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Gauge invariance and effective actions in $D=3$ at finite temperature

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For background gauge field configurations reducible to the form $A_\mu=(\widetilde{A}_3,\vec{A}(\vec{x}))$ where $\widetilde{A}_3$ is a constant, we provide an elementary derivation of the recently obtained result for the exact induced Chern-Simons (CS) effective action in QED$_3$ at finite temperature. The method allows us to extend the result in several useful ways: to obtain the analogous result for the ''mixed'' CS term in the Dorey-Mavromatos model of parity-conserving planar superconductivity, thereby justifying their argument for flux quantization in the model; to the induced CS term for a τ-dependent flux; and to the term of second order in $\vec{A}(\vec{x})$ (and all orders in $\widetilde{A}_3$) in the effective action. [S0556-2821(98)03502-4]

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I. INTRODUCTION

Recently, there has been significant progress [1–4] in resolving a puzzle concerning the gauge invariance of induced Chern-Simons (CS) terms at finite temperature, T. In the non-Abelian case, for example, general arguments imply that the coefficient of the CS term must be quantized at zero temperature [5], and also at nonzero temperature [6,7], if the action is to be invariant under topologically nontrivial (''large'') gauge transformations. On the other hand, simple perturbative calculations [8–17], give the result that the effect of moving to $T\neq 0$ is simply to multiply the zero-temperature CS term by a smoothly varying function of T [typically, $\tanh(\beta|M|/2)$, where $\beta=1/kT$ and M is the mass of the fermion(s) in the theory]. Plainly, the perturbative result contradicts the quantization requirement [18–20]. A similar difficulty arises in the Euclidean case for $T\neq 0$ due to the S^1 topology of the compactified Euclidean time.

Apart from its theoretical interest, the resolution of this puzzle is important in some physical applications. To give one specific example, consider the Dorey-Mavromatos (DM) model [21] of two-dimensional superconductivity without parity violation. This model employs two U(1) gauge fields, one the electromagnetic field A_μ, the other a ''statistical'' gauge field a_μ, which is also massless. There are $N_f\geqslant 2$ flavors of four-component fermions, the mass term is parity conserving, and A_μ and a_μ have opposite parity. At zero temperature a ''mixed Chern-Simons (MCS) term'' is generated by a fermion loop with one external A and one external a leg, the leading contribution to the action, in powers of derivatives, being

$$\Gamma_{\rm MCS}=N_f\,\frac{eq}{2\pi}\,\frac{M}{|M|}\int d^3x\,\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho, \tag{1}$$

where e and g are the couplings of A and a, respectively. Note that this term is not parity violating, though it is ''topological.'' One major novelty of this model is that it provides a mechanism for superconductivity in two space dimensions at $T\neq 0$, without the existence of an order parameter which is a nonsinglet under U(1)$_{\rm EM}$ (such a nonzero order parameter would violate the Coleman-Mermin-Wagner theorem [22]). Without such an order parameter, however, it is difficult to see how some familiar phenomenological features of superconductivity can arise—in particular flux quantization, which is conventionally derived from a Ginzburg-Landau Lagrangian, assuming the order parameter has charge $2e$ (pair field). In their model, DM argued as follows. At $T\neq 0$, (1) becomes

$$\Gamma_{\rm MCS}(T\neq 0)=iN_f\,\frac{M}{|M|}\,\frac{eg}{2\pi}\int_0^\beta d\tau\int d^2x\,\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho. \tag{2}$$

Consider now a configuration in which $a_\mu=[a_3(\tau),\vec{0}]$ and $A_\rho=[0,\vec{A}(\vec{x})]$. Its contribution to the action should be invariant under topologically nontrivial gauge transformations on a_3, of the form

$$a_3\rightarrow a_3+\partial_\tau\Omega(\tau) \tag{3}$$

with $\Omega(\beta)-\Omega(0)=2n\pi/g$, where n is an integer. Under (3), the variation of $\Gamma_{\rm MCS}(T\neq 0)$ is

$$\delta\Gamma_{\rm MCS}(T\neq 0)=iN_f\,\frac{M}{|M|}\,ne\int d^2x\,\epsilon_{ij}\partial_iA_j. \tag{4}$$

Considering then a superconducting annulus enclosing flux Φ, it follows that Φ has to have the value (restoring $\hbar$ and c)

$$\Phi=\frac{mhc}{N_fe} \tag{5}$$

where m is an integer, if $\delta\Gamma_{\rm MCS}(T\neq 0)$ is to be an integer multiple of $2\pi i$, for any n in (4). Equation (5) gives the required result for $N_f=2$, the value indicated in the DM model.

