A_\sigma\right]. \tag{3.7}$$

We use the gauge covariance of $\langle\tilde{J}\rangle$ in the following derivation. With Eq. (3.7) we have

$$\begin{aligned}\langle [G_\psi^a(x),G_A^b(y)]_{\rm ET}\rangle &= i\lim_{p_0\to\infty}\int dy_0 e^{ip_0(y_0-x_0)}D_j^{bd}(y)\frac{\delta}{\delta A_j^d(y)}\langle J_a^0(x)\rangle \\ &= i\lim_{p_0\to\infty}\int dy_0 e^{ip_0(y_0-x_0)}D_j^{bd}(y)\frac{\delta}{\delta A_j^d(y)}[\langle\tilde{J}_a^0(x)\rangle-\Delta J_a^0(x)].\end{aligned} \tag{3.8}$$

It follows from the covariance of $\langle\tilde{J}\rangle$ that the first term on the right-hand side does not contribute to Eq. (3.6). Thus, only taking into account terms which can contribute to Eq. (3.6), we get ($\epsilon^{ijk}=\epsilon^{0ijk}$)

$$\langle [G_\psi^a(x),G_A^b(y)]_{\rm ET}\rangle \sim -i\lim_{p_0\to\infty}\int dy_0 e^{ip_0(y_0-x_0)}D_j^{bd}(y)\frac{\delta}{\delta A_j^d(y)}\Delta J_a^0(x) \sim -\frac{i}{24\pi^2}\epsilon^{ijk}D_j^{bd}(y)\ {\rm Tr}[\{t^a,t^d\}A_k(x)]\partial_i^x\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{3.9}$$

This determines $q=i/24\pi^2$ and we have finally

$$\langle [G_A^a(x),G_A^b(y)]_{\rm ET}\rangle = i\langle G_A^{[a,b]}(x)\rangle\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{i}{24\pi^2}\epsilon^{ijk}D_i^{ac}(x)D_j^{bd}(y)\ {\rm Tr}[\{t^c,t^d\}A_k]\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{3.10}$$

Together with Eq. (3.3) this determines all commutators. The results coincide with the literature (e.g., [5,7]).

IV. DOUBLE COMMUTATORS

In the derivation of Eq. (3.10) we used relations only valid as expectation values [see Eqs. (3.5)–(3.10)]. Thus we expect that it is not possible to calculate the double commutator $[G_A^a,[G_A^b,G_A^c]]$ iteratively. Using the form of G_A [see Eq. (2.11)] we conclude

$$\begin{aligned}\langle [G_A^a(x),[G_A^b(y),G_A^c(z)]]_{\rm ET}\rangle &= \langle [G_A^a(x),iG_A^{[b,c]}(y)]_{\rm ET}\rangle\delta(\boldsymbol{y}-\boldsymbol{z}) \\ &\quad + i\int_{t,t',t''}\left\{\left[D_i^{ad}(t,\boldsymbol{x})\frac{\delta}{\delta A_i^d(t,\boldsymbol{x})}[D_j^{be}(t',\boldsymbol{y})D_k^{cf}(t'',\boldsymbol{z})]\right]\langle[\boldsymbol{\delta}_e^j(t',y),\boldsymbol{\delta}_f^k(t'',z)]\rangle\right. \\ &\quad \left. + D_i^{ad}(t,x)D_j^{be}(t',y)D_k^{cf}(t'',z)\langle[\boldsymbol{\delta}_d^i(t,x),[\boldsymbol{\delta}_e^j(t',y),\boldsymbol{\delta}_f^k(t'',z)]]\rangle\right\}.\end{aligned} \tag{4.1}$$

The first two terms follow easily with Eq. (3.10), since they only involve derivatives of $\widetilde{G}_A$ with respect to the gauge field. However, the last term cannot be calculated with the known commutators. It follows from algebraic considerations that it has to vanish if the Jacobi identity is fulfilled. The structure of the double commutator

$$K^{ijk}_{def}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = i\int_{t,t',t''} D^{ad}_i(t,x)D^{be}_j(t',y)D^{cf}_k(t'',z)\langle[\boldsymbol{\delta}^i_d(t,x),[\boldsymbol{\delta}^j_e(t',y),\boldsymbol{\delta}^k_f(t'',z)]]\rangle \tag{4.2}$$

follows by dimensional analysis as

$$K^{ijk}_{def}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = q\,\boldsymbol{\epsilon}^{ijk}D^{ad}_i(x)D^{be}_j(y)D^{cf}_k(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\,\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}). \tag{4.3}$$

