Initial value problem for maximally nonlocal actions

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We study the initial value problem for actions whose non-locality is ''maximal'' in the sense that there is no dependence upon the separation between points. In contrast with many other non-local actions, the classical solution set of these systems is at most discretely enlarged, and may even be restricted, with respect to that of a local theory. We show that the solutions are those of a local theory whose (spacetime constant) parameters vary with the initial value data according to algebraic equations. The various roots of these algebraic equations can be plausibly interpreted in quantum mechanics as different components of a multi-component wave function. It is also possible that the consistency of these algebraic equations imposes constraints upon the initial value data which appear miraculous from the context of a local theory. Although the discussion and examples are given in the context of simple mechanical systems the results should apply as well to field theory. [S0556-2821(98)00604-3]

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Most physicists probably recall wondering, at their first exposure to Lagrangians, why they are usually assumed to depend only upon the zeroth and first time derivatives of the dynamical variable. The answer is very simple: allowing higher time derivatives almost always leads to instabilities. This has been known since Ostrogradski's canonical formulation of such systems in the middle of the 19th Century [1].

To understand the problem, consider the dynamics of a point particle in one dimension whose position is $q(t)$ and whose Lagrangian $L(q,\dot{q},\ddot{q})$ includes second time derivatives. We assume only that the second derivatives cannot be removed by partial integration. This condition is known as *non-degeneracy* and amounts to the invertibility of the equation

$$P_2=\frac{\partial L(q,\dot{q},\ddot{q})}{\partial \ddot{q}}, \tag{1}$$

to solve for $\ddot{q}(q,\dot{q},P_2)$. Under the assumption of non-degeneracy the Euler-Lagrange equations

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}+\left(\frac{d}{dt}\right)^2\frac{\partial L}{\partial \ddot{q}}=0 \tag{2}$$

define time evolution by determining the fourth derivative of $q(t)$ in terms of q, $\dot{q}$, $\ddot{q}$ and $\dddot{q}$. One must therefore supply twice as much initial value data as in the usual case, and this entails a new set of canonically conjugate coordinates in the Hamiltonian formulation. In Ostrogradski's construction the canonical coordinates are

$$Q_1\equiv q, \quad Q_2\equiv \dot{q}, \tag{3}$$

and they are respectively conjugate to the following momenta:

$$P_1\equiv \frac{\partial L(q,\dot{q},\ddot{q})}{\partial \dot{q}}-\frac{d}{dt}\frac{\partial L(q,\dot{q},\ddot{q})}{\partial \ddot{q}}, \quad P_2\equiv \frac{\partial L(q,\dot{q},\ddot{q})}{\partial \ddot{q}}. \tag{4}$$

Note the equation for P_2 can be inverted to solve for $\ddot{q}$ in terms of just Q_1, Q_2 and P_2; P_1 is only needed to express $\dddot{q}$.

Ostrogradski's Hamiltonian is

$$H=\sum_i P_i\dot{Q}_i-L \tag{5}$$

$$=P_1Q_2+P_2\ddot{q}(Q_1,Q_2,P_2) -L(Q_1,Q_2,\ddot{q}(Q_1,Q_2,P_2)), \tag{6}$$

and his canonical equations for time evolution are the obvious ones suggested by the notation

$$\dot{Q}_i=\frac{\partial H}{\partial P_i}, \quad \dot{P}_i=-\frac{\partial H}{\partial Q_i}. \tag{7}$$

It is straightforward to verify that the evolution equations for Q_1, Q_2 and P_2 simply reproduce the definitions of Q_2, P_2 and P_1, respectively. The canonical expression of the Euler-Lagrange equation (2) is the evolution equation for P_1. So Ostrogradski's Hamiltonian generates time evolution; it is also conserved when the Lagrangian is free of explicit time dependence.

The problem with higher derivatives is apparent from expression (6): *the Hamiltonian is linear in* P_1. Such a Hamiltonian can never be bounded below; in fact there is not even

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any barrier to the system's decay. In a conventionally interacting system this means that one can excite positive energy degrees of freedom while conserving energy by exciting negative energy degrees of freedom. Since there are typically many more ways of exciting a degree of freedom than not, the system will tend to migrate to very highly excited states which have little in common with the physical reality we perceive.

