Finiteness conditions for light-front Hamiltonians

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In the context of simple models, it is shown that demanding finiteness for physical masses with respect to a longitudinal cutoff can be used to fix the ambiguity in the renormalization of fermions masses in the Hamiltonian light-front formulation. Difficulties that arise in applications of finiteness conditions to discrete light-cone quantization are discussed. [S0556-2821(98)01302-2]

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I. INTRODUCTION

Many advantages of the light-front (LF) formulation for bound state problems arise from the manifest boost invariance in the longitudinal direction [1–5]. The price for this advantage is that other symmetries, such as parity or rotational invariance (for rotations around a transverse axis) are no longer manifest [6,7]. From the technical point of view, the loss of manifest parity and full rotational invariance implies that LF Hamiltonians allow for a richer set of counterterms in the renormalization procedure, i.e. in general LF Hamiltonians contain more parameters than the underlying Lagrangian.

Of course, even though parity and full rotational invariance are not manifest symmetries in the LF formulation, a consistent calculation should still give rise to physical observables which are consistent with these symmetries. In Ref. [7] this fact has been used to determine one of these additional parameters by imposing parity covariance on the vector form factor of mesons. While such a procedure is practical, it is nevertheless desirable to have alternative procedures available for determining these ''additional'' parameters in the Hamiltonian. In this paper, finiteness conditions are exploited to develop algorithms for determining seemingly independent parameters in LF Hamiltonians.

As a specific example, let us consider a Yukawa model in 1+1 dimensions:

$$\mathcal{L}=\overline{\psi}(i\not\partial-m-g\phi)\psi-\frac{1}{2}\phi(\Box+\lambda^2)\phi. \tag{1.1}$$

In order to simplify the analysis further, we will in the following consider the Yukawa model in a planar approximation (formally this can easily be achieved by introducing ''color'' degrees of freedom) and by assuming an infinite number of ''colors.'' However, while a planar approximation will always be implicitly used, explicit color degrees of freedom will not be shown in order to keep the notation simple.

The main difference between scalar and Dirac fields in the LF formulation is that not all components of the Dirac field are dynamical: multiplying the Dirac equation

$$(i\not\partial-m-g\phi)\psi=0 \tag{1.2}$$

by γ^+ yields a constraint equation (i.e. an ''equation of motion'' without a time derivative)

$$2i\partial_-\psi_-=(m+g\phi)\gamma^+\psi_+, \tag{1.3}$$

where

$$\psi_\pm\equiv\frac{1}{2}\gamma^\mp\gamma^\pm\psi. \tag{1.4}$$

For the quantization procedure, it is convenient to eliminate ψ_- from the classical Lagrangian before imposing quantization conditions, yielding

$$\mathcal{L}=\sqrt{2}\psi_+^\dagger i\partial_+\psi_+-\frac{1}{2}\phi(\Box+\lambda^2)\phi-\psi_+^\dagger\frac{m_{kin}^2}{\sqrt{2}i\partial_-}\psi_+ -\psi_+^\dagger\left(\phi\frac{gm_V}{\sqrt{2}i\partial_-}+\frac{gm_V}{\sqrt{2}i\partial_-}\phi\right)\psi_+-\psi_+^\dagger\phi\frac{g^2}{\sqrt{2}i\partial_-}\phi\psi_+. \tag{1.5}$$

In anticipation of the results below, we have already introduced in Eq. (1.5) the so-called kinetic and vertex mass of the fermion (m_{kin} and m_V). Of course, in the canonical Lagrangian one has $m_{kin}=m_V=m$.

The rest of the quantization procedure very much resembles the procedure for self-interacting scalar fields. In particular, we must be careful about generalized tadpoles, which might cause additional counterterms in the LF Hamiltonian [8]. In the Yukawa model one usually (i.e. in a covariant formulation) does not think about tadpoles. However, after eliminating ψ_-, one is left with a four-point interaction in the Lagrangian, which does give rise to time-ordered diagrams that resemble tadpole diagrams. In fact, the four-point interaction gives rise to diagrams where a fermion emits a boson, which may or may not self-interact, and then reabsorb the boson at the same LF-time.[1] Such interactions cannot be generated by a LF Hamiltonian, i.e. the LF formalism generally defines such tadpoles to be zero. An exception are the so-called self-induced inertias, which arise from normal ordering the LF Hamiltonian. These terms, which are $\mathcal{O}(g^2)$, are usually kept.

[1]There are also tadpoles, where the fermions get contracted. But those only give rise to an additional boson mass counterterm, but not to the non-covariant fermion mass counterterm that is investigated here.