The iterative result obtained with Eq. (3.10) is $q=1/24\pi^2$. The Jacobi identity is only satisfied when $q=0$. Performing the BJL limit we get, for the last term in Eq. (4.1),

$$\begin{aligned} K^{ijk}_{def}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = &-\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int dx_0dy_0dz_0e^{ip_0x_0}e^{iq_0y_0}\theta(-z_0)D^{ad}_i(x)D^{be}_j(y)D^{cf}_k(z)\\ &\times\left[\frac{\delta}{\delta A^d_i(x)}\frac{\delta}{\delta A^e_j(y)}\frac{\delta}{\delta A^f_k(z)}W[A]\right]\\ &=0. \end{aligned} \tag{4.4}$$

The only terms of

$$\frac{\delta}{\delta A^d_i(x)}\frac{\delta}{\delta A^e_j(y)}\frac{\delta}{\delta A^f_k(z)}W[A]$$

which contribute to Eq. (4.4) are proportional to $\partial_0\delta(z_0-y_0)\,\delta(z_0-x_0),\partial_0\delta(x_0-z_0)\,\delta(y_0-z_0)$ and $\partial_0\delta(x_0-z_0)\,\delta(x_0-y_0)$. The group structure is similar to Eq. (3.10). It follows with the integrability of $W[A]$ that Eq. (4.4) is proportional to

$$\mathrm{Tr}[t^d\{t^e,t^f\}][\,\partial_0\delta(z_0-y_0)\,\delta(z_0-x_0)+\text{cycl. perms. of }(x_0,y_0,z_0)]=0. \tag{4.5}$$

In an iterative BJL limit one would only take into account one of the terms proportional to $\partial_0\delta(z_0-y_0)\,\delta(z_0-x_0)$ and $\partial_0\delta(x_0-z_0)\,\delta(x_0-y_0)$. However, only the sum of these two terms add up to zero in the limit, which is the reason for the violation of the Jacobi identity within an iterative calculation [the term proportional to $\partial_0\delta(x_0-z_0)\,\delta(y_0-z_0)$ is suppressed with $p_0/(p_0+q_0)$ in the BJL limit in Eq. (4.4)]. With Eq. (4.4) we have finally

$$\begin{aligned} \langle[G^a_A(x),[G^b_A(y),G^c_A(z)]]_{\mathrm{ET}}\rangle = &\langle[G^a_A(x),iG^{[b,c]}_A(y)]_{\mathrm{ET}}\rangle\,\delta(\boldsymbol{y}-\boldsymbol{z})\\ &+\frac{1}{24\pi^2}\boldsymbol{\epsilon}^{jkl}\int dtD^{ad}_i(x)\left[\frac{\delta}{\delta A^d_i(x)}[D^{be}_j(y)D^{cf}_k(z)]\right]\mathrm{Tr}[\{t^e,t^f\}A_l]\,\delta(\boldsymbol{y}-\boldsymbol{z})\\ =&\langle[G^a_A(x),[G^b_A(y),G^c_A(z)]]_{it}\rangle\\ &-\frac{1}{24\pi^2}\boldsymbol{\epsilon}^{ijk}D^{ad}_i(x)D^{be}_j(y)D^{cf}_k(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\,\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}). \end{aligned} \tag{4.6}$$

The double commutator with the subscript it is the iterative double commutator.

Now we proceed as in the case of the commutators. We use the anomaly and the double commutator Eq. (4.6) to calculate the other double commutators contributing to the algebra of G. As an important first step we prove, that double commutators with the structure $[G^a_{A/\psi},[G^b_{A/\psi},G^c]]$ agree with the iterative results. We derive, with the use of the notation Eq. (2.14),

$$\begin{aligned} \langle[G^a_{A/\psi}(x),[G^b_{\psi}(y),G^c(0)]]_{\mathrm{ET}}\rangle = &-\lim_{p_0\to\infty}\lim_{q_0\to\infty}p_0q_0\int dy_0dx_0e^{ip_0x_0}e^{iq_0y_0}\langle T^*G^a_{A/\psi}(x)G^b_{\psi}(y)G^c(0)\rangle\\ =&\,i\lim_{p_0\to\infty}p_0\int dx_0e^{ip_0x_0}\widetilde{G}^a_{A/\psi}(x)\lim_{q_0\to\infty}q_0\int dy_0e^{iq_0y_0}\langle T^*G^b_{\psi}(y)G^c(0)\rangle\\ =&\lim_{p_0\to\infty}p_0\int dx_0e^{ip_0x_0}\widetilde{G}^a_{A/\psi}(x)\lim_{q_0\to\infty}q_0\int dy_0e^{iq_0y_0}\widetilde{G}^b_{\psi}(y)\widetilde{G}^c(0)W[A]. \end{aligned} \tag{4.7}$$

