Electromagnetic fields in Schwarzschild and Reissner-Nordström geometry: Quantum corrections to the black hole entropy

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(Received 10 June 1997; published 5 December 1997)

Using standard coordinates, the Maxwell equations in Reissner-Nordström geometry are written in terms of a couple of scalar fields satisfying Klein-Gordon-like equations. The density of states is derived in the semiclassical approximation and the first quantum correction to black hole entropy is computed by using the brick-wall model. [S0556-2821(97)06124-9]

PACS number(s): 04.70.Dy, 04.62.+v, 11.10.Wx

I. INTRODUCTION

In the last decade, much effort has been made in order to understand the deep origin of black hole entropy [1–4] first introduced by Bekenstein [5–7] (for a recent review see Ref. [8]), but widely embraced only after Hawking's demonstration of black hole thermal radiation [9–11]. Some possible interpretations and methods of calculation have been proposed, but, at the moment, none of them seems to be the true answer [2,12–16]. Here we recall the 't Hooft proposal [2,17], in which the black hole entropy is identified with the entropy of the quantum fields surrounding the black hole itself. Since the density of states approaching the horizon diverges, in order to avoid divergences in the entropy, he has to introduce a cutoff parameter of the order of the Planck length, which is interpreted as the position of a ''brick wall'' (the brick wall model). He also computed the contribution to the entropy of a Schwarzschild black hole due to a scalar field using a semiclassical approximation. After this, quantum corrections to the Bekenstein-Hawking entropy, due to a scalar field, have been computed by different methods for Schwarzschild [18,19] Reissner-Nordström [20–23], and also for Kerr-Newman [24] black holes (for a recent review on quantum corrections to black hole entropy see, for example, Ref. [19]).

Also in the case of scalar fields, due to technical difficulties, one has to make some suitable approximation. Some authors directly consider the Rindler space, which can be considered as an approximation of the Schwarzschild case for very large mass. In the Rindler space, the contribution to the entropy due to scalar and also higher spin fields have been considered in Refs. [25–27], where in particular it has been shown that, depending on the method of calculation used, the contribution of the electromagnetic field is not just twice the scalar one, but it contains some unexpected anomalous surface terms.

In the present paper we focus our attention on the electromagnetic field in Reissner-Nordström background. We solve the Maxwell equations and then compute the contribution to the black hole entropy by using the brick wall model. All results are valid for the Schwarzschild black hole in the trivial limit of vanishing charge.

Here we do not expect anomalous terms similar to those which one has in the Rindler case [25–27], since, according to the brick wall method, we consider the system outside the horizon, while the possible anomalous surface terms are localized on the horizon itself. The analysis of anomalous contributions, which require more sophisticated techniques, is out of the aim of the present paper.

Maxwell equations in Schwarzschild, Reissner-Nordström, and also Kerr metric are usually solved by using Newman-Penrose formalism (see, for example, Ref. [28] and Ref. [29] for a complete treatment of solutions in such a formalism), which is not familiar to many readers. For this reason here we prefer to use a more conventional method, which consists in solving Maxwell equations for the electromagnetic potential in standard coordinates in a suitable gauge.

The paper is organized as follows. In Sec. II we consider the Maxwell equations for the electromagnetic potential in the Reissner-Nordström background and show that they reduce to a couple of independent scalar fields, which we solve in the semiclassical approximation, following 't Hooft's original work [2,17]. In this way we easily derive the expected contribution to the Bekenstein-Hawking entropy in Sec. III.

As usual we use natural units in which $G=\hbar=c=k=1$.

II. ELECTROMAGNETIC FIELDS IN SCHWARZSCHILD AND REISSNER-NORDSTRÖM BLACK HOLES

Here we study the electromagnetic waves in the Reissner-Nordström background (the solutions in the Schwarzschild geometry will be obtained as a limiting case for $Q\to 0$), that is the nonstatic solutions of the equations

$$\nabla_i F^{ij}=0, \quad i,j=0,\ldots,3, \tag{1}$$

in the metric g_{ij} given by

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$$ds^2=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 +r^2d\sigma^2, \tag{2}$$

F^{ij} being the electromagnetic field strength, ∇_k the covariant derivative, M and Q the mass and the charge of the black hole, respectively, and finally $d\sigma^2$ is the metric on the unit sphere, which is usually written in polar coordinates $\{\vartheta,\varphi\}$, but for our purposes the (complex) stereographic coordinates are more convenient. Then we write

$$d\sigma^2=d\vartheta^2+\sin^2\vartheta d\varphi^2=\frac{4}{(1+z\bar{z})^2}dzd\bar{z}, \tag{3}$$

where

$$z=\frac{\sin\vartheta e^{i\varphi}}{1-\cos\vartheta},\quad \bar{z}=\frac{\sin\vartheta e^{-i\varphi}}{1-\cos\vartheta}. \tag{4}$$