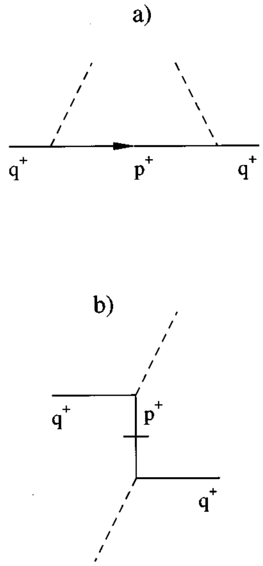

FIG. 1. $\mathcal{O}(g^2)$ contributions to the forward Compton amplitude. (a) Intermediate fermion line on mass shell. (b) Instantaneous fermion interaction contribution (denoted by a slashed line).

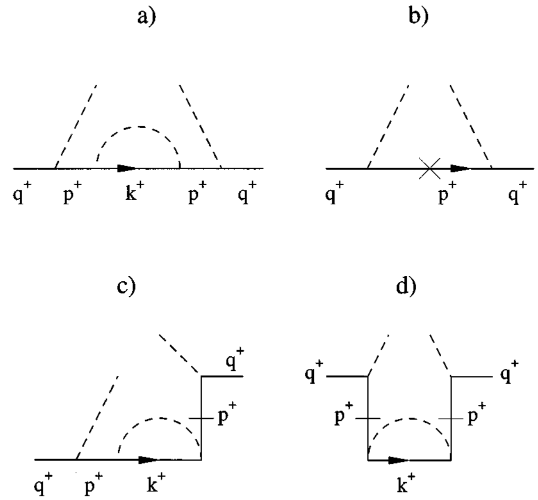

FIG. 2. $\mathcal{O}(g^4)$ contributions to the forward Compton amplitude. (a) All fermion lines on mass shell. (b) same as (a), but the loop replaced by the self-induced inertia. (c) One of the two diagrams with an instantaneous fermion interaction (denoted by a slashed line) adjacent to the self-energy insertion. (d) Both fermion propagators adjacent to the loop instantaneous.

II. PERTURBATIVE COUNTER-TERM ANALYSIS

At tree level, i.e. at order g^0, the kinetic mass and the vertex mass have to be the same. In order to see this, let us consider the two $\mathcal{O}(g^2)$ Compton scattering diagrams in Fig. 1. For simplicity we consider only forward scattering and we consider only diagrams which are singular.

The amplitude with an on-shell fermion intermediate state diverges as the p^+ momentum of its intermediate fermion line goes to zero

$$T_o=\frac{g^2}{q^+-p^+}\frac{\left(\frac{m_V}{q^+}+\frac{m_V}{p^+}\right)^2}{q^- - \frac{m_{kin}^2}{p^+}-\frac{\lambda^2}{q^+}} \tag{2.1}$$

(the subscript o stands for on-shell). This divergence is canceled exactly by the amplitude with an instantaneous fermion line (the subscript i stands for ''instantaneous'')

$$T_i=\frac{g^2}{q^+-p^+}\frac{1}{p^+} \tag{2.2}$$

if and only if $m_{kin}=m_V=m$. Note that this cancellation occurs if and only if the mass in the numerator (the ''vertex mass'') and the mass in the denominator (the ''kinetic mass'') are the same in Eq. (2.1). This is also the only choice of parameters that is consistent with parity invariance for Compton scattering at $\mathcal{O}(g^2)$.

Choosing the vertex mass equal to the kinetic mass is also crucial for a cancellation between the (momentum dependent) self-induced inertia (kinetic mass) counterterm [10]

$$\Delta m_{kin}^2=\frac{g^2}{4\pi}\int_0^{p^+}\frac{dk^+}{k^+} \tag{2.3}$$

and the divergent piece of the $\mathcal{O}(g^2)$ self-energy

$$\Delta^{(2)}p^-=\frac{g^2}{4\pi}\int_0^{p^+}\frac{dk^+}{p^+-k^+}\frac{\left(\frac{m_V}{p^+}+\frac{m_V}{k^+}\right)^2}{p^- - \frac{m_{kin}^2}{k^+}-\frac{\lambda^2}{p^+-k^+}}. \tag{2.4}$$

This well-known result has recently also been obtained using so-called ladder relations [11], by investigating divergences in the non-perturbative coupled Fock space equations for bound states.

While the self-induced inertia certainly cancels the divergent part of the $\mathcal{O}(g^2)$ self-energy, it has been questioned whether it also contains the correct finite part. In fact, in Ref. [7], parity invariance for physical observables has been used to determine the finite piece of the kinetic mass counterterm non-perturbatively.

However, the above analysis shows that the cancellation of divergences may also be used to determine the finite piece: if the tree level cancellation between instantaneous and on-shell amplitudes is spoiled by a wrong choice for the kinetic mass then higher order diagrams will contain a divergence of integrals over longitudinal momenta as a result of the incomplete cancellation. The question is—and this will be the subject of the rest of this paper—whether such ''finiteness conditions'' also arise at higher orders in the coupling constants and whether they can be used to determine the finite part of the kinetic mass counterterm.