Thus, the flux quantization does not come from a charge-$2e$ order parameter in this model, but precisely by requiring invariance, under topologically nontrivial gauge transformations, of a CS-like term. Unfortunately, however, (2) is only correct for T infinitesimally close to 0. Indeed, the standard lowest order derivative expansion calculation would yield (2) multiplied by $\tanh(\beta M/2)$, as stated above, and this T-dependent factor would destroy the result (5).

Nevertheless, by analogy with other CS-like terms at $T\neq 0$, there is reason to think that the quantization result (5) should be true after all. Perhaps the blame could be laid on using perturbation theory for the field undergoing the large gauge transformation. In an attempt to get away from perturbation theory, Cabra *et al.* [7], and Bralic *et al.* [23] considered how a more general ansatz, in which the tanh function is replaced by a general function $F(T)$, could be reconciled with gauge invariance, and concluded that F could only be a discrete-valued function of the temperature. The arguments of [7] and [23], however, depend crucially on the form of the ansatz assumed, and it turns out that it does not, in fact, represent the true nonperturbative structure.

A crucial advance was made by Dunne *et al.* [1], who considered a solvable model in $0+1$ dimensions, in which a CS-like term is generated at zero temperature. These authors found that the exact effective action at finite temperature was indeed gauge invariant, even though its perturbative expansion produced a result precisely analogous to that found in $2+1$ dimensions, namely, the gauge-non-invariant form "$\tanh(\beta M/2)\times\Gamma_{\rm CS}(T=0)$." As Dunne *et al.* remark [1], the exact result cannot be written as the integral of a density, in Euclidean spacetime, suggesting that the type of ansatz considered in [7] and [23] is not adequate.

Subsequently, nonperturbative calculations of the effective action in the $2+1$ Abelian case [2] and its explicit temperature-dependent parity-breaking part [3] have shown that the complete effective action is indeed invariant under both large and small gauge transformations, the result of [3], in particular, showing some remarkable similarity to that of Dunne *et al.* [1], as we shall discuss further in Sec. III below. The calculation of [3] has now been extended to the non-Abelian case [4].

It is important to note, however, that the explicit $(2+1)$-dimensional results have only been obtained for a particular class of background gauge field configuration. In the Abelian case, for example, they have the form

$$A_\mu=(\tilde{A}_3,\vec{A}(\vec{x})), \tag{6}$$

where $\tilde{A}_3=(1/\beta)\int_0^\beta d\tau A_3(\tau)$ [these configurations are gauge-equivalent to those in which $A_\mu=(A_3(\tau),\vec{A}(\vec{x}))$, as discussed in the following section]. For this case, the result of [3] and [4] is the following. For a theory with one two-component fermion of mass M, the zero temperature limit of the induced CS action is

$$\Gamma_{CS}(T\to 0)=\frac{ie}{2\pi}\,\frac{|M|}{M}\,\frac{e\beta\tilde{A}_3}{2}\int d^2x\,\epsilon_{jk}\partial_j A_k, \tag{7}$$

while at finite temperature the result is

$$\Gamma_{CS}(T\neq 0)=\frac{ie}{2\pi}\,\frac{|M|}{M}\,F\left(\frac{e\beta\tilde{A}_3}{2}\right)\int d^2x\,\epsilon_{jk}\partial_j A_k, \tag{8}$$

where

$$F(x)=\arctan\left[\tanh\left(\frac{\beta|M|}{2}\right)\tan(x)\right]. \tag{9}$$

Remarkably enough, one sees that F changes by $n\pi$ when A_3 undergoes a large gauge transformation with winding number n, granted that the branch of the arctan is understood to be shifted correspondingly. This is exactly the same behavior as in the $T\to 0$ limit (7), and consequently, quantization arguments are verified for $T\neq 0$ also, for these configurations. Note that such configurations would, in fact, be adequate for analyzing the Dorey-Mavromatos flux quantization argument.

The calculations of [3] and [4] were formulated in such a way as to make essential use of the result for an anomalous Fujikawa Jacobian [24] associated with a global chiral rotation on the fermionic variables, involving $\tilde{A}_3$. This result is intrinsically nonperturbative, and reveals the true structure of the odd-parity term, at least for the specified configuration. But, while certainly leading to a satisfactory outcome as regards gauge invariance, the method of [3] and [4] seems perhaps rather special, and possibly difficult to generalize to other situations, for example, the Dorey-Mavromatos problem. In addition, it would be interesting to know if the same result could be obtained by somehow summing up all powers of $\tilde{A}_3$ in a conventional perturbative (plus derivative) expansion approach, as actually envisaged by Dunne *et al.* [1]. After all, it was in the latter context that the gauge noninvariance puzzle emerged, and it would be nice to see its resolution there too. Such an approach is certainly capable of treating anomalylike terms correctly [25], provided a suitable regularization is performed. Nontopological contributions can be obtained this way too, of course, and the approach might be capable of handling more general background field configurations than those considered in [3] and [4].

