# Lattice asymmetry in finite temperature gluodynamics

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The role of the lattice asymmetry parameter $\xi$ in the phase structure description of SU(2) and SU(3) gluodynamics at finite temperature in the SU($N$)$\simeq$$Z$($N$) approach has been studied analytically. The properties of thermodynamic quantities have been investigated near the physical border. The effective action which includes the first nontrivial order from the spacelike part allows estimates to be made of the phase structure not only close to the physical border but in the whole area of couplings. We find that thermodynamic quantities depend on $\xi$ and this dependence may be strong enough, up to discontinuity over this parameter for some of them. [S0556-2821(97)04323-3]

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## I. INTRODUCTION

The purpose of this paper is to clarify the role of the lattice asymmetry parameter $\xi \equiv a_\sigma / a_\tau$ [where $a_\sigma(a_\tau)$ is the spatial (temporal) spacing] in the phase structure description of SU(2) and SU(3) lattice gluodynamics at finite temperature. We will work within the approach SU($N$)$\simeq$$Z$($N$) where the link variables are the SU($N$) gauge group center elements:

$$U_{x;\mu} \simeq \sigma_{x;\mu} = \exp\left(\frac{2\pi i q_{x;\mu}}{N}\right), \qquad q_{x;\mu} = 0,1,\ldots,N-1. \tag{1.1}$$

Although this approximation does not include all features of the SU($N$) gauge theories, it is commonly believed that the $Z(N)$ degrees of freedom are responsible for many important aspects of the SU($N$) phase structure. The models with discrete gauge groups have many nice features. First, the method of duality transformations is well elaborated for systems with discrete symmetry ([1] and references therein), although undoubtedly there are some achievements in its elaboration for non-Abelian theories (see, for example, [2]). Second, these models are also known to provide a transparent realization of 't Hooft algebra of order and disorder operators. And third, they are easier to handle—which is important if we want to get rigorous results.

Moreover, as is known, the effect of quantum fluctuations near the $Z(N)$ configurations on the symmetric lattice amounts only to a finite renormalization of the coupling constant $\kappa_{\rm old}$ [3]. The new coupling constant $\kappa_{\rm new}$ of the SU($N$) gauge theory will be

$$\kappa_{\rm new} = \kappa_{\rm old} - \frac{N^2-1}{4}. \tag{1.2}$$

We find this fact is one additional justification for the chosen approach. Since the additional term of Eq. (1.2) depends neither on $a_{\sigma,\tau}$ nor on $\kappa_{\rm old,new}$, one may hope that on an asymmetric lattice the effect of quantum fluctuations will also lead to a finite renormalization of the coupling constant.

Numerous Monte Carlo simulations and analytic calculations demonstrate that the $Z(2)$ gauge model possesses the two-phase structure [4,5] whose phases can be classified as the electric confinement and the confinement of magnetic charges, respectively. For $N > N_c$ one phase more appears in the $Z(N)$ lattice gauge theories [6–9]. These results were mainly obtained on the symmetric lattice (or at least on a lattice with definite asymmetry). However, in many cases (for example, in the Hamiltonian limit: $a_\tau \to 0, \xi \to \infty$), the studies are carried out on the asymmetric lattice. It is also a common case for the Monte Carlo experiments to work on lattices with various $\xi$. Therefore, it would be instructive to investigate how the picture may change on a lattice with arbitrary asymmetry.

To provide the proper transition of the Wilson theory

$$\mathcal{S} = \sum_{x;\mu\nu} \kappa_{\mu\nu} \mathrm{Re\ Tr}\left(1 - \frac{1}{N} U_{x;\mu} U_{x+\mu;\nu} U^{+}_{x+\nu;\mu} U^{+}_{x;\nu}\right) \tag{1.3}$$

into the continuous field theory in the "naive" limit

$$U_{x;\mu} \simeq 1 + i a_\mu g_\mu \mathcal{A}_{x;\mu}, \tag{1.4}$$

the following conditions should be imposed on the lattice coupling constants $\kappa_{\mu\nu}$:

$$\kappa_{\mu\nu} = \frac{2N}{g_\mu g_\nu} \frac{a_0 a_1 a_2 a_3}{a_\mu^2 a_\nu^2}. \tag{1.5}$$

Choosing the lattice to be symmetric in the spatial directions, i.e., $a_1 = a_2 = a_3 = a_\sigma$; $g_1 = g_2 = g_3 = g_\sigma$, we reduce

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the number of coupling constants to only two different couplings for the spacelike and timelike planes

$$\kappa_{mn}=\kappa_\sigma=\frac{2N}{g_\sigma^2}\frac{1}{\xi},$$

$$\kappa_{0n}=\kappa_\tau=\frac{2N}{g_\tau^2}\xi. \tag{1.6}$$

In the general case $g_\sigma^2\neq g_\tau^2$, so sometimes $\xi^\star\equiv\sqrt{\kappa_\tau/\kappa_\sigma}$ is considered as a proper lattice asymmetry parameter instead of $\xi$. However, in the weak coupling limit $\xi^\star$ is nearly the same as $\xi$, as has been shown in [10,11]:

$$\xi=\xi^\star\left\{1+N\frac{C_\sigma(\xi)-C_\tau(\xi)}{\zeta}+O(\zeta^{-2})\right\}, \tag{1.7}$$

$$\zeta=\sqrt{\kappa_\sigma\kappa_\tau}=\frac{2N}{g_\sigma g_\tau},\quad 4C_{\sigma,\tau}=\alpha^0_{\sigma,\tau}+\frac{\alpha^1_{\sigma,\tau}}{\xi}+O(\xi^{-2}),$$

$$\alpha^0_\tau=-0.27192,\quad \alpha^1_\tau=0.5,\quad \alpha^0_\sigma=0.39832,\quad \alpha^1_\sigma=0.$$

The numerical calculations for these relations [11,12] reveal that $\xi^\star$ is close to $\xi$ at intermediate couplings as well.

Following the heart point of the renormalization group theory, in the continuum limit thermodynamic quantities should be asymmetry independent, since asymmetry is a feature of the regularizations only. Such a dependence can be absorbed into the $\Lambda$ parameter which becomes a function of $\xi^\star$. The procedure of the corresponding renormalization group equations' construction on an asymmetric lattice has been carried out in [10–14]. However, this procedure works in the approximation where the links variables are close to fixed elements of the gauge group (the ''naive'' limit is an example of such an approximation). The center elements are also fixed in this approximation. The approximation $SU(N)\simeq Z(N)$ ''enlarges'' on the approximation of [10–14] in a sense. Certainly, in our approach thermodynamic quantities should also be asymmetry independent in the continuum limit. On the other hand, we should ensure that any model [in particular, $SU(N)\simeq Z(N)$] should be properly transformed in the ''naive'' limit into continuous field theory. In order to provide this requirement, $\kappa_\tau$, $\kappa_\sigma$ are chosen in precisely the same way as in [10–14].

This paper is organized as follows. Section II deals with the properties of thermodynamic quantities in the limiting cases when one of the coupling constants is close to the physical border. In Sec. III we suggest an expression for the effective action which takes into account the contributions of the spacelike part. This allows us to investigate the model under consideration not only at the physical border but a bit far from it as well. In Sec. IV. we study the behavior of the Wilson and 't Hooft loops average values in each of the coupling areas considered before. Section V contains our conclusions.

## II. LATTICE $Z(N)$ GLUODYNAMICS AT FINITE TEMPERATURE IN THE LIMITING CASES

In the approach $SU(N)\simeq Z(N)$ the Wilson action transforms into the Wegner one

$$\mathcal{S}=\sum_{\mu\nu}\kappa_{\mu\nu}\sum_x \mathrm{Re}\{1-\sigma_{x;\mu}\sigma_{x+\mu;\nu}\sigma^\star_{x+\nu;\mu}\sigma^\star_{x;\nu}\}+\text{c.c.} \tag{2.1}$$

Since the asymmetry exists only between the spatial and temporal directions, it is convenient to split the action into ''electric'' and ''magnetic'' parts:

$$\mathcal{S}=\mathcal{S}^E+\mathcal{S}^H,$$

$$\mathcal{S}^E=\kappa_\tau\sum_{n=1}^{3}\sum_{\vec{x},\tau}\mathrm{Re}(1-\sigma_{\vec{x},\tau;0}\sigma_{\vec{x},\tau+0;n}\sigma^\star_{\vec{x}+n,\tau;0}\sigma^\star_{\vec{x},\tau;n})+\text{c.c.}, \tag{2.2}$$

$$\mathcal{S}^H=\kappa_\sigma\sum_{n,m=1}^{3}\sum_x \mathrm{Re}(1-\sigma_{x;n}\sigma_{x+n;m}\sigma^\star_{x+m;n}\sigma^\star_{x;m})+\text{c.c.}$$

Let us first analyze the limiting cases:

$$\kappa_\sigma\to 0, \tag{2.3a}$$

$$\kappa_\tau\to 0, \tag{2.3b}$$

$$\kappa_\sigma\to\infty, \tag{2.3c}$$

$$\kappa_\tau\to\infty. \tag{2.3d}$$

In all of the above cases our original system reduces to the well-known Ising (Potts) model, so the critical values of these coupling constants can be found easily.