Here we used, that derivatives of $G^a_{A/\psi}$ with respect to the gauge field vanish in the BJL limit with at least $p_0/(p_0+q_0)\to 0$. Performing the limit $q_0\to\infty$ we get

$$\begin{aligned}\langle[G^a_{A/\psi}(x),[G^b_\psi(y),G^c(0)]]_{\mathrm{ET}}\rangle &= i\lim_{p_0\to\infty}\int dx_0 e^{ip_0x_0}\partial_0^x\widetilde{G}^a_{A/\psi}(x)\langle G^{[b,c]}_\psi(0)\rangle\delta(\boldsymbol{y})\\ &\quad-\lim_{p_0\to\infty}p_0\int dx_0 e^{ip_0x_0}\widetilde{G}^a_{A/\psi}(x)\left[\lim_{q_0\to\infty}q_0\int dy_0 e^{iq_0y_0}\widetilde{G}^b_\psi(y)\int dt\,\theta(-t)\mathcal{A}^c(t,0)\right]\\ &=\langle[G^a_{A/\psi}(x),iG^{[b,c]}_\psi(0)]_{\mathrm{ET}}\rangle\delta(\boldsymbol{y})+\int dx_0\partial_0\widetilde{G}^a_{A/\psi}(x)\int dy_0\frac{\delta}{\delta A^b_0(y)}\mathcal{A}^c(0).\end{aligned}\tag{4.8}$$

For $G^a_{A/\psi}=G^a_\psi$ the second term in the last line vanishes. For $G^a_{A/\psi}=G^a_A$ we have

$$\int dx_0\partial_0\widetilde{G}^a_A(x)=\int dx_0 D^{ad}_i(x)\frac{\delta}{\delta A^d_i(x)}.$$

Thus Eq. (4.8) is the iterative result. In the derivation we used

$$\widetilde{G}^c(x)W[A]=\int dt\,\theta(x_0-t)\left(\mathcal{A}^c(t,\boldsymbol{x})-f^{cde}A^d_0(t,\boldsymbol{x})\frac{\delta}{\delta A^e_0(t,\boldsymbol{x})}W[A]\right).\tag{4.9}$$

The Schwinger term in Eq. (4.8) is simply given by functional derivatives of the anomaly in contrast to the Schwinger term of $[G^a_A,[G^b_A,G^c_A]]$. It is directly related to the two-cocycle in the algebra of the Gauss law operator G.

Applying the derivation of Eq. (4.8) to the double commutator $[G^a_{A/\psi},[G^b_A,G^c]]$, we get

$$\begin{aligned}\langle[G^a_{A/\psi}(x),[G^b_A(y),G^c(0)]]_{\mathrm{ET}}\rangle &=-\lim_{p_0\to\infty}\lim_{q_0\to\infty}p_0q_0\int dy_0dx_0e^{ip_0x_0}e^{iq_0y_0}\langle T^*G^a_{A/\psi}(x)G^b_A(y)G^c(0)\rangle\\ &=i\lim_{p_0\to\infty}p_0\int dx_0e^{ip_0x_0}\widetilde{G}^a_{A/\psi}(x)\lim_{q_0\to\infty}q_0\int dy_0e^{iq_0y_0}\langle T^*G^b_A(y)G^c(0)\rangle\\ &\quad-\int dt\,\theta(-t)D^{[b,c]d}_i(t,0)\left[\frac{\delta}{\delta A^d_i(t,0)}\widetilde{G}^a_{A/\psi}(x)\right]W[A]\delta(\boldsymbol{y})\\ &=\langle[G_{A/\psi}(x),iG^{[b,c]}_A(0)]_{\mathrm{ET}}\rangle\delta(\boldsymbol{y})-\int dx_0\partial_0\widetilde{G}^a_{A/\psi}(x)\int dy_0D^{bd}_j(y)\frac{\delta}{\delta\partial_0A^d_j(y)}\mathcal{A}^c(0).\end{aligned}\tag{4.10}$$