The purpose of the present paper is to provide such an alternative derivation of the results of [3] and [4], based on straightforward effective action techniques; to extend it to the case in which the flux is allowed to depend on τ; to derive the corresponding result for the DM model, thus rescuing their flux quantization argument; and to present the result for the nontopological term which is quadratic in the magnetic field F_{ij}, and correct to all orders in $\tilde{A}_3$, for the case of a single gauge field A.

Before proceeding, one important remark needs to be made. The gauge field configurations which are considered in [3] and [4] are effectively τ-independent (or equivalent to τ-independent ones), and this is crucial in all the existing explicit calculations including our own which follow. In the nonstatic *time*-dependent case, terms arise (due to Landau damping) which are nonanalytic at the origin of momentum space, and which are therefore intrinsically nonlocal, as has been emphasized in [20]. Finding the explicit functional dependence on A_μ which ensures gauge invariance in the nonstatic case seems to be difficult. However, if one assumes that (after including the gauge field dynamics) only static quantities are going to be considered, namely, that the timelike momentum is always imaginary and discrete, then Landau damping cannot occur. This is what happens in the calculation of static quantities, such as the free energy.

The structure of this paper is as follows: In Sec. II we present the calculation of the term linear in the magnetic flux, discussing in particular the cases: (a) One two-component fermionic flavor and one gauge field, (b) many four-component flavors and two gauge fields (mixed CS term), and (c) extension to the case of a τ-dependent A_j. Section III deals with the next term in the expansion in powers of A_j.

II. TERMS LINEAR IN THE MAGNETIC FLUX

A. Two-component case: Parity-breaking term

We shall consider here the effective action $\Gamma(A)$ which is induced by integrating out a massive two-component fermion field coupled to an Abelian gauge field A_μ in $2+1$ dimensions at finite temperature,

$$e^{-\Gamma(A)}=\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp[-S_F(A)]. \tag{10}$$

The Euclidean action $S_F(A)$ for the fermion is given by

$$S_F(A)=\int_0^\beta d\tau\int d^2x\,\bar{\psi}(\not\partial+ie\not A+M)\psi. \tag{11}$$

In this section, we are using Euclidean Dirac matrices in the irreducible representation of the Dirac algebra (reducible representations will be considered in the next section):

$$\gamma_1=\sigma_1,\quad \gamma_2=\sigma_2,\quad \gamma_3=\sigma_3 \tag{12}$$

where σ_i are the usual Pauli matrices and $\beta=1/T$ is the inverse temperature. The label 3 is used to denote the Euclidean time coordinate τ. The fermionic fields obey antiperiodic boundary conditions in the timelike direction

$$\psi(\beta,x)=-\psi(0,x),\quad \bar{\psi}(\beta,x)=-\bar{\psi}(0,x),\quad \forall x, \tag{13}$$

with x denoting the two space coordinates. The gauge field satisfies periodic boundary conditions

$$A_\mu(\beta,x)=A_\mu(0,x),\quad \forall x. \tag{14}$$

We shall first consider configurations satisfying

$$A_3=A_3(\tau),\quad A_j=A_j(x),\quad j=1,2. \tag{15}$$

As stated in Ref. [3], and elaborated in Ref. [4], these configurations allow one to study gauge invariance under transformations with nontrivial windings around the compactified time coordinate:

$$\psi(\tau,x)\to e^{-ie\Omega(\tau,x)}\psi(\tau,x),\quad \bar{\psi}(\tau,x)\to e^{ie\Omega(\tau,x)}\bar{\psi}(\tau,x)$$

$$A_\mu(\tau,x)\to A_\mu(\tau,x)+\partial_\mu\Omega(\tau,x) \tag{16}$$

where

$$\Omega(\beta,x)=\Omega(0,x)+\frac{2\pi}{e}k \tag{17}$$

and k is an integer which labels the homotopy class of the gauge transformation.