In the case (2.3a) we may naturally disregard the ''magnetic'' part of action. As is known, at finite temperature an additional symmetry which arises due to periodic boundary conditions $\sigma_{\vec{x},\tau;\mu}=\sigma_{\vec{x},\tau+N_\tau;\mu}$ leads to the appearance of new invariants—the Polyakov loops $\Omega_{\vec{x}}\equiv\Pi_{\tau=0}^{N_\tau-1}\sigma_{\vec{x},\tau;0}$, and does not permit us to fix the Hamiltonian gauge $\sigma_{\vec{x},\tau;0}=1$ along the whole temporal axis. The common way to fix the gauge in this case is

$$\sigma_{\vec{x},\tau;0}=1+\delta^0_\tau[\Omega_{\vec{x}}-1]. \tag{2.4}$$

The ''electric'' part of action can be rewritten as

$$\mathcal{S}^E=\kappa_\tau\sum_{\vec{x},n}\left[\sum_{\tau=0}^{N_\tau-1}\mathrm{Re}(1-\sigma_{\vec{x},\tau;n}\sigma^\star_{\vec{x},\tau+1;n})\right.$$
$$\left.+\mathrm{Re}(1-\Omega_{\vec{x}}\sigma_{\vec{x},0;n}\Omega^\star_{\vec{x}+n}\sigma^\star_{\vec{x},1;n})\right]+\text{c.c.} \qquad (2.5)$$

By using the simple relation for the $Z(2)$ and $Z(3)$ gauge theories[1]

$$e^{\kappa(\sigma+\sigma^\star)/2}=e^{f(\kappa)}\left[1+\gamma(\kappa)\left(\frac{\sigma+\sigma^\star}{2}\right)\right], \qquad (2.6)$$

it is easy to sum over all the configurations[2] $\{\sigma_{x;n}\}$ thus obtaining the effective action

$$\widetilde{\mathcal{S}}=\widetilde{\kappa}_\tau\sum_{\vec{x},n}\mathrm{Re}\{1-\Omega_{\vec{x}}\Omega^\star_{\vec{x}+n}\}+\text{const}+\text{c.c.}, \qquad (2.7)$$

$$\widetilde{\gamma}_\tau(\widetilde{\kappa}_\tau)=[\gamma_\tau(\kappa_\tau)]^{N_\tau}.$$

For the $Z(2)$ theory $[\gamma_\tau(\kappa_\tau)=\mathrm{th}\kappa_\tau;\ \widetilde{\gamma}_\tau(\widetilde{\kappa}_\tau)=\mathrm{th}\widetilde{\kappa}_\tau]$ we get the expression for the Ising model action with the accuracy up to a constant. The partition function with action (2.7) is found to be singular at $\widetilde{\kappa}^c_\tau=\kappa_c$, where $\kappa_c=\mathrm{arth}(\gamma_c)=0.2217$ is the critical coupling constant of the three-dimensional Ising model. For our original theory it results in a singularity at $\gamma^c_\tau=\gamma_c^{1/N_\tau}$ (the transition of the second order).

For the $Z(3)$ gauge theory we get the Potts model action with the coupling

$$\widetilde{\gamma}_\tau(\widetilde{\kappa}_\tau)=t(\widetilde{\kappa}_\tau); \qquad \gamma_\tau(\kappa_\tau)=t(\kappa_\tau),$$
$$t(x)=2\frac{1-e^{-3x/2}}{1+2e^{-3x/2}}. \qquad (2.8)$$

We can relate the critical coupling value for our original theory (the transition of the first order) to the corresponding critical constant of the three-dimensional Potts model $\kappa_c=0.3664$:

$$\gamma^c_\tau=\gamma_c^{1/N_\tau}, \qquad \gamma_c=t(\kappa_c). \qquad (2.9)$$

In the other limiting case (2.3b) the ''electric'' part of action may be ignored. Since the partition function is split into $N_\tau$ equivalent independent contributions, each of which results from the plaquettes located in three-dimensional layers ($\tau=$const), we obtain the set of three-dimensional Wegner models:

$$\mathcal{S}_{\tau=\text{const}}=\kappa_\sigma\sum_{\vec{x};nm}\mathrm{Re}(1-\sigma_{\vec{x};n}\sigma_{\vec{x}+n;m}\sigma^\star_{\vec{x}+m;n}\sigma^\star_{\vec{x};m}). \qquad (2.10)$$

There is no interaction between the layers, so summing over $\sigma_{\vec{x},\tau;n}$ may be done independently for every $\tau=$const layer. It is more convenient to make the calculations on the dual lattice where the original set of three-dimensional Wegner models is transformed into the set of three-dimensional Ising (Potts) models with spins in the dual sites. New couplings are connected with the old ones by the relations

$$\widetilde{\gamma}_\sigma=\frac{1-\gamma_\sigma}{1+\gamma_\sigma}, \quad \gamma_\sigma=\mathrm{th}\kappa_\sigma, \quad \widetilde{\gamma}_\sigma=\mathrm{th}\widetilde{\kappa}_\sigma \quad \text{for } Z(2) \text{ theory},$$

$$\widetilde{\gamma}_\sigma=\frac{2-\gamma_\sigma}{1+\gamma_\sigma}, \quad \gamma_\sigma=t(\kappa_\sigma), \quad \widetilde{\gamma}_\sigma=t(\widetilde{\kappa}_\sigma) \quad \text{for } Z(3). \qquad (2.11)$$

These Ising (Potts) models are identical, and hence they exhibit the transition at $\widetilde{\gamma}_\sigma=\gamma_c$ simultaneously. Therefore, for the original $Z(2)$ gauge theory there is a singularity at $\gamma^c_\sigma=(1-\gamma_c)/(1+\gamma_c)$ [for $Z(3)$ at $\gamma^c_\sigma=(2-\gamma_c)/(1+\gamma_c)$].[3]

Before considering the rest of the limiting cases (2.3c) and (2.3d), it is worthy to note that under duality transformations the four-dimensional theory (2.1) transforms into itself, with new couplings (see Appendix A). The $Z(2)$ group couplings transform as

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[1] Expansion of any function $f(\sigma)$, where $\sigma\in Z(N)$, in a Taylor series contains only a finite number of terms. For $Z(2)$ theory even powers in this expansion are grouped in $\mathrm{ch}\kappa$, odd powers yield $\mathrm{sh}\kappa$, thus $\gamma(\kappa)=\mathrm{th}\kappa$ and $e^{f(\kappa)}=e^{\ln\mathrm{ch}\kappa}$. For $Z(3)$ theory

$$\frac{\sigma+\sigma^\star}{2}=\begin{cases}1\\-1/2\end{cases}.$$

Then

$$e^{\kappa(\sigma+\sigma^\star)/2}=\begin{cases}e^{\kappa}\\e^{-\kappa/2}\end{cases}=a+b\,\frac{\sigma+\sigma^\star}{2},$$

where

$$a=\frac{1+2e^{-3\kappa/2}}{3e^{-\kappa}}, \quad b=\frac{2(1-e^{-3\kappa/2})}{3e^{-\kappa}}.$$

By analogy for $Z(4)$ theory we have

$$e^{\kappa(\sigma+\sigma^\star)/2}=\begin{cases}e^{\kappa}\\1\\e^{-\kappa}\end{cases}=a+b\frac{\sigma+\sigma^\star}{2}+c\left(\frac{\sigma+\sigma^\star}{2}\right)^2,$$

where $a=1$, $b=\mathrm{sh}\kappa$, $c=\mathrm{ch}\kappa-1$.