Note the generality of the higher derivative instability. It does not depend upon any approximation scheme, nor any feature of the Lagrangian except non-degeneracy. Nor is quantization liable to prevent it because the instability is not confined to a small region of phase space. Note also the relation to the space of classical solutions: a non-degenerate higher derivative doubles the number of continuum degrees of freedom and at least half of the new degrees of freedom access negative energy.

Physicists are an inventive lot and such a bald no-go theorem provokes them to envisage tortuous evasions. It is impossible to prove a negative, so we will not assert that there is no way out for non-degenerate higher derivatives, but we do urge a little common sense. Ostrogradski's theorem should not seem surprising. It explains why every single system we have so far observed seems to be described, on the fundamental level, by a Lagrangian containing no higher than first time derivatives. The bizarre and incredible thing would be if this fact was simply an accident.

The relevance of Ostrogradski's theorem to non-local actions is that the instability grows worse as more higher derivatives are added. For Lagrangians which depend non-degenerately upon the Nth time derivative the associated Hamiltonian is linear in all but possibly one of the N canonical momenta [1]. When a particular non-local model can be represented as the limit of a sequence of non-degenerate higher derivative theories, then essentially half of the canonical degrees of freedom in the non-local model must access negative energy. This would seem to be a problem unless the dynamics can somehow contrive to prevent or at least suppress the excitation of these degrees of freedom.

The higher derivative representation is valid for non-locality that enters through entire functions of the derivative operator which cannot be subsumed into a field redefinition. A familiar example of such a system is string field theory [2]. On the other hand, the higher derivative representation is certainly not valid for many forms of non-locality. For example, one generally obtains a non-local Lagrangian by integrating out one or more of the fundamental dynamical variables. These Lagrangians contain poles of the derivative operator, so it is not correct to consider them as limits of higher derivative Lagrangians. And since it is of course valid to solve them in the original, local form, there is no extension of the space of classical solutions.

A local action is defined as the integral of a function of the dynamical variable and some finite number of its derivatives. What we will call a *maximally non-local action* involves non-trivial functions of such terms. To understand what we mean by the adjective ''maximal,'' consider a non-local functional of the position $q(t)$ of a one-dimensional particle:

$$F[q]=\int ds f(q(s),\dot{q}(s),\ldots)\int dt g(q(t),\dot{q}(t),\ldots)V(s,t). \tag{8}$$

Here f and g are ordinary functions of the dynamical variable and some finite number of its derivatives, and $V(s,t)$ is a C-number weighting function which controls the amount of non-locality. In a local theory the weighting function is infinitely peaked at $s=t$: $V(s,t)=\delta(s-t)$. The theory grows more non-local as the weighting function becomes less and less sharply peaked at $s=t$. For *maximal* non-locality the weighting function is just a constant. In this case the functional $F[q]$ breaks up into the product of two local functionals. The general maximally non-local action is an arbitrary function of local functionals of the dynamical variable.

As an example, consider the following generalization of the one-dimensional harmonic oscillator:

$$S[q]=\lim_{T\to\infty}\left\{\int_{-T}^{T}dt\,\frac{1}{2}m\dot{q}^2-\frac{1}{4}m\omega_0^2\ell_0^2T\left(\frac{1}{T}\int_{-T}^{T}dt\frac{q^2}{\ell_0^2}\right)^2\right\}, \tag{9}$$

where ω_0 and ℓ_0 are constants with the respective dimensions of frequency and length. An important fact about maximally non-local actions is that their equations of motion are local except for coupling ''constants'' which are functions of integrals of local functions of the dynamical variable. For the example just presented one finds

$$\frac{\delta S[q]}{\delta q(t)}=-m\ddot{q}(t)-m\omega^2[q]q(t)=0, \tag{10}$$

where the oscillator's frequency is

$$w^2[q]\equiv\lim_{T\to\infty}\frac{\omega_0^2}{T\ell_0^2}\int_{-T}^{T}dt q^2(t). \tag{11}$$

It has been argued that quantum field theoretic versions of maximally non-local actions might explain the apparent fine tuning of certain coupling constants [3]. Of course any such mechanism would be pointless if it inescapably entailed the Ostrogradskian instabilities. The aim of this paper is to show that it does not.