As shown in Ref. [3], the τ-dependence of the Dirac operator, which comes only from A_3, can be removed by a redefinition of the integrated fermionic fields with a gauge function

$$\Omega(\tau)=-\int_0^\tau d\tau' A_3(\tau')+\frac{1}{\beta}\int_0^\beta d\tau' A_3(\tau'), \tag{18}$$

without affecting the spatial components A_j. Such a transformation renders A_3 constant and equal to its mean value,

$$\tilde{A}_3=\frac{1}{\beta}\int_0^\beta d\tau A_3(\tau). \tag{19}$$

Following [3], the determinant is written as an infinite product of the corresponding $1+1$ Euclidean Dirac operators

$$\det(\not\partial+ie\not A+M)=\prod_{n=-\infty}^{n=+\infty}\det[\not d+M+i\gamma_3(\omega_n+e\tilde{A}_3)], \tag{20}$$

where $\omega_n=(2n+1)(\pi/\beta)$ is the usual Matsubara frequency for fermions and $\not d$ is the two-dimensional Euclidean Dirac operator

$$\not d=\gamma_j(\partial_j+ieA_j)=\not\partial+ie\not A, \tag{21}$$

where we have adopted the convention of using the slash to denote contraction with the two spatial Dirac matrices, when there is no risk of confusion. Then, the effective action $\Gamma(A)$ corresponding to this configuration will be

$$\Gamma(A)=-\sum_{n=-\infty}^{n=+\infty}\text{Tr}\log[\not d+i\gamma_3\tilde{\omega}_n+M] \tag{22}$$

where we have defined $\tilde{\omega}_n=\omega_n+e\tilde{A}_3$. This trace cannot, of course, be evaluated explicitly. But if we want to reproduce the result for the induced parity breaking term of [3] and [4], it is sufficient to evaluate it up to linear order in A_j, without making any expansion in $\tilde{A}_3$. A naive application of this approach leads, however, to an ambiguous result, as we will now see. Let us call this term $\Gamma^{(1)}(A)$. It is formally given by

$$\Gamma^{(1)}(A)=-ie\sum_{n=-\infty}^{n=+\infty}\text{Tr}[\not A(\not\partial+i\gamma_3\tilde{\omega}_n+M)^{-1}] \tag{23}$$

where the trace is evaluated over functional and Dirac indices. When written in momentum space, (23) becomes

$$\Gamma^{(1)}(A)=-ie\sum_{n=-\infty}^{n=+\infty}\int\frac{d^2p}{(2\pi)^2}\,\text{tr}[\tilde{\not A}(0)(i\not p+i\gamma_3\tilde{\omega}_n+M)^{-1}] \tag{24}$$

where tr is the Dirac trace, and $\tilde{A}_j(p)$ is the Fourier transform of $A_j(x)$ with respect to the two space variables

$$\tilde{A}_j(p)=\int d^2x\,e^{-ip\cdot x}A_j(x). \tag{25}$$

It is immediate to check that, by rationalizing the denominator and taking the Dirac trace in (24), we obtain 0, which is an unpropitious start to the calculation, and might seem to contradict the results of reference [3]. The way out of this impasse is to realize that, if we are going to deal with an A_j such that the associated magnetic flux is nonzero, then its zero momentum component is necessarily singular, and the result of (24) will be of the ambiguous form $0\times\infty$. Indeed, from the definition (25) of the Fourier transform, we see that the magnetic flux Φ can be expressed as the following limit

$$\Phi\equiv\int d^2x\,\epsilon_{jk}\partial_jA_k=i\lim_{q\to 0}\epsilon_{jk}q_j\tilde{A}_k(q). \tag{26}$$

It follows from (26) that, if we want to have a nonzero flux,[1] then the zero-momentum component of $\tilde{A}_j$ necessarily diverges. We somehow need to introduce another momentum (q) into the problem, and work with the finite quantity Φ.

In order to see how to do this let us consider, in fact, the leading term in the expansion of $\Gamma^{(1)}(A)$ in powers of $\tilde{A}_3$, which is just the perturbative action

$$\Gamma^{(1,1)}(A)=(ie)^2\sum_{n=-\infty}^{n=+\infty}\mathrm{Tr}[\gamma_3\tilde{A}_3(\not\partial+i\gamma_3\omega_n+M)^{-1}\times\not A(\not\partial+i\gamma_3\omega_n+M)^{-1}]. \tag{27}$$