For this purpose, let us consider the one-loop [$\mathcal{O}(g^4)$] corrections to the Compton amplitude. We will first assume that $m_V=m_{kin}=m$ and add corrections to m_{kin} later perturbatively. Again we restrict ourselves to planar diagrams. Since we are interested only in corrections to the $p^+\to 0$ singular contributions, it is also sufficient to consider only loop corrections to the fermion line which propagates between the two vertices. In LF-perturbation theory, we thus have to consider the four diagrams in Fig. 2.

Figures 2(a) and (b) together are finite (for finite p^+) and contribute

$$T_{oo}=\frac{g^4\left(\frac{m}{q^+}+\frac{m}{p^+}\right)^2}{4\pi(q^+-p^+)D_1^2}\int_0^{p^+}dk^+\left[\frac{\left(\frac{m}{p^+}+\frac{m}{k^+}\right)^2}{(p^+-k^+)D_2}+\frac{1}{k^+}\right], \tag{2.5}$$

where

$$D_1=q^- - \frac{m^2}{p^+}-\frac{\lambda^2}{q^+-p^+}$$

$$D_2=p^- - \frac{m^2}{k^+}-\frac{\lambda^2}{p^+-k^+} \tag{2.6}$$

(with $p^-\equiv q^- -(\lambda^2/q^+-p^+)$) are the energy denominators for the intermediate states. The diagrams with one or two instantaneous lines are finite without counterterms (for finite p^+) and yield, respectively,

$$T_{oi}=\frac{2g^4\left(\frac{m}{q^+}+\frac{m}{p^+}\right)}{4\pi(q^+-p^+)p^+D_1}\int_0^{p^+}dk^+\frac{\left(\frac{m}{p^+}+\frac{m}{k^+}\right)}{(p^+-k^+)D_2}$$

$$T_{ii}=\frac{g^4}{(q^+-p^+)p^{+2}}\int_0^{p^+}dk^+\frac{1}{4\pi(p^+-k^+)D_2}. \tag{2.7}$$

All three amplitudes diverge like $1/p^+$ as $p^+\to 0$. One finds

$$\lim_{p^+\to 0} p^+T_{oo}=\frac{g^4}{4\pi q^+}\left[\frac{1}{m^2}\ln\frac{\lambda^2}{m^2}-\int_0^1 dx\frac{2+x}{m^2(1-x)+\lambda^2 x}\right]$$

$$\lim_{p^+\to 0} p^+T_{oi}=\frac{g^4}{4\pi q^+}\int_0^1 dx\frac{2+2x}{m^2(1-x)+\lambda^2 x}$$

$$\lim_{p^+\to 0} p^+T_{ii}=-\frac{g^4}{4\pi q^+}\int_0^1 dx\frac{x}{m^2(1-x)+\lambda^2 x}. \tag{2.8}$$

The divergence at small p^+ does *not* cancel when one sums up the three terms.[2] In fact, what one finds is

$$\lim_{p^+\to 0} p^+(T_{oo}+T_{oi}+T_{ii})=\frac{g^4}{4\pi m^2 q^+}\ln\frac{\lambda^2}{m^2}. \tag{2.9}$$

Since there are no diagrams other than the ones listed in Fig. 2 which are singular at $\mathcal{O}(g^4)$, this implies that there is a problem: The $\mathcal{O}(g^4)$ self-energy of a fermion (Fig. 3) is obtained by integrating the $\mathcal{O}(g^4)$ forward Compton amplitude over p^+ and one obtains a logarithmic divergence. This divergence should not be there since the $(1+1)$-dimensional Yukawa model is super-renormalizable. Already in perturbation theory, the Yukawa model on the LF with only the self-induced inertias added as counterterms does not lead to finite answers.

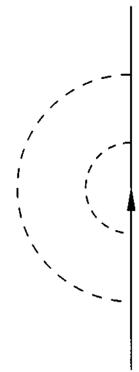

FIG. 3. $\mathcal{O}(g^4)$ contributions to the fermion self-energy, which is sensitive to the small p^+ behavior of the $\mathcal{O}(g^2)$ fermion self-energy.

Surprisingly, the resolution to this problem does *not* require one to add another infinite counterterm. In Ref. [7] a finite kinetic mass counterterm (in addition to the infinite self-induced inertias) was introduced and it was found to be necessary in order to obtain parity invariant form-factors. The effect of a $\mathcal{O}(g^2)$ kinetic mass counterterm is an additional $\mathcal{O}(g^4)$ term in the forward Compton amplitude [obtained by expanding Eq. (2.1)]:

$$T_{\Delta m^2}=\frac{g^2\left(\frac{m}{q^+}+\frac{m}{p^+}\right)^2}{(q^+-p^+)D_1^2}\frac{\Delta m_{kin}^2}{p^+}. \tag{2.10}$$

It can easily be verified that the choice

$$\Delta m_{kin}^2=\frac{g^2}{4\pi}\ln\frac{m^2}{\lambda^2} \tag{2.11}$$

leads to

$$\lim_{p^+\to 0} p^+(T_{oo}+T_{oi}+T_{ii}+T_{\Delta m^2})=0 \tag{2.12}$$

and hence the $\mathcal{O}(g^4)$ self-energy of a fermion is finite with this (and only this) particular choice for the kinetic mass counterterm. Note that exactly the same values for the $\mathcal{O}(g^2)$ kinetic mass counterterm also lead to parity invariant scattering amplitudes.