[2] The first term of the action (2.5) corresponds to the partition function of one-dimensional chain of spins along the temporal direction, $\Sigma_{\{\sigma_\tau\},\tau\neq 0,N_\tau-1}\Pi_{\tau=0}^{N_\tau-1}e^{\kappa_\tau\sigma_\tau\sigma_{\tau+1}}=e^{N_\tau f(\kappa_\tau)}[1+\gamma^{N_\tau}(\kappa_\tau)\sigma_0\sigma_{N_\tau-1}]$. Here both Eq. (2.6) and the relation $\Sigma_{\{\sigma_\tau\}}\times(\sigma_\tau)^j=\delta_j^{2k}$ have been taken into account. The structures $\sigma_{\tau-1}\sigma_\tau\sigma_\tau\sigma_{\tau+1}=\sigma_{\tau-1}\sigma_{\tau+1}$ appear in the product when running over $\tau$. In the corresponding product of $Z(3)$ theory the following structures appear: $\sigma_{\tau-1}\sigma^\star_\tau\sigma_\tau\sigma^\star_{\tau+1}=\sigma_{\tau-1}\sigma^\star_{\tau+1}$ and $\sigma^\star_{\tau-1}\sigma_\tau\sigma_\tau\sigma^\star_{\tau+1}=\sigma^\star_{\tau-1}\sigma^\star_\tau\sigma^\star_{\tau+1}$. In this case the relation $\Sigma_{\{\sigma_\tau\}}(\sigma_\tau)^j=\delta_j^{3k}$ is used. Finally, after summing over the spin configurations of first and last sites only the Polyakov loops will survive in the partition function.

[3] We consider $\gamma_\sigma$ and $\gamma_\tau$ independent, temporarily ignoring the dependence of $g_\sigma(g_\tau)$ on $a_\sigma(a_\tau)$ through the renormalization group relations, i.e., supposing that by varying $a_\sigma$ and $a_\tau$, we can get to any point of the square $(0\leqslant\gamma_\tau\leqslant 1;\ 0\leqslant\gamma_\sigma\leqslant 1)$.

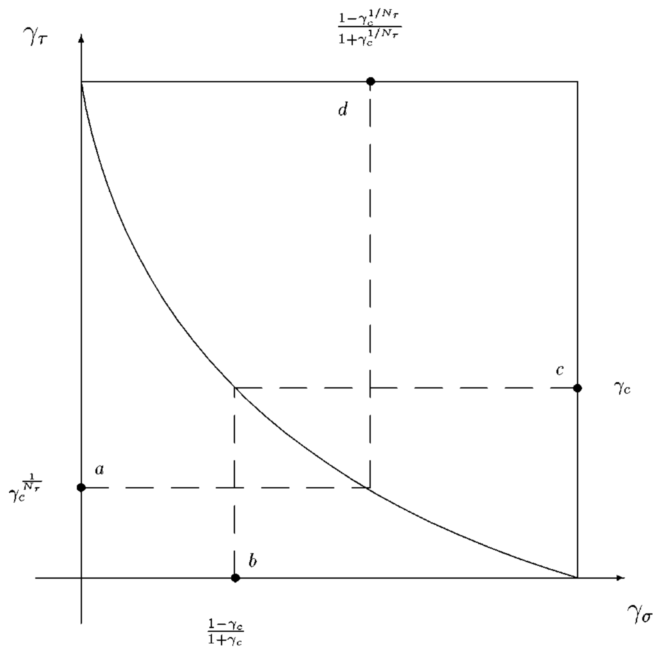


FIG. 1. Points $a$-$d$ and $b$-$c$ are dual symmetric by pairs.

$$\gamma_\sigma \to \gamma'_\sigma \equiv \frac{1-\gamma_\tau}{1+\gamma_\tau}, \quad \gamma_\tau \to \gamma'_\tau \equiv \frac{1-\gamma_\sigma}{1+\gamma_\sigma},$$

$$\gamma_\sigma = \mathrm{th}\kappa_\sigma, \quad \gamma'_\sigma = \mathrm{th}\kappa'_\sigma. \tag{2.12}$$

For the $Z(3)$ gauge group we have

$$\gamma_\sigma \to \gamma'_\sigma \equiv \frac{2-\gamma_\tau}{1+\gamma_\tau}, \qquad \gamma_\tau \to \gamma'_\tau \equiv \frac{2-\gamma_\sigma}{1+\gamma_\sigma},$$

$$\gamma_\sigma = t(\kappa_\sigma), \quad \gamma'_\sigma = t(\kappa'_\sigma). \tag{2.13}$$

For this dual representation we may examine again the two limiting cases ($\gamma'_\sigma \to 0$ and $\gamma'_\tau \to 0$) in precisely the same way as it has been done for our original theory. Following Eqs. (2.12) and (2.13), these correspond to the limits $\gamma_\sigma \to 1$ and $\gamma_\tau \to 1$ on the original lattice. We find singularities of the partition function on the original lattice at

| $Z(2)$ gauge system | | $Z(3)$ gauge system | |
|---|---|---|---|
| $\gamma_\sigma \to 1$ | $\gamma_\tau^c = \gamma_c$ | $\gamma_\sigma \to 1$ | $\gamma_\tau^c = \gamma_c$ |
| $\gamma_\tau \to 1$ | $\gamma_\sigma^c = \dfrac{1-\gamma_c^{1/N_\tau}}{1+\gamma_c^{1/N_\tau}}$ | $\gamma_\tau \to 1$ | $\gamma_\sigma^c = \dfrac{2-\gamma_c^{1/N_\tau}}{1+\gamma_c^{1/N_\tau}}$ |

where $\gamma_{\sigma,\tau}$ is a function of $\kappa_{\sigma,\tau}$ in accordance with Eqs. (2.8) and (2.11) for the $Z(2)$ and $Z(3)$ groups, respectively. We may depict all these critical points in the plane $[\gamma_\tau;\gamma_\sigma]$ (see Fig. 1).

## III. CONSTRUCTION OF THE EFFECTIVE ACTION

In order to investigate our model phase structure not only in the limiting cases but also at some distance from the singularity points, as well as in order to estimate the phase structure of the theory under consideration in the whole area of couplings (square $0 \leqslant \gamma_\tau \leqslant 1$ and $0 \leqslant \gamma_\sigma \leqslant 1$), let us construct the effective action taking into account the "electric" or "magnetic" contributions which have been neglected previously, considering them as perturbations. After defining $\langle A\rangle_E \equiv \Sigma_{\{\sigma\}} A e^{-\mathcal{S}_E}$ the partition function can be rewritten as

$$\mathcal{Z} = \langle e^{-\mathcal{S}_H}\rangle_E \equiv \left\langle \exp\left(\kappa_\sigma \sum_{x;nm} \sigma_{nm} + \mathrm{const}\right)\right\rangle_E = \sum_{r=0}^{\infty} \mathcal{Z}_r, \tag{3.1}$$

$$\mathcal{Z}_r \equiv \frac{\kappa_\sigma^r}{r!} \sum_{n,m=1}^{3} \sum_x \langle \sigma_{nm}^r \rangle_E,$$

where $\sigma_{nm} \equiv \mathrm{Re}\{\sigma_{x;n}\sigma_{x+n;m}\sigma^\star_{x+m;n}\sigma^\star_{x;m}\}$ is the spacelike plaquette variable.

The zero-order term of the expansion $\mathcal{Z}_0 = \langle 1\rangle_E$ corresponds to the case $\kappa_\sigma = 0$ already considered. It is easy to see that the first-order term

$$\mathcal{Z}_1 \equiv \kappa_\sigma \sum_{n,m=1}^{3} \sum_x \langle \sigma_{nm}\rangle_E$$

is equal to 0. This can be explained by the following considerations: $\Sigma_{\{\sigma\}}\sigma_{x;n} = 0$ and every spin $\sigma_{x;n}$ enters $\sigma_{nm}$ only once. On the other hand, structures containing $\sigma_{\vec{x},\tau';n}$ can be constructed out of plaquettes from $e^{-\mathcal{S}_E}$

$$e^{-\mathcal{S}_E} = \prod_{\vec{x},n} \left[ \prod_{\tau=0}^{N_\tau - 1} (1 + \gamma_\tau\{\sigma_{\vec{x},\tau;n}\sigma^\star_{\vec{x},\tau+1;n} + \mathrm{c.c.}\}) \times (1 + \gamma_\tau\{\Omega_{\vec{x}}\sigma_{\vec{x},0;n}\Omega^\star_{\vec{x}+n}\sigma^\star_{\vec{x},1;n} + \mathrm{c.c.}\}) \right]. \tag{3.2}$$

These structures survive after the summation over spins at $\tau \neq \tau', \tau'+1$ only as one-dimensional chains (along $\tau$) in the following form:

$$[1 + \gamma_\tau^{N_\tau}\{\Omega_{\vec{x}}\sigma_{\vec{x},\tau';n}\Omega^\star_{\vec{x}+n}\sigma^\star_{\vec{x},\tau'+1;n} + \mathrm{c.c.}\}].$$

However, none of the $\sigma_{nm}$ contain $\sigma_{\vec{x},\tau';n}$ and $\sigma_{\vec{x},\tau'+1;n}$ simultaneously, therefore, $\langle \sigma_{nm}\rangle_E$ is equal to zero.

