Yukawa coupling in three dimensions

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(Received 22 May 1997; published 19 December 1997)

We consider several renormalizable, scale free models in three space-time dimensions which involve scalar and spinor fields. The Yukawa couplings are bilinear in both the spinor and scalar fields and the potential is of sixth order in the scalar field. In a model with a single scalar field and a complex fermion field in three Euclidean dimensions, the couplings in the theory are both asymptotically free. This property is not retained in $(2+1)$-dimensional Minkowski space, as we illustrate by considering a renormalizable scale-free supersymmetric model. This is on account of the different properties of the Dirac matrices in Euclidean and Minkowski space. We also examine a model in $(2+1)$-dimensional Minkowski space in which two species of fermions, associated with the two unitarily inequivalent representations of the 2×2 Dirac matrices, couple in two different ways to two distinct scalar fields. There are two types of Yukawa couplings in this model, and either one or the other of them can be asymptotically free (but not both simultaneously). [S0556-2821(98)04402-6]

PACS number(s): 11.10.Kk, 11.30.Rd, 12.60.Jv

I. INTRODUCTION

Self-interacting scalar models are the simplest of all relativistic quantum theories to analyze. There are but three instances in which the self-interaction involves a dimensionless coupling λ (and is consequently perturbatively renormalizable): $\lambda\phi^4$ in four dimensions, $\lambda\phi^3$ in six dimensions, and $\lambda\phi^6$ in three dimensions. The renormalization group functions for these models have been determined beyond one-loop order using dimensional regularization [1]; see Refs. [2–4] respectively. In this paper we consider not just sixth order scalar couplings in three dimensions, but append to them Yukawa couplings which are bilinear in both the scalar and spinor fields. This interaction is renormalizable in three dimensions.

We compute the renormalization group functions associated with these couplings to first order in the scalar self-coupling and second order in the Yukawa coupling in a variety of models. First, a simple model in which there is a single scalar field ϕ and a two-component Dirac fermion field ψ is considered in Euclidean space and we find it to be asymptotically free. Next a supersymmetric model in Minkowski space is examined and, on account of the properties of Dirac matrices in this space, the model no longer possesses asymptotic freedom. Finally, a model containing two species of two-component Dirac Fermions and two types of scalars, coupled in a symmetric way that employs two distinct scalar self-couplings and two Yukawa couplings, is analyzed. It turns out that it is possible to arrange for one of the Yukawa couplings to be asymptotically free.

Dimensional regularization [1] is employed to compute Feynman integrals for several reasons. First of all, massless tadpoles vanish when using this technique, eliminating a number of Feynman graphs. Second, divergences in an odd number of dimensions occur only beyond one-loop order. Third, minimal subtraction [5] is an efficient renormalization scheme which permits one to compute the renormalization group functions with relative ease; this technique is contingent upon using dimensional regularization. Fourth, if one uses dimensional regularization in a three dimensional model that initially is free of dimensionful parameters, then one is not forced by consideration of radiatively induced divergences to introduce couplings involving massive parameters (such as masses for the scalar and spinor fields and quartic scalar couplings). We now consider the various models.

II. YUKAWA MODEL

Initially, let us consider the model defined by the three-dimensional Euclidean action

$$S=\int d^3x\left[-\tfrac{1}{2}\phi p^2\phi-\frac{\lambda}{6!}\phi^6-\overline{\psi}(\not{p}+\tfrac{1}{2}g\phi^2)\psi\right]. \qquad (1)$$

The conventions used are given in the Appendix.

In three dimensions, the couplings λ and g are dimensionless. When dimensional regularization [1] is used, only these couplings and the wave functions ψ and ϕ need to be renormalized. In computing the renormalization group functions, it is necessary to evaluate the divergent parts of the two-point functions $\langle\phi\phi\rangle$, and $\langle\psi\overline{\psi}\rangle$, the four-point function $\langle\psi\psi\phi\phi\rangle$, and the six-point functions $\langle\phi\phi\phi\phi\phi\phi\rangle$.

As was mentioned in the Introduction, using dimensional regularization with this model means that tadpole and one-loop diagrams do not contribute to the renormalization group functions. Consequently, to order λ and g^2, we only consider the diagrams in Fig. 1, using the Feynman rules of Fig. 2. In Table I, we list the number of distinct diagrams associated with each of the graphs in Fig. 1, the associated combinatoric factors, and the pole part of the diagram (in $3-\epsilon$ dimensions).

The divergences computed in Table I do not contribute to vertices that occur in one-loop diagrams to the order which we are considering.