Note that while the calculations presented above had been done for a scalar Yukawa theory, very similar results hold for models with similar interactions, such as pseudoscalar Yukawa theory or fermions coupled to the $\perp$ component of a vector field.

III. A NON-PERTURBATIVE EXAMPLE

For a non-perturbative example, let us consider the model introduced in Ref. [9].[3] Of course, the perturbative results from Sec. II prove that for non-perturbative finiteness it will be necessary to introduce independent vertex and kinetic

[2] An exception is the ''supersymmetric'' case $m^2=\lambda^2$.

[3] For more details the reader is referred to this paper.

masses, but it is not obvious whether this will also be sufficient. Furthermore, it is not immediately clear how the arguments generalize to $3+1$ dimensions. The model from Ref. [10] not only gives an example that the vertex/kinetic mass renormalization is also sufficient but also gives a 3+1 dimensional example at the same time. This proves that the results from Sec. II, which had been kept simple in order to be easily comprehensible, are rather general and apply to much more general conditions. Even though the model has already been solved in Ref. [9], what is new here is the demonstration that demanding finiteness yields the same solution as the one obtained in Ref. [9] by comparison with the Schwinger-Dyson equations.

The model describes fermions in 3+1 space-time dimensions coupled to the $\perp$ components of a massive vector field in planar approximation:

$$\mathcal{L}=\bar{\psi}\left(i\not{\partial}-m-\frac{g}{\sqrt{N_C}}\vec{\gamma}_\perp\vec{A}_\perp\right)\psi-\frac{1}{2}\operatorname{tr}(\vec{A}_\perp\Box\vec{A}_\perp+\lambda^2\vec{A}_\perp^2). \tag{3.1}$$

With ''$\perp$ component'' we mean here the x and y components. Furthermore we will impose a transverse momentum cutoff on the fields and we will consider the model at fixed $\perp$ momentum cutoff. With a fixed $\perp$ momentum cutoff in place, the model becomes super-renormalizable as far as longitudinal divergences are concerned.

In the large N_C limit, the model described by the above Lagrangian (3.1) can be solved in the rainbow approximation. The non-perturbative Green's function for a fermion in this model can be written in the form

$$G(p^\mu)=\gamma^+p^-G_+(2p^+p^-,\vec{p}_\perp^2)+\gamma^-p^+G_-(2p^+p^-,\vec{p}_\perp^2)$$
$$+\not{k}_\perp G_\perp(2p^+p^-,\vec{p}_\perp^2)+G_0(2p^+p^-,\vec{p}_\perp^2), \tag{3.2}$$

where each of the G_i has a spectral representation

$$G_i(2p^+p^-,\vec{p}_\perp^2)=\int_0^\infty dM^2\,\frac{\rho_i^{LF}(M^2,\vec{p}_\perp^2)}{2p^+p^--M^2+i\varepsilon}. \tag{3.3}$$

Note that $\operatorname{tr}(\gamma^-G)$ cannot contain a term proportional to $(1/p^+)\int dM^2\rho(M^2,p_\perp^2)/(2p^+p^--M^2+i\varepsilon)$ because this would lead to logarithmic small p^+ divergences, which are not canceled by the self-induced inertias, when the Green's function is used to calculate the self-energy self-consistently (see below).

From the fermion Green's function, one computes the self-energy self-consistently via

$$G^{-1}=\not{p}-m-\Sigma, \tag{3.4}$$

where

$$\Sigma=\gamma^+\Sigma_++\gamma^-\Sigma_-+\Sigma_0. \tag{3.5}$$

For the LF components of the self-energy one finds

$$\Sigma_i=g^2\int_0^\infty dM^2\int_0^{p^+}\frac{dk^+}{k^+}\int\frac{d^2k_\perp}{16\pi^3}f_i, \tag{3.6}$$

where

$$f_+=\left[\frac{\tilde{k}^-}{D(p^+-k^+)}-1\right]\rho_+(M^2,\vec{k}_\perp^2)$$
$$f_-=\frac{k^+\rho_-(M^2,\vec{k}_\perp^2)}{D(p^+-k^+)}$$
$$f_0=-\frac{\rho_0(M^2,\vec{k}_\perp^2)}{D(p^+-k^+)} \tag{3.7}$$

and

$$D=p^--\frac{M^2}{2k^+}-\frac{\lambda^2+(\vec{p}_\perp-\vec{k}_\perp)^2}{2(p^+-k^+)}$$
$$\tilde{k}^-=p^--\frac{\lambda^2+(\vec{p}_\perp-\vec{k}_\perp)^2}{2(p^+-k^+)}. \tag{3.8}$$

Because of the large N_C limit, such a truncation of the LF Schwinger-Dyson equations is exact. The non-perturbative solution is then obtained by solving Eqs. (3.4) and (3.6) iteratively until self-consistency is achieved.