In this expression it is clear that the trace will involve a second momentum integration if $\tilde{A}_3$ is allowed to depend on x as well as on τ:

$$\begin{aligned}\Gamma^{(1,1)}[\tilde{A}_3(x),\vec{A}(\vec{x})]=(ie)^2\sum_{n=-\infty}^{n=+\infty}\int d^2p\,d^2q\\ \times\mathrm{tr}[\langle p+q|\gamma_3\tilde{A}_3|p\rangle\\ \times(i\not p+i\gamma_3\omega_n+M)^{-1}\langle p|\not A|p+q\rangle\\ \times(i(\not p+\not q)+i\gamma_3\omega_n+M)^{-1}],\end{aligned} \tag{28}$$

where tr means trace over the Dirac indices only. The case of x-independent $\tilde{A}_3$ may then be obtained via the limiting process in which the x dependence of $\tilde{A}_3(x)$ is removed, leading to

$$\begin{aligned}\Gamma^{(1,1)}[\tilde{A}_3,\vec{A}(x)]=(ie)^2\tilde{A}_3\sum_{n=-\infty}^{+\infty}\int\frac{d^2p}{(2\pi)^2}\\ \times\lim_{q\to 0}\mathrm{tr}\{\gamma_3(i\not p+i\gamma_3\omega_n+M)^{-1}\tilde{\not A}(q)\\ \times(i(\not p+\not q)+i\gamma_3\omega_n+M)^{-1}\}.\end{aligned} \tag{29}$$

Rationalizing the second denominator and taking the Dirac trace reveals the presence of a term proportional to Φ, whose contribution is precisely the odd-parity perturbative action, for our special field configuration. One can proceed in this way, with higher powers of $\tilde{A}_3$, still retaining only one power of $\vec{A}$. At the term of order $(\tilde{A}_3)^2\vec{A}$, for example, one will have three momentum integrals before the x-dependence of $\tilde{A}_3$ is removed, and one might wonder if a further and more complicated limiting procedure is required. But it turns out that a well-regulated expression is obtained by retaining an extra momentum dependence in only the ''final'' propagator as in (29),

$$\begin{aligned}\Gamma^{(2,1)}[\tilde{A}_3,\vec{A}(\vec{x})]=-(ie)^3\tilde{A}_3^2\sum_{n=-\infty}^{n=+\infty}\int\frac{d^2q}{(2\pi)^2}\lim_{q\to 0}\mathrm{tr}\{\gamma_3(i\not p\\ +i\gamma_3\omega_n+M)^{-1}\gamma_3(i\not p+i\gamma_3\omega_n\\ +M)^{-1}\tilde{\not A}(q)[i(\not p+\not q)+i\gamma_3\omega_n+M]^{-1}\},\end{aligned} \tag{30}$$

and the same is true for higher orders in $\tilde{A}_3$.

But clearly the prospect of evaluating the general term $\Gamma^{(n,1)}$, and then trying to sum up the answer so as to obtain the full $\Gamma^{(1)}$, which is nonperturbative in $\tilde{A}_3$, is unappealing. Fortunately, this is not necessary. Recall that we resorted to perturbation theory in $\tilde{A}_3$ in order to gain insight into the IR behavior—we were not otherwise forced to expand the denominator in (24) in powers of $\tilde{A}_3$. Indeed, the formulas (29) and (30) indicate that (24) should be interpreted in terms of a limit in which the argument of the trace is replaced by $\tilde{\not A}(q)[i(\not p+\not q)+i\gamma_3\tilde{\omega}_n+M]^{-1}$, and then the limit $q\to 0$ is taken. At this point, however, one realizes that such an expression, though IR regular, is UV divergent. To regulate the UV divergence, we work instead with the derivative of $\Gamma(A)$ with respect to $\tilde{A}_3$, which improves the large momentum dependence of the integrand by adding an extra propagator, and amounts to subtracting the value at $\tilde{A}_3=0$. These considerations then lead to the well-behaved expression

$$\begin{aligned}\frac{\partial}{\partial\tilde{A}_3}\Gamma(A)=-e^2\sum_{n=-\infty}^{n=+\infty}\lim_{q\to 0}\int d^2p\,\mathrm{tr}\{\gamma_3\langle p|(\not\partial+i\gamma_3\tilde{\omega}_n\\ +M)^{-1}\tilde{\not A}(\not\partial+i\gamma_3\tilde{\omega}_n+M)^{-1}|p+q\rangle\}.\end{aligned} \tag{31}$$

We are now ready to evaluate (31). After taking the trace, one finds that the only nonvanishing contribution to the derivative of $\Gamma(A)$ is

$$\begin{aligned}\frac{\partial}{\partial\tilde{A}_3}\Gamma(A)=2Me^2\sum_{n=-\infty}^{n=+\infty}\lim_{q\to 0}[\epsilon_{jk}q_j\tilde{A}_k(q)]\\ \times\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+\tilde{\omega}_n^2+M^2)^2},\end{aligned} \tag{32}$$

which, by using (26), may be put as

$$\frac{\partial}{\partial\tilde{A}_3}\Gamma(A)=-2iMe^2\Phi\sum_{n=-\infty}^{n=+\infty}\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+\tilde{\omega}_n^2+M^2)^2}. \tag{33}$$

[1]Zero-flux configurations give $\Gamma=0$ without any ambiguity. This is consistent with the result of [3].