The second-order contribution

$$\mathcal{Z}_2 \equiv \frac{\kappa_\sigma^2}{2!} \sum_{n,m=1}^{3} \sum_x \langle \sigma_{nm}^2\rangle_E$$

in the expansion (3.1) consists of plaquettes $\sigma_{nm}$ having the same spatial coordinates and positioned at different points $\tau$ and $\tau+\Delta$ along the temporal axis (see Fig. 2). Corresponding one-dimensional chains

$$\mathcal{J}_\Delta \equiv [1 + \gamma_\tau^\Delta\{\sigma_{\vec{x},\tau;n}\sigma^\star_{\vec{x},\tau+\Delta;n} + \mathrm{c.c.}\}]$$

and

$$\mathcal{J}'_\Delta \equiv [1 + \gamma_\tau^{N_\tau - \Delta}\{\Omega_{\vec{x}}\sigma_{\vec{x},\tau;n}\Omega^\star_{\vec{x}+n}\sigma^\star_{\vec{x},\tau+\Delta;n} + \mathrm{c.c.}\}]$$

along the plaquettes mentioned above give nonzero contributions to the partition function

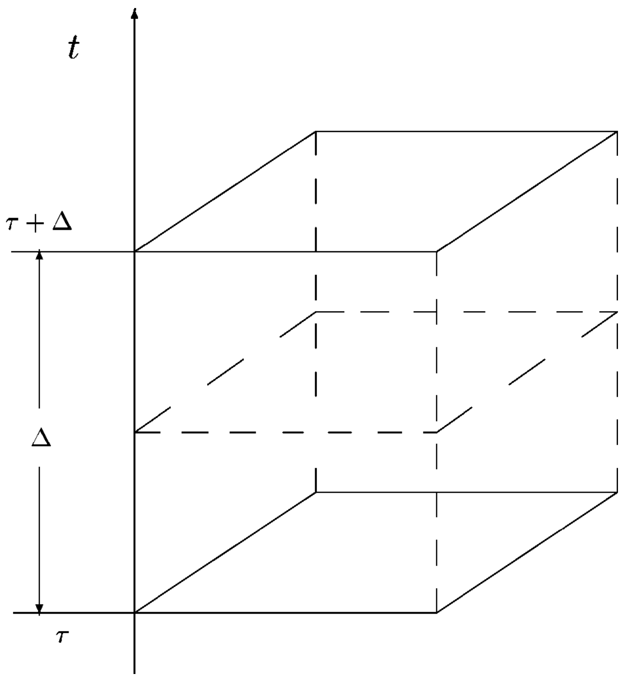


FIG. 2. ''Ladder'' of plaquettes $\sigma_{nm}$ having the same spatial coordinates and positioned at different points $\tau$ along the temporal axis.

$$\mathcal{Z}_2=\frac{\gamma_\sigma^2}{2}\sum_{n,m=1}^{3}\sum_{\vec{x}}\sum_{\Delta=1}^{N_\tau-1}\sigma_{nm}(\tau+\Delta)\sigma_{nm}(\tau)(\mathcal{J}_\Delta+\mathcal{J}'_\Delta). \tag{3.3}$$

Summing over $\Delta$ we get

$$\mathcal{Z}_2=\frac{\gamma_\sigma^2}{2}\sum_{\vec{x},n,m}[c_0+c_1W'_{\vec{x};nm}+c_2W''_{\vec{x};nm}], \tag{3.4}$$

where

$$W'_{\vec{x};nm}=\Omega_{\vec{x}}\Omega^\star_{\vec{x}+n}+\Omega_{\vec{x}+n}\Omega^\star_{\vec{x}+n+m}+\Omega_{\vec{x}+n+m}\Omega^\star_{\vec{x}+m}+\Omega_{\vec{x}+m}\Omega^\star_{\vec{x}},$$

$$W''_{\vec{x};nm}=\Omega_{\vec{x}}\Omega^\star_{\vec{x}+n+m}+\Omega_{\vec{x}+n}\Omega^\star_{\vec{x}+m}+\Omega_{\vec{x}}\Omega^\star_{\vec{x}+n}\Omega_{\vec{x}+n+m}\Omega^\star_{\vec{x}+m},$$

and the coefficients have been obtained after lengthy calculation

$$c_0=\alpha_0\beta_0+4\alpha_1\beta_1+3\alpha_2\beta_2,$$

$$c_1=\alpha_0\beta_1+\alpha_1\beta_0+3(\alpha_1\beta_2+\alpha_2\beta_1),$$

$$c_2=\alpha_0\beta_2+\alpha_2\beta_0+4\alpha_1\beta_1+2\alpha_2\beta_2,$$

$$\beta_0=1+\tilde{\gamma}_\tau^4,$$

$$\beta_1=-\tilde{\gamma}_\tau-\tilde{\gamma}_\tau^3,$$

$$\beta_2=2\tilde{\gamma}_\tau^2,$$

$$\alpha_0=2\frac{\gamma_\tau^{4N_\tau}-\gamma_\tau^4}{\gamma_\tau^4-1},$$

$$\alpha_1=2\gamma_\tau^{N_\tau}\frac{\gamma_\tau^{2N_\tau}-\gamma_\tau^2}{\gamma_\tau^2-1},$$

$$\alpha_2=2\gamma_\tau^{2N_\tau}(N_\tau-1).$$

$W'_{\vec{x};nm}$ consists of Ising-type terms; $W''_{\vec{x};nm}$ contains both Ising-type terms (between the diagonal neighbors however) and four-interaction terms. Using precisely the same method as in [15] we can replace all spins $\Omega_{\vec{x}}$ with ''average spins'' $\Omega=(1/N)\Sigma_{x=1}^{N}\Omega_{\vec{x}}$. The partition function which includes the first nontrivial contribution over $\gamma_\sigma$ in addition to the ''electric'' part of action has the following form:

$$\mathcal{Z}=\sum_{\{\Omega\}}e^{\tilde{\gamma}_\tau\Omega^2Nd+\gamma_\sigma^2[(c_1+c_2)\Omega^2+c_2\Omega^4]Nd(d-1)+O(\gamma_\sigma^3)-\mathcal{L}(\Omega)N},$$

$$N=N_\sigma^3N_\tau. \tag{3.5}$$

The first term of the exponential power in Eq. (3.5) comes from the ''electric'' part of the action; the second one is the contribution from the ''magnetic'' part. Appendix B presents the calculation of the ''Jacobian'' of transformation $e^{\mathcal{L}(\Omega)N}$ to new variables—''average spins.'' In the area considered ($\gamma_\sigma\rightarrow 0$), the value of $\Omega$ is close to 0, so $\Omega^4$ contributions can be omitted in practice. The singularity line starting at the point $a$ has been calculated numerically (see Fig. 3). Using the same technique for the other three cases, we have calculated and drawn lines starting from the points $b$-$d$, thus splitting the plane $[\gamma_\tau;\gamma_\sigma]$ into four areas, shown schematically in Fig. 4.

## IV. STUDYING THE WHOLE PHASE PLANE $[\gamma_\tau;\gamma_\sigma]$

We offer some suggestions concerning the phase structure in the whole area of coupling constants and try to clarify the nature of the phases previously obtained. In the case $\kappa_\tau\simeq 0$ the four-dimensional system transforms into the set of independent three-dimensional subsystems with $t=t_j$ as already mentioned. The probe sources (the potential between them has been calculated on the dual lattice, see Appendix C) correspond to the magnetic charges placed inside the cubes of the original lattice.