It is simplest to begin by solving the maximally non-local harmonic oscillator (9). From its definition (11), $\omega^2[q]$ must be positive semi-definite, so the general solution for fixed ω is

$$q(t)=q_0\cos(\omega t)+\frac{\dot{q}_0}{\omega}\sin(\omega t), \tag{12}$$

where q_0 and $\dot{q}_0$ are the initial value data. Substituting this into the definition of ω^2 gives

$$\omega^2=\frac{\omega_0^2}{\ell_0^2}\left(q_0^2+\frac{\dot{q}_0^2}{\omega^2}\right). \tag{13}$$

This equation has two solutions but only the positive root is consistent with the explicit positive semi-definiteness of Eq. (11). We can therefore write

$$\omega^2=\frac{\omega_0}{2\ell_0^2}(\omega_0 q_0^2+\sqrt{\omega_0^2 q_0^4+4\ell_0^2\dot{q}_0^2}). \tag{14}$$

Since this theory is time translation invariant, it makes no difference if we replace q_0 and $\dot{q}_0$ in $\omega^2[q]$ by $q(t)$ and $\dot{q}(t)$, respectively. The vanishing of $d\omega^2/dt$ follows from the equation of motion (10). The conserved energy associated with time translation invariance is

$$E=\frac{1}{2}m\dot{q}^2+\frac{1}{2}m\omega^2 q^2. \tag{15}$$

To make it generate time evolution we define the Poisson brackets as follows:

$$\{q,\dot{q}\}=\dot{q}\Big/\frac{\partial E}{\partial\dot{q}} \tag{16}$$

$$=\frac{1}{m}\frac{\sqrt{\omega_0^2 q^4+4\ell_0^2\dot{q}^2}}{\sqrt{\omega_0^2 q^4+4\ell_0^2\dot{q}^2}+\omega_0 q^2}. \tag{17}$$

This makes the Poisson bracket of q with E give $\dot{q}$. The other evolution equation

$$\ddot{q}=\{\dot{q},E\}=-\frac{\partial E}{\partial q}\times\{q,\dot{q}\}, \tag{18}$$

follows by taking the time derivative and using energy conservation. It can also be checked explicitly.

Since the energy (15) of this system is bounded below we have shown by explicit example that the Ostrogradskian instabilities are not inevitable for maximally non-local actions. For the system considered there was not even any enlargement of the solution set of its local cognate — there is a unique solution for every choice of q_0 and $\dot{q}_0$. This feature is not generic; more complicated models can show a discrete enlargement of the solution set. Consider, for example, the following maximally non-local action:

$$S[q]=\lim_{T\to\infty}\left\{\int_{-T}^{T}dt\left[\frac{1}{2}m\dot{q}^2-\frac{1}{4}m\omega_0^2 q^2\right]-\frac{1}{8}m\omega_0^2\ell_0^2 T\sin\left[\frac{2}{T}\int_{-T}^{T}dt\frac{q^2}{\ell_0^2}\right]\right\}. \tag{19}$$

The field equations are again those of a harmonic oscillator but with a slightly different form for the frequency-squared:

$$\frac{\delta S[q]}{\delta q(t)}=-m\{\ddot{q}(t)+\omega^2[q]q(t)\}, \tag{20}$$

$$\omega^2[q]\equiv\omega_0^2\cos^2\left[\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}dt\frac{q^2}{\ell_0^2}\right]. \tag{21}$$

For fixed ω the general initial value solution is still Eq. (12) so ω^2 is any positive root of the following transcendental equation:

$$\omega^2=\omega_0^2\cos^2\left[\frac{q_0^2}{\ell_0^2}+\frac{\dot{q}_0^2}{\omega^2\ell_0^2}\right]. \tag{22}$$

For $\dot{q}_0\neq 0$ there are a countable infinity of solutions. No closed form can be obtained, but for large integers N they are, approximately,

$$\omega_N^2\approx\frac{2}{2N+1}\frac{\dot{q}_0^2}{\pi\ell_0^2}. \tag{23}$$

Of course the energy is still Eq. (15) — with the new meaning of ω^2 — and it is positive semi-definite for each root.