It follows with Eqs. (4.8) and (4.10) that double commutators of the form $[G^a_{A/\psi},[G^b_{A/\psi},G^c]]$ agree with the iterative results. Therefore they can be calculated from the known commutators, Eqs. (3.3) and (3.10). The algebra for arbitrary combinations of $G_{A/\psi}$ is determined with the double commutators, Eqs. (4.6), (4.8), (4.10), and the result for $[G^a_\psi,[G^b_A,G^c_A]]$. We calculate

$$\begin{aligned}\langle[G^a_\psi(x),[G^b_A(y),G^c_A(0)]]_{\mathrm{ET}}\rangle &=\langle[G^a_\psi(x),iG^{[b,c]}_A(0)]_{\mathrm{ET}}\rangle\delta(\boldsymbol{y})-\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int dx_0dy_0dz_0e^{ip_0x_0}e^{iq_0y_0}\theta(-z_0)\\ &\quad\times D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}\partial_0\frac{\delta}{\delta A^a_0(x)}W[A].\end{aligned}\tag{4.11}$$

Using

$$\partial_0\frac{\delta}{\delta A^0_a(x)}W[A]=\mathcal{A}^a(x)-\left[D^{af}_k(x)\frac{\delta}{\delta A^f_k(x)}+f^{agf}A^g_0(x)\frac{\delta}{\delta A^f_0(x)}\right]W[A]$$

it follows that

$$\langle[G^a_\psi(x),[G^b_A(y),G^c_A(0)]]_{\rm ET}\rangle=\langle[G^a_\psi(x),iG^{[b,c]}_A(0)]_{\rm ET}\rangle\,\delta(\boldsymbol{y})-\int dx_0dy_0D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta\partial_0A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}\mathcal{A}^a(x)$$
$$+\int dx_0dy_0D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta\partial_0A^e_j(z)}\mathcal{A}^a(x)$$
$$+\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int dx_0dy_0dz_0e^{ip_0x_0}e^{iq_0y_0}\theta(-z_0)$$
$$\times D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}D^{af}_k(x)\frac{\delta}{\delta A^f_k(x)}W[A]. \tag{4.12}$$

The terms containing functional derivatives of the anomaly have the form

$$\int dx_0\int dy_0D^{bd}_i(y)D^{ce}_j(z)\left[\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta\partial_0A^e_j(z)}-\frac{\delta}{\delta\partial_0A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}\right]\mathcal{A}^a(x)$$
$$=\frac{1}{24\pi^2}\epsilon^{ijk}D^{bd}_i(y)D^{ce}_j(z)\partial^x_k\,{\rm Tr}[t^a\{t^d,t^e\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{x}-\boldsymbol{z}). \tag{4.13}$$

The last term on the right-hand side of Eq. (4.12) consists of equal-time commutators of the form $[\boldsymbol{\delta}_{[a,g]},\boldsymbol{\delta}_h]$ [see Eq. (3.10)] and terms proportional to $\delta^3_AW[A]$, which vanish in the BJL limit. Performing the BJL limit, we are led to

$$\langle[G^a_\psi(x),[G^b_A(y),G^c_A(z)]]_{\rm ET}\rangle=\langle[G^a_\psi(x),iG^{[b,c]}_A(y)]_{\rm ET}\rangle\,\delta(\boldsymbol{y}-\boldsymbol{z})+\frac{1}{24\pi^2}\epsilon^{ijk}D^{bd}_i(y)D^{ce}_j(z)\partial^x_k$$
$$\times\,{\rm Tr}[t^a\{t^d,t^e\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{x}-\boldsymbol{z})-i\int dtdt'D^{bd}_i(t,\boldsymbol{y})D^{ce}_j(t',\boldsymbol{z})$$
$$\times\{\langle[\boldsymbol{\delta}^i_{[a,d]}(t,y),\boldsymbol{\delta}^j_e(t',z)]\rangle\,\delta(\boldsymbol{x}-\boldsymbol{y})-\langle[\boldsymbol{\delta}^j_{[a,e]}(t',z),\boldsymbol{\delta}^i_d(t,y)]\rangle\,\delta(\boldsymbol{x}-\boldsymbol{z})\}$$
$$=\langle[G^a_\psi(x),iG^{[b,c]}_A(y)]_{\rm ET}\rangle\,\delta(\boldsymbol{y}-\boldsymbol{z})+\frac{1}{24\pi^2}\epsilon^{ijk}D^{bd}_i(y)D^{ce}_j(z)\partial^x_k\,{\rm Tr}[t^a\{t^d,t^e\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{x}-\boldsymbol{z})$$
$$-\epsilon^{ijk}\frac{1}{24\pi^2}D^{bd}_i(y)D^{ce}_j(z)\,{\rm Tr}[(\{[t^a,t^d],t^e\}+\{[t^a,t^e],t^d\})A_k(x)]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{x}-\boldsymbol{z})$$
$$=\langle[G^a_\psi(x),iG^{[b,c]}_A(y)]_{\rm ET}\rangle\,\delta(\boldsymbol{y}-\boldsymbol{z})$$
$$+\frac{1}{24\pi^2}\epsilon^{ijk}D^{bd}_i(y)D^{ce}_j(z)D^{af}_k(x)\,{\rm Tr}[t^f\{t^d,t^e\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{x}-\boldsymbol{z}). \tag{4.14}$$