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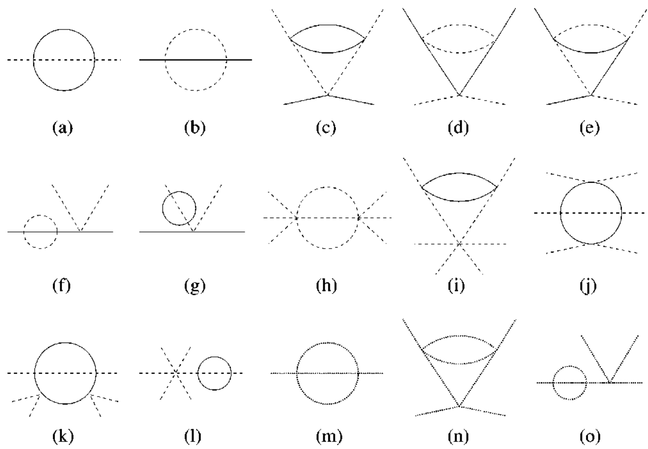

FIG. 1. Feynman diagrams for the Euclidean, supersymmetric, and two-component models.

TABLE I. Two-loop poles in the Euclidean model.

Graph	Number of diagrams	Combinatoric factor	Pole part
a	1	1	$g^2p^2/96\pi^2\epsilon$
b	1	1/2	$g^2\not{p}/192\pi^2\epsilon$
c	1	1	$-g^3/32\pi^2\epsilon$
d	1	1/2	$-g^3/64\pi^2\epsilon$
e	4	1	$-g^3/16\pi^2\epsilon$
f	2	1/2	$-g^3/96\pi^2\epsilon$
g	2	1	$-g^3/48\pi^2\epsilon$
h	10	1/6	$5\lambda^2/96\pi^2\epsilon$
i	15	1	$-15\lambda g^2/32\pi^2\epsilon$
j	90	1	$-45g^4/8\pi^2\epsilon$
k	180	1	$-45g^4/8\pi^2\epsilon$
l	6	1	$-\lambda g^2/16\pi^2\epsilon$

Analogous calculations are easily performed in the supersymmetric version of the model of Eq. (1). The Minkowski space superfield action for this model is (using notation explained in the Appendix)

$$S=\int d^3x d^2\theta\left(\tfrac{1}{2}\Phi D^2\Phi+\frac{\lambda}{4!}\Phi^4\right) \tag{2}$$

which in component form becomes

$$S=\int d^3x\left[\tfrac{1}{2}(A\Box A+i\psi^\alpha\partial_\alpha{}^\beta\psi_\beta+F^2)+\frac{\lambda}{4!}(12A^2\psi^2+4A^3F)\right], \tag{3}$$

where ψ is a Majorana two-component spinor in this theory; A and F are both real scalar fields. The equation of motion for the auxiliary field F may be used to reexpress Eq. (3) in the form

$$S=\int d^3x\left[\frac{1}{2}(A\Box A+i\psi^\alpha\partial_\alpha{}^\beta\psi_\beta)+\frac{\lambda}{2}A^2\psi^2-\frac{\lambda^2}{72}A^6\right] \tag{4}$$

which is closely related to the action of Eq. (1).

We determine the lowest order radiative corrections to the coupling λ by computing the divergent parts of the integrals corresponding to the diagrams in Figs. 1(c), 1(d), 1(e), 1(f), and 1(g) using the appropriate Feynman rules given in Fig. 2. The results are given in Table II.

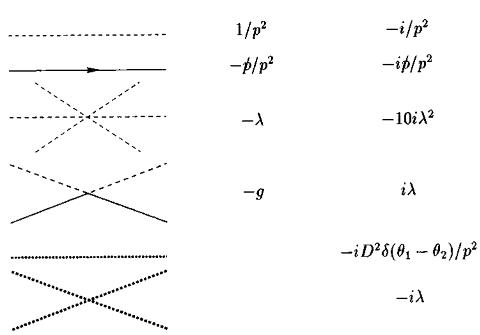

FIG. 2. Feynman rules in the Euclidean and supersymmetric models.

We have also provided the results of computing the superfield diagrams of Figs. 1(n) and 1(o). We note that we could have examined the renormalization of the coupling λ by computing the divergent part of the six-point diagrams rather than the four-point diagrams in Fig. 1; supersymmetry ensures that the results of this approach would yield an identical renormalization of λ.

A third model will now be considered. This will be done in Minkowski space and will incorporate two complex two-component spinor fields ψ_1 and ψ_2 which form the upper and lower components of a four component spinor field ψ, as well as a pair of real scalar fields A and B. The 2×2 Dirac matrices γ_a in three dimensions, which are discussed in the Appendix, are unitarily inequivalent to the matrices $-\gamma^a$ (which of course satisfy the same Dirac algebra); the Dirac matrices we will use in this model are given by

$$\Gamma^a=\begin{pmatrix}\gamma^a & 0\\ 0 & -\gamma^a\end{pmatrix}\quad (a=0,1,2). \tag{5}$$

We define two additional anticommuting Dirac matrices

$$\Gamma_4=\begin{pmatrix}0 & -i\mathbb{1}\\ i\mathbb{1} & 0\end{pmatrix}\quad\text{and}\quad \Gamma_5=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}. \tag{6}$$

The action which we will consider is initially expressed in the form

TABLE II. Two-loop poles in the supersymmetric model.