The contribution from the spacelike cube plaquettes can be associated with the magnetic field flux through the cube surface,

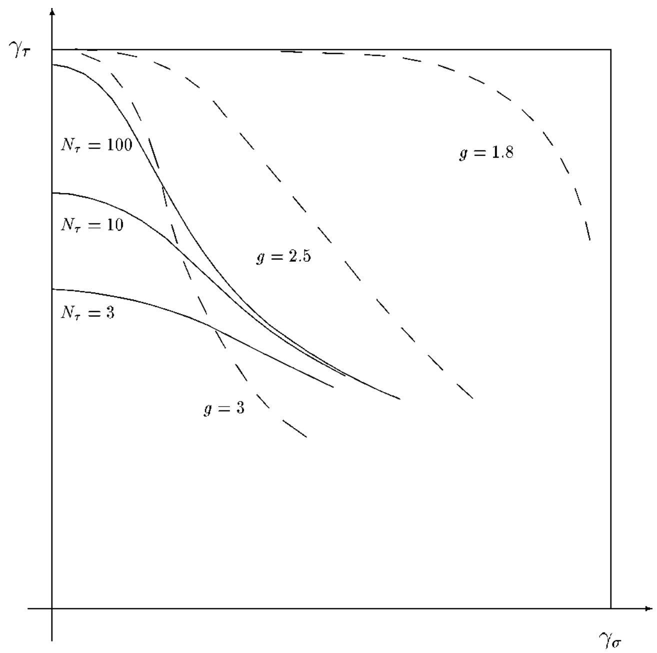


FIG. 3. Singularity lines leaving the point $a$ at different $N_\tau$ (solid lines). Lines $\kappa_\tau \kappa_\sigma = \text{const}$ at different $g$ (dashed lines).

$$\prod_{\text{cube}} \Box_\sigma = \exp\left( \text{const} \times \sum_{\text{cube}} \vec{B} \cdot \vec{n} \right), \tag{4.1}$$

$$B_k = \frac{1}{2} \epsilon_{kmn} \mathcal{F}_{mn},$$

and is not equal to zero when the probe source is placed in the corresponding dual site. In other words, the probe source of ''electric'' charge $e_x$ in the site of the dual lattice corresponds to the monopole (''magnetic'' charge $m_x$) on the original lattice.

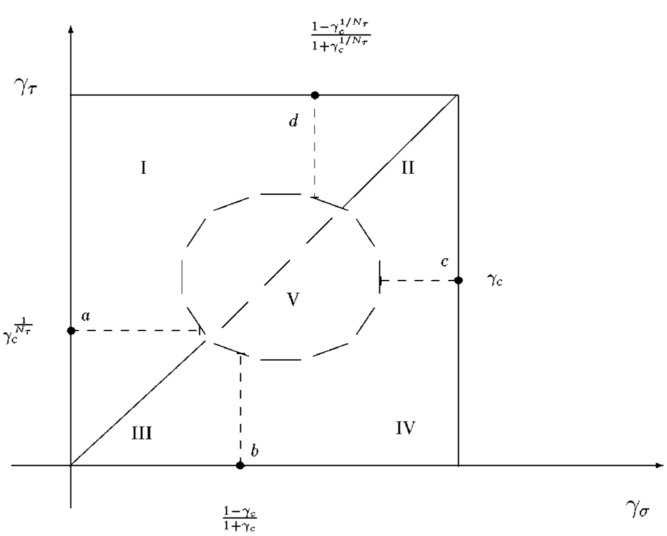


FIG. 4. Four areas in the plane $[\gamma_\tau;\gamma_\sigma]$: I, deconfinement of electric and magnetic charges; II, magnetic confinement, electric deconfinement; III, electric confinement, magnetic deconfinement; IV, electric confinement, magnetic confinement; V, the area we cannot investigate analytically.

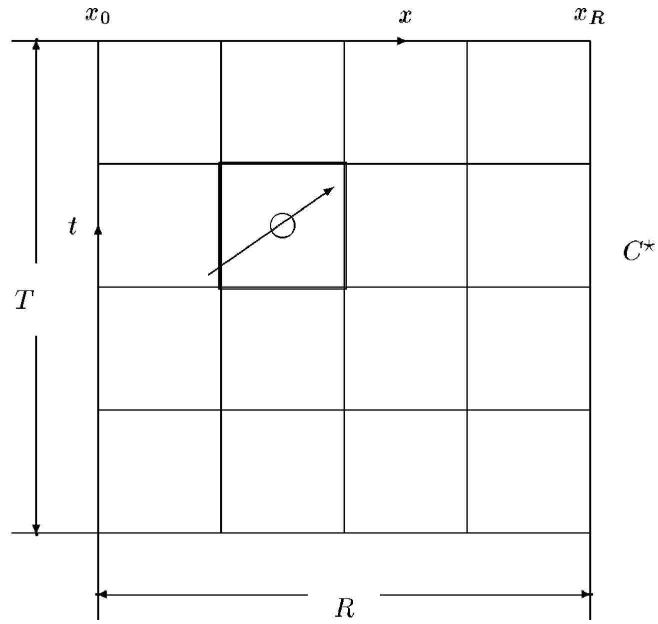


FIG. 5. If the Wilson loop $C^\star$ in the plane $[t;x]$ is intersected with the plane $t = t_j$, then the loop will pierce the plane $[z;y]$ at two points (monopole–antimonopole).

It is known that if the Wilson loop $C^\star$ in the plane $[t;x]$ of the four-dimensional dual lattice is intersected with the plane $t = t_j$, then the loop will pierce (see Fig. 5) the plane $[z;y]$ at two points (monopole-antimonopole). The Dirac string which ties them together is in the plane $t = t_j$ (see Fig. 6) [16]. If the potential between probe sources in each plane increases linearly with $R$ (in the region of coupling $\gamma_\tau < \gamma_\tau^c$), then the Wilson loop average value[4] $\langle E \rangle = \Pi_{t=1}^{T} \langle e_{x_0} e_{x_R} \rangle$ will decrease exponentially according to the ''area law.'' The average value of the corresponding 't Hooft loop $\langle M \rangle$ must behave in the same way in the region $\gamma_\sigma > \gamma_\sigma^c$:

$$\langle M \rangle_{\text{original}} = \prod_{t=1}^{T} \langle m_{x_0}, m_{x_R} \rangle = \langle E \rangle_{\text{dual}} = \prod_{t=1}^{T} \langle e_{x_0} e_{x_R} \rangle \sim e^{-\lambda TR}. \tag{4.2}$$

It is obvious that the parameters area ($\gamma_\tau$ and $\gamma_\sigma$) is split into four sectors (Fig. 4) with different behavior of the Wilson and 't Hooft loops average values.

| | | | | |
|---|---|---|---|---|
| I | $\gamma_\tau > \gamma_\tau^c$ | $\gamma_\sigma < \gamma_\sigma^c$ | $\langle E \rangle \sim e^{-\alpha L_C}$ | $\langle M \rangle \sim e^{-\alpha' L_{C'}}$ |
| II | $\gamma_\tau > \gamma_\tau^c$ | $\gamma_\sigma > \gamma_\sigma^c$ | $\langle E \rangle \sim e^{-\alpha L_C}$ | $\langle M \rangle \sim e^{-\lambda' \Sigma_{C'}}$ |
| III | $\gamma_\tau < \gamma_\tau^c$ | $\gamma_\sigma < \gamma_\sigma^c$ | $\langle E \rangle \sim e^{-\lambda \Sigma_C}$ | $\langle M \rangle \sim e^{-\alpha' L_{C'}}$ |
| IV | $\gamma_\tau < \gamma_\tau^c$ | $\gamma_\sigma > \gamma_\sigma^c$ | $\langle E \rangle \sim e^{-\lambda \Sigma_C}$ | $\langle M \rangle \sim e^{-\lambda' \Sigma_{C'}}$ |

This picture covers all four types of possible behavior of the averages under consideration which were found by G. 't Hooft [17] from the commutation relations analysis. It seems impossible to ''see'' all four phases on a lattice with fixed asymmetry ($\kappa_\tau = \text{const} \kappa_\sigma$) including the symmetric

[4]The contributions of all planes are independent.

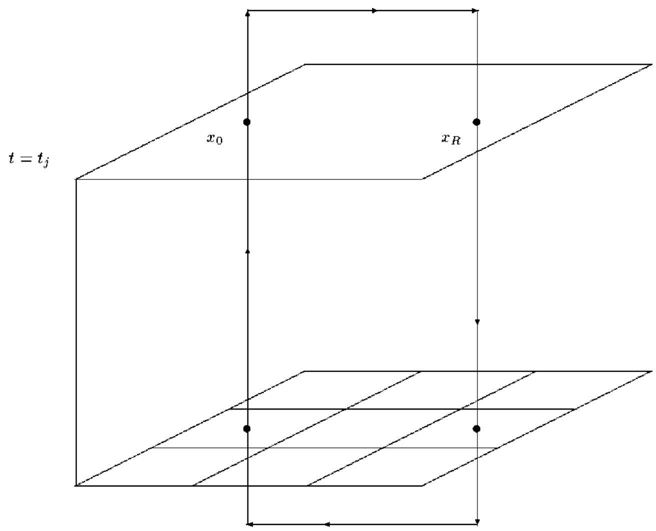


FIG. 6. Dirac string is in the plane $t=t_j$.

one. Our results are also in good agreement with [16] in the areas studied on the symmetric lattice (line $\gamma_\tau=\gamma_\sigma$).