In order to be able to investigate whether the self-induced inertias cancel the infinite part of the self-energy one needs to know the small p^+ behavior of G and thus the small p^+ behavior of Σ.

As an example, let us suppose that $\Sigma=c\gamma^+/p^+$, in which case

$$G=\frac{1}{\not{p}-m-\Sigma}\xrightarrow{p^+\to 0}\frac{c}{m^2+2c}\frac{\gamma^+}{p^+}, \tag{3.9}$$

which diverges for $p^+\to 0$, while the propagator for $c=0$ remains finite in this limit. The self induced inertias cancel the infinite part of the self-energy in the case where the fermion propagator inside the loop is a free propagator. If one desires that the same cancellation occurs with the full propagator, it is necessary that the self-energy which modifies the propagator remains finite as $p^+\to 0$.

We shall now investigate the consequences of this fact for the model described by Eq. (3.1). In particular, we shall focus on the γ^+ component of Σ, which is the most singular term as $p^+\to 0$. Including only the self-induced inertia counterterm, one finds [9]

$$\Sigma_+=g^2p^-\int_0^{p^+}\frac{dk^+}{p^+}\int\frac{d^2k_\perp}{8\pi^3}G_+(2k^+\tilde{k}^-,\vec{k}_\perp^2)$$
$$-\frac{g^2}{2p^+}\int_0^\infty dM^2\int\frac{d^2k_\perp}{8\pi^3}\rho_+(M^2,\vec{k}_\perp^2)\ln\frac{M^2}{\lambda^2+\vec{k}_\perp^2}. \tag{3.10}$$

The first term on the right-hand-side (RHS) of Eq. (3.10) is finite as $p^+\to 0$, but the second term diverges in this limit like const/p^+.[4] When one inserts Eq. (3.10) into Eq. (3.4),

[4]This second term is the non-perturbative analog of the $\ln m^2/\lambda^2$ term (2.11), which was necessary to render the two-loop self-energy finite.

one finds that G itself diverges as $p^+\rightarrow 0$ [as illustrated by Eq. (3.9)]. This in turn leads to a divergence in Eq. (3.10) when one tries to calculate G in a self-consistent procedure. The only way to avoid this dilemma is to add a kinetic mass counterterm

$$\Delta m_{kin}^2 = g^2\int_0^\infty dM^2\int \frac{d^2k_\perp}{8\pi^3}\rho_+(M^2,\vec{k}_\perp^2)\ln\frac{M^2}{\lambda^2+\vec{k}_\perp^2} \tag{3.11}$$

which exactly cancels the second term in Eq. (3.10). It turns out that the kinetic mass counterterm which we have thus obtained by demanding finiteness of the longitudinal momentum integrals agrees exactly with the one obtained in Ref. [9] by comparison with the Schwinger-Dyson solution for the same model.

In summary, what we have found is that the two-loop result generalizes directly to all orders in this non-perturbative example. In fact, as long as one works with a fixed $\perp$ momentum cutoff, one can show that the result generalizes to an entire class of models with Yukawa (scalar and pseudo-scalar) interactions as well as models with couplings to transverse components of vector fields. However, I was not able to show that the result generalizes to all orders to models with couplings to longitudinal components of a vector field (i.e. gauge theories). Semi-perturbative considerations suggest that the results also apply to dimensionally reduced models for QCD [11] as well as $\perp$ lattice QCD, but I could not find a general proof (beyond perturbative calculations).

When discussing the issue of finiteness, it is very important to discuss the cutoff scheme dependence. So far, we have purposely avoided specifying a cutoff procedure—which one always has to do when dealing with divergent (or potentially divergent) quantities. The reason we did not have to specify the cutoff procedure is that the one-loop divergence is canceled locally (the singularities of the integrand cancel) by the self-induced inertia and higher order divergences are also canceled locally by the finite kinetic mass counter terms. However, we still assumed implicitly that the result for the inner loop was (apart from trivial kinematical factors) momentum independent—otherwise it would not have been sufficient to merely add a number (not a function) as a counterterm.

It is easily possible to introduce cutoffs which have this property, for example a $\perp$ momentum cutoff combined with a boost invariant longitudinal momentum cutoff, such as an invariant mass difference cutoff at each 3-point vertex (and a cutoff for the instantaneous fermion exchange diagrams which is consistent with cutoffs on iterated 3-point vertices), or even more simply just a cutoff on momentum fractions at each vertex.