Graph	Number of diagrams	Combinatoric factor	Pole part
c	1	1/2	$-i\lambda^3/64\pi^2\epsilon$
d	1	1/2	$-i\lambda^3/64\pi^2\epsilon$
e	4	1	$-i\lambda^3/16\pi^2\epsilon$
f	2	1/2	$-i\lambda^3/96\pi^2\epsilon$
g	2	1/2	$-i\lambda^3/96\pi^2\epsilon$
n	6	1/2	$-3i\lambda^3/32\pi^2\epsilon$
o	4	1/6	$-i\lambda^2/48\pi^2\epsilon$

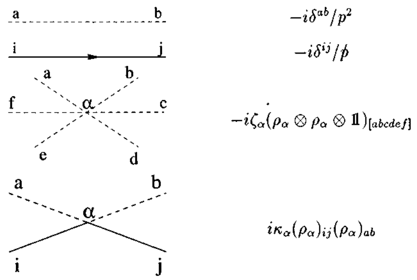

FIG. 3. Feynman rules for the two-component model.

$$S=\int d^3x\bigg[-\tfrac{1}{2}(Ap^2A+Bp^2B)-\bar{\psi}\not{p}\,\psi+\frac{g}{2}\,\bar{\psi}\psi(A^2-B^2)$$
$$-iG\bar{\psi}\Gamma_5\psi AB-\frac{\lambda}{6!}(A^6+B^6)-\frac{\sigma}{4!2!}(A^2B^4+A^4B^2)\bigg] \tag{7}$$

where $p=i\partial$ and $\bar{\psi}=\psi^{+}\Gamma_0$. There are two discrete symmetries present in this Lagrangian,

$$A\rightarrow B,\quad B\rightarrow -A,\quad \psi\rightarrow\Gamma_5\psi \tag{8a}$$

and

$$A\rightarrow B,\quad B\rightarrow A,\quad \psi\rightarrow\Gamma_4\psi. \tag{8b}$$

These symmetries exclude couplings of the form $\bar{\psi}\psi(A^2+B^2)$, $\bar{\psi}\psi AB$, $\bar{\psi}\Gamma_5\psi(A^2\pm B^2)$, $\bar{\psi}\Gamma_4\psi AB$, $\bar{\psi}\Gamma_4\psi(A^2\pm B^2)$, $A^5B\pm AB^5$, A^3B^3, $A^2B^4-A^4B^2$, and A^6-B^6. They do not exclude $\bar{\psi}\Gamma_4\Gamma_5\psi(A^2+B^2)$, but this interaction is not perturbatively generated by radiative corrections induced by the action of Eq. (7).

If in Eq. (7) we were to set $G=g$ and $\sigma=\lambda/5$, then the action would become

$$S=\int d^3x\bigg[-\phi^{*}p^2\phi-\bar{\psi}\not{p}\,\psi+g\bar{\psi}(\phi^2P_{+}+(\phi^{*})^2P_{-})\psi$$
$$-\frac{\lambda}{90}(\phi^{*}\phi)^3\bigg] \tag{9}$$

where $\phi=(A+iB)/\sqrt{2}$ and $P_{\pm}=(1\pm\Gamma_5)/2$. The symmetry of Eq. (8a) now becomes continuous:

$$\phi\rightarrow e^{i\theta}\phi,\quad \psi\rightarrow e^{-i\theta\Gamma_5/2}\psi \tag{10}$$

as $P_{\pm}e^{-i\theta\Gamma_5/2}=P_{\pm}e^{\mp i\theta/2}$.

The interaction terms of Eq. (7) are invariant under the transformations

$$A=(A'+B')/\sqrt{2},\quad B=(A'-B')/\sqrt{2},$$

$$\psi=(i\Gamma_5)^{1/2}\psi'=\frac{1+i\Gamma_5}{\sqrt{2}}\psi', \tag{11}$$

provided that we simultaneously redefine the couplings so that

$$g'=G,\quad G'=g, \tag{12a}$$

$$\sigma'=\frac{\lambda-\sigma}{4},\quad \lambda'=(15\sigma+\lambda)/4. \tag{12b}$$

By exploiting the symmetries of Eqs. (12), we can determine how G and σ are renormalized to a given order in perturbation theory from results which give how g and λ are renormalized. To do this directly we rewrite Eq. (7) in the form

$$S=\int d^3x\bigg[-\tfrac{1}{2}\Phi^Tp^2\Phi-\bar{\psi}_i\not{p}\,\psi_i+\tfrac{1}{2}\sum_{\alpha=1}^{2}\kappa_\alpha(\Phi^T\rho_\alpha\Phi)(\bar{\psi}\rho_\alpha\psi)$$
$$-\frac{1}{6!}\sum_{\alpha=1}^{2}\zeta_\alpha(\Phi^T\rho_\alpha\Phi)^2\Phi^T\Phi\bigg] \tag{13}$$

where we have reverted to using the two-component spinors ψ_i and have defined

$$\Phi^T=(A,B), \tag{13a}$$

$$\kappa_\alpha=(G,g), \tag{13b}$$

$$\zeta_\alpha=\left(\frac{15\sigma+\lambda}{4},\lambda\right), \tag{13c}$$

and

$$\rho_\alpha=(\sigma_1,\sigma_3). \tag{13d}$$

The transformations of Eqs. (11) and (12) become

$$\psi\rightarrow\frac{1}{\sqrt{2}}(\rho_1+\rho_2)\psi, \tag{14a}$$

$$\Phi\rightarrow\frac{1}{\sqrt{2}}(\rho_1+\rho_2)\Phi, \tag{14b}$$

and

$$\kappa_1\leftrightarrow\kappa_2\quad \zeta_1\leftrightarrow\zeta_2, \tag{15}$$

under which the action of Eq. (13) is invariant. The Feynman rules associated with the action in Eq. (13) are given in Fig. 3.