The noniterative part in Eq. (4.14) is exactly the same as for the double commutator, Eq. (4.6), with a relative minus sign. We conclude

$$\langle[G^a(x),[G^b_A(y),G^c_A(z)]]_{\rm ET}\rangle=\langle[G^a(x),[G^b_A(y),G^c_A(z)]]_{it}\rangle. \tag{4.15}$$

We want to emphasize that the noniterative part of Eq. (4.14) is also connected to the fact that the result for the commutator $[G^a_A,G^b_A]$ [see Eq. (3.10)] is not valid on the operator level. With Eqs. (4.8), (4.10), and (4.15) it follows that double commutators containing at least one full G are given by the iterative results. We collect the results

$$\langle[G^a_{A/\psi}(x),[G^b_{A/\psi}(y),G^c(z)]]_{\rm ET}\rangle=\langle[G^a_{A/\psi}(x),[G^b_{A/\psi}(y),G^c(z)]]_{it}\rangle,$$
$$\langle[G^a(x),[G^b_{A/\psi}(y),G^c_{A/\psi}(z)]]_{\rm ET}\rangle=\langle[G^a(x),[G^b_{A/\psi}(y),G^c_{A/\psi}(z)]]_{it}\rangle. \tag{4.16}$$

Equation (4.16), the commutators, Eqs. (3.3) and (3.10), and the double commutator, Eq. (4.6), fix all double commutators, as already mentioned. The algebra of two G_A's and one G_ψ is given by

$$\langle[G_A^a(x),[G_A^b(y),G_\psi^c(z)]]_{\mathrm{ET}}\rangle=\langle[G_A^a(x),[G_A^b(y),G_\psi^c(z)]]_{it}\rangle+\frac{1}{24\pi^2}\epsilon^{ijk}D_i^{ad}(x)\times D_j^{be}(y)D_k^{cf}(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}),$$
$$\langle[G_\psi^a(x),[G_A^b(y),G_A^c(z)]]_{\mathrm{ET}}\rangle=\langle[G_\psi^a(x),iG_A^{[b,c]}(y)]_{\mathrm{ET}}\rangle+\frac{1}{24\pi^2}\epsilon^{ijk}D_i^{ad}(x)\times D_j^{be}(y)D_k^{cf}(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}). \tag{4.17}$$

The algebra of two G_ψ and one G_A is given by

$$\langle[G_\psi^a(x),[G_A^b(y),G_\psi^c(z)]]_{\mathrm{ET}}\rangle=-\frac{1}{24\pi^2}\epsilon^{ijk}D_i^{ad}(x)D_j^{be}(y)D_k^{cf}(z)\times\mathrm{Tr}[t^d\{t^e,t^f\}]\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}),$$
$$\langle[G_A^a(x),[G_\psi^b(y),G_\psi^c(z)]]_{\mathrm{ET}}\rangle=\langle[G_A^a(x),[G_\psi^b(y),G_\psi^c(z)]]_{it}\rangle-\frac{1}{24\pi^2}\epsilon^{ijk}D_i^{ad}(x)D_j^{be}(y)D_k^{cf}(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\times\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}). \tag{4.18}$$

The algebra of the fermionic currents is given by

$$\langle[G_\psi^a(x),[G_\psi^b(y),G_\psi^c(z)]]_{\mathrm{ET}}\rangle=\langle[G_\psi^a(x),iG_\psi^{[b,c]}(y)]_{\mathrm{ET}}\rangle\,\delta(\boldsymbol{y}-\boldsymbol{z})+\frac{1}{24\pi^2}\epsilon^{ijk}D_i^{ad}(x)D_j^{be}(y)D_k^{cf}(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\times\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(\boldsymbol{y}-\boldsymbol{z}). \tag{4.19}$$

It can be shown explicitly in a long, but straightforward calculation, that the Jacobi identity is fulfilled for all these results. Since we have related the double commutators to derivatives of $W[A]$ with respect to the gauge field, this provides a convenient way to prove the Jacobi identity (integrability of $W[A]$).