One of the most popular cutoffs used in non-perturbative LF-calculations is discrete light-core quantization (DLCQ) [10], where all momenta are discretized and thus a cutoff on the longitudinal momenta is provided by the spacing of the grid in momentum space. With such a cutoff procedure the self energy of a fermion does depend on its momentum (beyond the trivial $1/p^+$ dependence). This point will be elaborated in Sec. V. However, before we discuss numerical implications in DLCQ, let us first consider finiteness relations derived by using perturbative relations between Fock space components in non-perturbative bound state problems.

IV. FINITENESS CONDITIONS AND LADDER RELATIONS

In bound state problems it is often possible to relate Fock space components which are highly off energy shell to lower Fock components using perturbation theory. This fact has been used within a dimensionally reduced model for QCD in Ref. [11] to relate the end-point behavior of Fock space amplitudes with n+1 quanta to Fock space amplitudes with n quanta, using tree level ladder relations, yielding[5]

$$\psi_{n+1}(x_1,x_2,\ldots,x_{n-1},0)\propto\frac{1}{m\sqrt{x_{n-1}}}\psi_n(x_1,x_2,\ldots,x_{n-1}). \tag{4.1}$$

Note that Eq. (4.1) shows that wave functions in higher Fock components do not vanish near the end-point (i.e. for vanishing fermion momenta), which leads to divergent matrix elements of the kinetic energy as well as the interaction. The divergence that arises when only the fermion momentum goes to zero is canceled exactly by the self-induced inertias [Eq. (2.3)] if and only if the vertex mass m_V and the kinetic mass m_{kin} are the same.

In Ref. [11] it is thus claimed that the bound state equation (with $m_V = m_{kin}$) is finite. This claim is false: the Hamiltonian studied in Ref. [11] is in general not finite. The point is that both Eq. (4.1) as well as the cancellation conditions need to be modified when two momenta go to zero simultaneously. The best way to see that without going into too much detail is to consider the matrix element which connects states which differ by one boson. Such a matrix element involves the inverse of the momenta of both the incoming and outgoing fermion. If only the outgoing momentum goes to zero, then the term with the inverse of the momentum of the incoming fermion can obviously be neglected. However, this is not the case if both incoming and outgoing momentum go to zero simultaneously.[6] Since the vanishing of both incoming and outgoing fermion momenta also implies that the momentum of the emitted boson also vanishes, one can therefore conclude that the end-point behavior gets modified if the momenta of both the fermion and a boson vanish simultaneously.

Furthermore, the cancellation conditions also get modified when proper care is taken for the case where several momenta vanish simultaneously. In particular, in order for the Hamiltonian to give finite results one does in general need to keep $m_V \neq m_{kin}$.

The two-loop example considered above can be considered a formal proof (by counterexample) for these intuitively obvious facts.

In Ref. [11], numerical evidence is offered for the finiteness claim made in the same paper. Below, in Sec. V, it will

[5]No distinction between vertex and kinetic masses has been made in Ref. [11].

[6]After this paper was submitted, improved ladder relations were introduced, which seem to resolve this problem [12].

be demonstrated that the (logarithmic) divergence arising from the two-loop diagram shows up only for very large values of the DLCQ parameter K. This is probably the main reason why the divergence did not show up in the numerical results presented in Ref. [11].

V. FINITENESS CONDITIONS IN DLCQ

It is very easy to see that discretization in momentum space leads to a momentum dependent self-mass. Compared to a continuum calculation, integrals are approximated by sums and the number of points over which the summation is performed is determined by the total momentum. In this section, we will investigate the implications of this obvious fact for finiteness conditions.

In order to simplify the discussion, let us consider a cutoff which is very similar to the DLCQ cutoff, namely a sharp momentum cutoff (in the continuum) on all momenta that are smaller than an arbitrary constant ε.

The point is that since the cutoff acts both on the boson and on the fermion line, self-energy corrections to the $\mathcal{O}(g^4)$ Compton amplitude are absent for $p^+ < 2\varepsilon$ and they are suppressed for p^+ near that value. On the other hand, a (momentum independent) kinetic mass counterterm would contribute all the way down to the cutoff, namely $p^+ = \varepsilon$. For the self-energy this implies that there is an incomplete cancellation between terms that would cancel if the cutoff on the inner loop would be sent to zero *before* the outer loop integration is performed.