In Table III we provide the results of computing the relevant diagrams in Fig. 1 in the context of the action of Eq. (13). We have verified that these results can be obtained by using the action of Eq. (7) directly, and also that the model of Eq. (9) yields results that coincide with those following from Eqs. (7) and (13) in the limit $g=G$, $\sigma=\lambda/5$.

We are now in a position to use these results to determine the renormalization group functions associated with these models to lowest order.

III. RENORMALIZATION GROUP FUNCTIONS

The pole parts of the diagrams in Fig. 1 as tabulated in Tables I–III serve to fix the relationship between the renormalized and bare couplings, and hence the renormalization group functions, in the three models which we have considered [1–5]. In other words, we can determine the functions a_ν^λ and a_ν^g in the equations

TABLE III. Two-loop poles in the two-component model.

Graph	Number of diagrams	Combinatoric factor	Pole part
a	1	1	$\frac{-i}{48\pi^2\epsilon}(\kappa_1^2+\kappa_2^2)\mathbb{1}p^2$
b	1	1/2	$\frac{-i}{96\pi^2\epsilon}(\kappa_1^2+\kappa_2^2)\mathbb{1}\not{p}$
c	1	1	$\frac{-i}{16\pi^2\epsilon}(\kappa_1^3-\kappa_1\kappa_2^2)(\rho_1\otimes\rho_1)+(1\leftrightarrow 2)$
d	1	1/2	$\frac{-i}{32\pi^2\epsilon}(\kappa_1^3-\kappa_1\kappa_2^2)(\rho_1\otimes\rho_1)+(1\leftrightarrow 2)$
e	4	1	$\frac{-i}{16\pi^2\epsilon}(\kappa_1^3-\kappa_1\kappa_2^2)(\rho_1\otimes\rho_1)+(1\leftrightarrow 2)$
h	10	1/6	$\frac{i}{1440\pi^2\epsilon}(76\zeta_1^2-8\zeta_1\zeta_2+16\zeta_2^2)(\rho_1\otimes\rho_1\otimes\mathbb{1})+(1\leftrightarrow 2)$
i	15	1	$\frac{15i}{16\pi^2\epsilon}(\zeta_1\kappa_1^2+\frac{4}{15}\zeta_2\kappa_2^2-\frac{1}{15}\zeta_1\kappa_2^2)(\rho_1\otimes\rho_1\otimes\mathbb{1})+(1\leftrightarrow 2)$
j	45	2	$\frac{-45i}{4\pi^2\epsilon}(\kappa_1^4-\kappa_1^2\kappa_2^2)(\rho_1\otimes\rho_1\otimes\mathbb{1})+(1\leftrightarrow 2)$
k	45	4	$\frac{-45i}{4\pi^2\epsilon}(\kappa_1^4+\kappa_1^2\kappa_2^2)(\rho_1\otimes\rho_1\otimes\mathbb{1})+(1\leftrightarrow 2)$

$$\lambda_B=\mu^{2\epsilon}\left[\lambda_R+\sum_{\nu=1}^{\infty}\frac{a_\nu^\lambda(\lambda_R,g_R)}{\epsilon^\nu}\right] \tag{16a}$$

$$g_B=\mu^{\epsilon}\left[g_R+\sum_{\nu=1}^{\infty}\frac{a_\nu^g(\lambda_R,g_R)}{\epsilon^\nu}\right] \tag{16b}$$

where, for concreteness, we are using the model of Eq. (1). Similarly, the wave-function renormalizations are given by

$$Z_\phi=1+\sum_{\nu=1}^{\infty}\frac{c_\nu^\phi(\lambda_R,g_R)}{\epsilon^\nu} \tag{17a}$$

and

$$Z_\psi=1+\sum_{\nu=1}^{\infty}\frac{c_\nu^\psi(\lambda_R,g_R)}{\epsilon^\nu}. \tag{17b}$$