Performing the (convergent) momentum integral in (33), we obtain

$$\frac{\partial}{\partial \widetilde{A}_3}\Gamma(A) = -\frac{iMe^2}{2\pi}\Phi\sum_{n=-\infty}^{n=+\infty}\frac{1}{\widetilde{\omega}_n^2+M^2}. \tag{34}$$

When this expression is integrated term by term over $\widetilde{A}_3$, it yields

$$\Gamma(A) = \frac{ie}{2\pi}\Phi\sum_{n=-\infty}^{n=+\infty}\arctan\left[\frac{\omega_n+e\widetilde{A}_3}{M}\right]. \tag{35}$$

This series is exactly equal to the one that appears in Ref. [3], and indeed, it shows already the identity of this result to the one obtained there by the ''decoupling'' change of variables. Thus we conclude that, for one fermionic flavor, and keeping terms linear in A_j, the result for the induced action Γ is

$$\Gamma(A) = \frac{ie}{2\pi}\Phi\arctan\left[\tanh\frac{\beta M}{2}\tan\left(\frac{e}{2}\int_0^\beta d\tau A_3(\tau)\right)\right]. \tag{36}$$

As in [3,4], the branch of the arctan in (35) and (36) is defined so as to make Γ a continuous function of $e\beta\widetilde{A}_3$ as it moves continuously through the values $2\pi k$, with k integer. Also note that, for the non-Abelian configurations considered in (4), the steps that lead to our final expression (36) hold true, with the only addition of a trace over colour indices, in agreement with [4].

B. Four-component case: Mixed CS term

In this section we show what the consequences of the procedure of the previous section are for the case of the Lagrangian appearing in the DM model. In that model, the fermionic action is defined to be

$$S_f = \sum_{a=1}^{N_f}\int_0^\beta d\tau\int d^2x\,\overline{\psi_a}(\partial\!\!\!/+ieA\!\!\!/+iga\!\!\!/\tau_3+M)\psi_a \tag{37}$$

where the Dirac matrices are in the reducible 4×4 representation:

$$\gamma_\mu = \begin{pmatrix}\sigma_\mu & 0\\ 0 & -\sigma_\mu\end{pmatrix},\quad \tau_3 = \begin{pmatrix}I_{2\times2} & 0\\ 0 & -I_{2\times2}\end{pmatrix}, \tag{38}$$

where $\mu=1,2,3$, $I_{2\times2}$ is the 2×2 identity matrix and a is the flavour index. It is easy to see that the mass term is parity conserving; indeed, rewriting the action in terms of two-component fermions, we get

$$S_f = \sum_{a=1}^{N_f}\int_0^\beta d\tau\int d^2x[\overline{\chi_{a1}}(\partial\!\!\!/+ieA\!\!\!/+iga\!\!\!/+M)\chi_{a1} + \overline{\chi_{a2}}(\partial\!\!\!/+ieA\!\!\!/-iga\!\!\!/-M)\chi_{a2}] \tag{39}$$

where χ_{a1} and χ_{a2} are the upper and lower two-component spinors corresponding to ψ_a.

For the particular configurations $a_3=a_3(\tau)$, and $A_j=A_j(x)$ (all the other components$=0$), we can directly apply the results of the previous section to each of the two terms in the action, just by replacing $\widetilde{A}_3$ by $\widetilde{a}_3$. This amounts to using an $\widetilde{\omega}_n=\omega_n+e\widetilde{a}_3$ in the propagators. As a consequence, for N_f four-component fermions we get

$$\Gamma(A,a) = \frac{ie}{\pi}N_f\Phi\arctan\left[\tanh\frac{\beta M}{2}\tan\left(\frac{g}{2}\int_0^\beta d\tau a_3(\tau)\right)\right], \tag{40}$$

which is the result that assures the validity of the flux quantization argument for any nonzero temperature, as explained in the Introduction.

C. τ-dependent A_j

As an example of an extension of the kind of configuration that can be treated with this method, we shall consider the case of a gauge field where the constraint of τ-independence of A_j is relaxed, namely,

$$A_3=A_3(\tau),\quad A_j=A_j(\tau,x). \tag{41}$$

We will, however, calculate this within the approximation of keeping terms linear in A_j, and everything shall be discussed for the single flavor two-component case. The necessary changes to make it appropriate for the many flavors four-component case are analogous to the ones performed in Sec. II B.