V. DISCUSSION

In this paper a purely algebraic BJL-type derivation of the algebra of the Gauss law operator G in chiral gauge theories was given. The theory has been defined in an external gauge field (adiabatic solution). The expectation values of commutators and double commutators can be expressed as derivatives of $W[A]$ with respect to the gauge field. All Schwinger terms follow with the anomaly without any further assumption.

The Schwinger terms depend on the properties of the time evolution operator U, which defines the underlying theory. If the representation of the gauge group is projective (on the Fock space), we have $D^i(\delta/\delta A^i)U(x_0,x_0)\neq 0$ (see Sec. II). This is the reason for the Schwinger term in the commutator $[G_A^a,G_A^b]$ [Eq. (3.10)]. We want to emphasize that the explanation of this Schwinger term is based purely on an adiabatic solution of the theory in an external gauge field. There is no need to introduce a quantized gauge field to obtain this result.

The commutators [Eqs. (3.3) and (3.10)] coincide with the BJL results obtained via the explicit calculation of diagrams [5], or path integral methods [7], where only fermionic loops give contributions. In contrast with these approaches we do not need to calculate diagrams nor do we need further assumptions as in [7]. The algebraic derivation in [7] was based on the assumption that $[G_A^a,G_A^b]_{\mathrm{ST}}=0$ for G defined with the covariant current. Adding the Bardeen-Zumino polynomial ΔJ^0 to G_ψ we get the results for G defined with the covariant current (e.g., [7–10]).

It is also possible to relate the Schwinger terms of $G_{A/\psi}$ obtained in the present paper to those in [13,14]. Although the consistent current was used in [13,14] and the algebra of G coincides with Eq. (3.3), the Schwinger terms in the algebra of $G_{A/\psi}$ are different. This can be explained by studying the time evolution operator U. If we assume that

$$U(x_0,x_0)=1, \tag{5.1}$$

in Eq. (2.18) we would arrive at the results of [13]. In [13], Eq. (5.1) was used implicitly, since the commutators of $G_{A/\psi}$ were calculated as defined in Eq. (2.18). Condition (5.1) is equivalent to gauge invariance of the underlying theory. In [14] such a theory was constructed by adding a Wess-Zumino field. In this theory G is not the full Gauss law operator, since G does not generate gauge transformations on the Wess-Zumino field. The results in [13] coincide with those of [14], which confirms the interpretation.

The double commutators have been derived by using the integrability of $W[A]$. Double commutators, which contain at least one G agree with the iterative results. The Jacobi identity is satisfied by our results [Eqs. (4.6), (4.17), (4.18), and (4.19)]. This is an outcome of the integrability (consistency) of $W[A]$, since the double commutators are related to derivatives of $W[A]$ with respect to the gauge field. In an iterative calculation one drops terms ensuring the integrability [see Eq. (4.4) and the following discussion], which leads to the violation of the Jacobi identity. Since double commutators containing at least one G agree with the iterative result, there is only one noniterative term (up to a minus sign)

$$\frac{1}{24\pi^2}\epsilon^{ijk}D_i^{ad}(x)D_j^{be}(y)D_k^{cf}(z)\ \mathrm{Tr}[t^d\{t^e,t^f\}]\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}). \tag{5.2}$$

This can be traced back to the fact that the gauge group representation is projective and is related directly to the Schwinger term in the commutator $[G_A^a,G_A^b]$.

The noniterative terms of double commutators, which are related via cyclic permutation of the operators, are identical (no relative minus sign) [see Eqs. (4.17)–(4.19)]. The BJL-type prescription of [11] only works in this case. Therefore the results here confirm the validity of the prescription in the case of $G_{A/\psi}$. Moreover, the connection of the noniterative term, Eq. (5.2), to the integrability of the generating functional $W[A]$ indicates that this should hold for arbitrary double commutators.

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