Altering the mass scale μ so that

$$\mu'=\mu(1+\rho)\quad(\rho\approx 0) \tag{18}$$

leads to expressions for λ_B and g_B that contain contributions linear in ϵ which can be eliminated by setting

$$\tilde{\lambda}_R=\lambda_R(1-2\epsilon\rho) \tag{19a}$$

$$\tilde{g}_R=g_R(1-\epsilon\rho). \tag{19b}$$

If we now write λ_B and g_B in terms of $\tilde{\lambda}_R$ and $\tilde{g}_R$ and identify λ_R' and g_R' (the renormalized couplings at scale μ') with terms independent of poles in ϵ, we find that

$$\lambda_R'=\tilde{\lambda}_R-2\rho a_1^\lambda+2\rho\tilde{\lambda}_R\frac{\partial a_1^\lambda}{\partial\tilde{\lambda}_R}+\rho\tilde{g}_R\frac{\partial a_1^\lambda}{\partial\tilde{g}_R}, \tag{20a}$$

$$g_R'=\tilde{g}_R-\rho a_1^g+2\rho\tilde{\lambda}_R\frac{\partial a_1^g}{\partial\tilde{\lambda}_R}+\rho\tilde{g}_R\frac{\partial a_1^g}{\partial\tilde{g}_R}, \tag{20b}$$

so that

$$\mu\frac{\partial\lambda}{\partial\mu}=-2a_1^\lambda+2\lambda\frac{\partial a_1^\lambda}{\partial\lambda}+g\frac{\partial a_1^\lambda}{\partial g}, \tag{21a}$$

$$\mu\frac{\partial g}{\partial\mu}=-a_1^g+2\lambda\frac{\partial a_1^g}{\partial\lambda}+g\frac{\partial a_1^g}{\partial g}. \tag{21b}$$

In determining a_1^λ to lowest order, one must keep in mind that a diagram involving a self-energy on an external propagator (so that the diagram is one particle reducible) has its divergence shared between the external wave function and the coupling constant characterizing the vertex. This has the effect of reducing the contribution of the pole parts of the diagrams in Figs. 1(f), 1(g), and 1(l) to the appropriate coupling constant renormalizations by a factor of $\frac{1}{2}$ in each of the models we are examining.

Using Table I, we find that, for the Euclidean model of Eq. (1),

$$a_1^\lambda=\frac{1}{\pi^2}\left(\frac{5\lambda^2}{96}-\frac{\lambda g^2}{2}-\frac{45}{4}g^4\right), \tag{22a}$$

$$a_1^g=-\frac{g^3}{8\pi^2}, \tag{22b}$$

while, for the supersymmetric model of Eqs. (2) and (3),

$$a_1^{\lambda}=\frac{5\lambda^3}{48\pi^2}, \tag{23}$$

as can be seen from Table II. Finally, for the two component model of Eqs. (7) and (13), Table III gives the result

$$a_1^{g}=\frac{g}{16\pi^2}(3g^2-2G^2), \tag{24a}$$

$$a_1^{G}=\frac{G}{16\pi^2}(3G^2-2g^2), \tag{24b}$$

$$a_1^{\lambda}=\frac{1}{16\pi^2}\left(\frac{5}{6}\lambda^2+\frac{5}{2}\sigma^2+16\lambda g^2+\lambda G^2+15\sigma G^2-360g^4\right), \tag{24c}$$

$$a_1^{\sigma}=\frac{1}{16\pi^2}\left(3\sigma^2+\frac{1}{3}\sigma\lambda+\lambda G^2+15\sigma G^2+24g^4-96G^4\right). \tag{24d}$$

The results of Eq. (24) are consistent with the symmetries of Eq. (12).

Together, Eq. (21) and the functions a_1 in Eqs. (22)–(24) show that the rate of change of any coupling to lowest order is given by twice the corresponding a_1; viz. for the Euclidean model

$$\mu\frac{\partial\lambda}{\partial\mu}=\frac{1}{\pi^2}\left(\frac{5\lambda^2}{48}-\lambda g^2-\frac{45}{2}g^4\right) \tag{25a}$$

$$\mu\frac{\partial g}{\partial\mu}=\frac{-g^3}{4\pi^2} \tag{25b}$$

for the supersymmetric model

$$\mu\frac{\partial\lambda}{\partial\mu}=\frac{5\lambda^3}{24\pi^2} \tag{26}$$

and for the two-component model

$$\mu\frac{\partial g}{\partial\mu}=\frac{g}{8\pi^2}(3g^2-2G^2), \tag{27a}$$

$$\mu\frac{\partial G}{\partial\mu}=\frac{G}{8\pi^2}(3G^2-2g^2), \tag{27b}$$

$$\mu\frac{\partial\lambda}{\partial\mu}=\frac{1}{8\pi^2}\left(\frac{5}{6}\lambda^2+\frac{5}{2}\sigma^2+16\lambda g^2+\lambda G^2+15\sigma G^2-360g^4\right), \tag{27c}$$

$$\mu\frac{\partial\sigma}{\partial\mu}=\frac{1}{8\pi^2}\left(3\sigma^2+\frac{1}{3}\sigma\lambda+\lambda G^2+15\sigma G^2+24g^4-96G^4\right). \tag{27d}$$

We are now in a position to discuss the properties of these couplings.