In order to illustrate what consequences this might have, let us consider a simple mathematical model which has the right qualitative features: let us assume that the sum of amplitudes in Fig. 2 in the presence of a cutoff is given by

$$p^+ T = \frac{c}{q^+}\Theta(p^+ - 2\varepsilon). \tag{5.1}$$

Including a kinetic mass counterterm Δm^2_{kin}, the two loop self-energy is then given by

$$\Delta^{(4)} q^- \propto \int_\varepsilon^{q^+} \frac{dp^+}{p^+}[c\Theta(p^+ - 2\varepsilon) - \Delta m^2_{kin}]. \tag{5.2}$$

Despite the fact that the integral over the self-energy piece starts at $p^+ = 2\varepsilon$, while the integral over the mass counterterm contribution starts at $p^+ = \varepsilon$, the unique choice for Δm^2_{kin} which yields a finite two loop self-energy as $\varepsilon \to 0$ is $\Delta m^2_{kin} = c$. And the result of the integral in this case is $-c \ln 2$ (independent of ε). Had we taken the limit $\varepsilon \to 0$ in the integrand, then the integrand would identically vanish and the integral would be zero. In other words, the finiteness condition would have given us the correct value for the kinetic mass counterterm at $\mathcal{O}(g^2)$, but the wrong result for the physical mass at $\mathcal{O}(g^4)$.

In order to demonstrate that this problem does indeed occur in DLCQ, let us consider a concrete problem, namely the $\mathcal{O}(g^4)$ self-mass $\Delta M^2 \equiv q^+ \delta^{(4)} q^-$ resulting from the rainbow diagram (Fig. 3). Even though we know the correct kinetic mass counterterm for this case from Eq. (2.11), let us pretend here that we do not know it and let us consider the

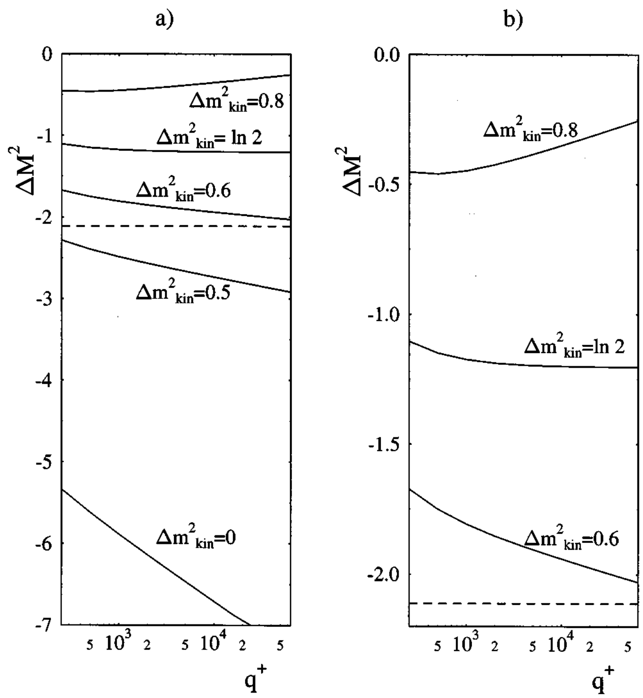

FIG. 4. (a) Two loop self energy of a fermion calculated using DLCQ (with anti-periodic boundary conditions for the fermion) as a function of the momentum of the fermion. The different curves represent different kinetic mass counterterms Δm^2_{kin}. (b) Same as (a) but a smaller interval of the y-axis is shown to simplify determination of the kinetic mass counterterm for which the result is stable for large momenta. The covariant result is indicated as a dashed line. Note that, while $\Delta m^2_{kin} = \ln 2$ leads to a convergent result, it does not converge towards the covariant answer.

two loop self-energy both as a function of the momentum q^+ (in discrete units) and the kinetic mass counterterm. The coupling constant is set to $g = \sqrt{4\pi}$, and for the masses we choose $\lambda^2 = 1$ and $m^2 = 2$. Figure 4 shows $4\pi q^+$ times the self-energy (including the kinetic mass counterterm) of the fermion as a function of q^+ for different values of the parameter Δm^2_{kin}. There are several things one can learn from this calculation.

First of all, Fig. 4 clearly shows that a kinetic mass counterterm Δm^2_{kin} (in addition to the self-induced inertia) is necessary in order to obtain finite results: the two-loop result for ΔM^2 obviously diverges when one sets $\Delta m^2_{kin} = 0$.

Secondly, the procedure is not very sensitive since the divergence is only logarithmic and the coefficient of the divergent piece is not very large. In order to obtain a precise picture about which value for the kinetic mass parameter leads to a convergent piece one has to go to values of $q^+ > 1000$, which is forbiddingly large for a non-perturbative calculation, but a reasonable estimate can already be obtained at lower values.

Thirdly, the finiteness condition does give the correct value for the kinetic mass counterterm. Only for $\delta m^2_{kin} \approx \ln 2$ (for $m^2 = 2$ and $\lambda^2 = 1$) one finds no noticeable q^+ dependence of the self-energy for large q^+. Even small deviations lead to a $\log q^+$ divergence proportional to that deviation.