Another curious feature of maximally non-local actions is the possibility for *restrictions* on the initial value data. It is this property which might offer an explanation for otherwise miraculous fine tunings [3]. To understand it, consider the following maximally non-local harmonic oscillator:

$$S[q]=\lim_{T\to\infty}\left\{-\frac{1}{2}m\omega_0^2\ell_0^2\exp\left[-\frac{1}{T}\int_{-T}^{T}dt\frac{\dot{q}^2}{\omega_0^2\ell_0^2}\right]-\frac{1}{2}m\omega_0^2 q^2\right\}. \tag{24}$$

The equation of motion is still that of a harmonic oscillator but with its frequency-squared defined as

$$\omega^2[q]\equiv\omega_0^2\exp\left[\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}dt\frac{\dot{q}^2}{\omega_0^2\ell_0^2}\right]. \tag{25}$$

One determines ω^2 as a function of the initial value data from the following transcendental equation:

$$\omega^2=\omega_0^2\exp\left[\frac{\omega^2 q_0^2}{\omega_0^2\ell_0^2}+\frac{\dot{q}_0^2}{\omega_0^2\ell_0^2}\right]. \tag{26}$$

When the ratio q_0/ℓ_0 is much less than 1 the exponential grows slowly enough to intersect the quadratic, and there are two solutions. However, when q_0/ℓ_0 is larger than 1 there is no solution.

In fact the analysis we have just gone through applies as well to field theory. The field equations of maximally non-local actions are related to local ones whose couplings, $\lambda_1,\lambda_2,\ldots$, are really non-dynamical constants. Suppose that the local cognate of a maximally non-local action has the following general initial value solution:

$$\phi(t,\vec{x})=\Phi[\phi_0,\dot{\phi}_0](t,\vec{x},\lambda_1,\lambda_2,\ldots). \tag{27}$$

Since its maximally non-local cousin has the same field equation (by definition) its solutions must be the same, except that the couplings, $\lambda_1,\lambda_2,\ldots$, are spacetime constant integrals of the dynamical variable:

$$\lambda_i=\Lambda_i[\phi]. \tag{28}$$

Substituting the general initial value solution (27) into these integrals gives algebraic equations for the couplings:

$$\lambda_i = \Lambda_i[\Phi[\phi_0, \dot{\phi}_0](t, \vec{x}, \lambda_1, \lambda_2, \ldots)]. \tag{29}$$

There will in general be more than one root, the choice of which represents discrete degrees of freedom not present in the local theory. It is also possible that some or all of the roots may disappear unless the initial value data lie within certain regions.

If its local cognate has a conserved energy the same functional will be conserved for a maximally non-local action. One will also be able to impose Poisson brackets to make it generate time evolution. Whether or not this energy is bounded below will depend upon what the various solutions do to the energy functional of the local cognate. But our explicit example of the non-local oscillator shows that the Hamiltonian can be bounded below, so there is no generic instability of the Ostrogradskian type.

We close by proposing a physical interpretation for the discrete degrees of freedom associated with multiple roots of the parameter equations. We do not feel one is entitled to select a particular root and ignore the others. Instead, we believe that in quantum mechanics a natural interpretation for the various roots is as different components of a multi-component wave function. One could then treat the various roots the way one works with spin or internal quantum numbers such as isospin. Note that the dynamics of each component would be the same except for different couplings. It is tempting to speculate that such a formalism might be used to unify the mysterious generational structures which appear in elementary particle physics.

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