First of all, by Eq. (25b), we see that

$$g^2=\frac{g_0^2}{1+\frac{g_0^2}{2\pi^2}\ln\left(\frac{\mu}{\mu_0}\right)}\equiv\frac{2\pi^2}{\ln\mu/\Lambda}\quad\text{or}\quad g=0 \tag{28}$$

where Λ is some renormalization invariant scale and $g=g_0$ when $\mu=\mu_0$. Consequentially, the theory is asymptotically free, a property shared by non-Abelian gauge theory in four dimensions [6], ϕ^3 theory in six dimensions [3] and ϕ^4 theory in four dimensions when the coupling is of the ''wrong'' sign [7]. In the latter two cases, the models are unacceptable as they are energetically unstable.

Since Eqs. (25) are homogeneous in λ, g^2 and $\ln(\mu/\Lambda)^{-1}$ we one can solve explicitly for the renormalization group (RG) flow in the λ-g plane and thus obtain a full solution to the scaling behavior of both couplings at lowest order. In order to do this, we multiply Eq. (25b) by g and then divide by Eq. (25a) to obtain the ordinary differential equation (ODE)

$$\frac{dg^2}{d\lambda}=-\frac{1}{2}\left[\frac{g^4}{\frac{5\lambda^2}{48}-\lambda g^2-\frac{45}{2}g^4}\right]. \tag{29}$$

Then, by upon setting $g^2=\lambda z$, it is straightforward to solve for $z(\lambda)$. Unless $g=0$, the general solutions are

$$\lambda=\frac{g^2[(g^2)^{\sqrt{77/2}}-\kappa]}{z_+(g^2)^{\sqrt{77/2}}-\kappa z_-}\quad\text{for some}\quad\kappa\neq 0 \tag{30a}$$

and

$$g^2=\lambda z_{\pm}, \tag{30b}$$

where $z_{\pm}=(\pm\sqrt{77/2}-1)/90$. Notice that the generic solution (30a) appears to admit more than one phase; if $\kappa>0$, then λ varies smoothly with g and approaches zero in the ultraviolet limit. However, if $\kappa<0$, then λ can branch outside of the perturbative regime when the denominator of Eq. (30a) is sufficiently small. Whether this occurs will depend on the particular choice of renormalization conditions.

For the supersymmetric model, Eq. (26) implies that

$$\lambda=\frac{\lambda_0^2}{1-\frac{5}{12\pi^2}\lambda_0^2\ln\left(\frac{\mu}{\mu_0}\right)}\equiv\frac{12\pi^2/5}{\ln(\Lambda/\mu)}. \tag{31}$$

This model is not asymptotically free, in accordance with the general result of Ref. [8].

Finally, we consider the renormalization group equations for the two-component model in Eqs. (27). By Eqs. (27a) and (27b) we see that

$$\frac{dG^2}{dg^2}=\frac{3G^4-2g^2G^2}{3g^4-2g^2G^2}. \tag{32}$$

This is a homogeneous equation; upon setting $G^2=zg^2$ it can be solved easily to give either

$$g^2=G^2=\frac{4\pi^2}{\ln(\Lambda/\mu)} \tag{33a}$$

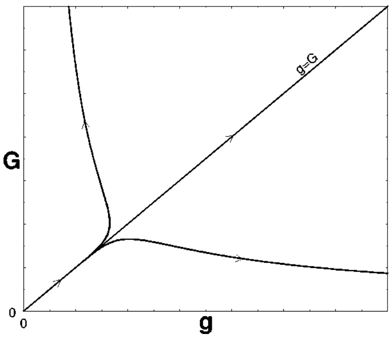

FIG. 4. Leading order flows for the Yukawa couplings in the two-component model of Eq. (7). (The arrows represent the direction of increasing energy scale.)

[i.e. we have the Yukawa couplings of Eq. (9)] or

$$g^6G^6 \propto |g^2-G^2|. \tag{33b}$$

Referring to Fig. 4, the graph of Eq. (33), we see that the Yukawa sector has three distinct phases. If $G=g$, then both couplings increase with scale according to Eq. (33a) with their equality preserved by the renormalization group flow. If $g\neq G$, then Eq. (33b) holds, and the smaller of the two couplings becomes asymptotically free while the larger evolves outside of the perturbative region with increasing momentum scale, in such a way that the combination G^4+g^4 always increases with μ. To see this we set $g^2=r\cos\theta$ and $G^2=r\sin\theta$ in Eq. (32) to obtain

$$\frac{5}{r}\frac{dr}{d\theta}=\left[\frac{3-5\sin\theta\cos\theta}{\sin\theta-\cos\theta}\right]\left[\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}\right]. \tag{34}$$

Another dominant feature of Fig. 4 occurs in the infrared region where the Yukawa couplings approach zero along the asymptote $g=G$. In fact, we have completed several numerical solutions to the full four-parameter renormalization group flow of Eqs. (27); all solutions indicate that coupling configuration $G=g$, $\lambda=5\sigma$ [which leads to the chirally symmetric model of Eqs. (9) and (10)] is asymptotically realized in the infrared domain.

In the special case when the Yukawa couplings g^2 and G^2 are weak, then, in the ultraviolet limit, the six-point couplings tend to branch outside of the perturbative region parallel to one of the two UV-stable lines $\lambda=\sigma$ or $\sigma=0$.