Finally, and this is very important, despite the fact that the finiteness condition yields the correct value for δm_{kin}^2, the final result of the $\mathcal{O}(g^4)$ differs from the covariant result: For $\Delta m_{kin}^2 \approx \ln 2$ one finds $\lim_{q^+\to\infty} \Delta M^2 \approx -1.204$, while the correct (covariant) result for the two loop diagram (Fig. 3) is given by $\Delta M^2 \approx -2.112$ for the same masses and couplings. As we discussed above, this is because in DLCQ the momentum of a line that enters a sub-loop is not necessarily high above the cutoff inside that sub-loop. Therefore, the sensitivity to the cutoff never goes away—not even when the overall momentum is sent to infinity. Another way to look at this result is to conclude that in DLCQ one cannot introduce just one kinetic mass counterterm, but instead one needs to introduce a kinetic mass which depends on the momentum. Formally, this should not come as a surprise, since the boost invariance (which is normally manifest in LF quantization) is broken by the DLCQ regulator. However, in a number of examples, such as $1+1$ dimensional QED/QCD and theories with only self-interacting scalar fields, momentum dependent counterterms are not necessary and DLCQ workers have become accustomed to assume momentum independence of all counterterms as a starting point. Unfortunately, the Yukawa model that we have considered here is a clear counterexample to this simplified picture.

Of course, for a perturbative diagram one can always calculate the proper momentum dependence, but this seems impossible to do analytically in a non-perturbative context. An alternative procedure is the one employed in Refs. [7], where a momentum dependent kinetic mass is introduced such that the physical mass of the lightest states is independent of the momentum. The physical mass then replaces the bare kinetic mass as a renormalization parameter. In Refs. [7] the new parameters were determined by imposing parity invariance on physical amplitudes or by comparison with a covariant calculation. However, it is not obvious how to translate the finiteness condition for kinetic masses into a condition for the physical masses.

The fact that a simple (i.e. momentum independent) kinetic mass counterterm yields incorrect results also means that the ansatz for the LF Hamiltonian in theories with fermions and Yukawa type interactions (this includes QED/QCD) used by DLCQ workers (see for example Refs. [10, 11]) is insufficient.

There are several obvious patches that one can apply to the DLCQ calculations, but they all seem to have one feature in common: one needs to introduce another cutoff—beyond DLCQ—which has the feature that it gives momentum independent results. Typical examples are a Pauli-Villars regulator [6,13] or a cutoff on the invariant energy transfer. Of course, even with a cutoff that gives momentum independent results, one still needs to keep the kinetic mass as an ‘‘independent parameter,’’ [7] which then needs to be determined using for example parity or finiteness conditions, but at least one does not have to introduce a kinetic mass which is a function of the momentum. It is not clear whether adding an $\mathcal{O}(g^4)$ kinetic mass counterterm to correct for the artifacts introduced by the DLCQ cutoff leads to a consistent procedure at $\mathcal{O}(g^6)$ or higher.

VI. SUMMARY

We have investigated the conditions under which light-front Hamiltonians with fermions interacting via Yukawa type interactions (including interactions to the transverse component of a vector field) lead to convergent loop integrals at small values of the LF momentum $p^+ \equiv p^0 + p^3$. In the continuum, it was found that it is both necessary and sufficient to add a kinetic mass counterterm (in addition to the self-induced inertias) to the Hamiltonian in order to obtain finite results with respect to the small p^+ cutoff for higher order diagrams. Counterterm functions were not necessary. The additional parameter (i.e. the kinetic mass counterterm) is determined by demanding finiteness for the p^+ integrals. Imposing such a finiteness condition makes sense, since the small p^+ divergence is an artifact of the LF approach. It turns out that the kinetic mass counterterm thus obtained is identical to the one determined by imposing parity invariance for physical observables.

Unfortunately, there are several obstacles before one can apply this ‘‘finiteness condition’’ in practical calculations—particularly in DLCQ. One obstacle is that the divergences that one needs to look for are only logarithmic, which makes them hard to detect numerically. Furthermore, the situation in DLCQ is not quite as simple as it is in the continuum. DLCQ breaks manifest boost invariance, and we have in fact shown that a simple ansatz, where the kinetic mass counterterm is *not* a function of the momentum, is inconsistent in DLCQ already in perturbation theory in a super-renormalizable model. However, it is conceivable that a DLCQ calculation with additional cutoffs (such that momentum independence of the results is achieved) can be based on Hamiltonians with momentum independent mass counterterms. These counterterms can then, at least in principle, be determined using the finiteness condition that was derived in this paper.

More important from the practical point of view is the finding that there exist examples where finiteness without counterterm functions can be achieved. However, in order to obtain such a result it seemed necessary to keep longitudinal and transverse cutoffs separate and to keep the $\perp$ cutoff fixed while taking the longitudinal continuum limit. An example for such a regularization is given by the $\perp$ lattice formulation for QCD [14], where the transverse spatial coordinate is discretized and the longitudinal (x^-) direction is kept continuous. This formulation has the additional advantage that $p^+ \to 0$ singularities arising from the longitudinal components of the gauge fields cancel manifestly and therefore do not lead to further divergences.

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