As for the previous configurations, we first go to a gauge where $A_3(\tau)\to\widetilde{A}_3$ becomes a constant. After this, A_j will remain τ-dependent. In spite of the fact that there is no τ translation invariance, we perform a Fourier transformation with respect to the imaginary time in (11), obtaining

$$S_F(A) = \frac{1}{\beta}\sum_{m=-\infty}^{m=+\infty}\sum_{n=-\infty}^{n=+\infty}\int d^2x\,\overline{\psi_m}(x)\{\delta_{m,n}[\partial\!\!\!/+i\gamma_3\widetilde{\omega}_n+M] + ie\widetilde{A\!\!\!/}^{(m-n)}(x)\}\psi_n(x) \tag{42}$$

where

$$\widetilde{A}_j^{(k)} = \frac{1}{\beta}\int_0^\beta d\tau e^{-i2k r/\beta}A_j(x,\tau). \tag{43}$$

The only change that we have to make in the calculation corresponding to the static A_j case is that now the matrix whose determinant we are evaluating is not diagonal in the space of Matsubara frequencies. Then

$$\Gamma(A) = -\mathrm{Tr}\,\log\{\delta_{m,n}[\partial\!\!\!/+i\gamma_3\widetilde{\omega}_n+M]+ie\widetilde{A\!\!\!/}^{(m-n)}(x)\} \tag{44}$$

where now, of course, the trace also affects the discrete frequencies. There is an important simplification which occurs because we are actually dealing with the first order term in A_j. When considering this first order term in the derivative of Γ with respect to $\widetilde{A}_3$, we obtain

$$\frac{\partial\Gamma(A)}{\partial\widetilde{A}_3} = -e^2 \, \mathrm{Tr}\{\gamma_3(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_m + M)^{-1}\widetilde{A\!\!\!/}^{(m-n)}$$
$$\times(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_n + M)^{-1}\}$$
$$= -e^2 \sum_{n=-\infty}^{n=+\infty} \mathrm{Tr}\{\gamma_3(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_n + M)^{-1}$$
$$\times\widetilde{A\!\!\!/}^{(0)}(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_n + M)^{-1}\}. \tag{45}$$

Note that only the zero-frequency component of A_j appears in this expression, which, on the other hand, can now be evaluated analogously to the static-A_j case, by replacing A_j by its zero-frequency component. The final result is then:

$$\Gamma(A) = \frac{ie}{2\pi}\frac{1}{\beta}\int_0^\beta d\tau\Phi(\tau)$$
$$\times \arctan\left[\tanh\left(\frac{\beta M}{2}\right)\tan\left(\frac{e}{2}\int_0^\beta d\tau' A_3(\tau')\right)\right], \tag{46}$$

where $\Phi(\tau) \equiv \int d^2x \epsilon_{jk}\partial_j A_k(x,\tau)$.

III. THE TERM QUADRATIC IN $\vec{A}$

The previous section considered various examples of what is, in fact, the imaginary part of the full effective action in Euclidean space. It is this part, linear in the flux, which exhibits the interesting properties under large gauge transformations. The real part of the effective action is not anomalous in that sense, and the terms of second and higher order in A_j should be straightforwardly gauge invariant. Nevertheless, we think that a calculation of the $\mathcal{O}(A_j^2)$ term has some interest in the present context, for the following reason. In the soluble $(0+1)$ model considered by Dunne *et al.* [1] the complete effective action (for one flavor) is

$$\Gamma(A) = \log\left[\cos x - i \tanh\left(\frac{\beta M}{2}\right)\sin x\right] \tag{47}$$

where $x = \beta\widetilde{A}/2$ and $\widetilde{A} = (1/\beta)\int_0^\beta d\tau A(\tau)$. The imaginary part of this is, of course, just the arctan function found in [3] and [4] and in Sec. II above. An obvious question to ask is whether the real part of the action in our $2+1$ case (always for our special field configuration) bears any relation to the real part of (47). We therefore calculate the first nonvanishing contribution to the real part, that of order A_j^2, retaining all powers of $\widetilde{A}_3$.

We consider one two-component fermion and start as before, from the exact expression for the derivative of the effective action with respect to $\widetilde{A}_3$. The term of order two in A_j [denoted $(\partial\Gamma^{(2)}/\partial\widetilde{A}_3)(A)$] is

$$\frac{\partial\Gamma^{(2)}}{\partial\widetilde{A}_3}(A) = ie^2 \sum_{n=-\infty}^{+\infty} \mathrm{Tr}\{\gamma_3(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_n + M)^{-1}A\!\!\!/(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_n$$
$$+ M)^{-1}A\!\!\!/(\partial\!\!\!/ + i\gamma_3\widetilde{\omega}_n + M)^{-1}\}, \tag{48}$$