We can slightly alter the model of Eq. (7) so that there are $N_g\geqslant 1$ fermions coupling with strength g to A^2-B^2, and $N_G\geqslant 1$ distinct fermions coupling with strength G to AB. In this case the renormalization group equations of Eqs. (27a) and (27b) are replaced by

$$\mu\frac{\partial G}{\partial\mu}=\frac{G}{24\pi^2}[(4N_G+5)G^2-2N_g g^2] \tag{35a}$$

$$\mu\frac{\partial g}{\partial\mu}=\frac{g}{24\pi^2}[(4N_g+5)g^2-2N_G G^2]. \tag{35b}$$

Again we find that there are special solutions in which the couplings flow in the following straight lines away from the origin as the energy scale increases:

$$G=0 \quad\text{and}\quad g^2=\frac{12\pi^2}{(4N_g+5)\ln(\Lambda/\mu)}, \tag{36a}$$

$$g=0 \quad\text{and}\quad G^2=\frac{12\pi^2}{(4N_G+5)\ln(\Lambda/\mu)}, \tag{36b}$$

or

$$\frac{G^2(\mu)}{g^2(\mu)}=\frac{6N_G+5}{6N_g+5}. \tag{36c}$$

In addition there are generic solutions similar to the curves depicted in Fig. 4 in which either g or G is asymptotically free depending on whether the fraction G^2/g^2 is larger or smaller than its critical value in Eq. (36c).

IV. DISCUSSION

We have considered a number of Yukawa couplings in three dimensions as well as the associated radiatively induced six-point scalar couplings. The lowest order contributions to the renormalization group functions for the couplings constants in these models have been computed, complementing the work of Ref. [8], where a global SU(N) flavor symmetry is imposed on the interactions.

One curious feature of these results is that the Yukawa coupling is asymptotically free in the model defined in Euclidean space by Eq. (1), while the supersymmetric model in Minkowski space [with metric $g_{\mu\nu}=(++-)$] whose action is given in Eqs. (2) and (3) is not. This apparent discrepancy is a consequence of the different properties of the Dirac matrices γ^a in the two models. In Euclidean space the kinetic term $\bar{\psi}\not{p}\psi$ in Eq. (1) is Hermitian provided γ^a is identified with some unitary equivalent representation of the Pauli spin matrices σ^a (or $-\sigma^a$), in which case $\not{p}^2=p^2$. In the supersymmetric (Minkowski) model of Eqs. (2) and (3), the kinetic term for the spinor field is Hermitian provided γ^a is represented, for example, by Eq. (A6), so that $\not{p}^2=-p^2$. This point of distinction is sufficient to alter the ultraviolet behavior of the models, as the integrals associated with Figs. 1(c)–1(g) each contain spinor propagators. If, for instance, we were to compute the Feynman integrals associated with diagrams 1(a)–1(l) in the Minkowski space version of model (1), then it is an easy exercise to show that Eqs. (25a) and (25b) become

$$\mu\frac{\partial\lambda}{\partial\mu}=\frac{1}{\pi^2}\left(\frac{5\lambda^2}{48}+\lambda g^2-\frac{45}{2}g^4\right) \tag{37a}$$

$$\mu\frac{\partial g}{\partial\mu}=\frac{g^3}{4\pi^2}, \tag{37b}$$

clearly demonstrating that in Minkowski space the Yukawa coupling is not asymptotically free.

The two component model of Eq. (7) has the peculiar feature that unless $g=G$, one of the couplings becomes small while the other grows as the renormalization group scale increases. This differs from the analogous situation in four dimensions where none of the couplings involving scalars and spinors is ever asymptotically free. We are pursuing an analysis of this model along the lines of Ref. [9] to see how it behaves in Euclidean space.

The stability of these models is clearly a problem of interest; computation of the effective potential to two-loop order could shed some light on this problem. Stability would be maintained if the renormalization group improved effective potential is positive definite.

ACKNOWLEDGMENTS

We wish to thank NSERC for financial assistance. M. McInnis provided help and encouragement.

APPENDIX: CONVENTIONS

In this appendix the notation and conventions that we have been using are outlined. In the Euclidean model, we work in Euclidean space with $p=-i\partial$, and the Dirac gamma matrices are identified with the Pauli spin matrices so that $\gamma^a\gamma^b=\delta^{ab}+i\epsilon^{abc}\gamma^c$. This ensures that the kinetic term $\overline{\psi}=\psi^+$. Feynman integrals are computed using

$$\int \frac{d^nk}{(2\pi)^n}\frac{(k^2)^a}{(k^2+m^2)^b}=\frac{1}{(4\pi)^{n/2}}(m^2)^{n/2+a-b}\times\frac{\Gamma(n/2+a)\Gamma(b-a-n/2)}{\Gamma(n/2)\Gamma(b)}. \tag{A1}$$