## V. CONCLUSIONS AND DISCUSSION

Until now we regarded the couplings $\gamma_\sigma(\gamma_\tau)$ as independent. However, the underlying constants $g_\sigma(g_\tau)$ depend on the lattice spacings through the renormalization group relations. This dependence may make some areas of the square $(0\leqslant\gamma_\sigma\leqslant 1,\ 0\leqslant\gamma_\tau\leqslant 1)$ inaccessible. To find the exact borders of the accessible area we should construct the renormalization group relations on an asymmetric lattice and substitute $g_\sigma^2(g_\tau^2)$ into $\gamma_\sigma(\gamma_\tau)$. It should be noted that in the limit $(a_{\sigma,\tau}\rightarrow 0)$, $g_\sigma^2\simeq g_\tau^2\simeq g^2\ll 1$ [10], where $g\equiv\sqrt{g_\tau g_\sigma}, g_\sigma^2 \simeq g^2+O(g^4)$, $g_\tau^2\simeq g^2+O(g^4)$. So, $\kappa_\tau\kappa_\sigma=4N^2/g^4\gg 1$ (shaded region at Fig. 7). Say, at $\kappa_\sigma\rightarrow 0$; $\kappa_\tau\rightarrow\infty$ as $1/g^4\kappa_\sigma$ and $\xi\rightarrow\infty$ as $1/g^2\kappa_\sigma$. This makes the accessible parameters area narrower. Moreover, at $N_\tau\rightarrow\infty$ the points $a$ and $d$ move to the point $(\gamma_\tau=1;\ \gamma_\sigma=0)$ (see Fig. 7).

If $\xi>1$ the relations between $\kappa_\sigma(\kappa_\tau)$ and $\xi$ have the following form [11,14] [and result from Eq. (1.7)]:

$$\kappa_\sigma=\frac{1}{\xi}\left(\frac{4}{g^2}+\alpha_\sigma^0+O(\xi^{-2})\right),$$

$$\kappa_\tau=\xi\left(\frac{4}{g^2}+\alpha_\tau^0+\frac{1}{2\xi}+O(\xi^{-2})\right). \tag{5.1}$$

At a given $g^2$ we get in the plane $[\kappa_\tau;\kappa_\sigma]$ the curve

$$\left(\kappa_\tau-\frac{1}{2}\right)\kappa_\sigma\simeq\left(\frac{4}{g^2}+\alpha_\tau^0\right)\left(\frac{4}{g^2}+\alpha_\sigma^0\right), \tag{5.2}$$

which is nearly the same as the "classical" curve $\kappa_\tau\kappa_\sigma\simeq(2N/g^2)^2$ in the region $\xi>1, g^2<1$ we are interested in. Changing $\xi$ we can get to any point of this curve. As it is shown in Fig. 3, only at very big $N_\tau$ ($N_\tau\gg 100$) and small $g^2$ the curve (5.2) may cross the critical curve $\gamma_\tau=\gamma_\tau^{(a)}(\gamma_\sigma)$. Weak dependence of thermodynamic quantities on $\xi$ around $\gamma_\tau=\gamma_\tau^{(a)}(\gamma_\sigma)$ was pointed out in [14]. The dependence of

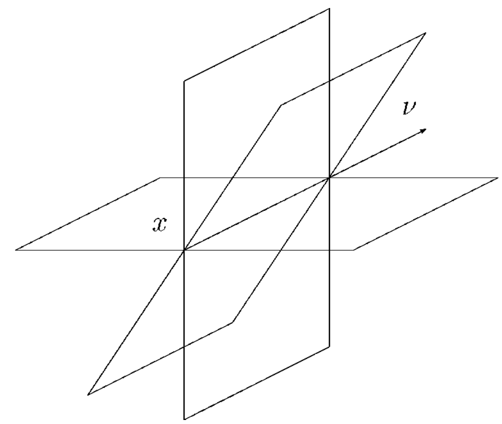


FIG. 8. Six plaquettes which adjoin the link $x;\nu$.

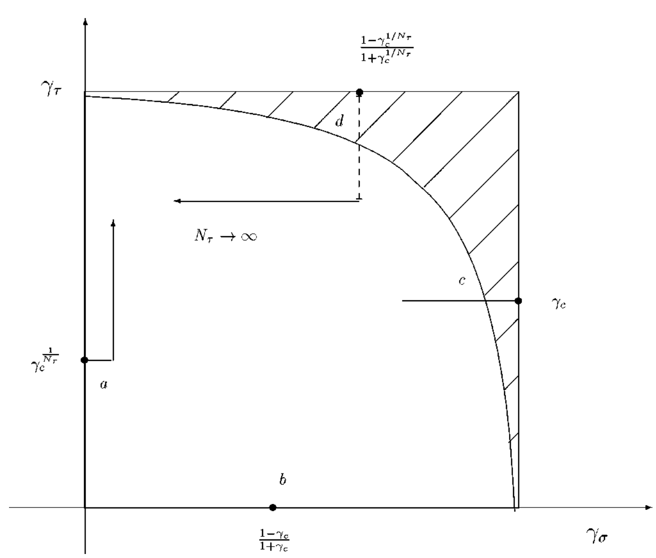


FIG. 7. Shaded region corresponds to $\kappa_\tau\kappa_\sigma\gg 1$. At $N_\tau\rightarrow\infty$ the points $a$ and $d$ move to the point $(\gamma_\tau=1;\ \gamma_\sigma=0)$.

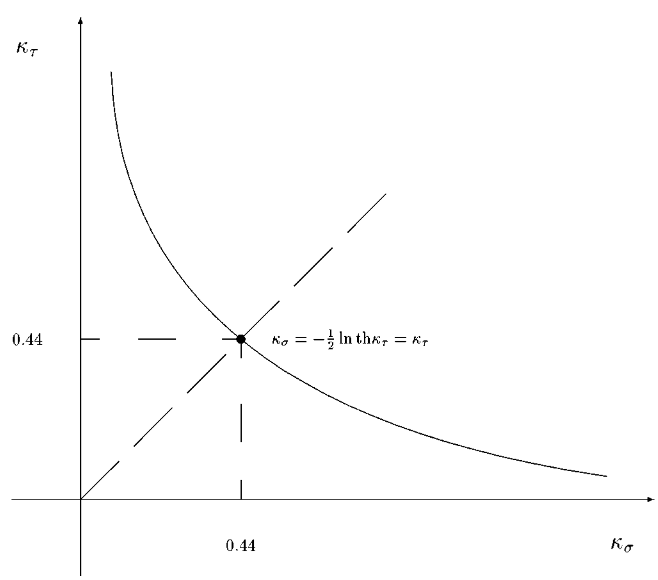


FIG. 9. In the plane $[\kappa_\tau;\kappa_\sigma]$ there is the self-duality line. $\kappa_c=0.44$ is the critical coupling value for the four-dimensional $Z(2)$ pure gauge theory.

$\gamma_\tau^{(a)}$ on $N_\tau$ is getting weaker with increasing $\gamma_\sigma$, which is an indirect confirmation of the weak dependence on $\xi$. Close to the critical curve $\gamma_\tau=\gamma_\tau^{(d)}(\gamma_\sigma)$ at $\kappa_\sigma\simeq-\ln\gamma_c/2N_\tau$ we get a condition on $\xi$ for the curve (5.2) to cross the critical curve: $\xi\simeq(1/\kappa_\sigma)(4/g^2+\alpha_\sigma^0)\simeq(N_\tau/g^2)$const. With increasing $N_\tau$ [as well as with decreasing $a_\sigma(a_\tau)$] the phase border crossing will be at ever greater $\xi$. It increases the expansion reliability (5.1) for $\kappa_\tau(\kappa_\sigma)$.

The parameter $\xi\simeq\sqrt{\kappa_\tau/\kappa_\sigma}$ is usually chosen arbitrarily ($\xi_{\rm Hamilt}=\infty$ and $\xi_{\rm Eucl}=1$). So, if this parameter is not restricted with an additional condition then by changing it we may reach any point of the curve $\kappa_\sigma\kappa_\tau=$const at any small $g^2$, thereby crossing at least one line between phases (II and IV). This confirms that the thermodynamic quantities depend on $\xi$ and, moreover, the discontinuity over this parameter is possible for some of them. It is commonly believed that changing the parameter $\xi$ should not result in any observable effects. An attempt to exclude the undesirable dependence of thermodynamic quantities on $\xi$ is in contradiction either with renormalization group relations [10,11,14] or with restrictions on the coupling constants imposed by Eq. (1.5). To keep the independence of the observed quantities on $\xi$ in the "naive" limit, we chose $\kappa_\tau,\kappa_\sigma$ in precisely the same way as in [10,11,14]. So, we suggest that lattice gauge theories need an additional condition which fixes $\xi$.