which needs no IR regularization. Evaluating the Dirac trace and the functional trace in momentum space, we can write this term as

$$\frac{\partial\Gamma^{(2)}}{\partial\widetilde{A}_3}(A) = ie^3\int\frac{d^2p}{(2\pi)^2}\widetilde{A}_j(p)\Gamma_{jk}(p)\widetilde{A}_k(-p) \tag{49}$$

where

$$\Gamma_{jk}(p) = 2\sum_{n=-\infty}^{+\infty} i\widetilde{\omega}_n \int\frac{d^2q}{(2\pi)^2}\left[\frac{(p^2 - q^2 - \widetilde{\omega}_n^2 - M^2)\delta_{jk} + 4q_jq_k + 2(p_jq_k + q_jp_k)}{[(p+q)^2 + \widetilde{\omega}_n^2 + M^2]^2(q^2 + \widetilde{\omega}_n^2 + M^2)}\right]. \tag{50}$$

All the momentum integrals appearing in the last expression are convergent, and moreover, by a lengthy but straightforward calculation we can recast it into the following explicitly gauge invariant form

$$\Gamma_{jk}(p) = 2\sum_{n=-\infty}^{+\infty} i\widetilde{\omega}_n\left[\frac{1}{4\pi}\int_0^1\frac{dx}{\mathcal{D}_n^2}(x + x^2 - 2x^3)\right](p^2\delta_{jk} - p_jp_k), \tag{51}$$

where

$$\mathcal{D}_n = M^2 + \widetilde{\omega}_n^2 + x(1-x)p^2. \tag{52}$$

On the other hand, this may also be written as

$$\Gamma_{jk}(p) = \frac{-i}{4\pi e^2}\int_0^1 dx\,\frac{x + x^2 - 2x^3}{\sqrt{M^2 + x(1-x)p^2}}\frac{\partial^2}{\partial\widetilde{A}_3^2}\sum_{n=-\infty}^{+\infty}\arctan\left[\frac{\widetilde{\omega}_n}{\sqrt{M^2 + x(1-x)p^2}}\right](p^2\delta_{jk} - p_jp_k). \tag{53}$$

The summation over frequencies can be obtained by borrowing the result appearing in [3,4]. Inserting this into the expression for the second order term in the derivative of the effective action, and integrating over $\widetilde{A}_3$, yields

$$\Gamma^{(2)}(\widetilde{A}_3, A_j) - \Gamma^{(2)}(0, A_j) = ie^3\int\frac{d^2p}{(2\pi)^2}\widetilde{A}_j(p)G_{jk}(p)\widetilde{A}_k(-p) \tag{54}$$

where

$$G_{jk}(p)=-\frac{i\beta}{8\pi e}(p^2\delta_{jk}-p_jp_k)\int_0^1 dx\,\frac{x+x^2-2x^3}{\sqrt{M^2+x(1-x)p^2}}\,\frac{\tanh\left[\frac{\beta}{2}\sqrt{M^2+x(1-x)p^2}\right]}{\cos^2\left(\frac{e\beta\widetilde{A}_3}{2}\right)+\tanh^2\left[\frac{\beta}{2}\sqrt{M^2+x(1-x)p^2}\right]\sin^2\left(\frac{e\beta\widetilde{A}_3}{2}\right)}. \quad (55)$$

We do not write the explicit form of $\Gamma^{(2)}(0,A_j)$ because it is perturbative and insensitive to large gauge transformations. Indeed, it can be obtained, for example, by putting $A_3=0$ and $A_j=A_j(x)$ in the result for the induced parity conserving term presented in [19]. The expression (55) would of course become nonlocal if converted to the coordinate-space representation; nevertheless it is, in fact, invariant under large gauge transformations on $\widetilde{A}_3$. A derivative expansion of (55) gives a series of local terms, the leading one of which is

$$\Gamma^{(2)}(\widetilde{A}_3,A_j)-\Gamma^{(2)}(0,A_j)\simeq\frac{e^2\beta}{48\pi M}\,\frac{\tanh\left(\frac{\beta M}{2}\right)}{\cos^2\left(\frac{e\beta\widetilde{A}_3}{2}\right)+\tanh^2\left(\frac{\beta M}{2}\right)\sin^2\left(\frac{e\beta\widetilde{A}_3}{2}\right)}\int d^2xF_{jk}F_{jk}. \quad (56)$$

It is indeed amusing, and perhaps of some significance, that the denominator function appearing in (56) is just the modulus squared of the complex function whose logarithm is the result (47) of Dunne *et al.* [1], just as the imaginary part of our effective action involves the phase of that function. We have, however, not been able to explore this possible connection any further as yet.

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