In the supersymmetric model, the conventions of [10] are used. This means that an extra factor of (i) appears in Eq. (A1), as the metric is given by $g_{\mu\nu}=(++-)$. Indices are raised and lowered by the use of an antisymmetric tensor $C_{\alpha\beta}$, so that

$$\psi^\alpha=C^{\alpha\beta}\psi_\beta,\quad \psi_\alpha=\psi^\beta C_{\beta\alpha} \tag{A2}$$

where $C_{\alpha\beta}=-C^{\alpha\beta}=\left(\begin{smallmatrix}0&-i\\ i&0\end{smallmatrix}\right)$. Consequently, if $\delta_\alpha{}^\beta=\left(\begin{smallmatrix}1&0\\ 0&1\end{smallmatrix}\right)$, then $\delta^\alpha{}_\beta=-\delta_\alpha{}^\beta$. A rank-two spinor $V_{\alpha\beta}$ is identified with a vector [viz. Eq. (A5)] provided $V_{\alpha\beta}=V_{\beta\alpha}$ or, equivalently, $V_\alpha{}^\alpha=0$. Spinorial derivatives are defined by $\{\partial_\alpha,\theta^\beta\}=\{-ip_\alpha,\theta^\beta\}=\delta_\alpha{}^\beta$ and the covariant derivative $D_\alpha=\partial_\alpha+i\theta^\beta\partial_{\alpha\beta}$ satisfies

$$(D^2)^2=\Box$$

$$D^2\delta(\theta)|_{\theta=0}=1$$

$$\int d^3x\int d^2\theta[D^2f(\theta)]g(\theta)=\int d^3x\int d^2\theta f(\theta)[D^2g(\theta)] \tag{A3}$$

where $D^2=\frac{1}{2}D^\alpha D_\alpha=-\frac{1}{2}D_\alpha D^\alpha$. We also have

$$\partial^{\alpha\lambda}\partial_{\beta\lambda}=\delta_\beta{}^\alpha\Box. \tag{A4}$$

These conventions are all consistent with replacing a vector $V_\alpha{}^\beta$ by

$$V_\alpha{}^\beta=(\gamma^a)_\alpha{}^\beta V_a \tag{A5}$$

where V_a is the usual vector in $2+1$ dimensions and

$$(\gamma^a)_\alpha{}^\beta=\left[\begin{pmatrix}i&0\\0&-i\end{pmatrix},\begin{pmatrix}0&i\\i&0\end{pmatrix},\begin{pmatrix}0&i\\-i&0\end{pmatrix}\right]. \tag{A6}$$

With these conventions

$$(\gamma^a)_\alpha{}^\alpha=0,\quad (\gamma^a)_\alpha{}^\beta=-(\gamma^a)_\alpha{}^{\beta *}$$

and

$$(\gamma^a)_\alpha{}^\lambda(\gamma^b)_\lambda{}^\beta+(\gamma^b)_\alpha{}^\lambda(\gamma^a)_\lambda{}^\beta=-2g^{ab}\delta_\alpha{}^\beta\,. \tag{A7}$$

The wave equation for ψ_α is now consistent with ψ_α being real. The kinetic term $\psi^\alpha(i\partial_\alpha{}^\beta)\psi_\beta=\psi^\alpha(\not{p})_\alpha{}^\beta\psi_\beta$ is Hermitian.

In the two-component model (13), some useful identities for the matrices ρ_α in Eq. (13d) are

$$\begin{aligned}45(\rho_\alpha\otimes\rho_\alpha\otimes\mathbb{1})_{[bcaaaa]}={}&6(\rho_\alpha)_{bc}(\rho_\alpha)_{aa}\delta_{aa}\\&+12(\rho_\alpha)_{ba}(\rho_\alpha)_{ca}\delta_{aa}\\&+12(\rho_\alpha)_{ba}(\rho_\alpha)_{aa}\delta_{ca}\\&+12(\rho_\alpha)_{ca}(\rho_\alpha)_{aa}\delta_{ba}\\&+3(\rho_\alpha)_{aa}(\rho_\alpha)_{aa}\delta_{bc}\end{aligned}$$

and

$$\begin{aligned}45(\rho_\alpha\otimes\rho_\alpha\otimes\mathbb{1})_{[bbbaaa]}={}&9(\rho_\alpha)_{bb}(\rho_\alpha)_{ba}\delta_{aa}\\&+9(\rho_\alpha)_{bb}(\rho_\alpha)_{aa}\delta_{ba}\\&+18(\rho_\alpha)_{ba}(\rho_\alpha)_{ba}\delta_{ba}\\&+9(\rho_\alpha)_{ba}(\rho_\alpha)_{aa}\delta_{bb}\,.\end{aligned}$$

No summation is implied in these latter equations; square parentheses $[\cdots]$ indicate symmetrization of all enclosed indices, while repeated Roman indices *a* and *b* indicate symmetrization over specific indices (which are thus rendered indistinguishable). The coefficient 45 is shown on the left hand sides to elucidate the 45 distinguishable permutations of the indices of $\rho_\alpha\otimes\rho_\alpha\otimes\mathbb{1}$ given that both ρ_α and $\mathbb{1}$ are already symmetric.

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