There are reasons to hope that the estimates for the SU(2) and SU(3) gauge groups will be similar to those obtained in the current paper, at least within approximations of [3]. In the future we will take the corrections to our approximation into account. This will allow us to clarify the picture in detail for the SU(2) and SU(3) groups.

## APPENDIX A: DUALITY TRANSFORMATIONS IN FOUR DIMENSIONS

When considering the duality transformation in four-dimensional space-time it should be pointed out that space-like plaquettes transform into timelike ones. In other words,

$$\kappa'_\sigma=-\frac{1}{2}\ln\mathrm{th}\kappa_\tau \quad \text{or} \quad \gamma'_\sigma=\frac{1-\gamma_\tau}{1+\gamma_\tau} \tag{A1}$$

and vice versa, $\kappa'_\tau=-1/2\ln\mathrm{th}\kappa_\sigma$.

This statement becomes clear from the following. Let us rewrite the partition function of the $Z(2)$ system in the form

$$\mathcal{Z}=\sum_{\{\sigma\}}\exp\left(\sum_{x,\mu\nu}\kappa_{\mu\nu}\Box_{x,\mu\nu}\right)=\sum_{\{\sigma\}}\prod_{x,\mu\nu}\mathrm{ch}\kappa_{\mu\nu}(1+\Box_{x,\mu\nu}\mathrm{th}\kappa_{\mu\nu})=e^{-f}\sum_{\{\sigma\}}\prod_{x,\mu\nu}\sum_{q=\pm1}(\Box_{x,\mu\nu}\mathrm{th}\kappa_{\mu\nu})^{(q_{x,\mu\nu}+1)/2}$$
$$=e^{-f}\sum_{\{q\}}\prod_{x,\mu\nu}(e^{\ln\mathrm{th}\kappa_{\mu\nu}})^{(q_{x,\mu\nu}+1)/2}\prod_{\rm links}\sum_{\sigma=\pm1}(\sigma)^Q=e^{-f}\sum_{\{q\}}\prod_{x,\mu\nu}(e^{\ln\mathrm{th}\kappa_{\mu\nu}})^{(q_{x,\mu\nu}+1)/2}\prod_{\rm links}2\,\delta_2(Q), \tag{A2}$$

where

$$\Box_{x,\mu\nu}=\sigma_{x;\mu}\sigma_{x+\mu;\nu}\sigma^\star_{x+\nu;\mu}\sigma^\star_{x;\nu},$$

$$Q=\sum_{\mu=-3,\mu\neq\pm\nu}^{3}\frac{q_{x,\mu\nu}+1}{2},$$

$$-f=N_\tau N_\sigma^3\sum_{\mu\nu}\ln\mathrm{ch}\kappa_{\mu\nu}.$$

We introduce a new set of variables $\{q\}$—one for each plaquette. This partition function is not equal to zero only if $q_{x,\mu\nu}$ satisfies the following condition on the sum over six $q_{x,\mu\nu}$ [associated with six plaquettes which adjoin the link $x;\nu$ (see Fig. 8)]:

$$\frac{1}{2}\sum_{\mu=-3;\mu\neq\pm\nu}^{3}(q_{x,\mu\nu}+1)=0_{\rm mod2}$$

or

$$\sum_{\mu=-3;\mu\neq\pm\nu}^{3}q_{x,\mu\nu}=2_{\rm mod4}. \tag{A3}$$

The solution of last equation can be found if we associate every $q_{x,\mu\nu}$ with one of the cube planes and

$$q_{x,\mu\nu}=s_{x;\rho}s_{x+\rho;\lambda}s^\star_{x+\lambda;\rho}s^\star_{x;\lambda},$$
$$\nu\neq\mu\neq\rho\neq\lambda, \tag{A4}$$

where the dual link variable $s_{x;\rho}$ is an element of the $Z(2)$ group. It becomes intuitively evident, if we consider the starting case when all $s$ are equal to 1. This dictates that $\Sigma_{\mu\neq\pm\nu}q_{x,\mu\nu}$ must be equal to $6_{\rm mod4}=2_{\rm mod4}$. Every link enters the solution twice (because the plaquettes form a cube) and changing the sign of a link to opposite results in changing $\Sigma^3_{\mu=-3;\mu\neq\pm\nu}q_{x,\mu\nu}$ only by $\pm4$.

Consequently, in the plane $[\kappa_\tau;\kappa_\sigma]$ there is the self-duality line (see Fig. 9). Balian, Drouffe, and Itzykson [4] pointed out the possibility of the critical behavior at $\kappa_c=0.44$ for the four-dimensional $Z(2)$ pure gauge theory on a symmetric lattice supposing this critical point is the only one.

By analogy this can be shown for the $Z(3)$ gauge theory

$$\mathcal{Z}=\sum_{\{\sigma\}}\exp\left[\sum_{x,\mu\nu}\kappa_{\mu\nu}\left(\frac{\Box_{x,\mu\nu}+\Box^{\star}_{x,\mu\nu}}{2}\right)\right]=e^{-f}\sum_{\{\sigma\}}\prod_{x,\mu\nu}\left[1+\gamma(\kappa_{\mu\nu})\left(\frac{\Box_{x,\mu\nu}+\Box^{\star}_{x,\mu\nu}}{2}\right)\right]$$

$$=e^{-f}\sum_{\{\sigma\}}\prod_{x,\mu\nu}\sum_{\{q\in Z(3)\}}\left[\gamma(\kappa_{\mu\nu})\left(\frac{\Box_{x,\mu\nu}+\Box^{\star}_{x,\mu\nu}}{2}\right)\right]^{[(q_{x,\mu\nu}+q^{\star}_{x,\mu\nu})+1]/3}$$

$$=e^{-f}\sum_{\{q\}}\prod_{x,\mu\nu}(e^{\ln\gamma(\kappa_{\mu\nu})})^{[(q_{x,\mu\nu}+q^{\star}_{x,\mu\nu})+1]/3}\prod_{\text{links}}\sum_{\{\sigma\in Z(3)\}}\left(\frac{\sigma+\sigma^{\star}}{2}\right)^{Q}=e^{-f}\sum_{\{q\}}\prod_{x,\mu\nu}(e^{\ln\gamma(\kappa_{\mu\nu})})^{[(q_{x,\mu\nu}+q^{\star}_{x,\mu\nu})+1]/3}\prod_{\text{links}}3\,\delta_3(Q),$$

where

$$\Box_{x,\mu\nu}=\sigma_{x;\mu}\sigma_{x+\mu;\nu}\sigma^{\star}_{x+\nu;\mu}\sigma^{\star}_{x;\nu},$$

$$Q=\sum_{\mu=-3,\mu\neq\pm\nu}^{3}\frac{(q_{x,\mu\nu}+q^{\star}_{x,\mu\nu})+1}{3},$$

$$-f=N_\tau N_\sigma^3\sum_{\mu\nu}\ln\frac{1+2e^{-3\kappa_{\mu\nu}/2}}{3e^{-\kappa_{\mu\nu}}},$$

$$\gamma(\kappa_{\mu\nu})=2\frac{1-e^{-3\kappa_{\mu\nu}/2}}{1+2e^{-3\kappa_{\mu\nu}/2}}.$$

If $q_{x,\mu\nu}$ is the same as in Eq. (A4), but $s_{x;\rho}$ is now an element of the $Z(3)$ group, then $q_{x,\mu\nu}$ is the solution of the equation: $\frac{1}{3}\sum_{\mu=-3,\mu\neq\pm\nu}^{3}[(q_{x,\mu\nu}+q^{\star}_{x,\mu\nu})+1]=0_{\text{mod}3}$.

## APPENDIX B: ''JACOBIAN'' OF TRANSFORMATION TO COLLECTIVE VARIABLES: ''AVERAGE SPINS''

The quasiaverage for a dynamical quantity $f(s_x)$ is defined as

$$\langle f(s_x)\rangle_{\overline{s}}=\frac{\left\langle f(s_x)\,\delta\left(\overline{s}N-\sum_{x=1}^{N}s_x\right)\right\rangle}{\left\langle \delta\left(\overline{s}N-\sum_{x=1}^{N}s_x\right)\right\rangle}. \tag{B1}$$

The conventional expectation value in statistical mechanics will then be

$$\langle f(s_x)\rangle=\int\langle f(s_x)\rangle_{\overline{s}}\,d\overline{s}. \tag{B2}$$

We should replace all spins with ''average spins''

$$\overline{s}=\frac{1}{N}\sum_{x=1}^{N}s_x. \tag{B3}$$

In this Appendix we compute the ''Jacobian'' $e^{\mathcal{L}N}=\Xi(\overline{s})=\mathrm{Tr}_{\{s\}}\delta(\overline{s}N-\sum_{x=1}^{N}s_x)$ of the transformation to the new variables $\overline{s}$. For the $Z(2)$ theory we have

$$\Xi=\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{-i\overline{s}N\phi+NL}=\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{-iN\overline{s}\phi}[e^{-i\phi}+e^{i\phi}]^N=\sum_{k=0}^{N}\binom{N}{k}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{-iN\phi(1+\overline{s})}e^{2i\phi k}=\sum_{k=0}^{N}\binom{N}{k}\delta_{2k}^{N(1+\overline{s})}$$

$$=\frac{N!}{\{N[(1+\overline{s})/2]\}!\{N[(1-\overline{s})/2]\}!}, \tag{B4}$$

and taking into account that $\ln N!\sim N\ln N-N$ for $\mathcal{L}$ we get

$$\mathcal{L}=-\frac{1}{2}\ln(1-\overline{s}^{\,2})+\frac{\overline{s}}{2}\ln\frac{1-\overline{s}}{1+\overline{s}}. \tag{B5}$$

By analogy we proceed with the ''Jacobian'' for the $Z(3)$ theory. For an element of the $Z(3)$ group we have

$$z=z_1+iz_2=\frac{1}{2}s_1+i\frac{\sqrt{3}}{2}s_2,$$
$$(s_1,s_2)=\{(2,0);\ (-1,1);\ (-1,-1)\},\tag{B6}$$

$$\Xi=\sum_{\{s_{1x},s_{2x}\}}\delta\left(N\overline{s}_1-\sum_{x=1}^{N}s_{1x}\right)\delta\left(N\overline{s}_2-\sum_{x=1}^{N}s_{2x}\right)=\int_{-\pi}^{\pi}\frac{d\phi_1 d\phi_2}{(2\pi)^2}e^{-iN(\overline{s}_1\phi_1+\overline{s}_2\phi_2)+NL}$$
$$=\int_{-\pi}^{\pi}\frac{d\phi_1 d\phi_2}{(2\pi)^2}e^{-iN(\phi_1\overline{s}_1+\phi_2\overline{s}_2)}e^{2i\phi_1 N}[1+e^{-3i\phi_1}(e^{i\phi_2}+e^{-i\phi_2})]^N.\tag{B7}$$

In view of

$$(a+b)^n=\sum_{j=0}^{n}a^{n-j}b^j\binom{n}{j}$$

we get

$$\Xi=\sum_{k=0}^{N}\binom{N}{k}\int_{-\pi}^{\pi}\frac{d\phi_1 d\phi_2}{(2\pi)^2}e^{-iN\phi_1(\overline{s}_1-2)-iN\overline{s}_2\phi_2}e^{-3ik\phi_1}(e^{i\phi_2}+e^{-i\phi_2})^k$$

$$=\binom{N}{N\frac{2-\overline{s}_1}{3}}\int_{-\pi}^{\pi}\frac{d\phi_2}{2\pi}e^{-iN\overline{s}_2\phi_2}(e^{i\phi_2}+e^{-i\phi_2})^{N(2-\overline{s}_1)/3}=\binom{N}{N\frac{2-\overline{s}_1}{3}}\sum_m\binom{N\frac{2-\overline{s}_1}{3}}{m}\delta_{N\overline{s}_2}^{N(2-\overline{s}_1)/3-2m}$$

$$=\binom{N}{N\frac{2-\overline{s}_1}{3}}\binom{N\frac{2-\overline{s}_1}{3}}{N\left(\frac{2-\overline{s}_1}{6}-\frac{\overline{s}_2}{2}\right)}=\frac{N!}{\{N[(1+\overline{s}_1)/3]\}!\,\{N[(2-\overline{s}_1-3\overline{s}_2)/6]\}!\,\{N[(2-\overline{s}_1+3\overline{s}_2)/6]\}!},\tag{B8}$$

or

$$\mathcal{L}=\frac{2}{3}(1-\overline{z}_1)\ln\frac{2}{3}(1-\overline{z}_1)+\frac{1-\overline{z}_1-\overline{z}_2\sqrt{3}}{3}\ln\frac{1-\overline{z}_1-\overline{z}_2\sqrt{3}}{3}+\frac{1-\overline{z}_1+\overline{z}_2\sqrt{3}}{3}\ln\frac{1-\overline{z}_1+\overline{z}_2\sqrt{3}}{3}.\tag{B9}$$

## APPENDIX C: POTENTIAL BETWEEN TWO PROBE SOURCES

We would like to estimate the connected correlation function $\langle\sigma_x\sigma_{x+R}\rangle$ for the Ising model with different couplings in each direction within the spherical model. The crucial point is the following condition:

$$\frac{1}{N_\sigma^3}\sum_x\sigma_x^2=1.\tag{C1}$$

Then

$$\mathcal{Z}=\sum_{\{\sigma\}}\int_{c-i\infty}^{c+i\infty}\frac{d\alpha}{2\pi i}\exp\left(\alpha N_\sigma^3-\alpha\sum_x\sigma_x^2+\frac{1}{2}\sum_{x,n}\zeta_n\sigma_x\sigma_{x+n}\right),\tag{C2}$$

where $\zeta_n = \kappa_{mk}$, and the constant **c** is chosen to ensure the correctness of interchanging the integration and summation order. It means that **c** is a line to the right of all $\alpha$-singularities.

We can rewrite the partition function as

$$\mathcal{Z} = \int \frac{d\alpha}{2\pi i} e^{\alpha N_\sigma^3} \sum_{\sigma_x} e^{-(1/2)\sigma_x \mathcal{A}_{x-x'} \sigma_{x'}}, \tag{C3}$$

where

$$\mathcal{A}_{x-x'} = \alpha \delta_x^{x'} - \sum_{n=1}^{3} \zeta_n \delta_x^{x+n} = \int \left( \alpha - \sum_{n=1}^{3} \zeta_n \cos\phi_n \right) e^{i\phi(x-x')} \frac{d^3\phi}{(2\pi)^3}. \tag{C4}$$

The correlation function $\langle \sigma_x \sigma_{x+R} \rangle$ can be calculated as the derivative of the generation function over source variables,

$$\langle \sigma_x \sigma_{x+R} \rangle = \langle \sigma_0 \sigma_R \rangle = \frac{1}{\mathcal{Z}} \frac{\partial}{\partial \eta_0} \frac{\partial}{\partial \eta_R} \int d\alpha \sum_{\{\sigma\}} e^{\alpha N_\sigma^3 - (1/2)\sigma_x \mathcal{A}_{x-x'} \sigma_{x'} + \eta_x \sigma_x}, \tag{C5}$$

and after shifting integration variables we have

$$\langle \sigma_x \sigma_{x+R} \rangle = \frac{\partial}{\partial q_0} \frac{\partial}{\partial q_R} \int d\alpha e^{(1/4)\eta_x \mathcal{A}_{x-x'}^{-1} \eta_{x'}} = \mathcal{A}_R^{-1} = \int \frac{e^{-i\vec{\phi}\vec{R}}}{\alpha_0 - \Sigma_{n=1}^3 \zeta_n \cos\phi_n} \frac{d^3\phi}{(2\pi)^3} = \int_0^\infty dt e^{-\alpha_0 t} I_{R_1}(t\zeta_1) I_{R_2}(t\zeta_2) I_{R_3}(t\zeta_3), \tag{C6}$$

where $I_{R_n}(t\zeta_n)$ is the modified Bessel function of order $R_n$; $\alpha_0$ is the saddle point which is determined by the condition

$$\int \frac{d^3\phi}{\alpha_0 - \Sigma_{n=1}^3 \zeta_n \cos\phi_n} = \langle \sigma_0^2 \rangle = 1. \tag{C7}$$

At $R_n \to \infty$ the correlation function will be

$$\langle \sigma_x \sigma_{x+R} \rangle \simeq \exp\left\{ -\sqrt{2\left( \alpha_0 - \sum_{n=1}^{3} \zeta_n \right) \sum_{n=1}^{3} \frac{r_n}{a_n^2 \zeta_n}} \right\}, \tag{C8}$$

where $R_n = r_n / a_n$.

It is easily seen that the asymmetry dependence evidence (different behavior of the potential between two probe sources in different directions $n$) does not disappear even in the vicinity of the critical point ($\alpha_0^c = \Sigma_{n=1}{}^3 \zeta_n$).


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