In such coordinates the nonvanishing components of the metric read

$$g_{00}\equiv g_{tt}=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right),$$

$$g_{11}\equiv g_{rr}=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1},$$

$$g_{23}\equiv g_{z\bar{z}}=\frac{2r^2}{(1+z\bar{z})^2}=g_{32}\equiv g_{\bar{z}z}. \tag{5}$$

In terms of the electromagnetic potential A_k, Eqs. (1) become

$$\Box A_k-\nabla_j\nabla_kA^j=\Box A_k-\nabla_k\nabla_jA^j-R_{kj}A^j=0 \tag{6}$$

and finally, after some calculations we can put them in the more useful form

$$LA_k=A_j\partial_k\Gamma^j+2g^{rs}\Gamma^j_{rk}\partial_sA_j+\partial_k\nabla_jA^j, \tag{7}$$

where $\Gamma^j=g^{rs}\Gamma^j_{rs}$, $\Box=g^{ij}\nabla_i\nabla_j$ is the Dalembertian's operator, while L represents the Dalembert's operator acting on functions, that is

$$L=\frac{1}{\sqrt{|g|}}\partial_i\sqrt{|g|}g^{ij}\partial_j. \tag{8}$$

From Eq. (7), after straightforward calculations we get

$$LA_0=-2\left(\frac{M}{r^2}-\frac{Q^2}{r^3}\right)(\partial_0A_1-\partial_1A_0)+\partial_0\nabla_jA^j, \tag{9}$$

$$LA_1=-2\left(\frac{M}{r^2}-\frac{Q^2}{r^3}\right)[\partial_1A_1-(g^{00})^2\partial_0A_0]+\frac{2}{r^2}\left(1-\frac{2M}{r}\right)A_1 +\frac{2g^{23}}{r}(\partial_2A_3+\partial_3A_2)+\partial_1\nabla_jA^j, \tag{10}$$

$$LA_2=-\frac{2\bar{z}(1+z\bar{z})}{r^2}\partial_3A_2+\frac{2g^{11}}{r}(\partial_1A_2-\partial_2A_1)+\partial_2\nabla_jA^j, \tag{11}$$

$$LA_3=-\frac{2z(1+z\bar{z})}{r^2}\partial_3A_3+\frac{2g^{11}}{r}(\partial_1A_3-\partial_3A_1)+\partial_3\nabla_jA^j. \tag{12}$$

In order to select the physical degrees of freedom, now we fix the gauge $A_0=0$. In this way Eq. (9) gives the constraint

$$\nabla_jA^j-2\left(\frac{M}{r^2}-\frac{Q^2}{r^3}\right)A_1=\frac{g^{rr}}{r^2}\partial_r(r^2A_r)+g^{z\bar{z}}(\partial_zA_{\bar{z}}+\partial_{\bar{z}}A_z) =0, \tag{13}$$

while Eqs. (10)–(12) simplify to

$$LA_r=-\frac{2}{r^2}\partial_r(rg^{rr}A_r), \tag{14}$$

$$\left(L+\frac{2\bar{z}(1+z\bar{z})}{r^2}\partial_{\bar{z}}\right)A_z=\frac{2g^{rr}}{r}\partial_rA_z -\frac{2}{r}\left(1-\frac{3M}{r}+\frac{2Q^2}{r^2}\right)\partial_zA_r, \tag{15}$$

$$\left(L+\frac{2z(1+z\bar{z})}{r^2}\partial_z\right)A_{\bar{z}}=\frac{2g^{rr}}{r}\partial_rA_{\bar{z}} -\frac{2}{r}\left(1-\frac{3M}{r}+\frac{2Q^2}{r^2}\right)\partial_{\bar{z}}A_r. \tag{16}$$

It has to be noted that for any function $\psi(t,r,z,\bar{z})$ one has

$$\left(L+\frac{2\bar{z}(1+z\bar{z})}{r^2}\partial_{\bar{z}}\right)\partial_z\psi=\partial_zL\psi,$$

$$\left(L+\frac{2\bar{z}(1+z\bar{z})}{r^2}\partial_z\right)\partial_{\bar{z}}\psi=\partial_{\bar{z}}L\psi.$$

This means that the variables can be easily separated by putting $A_z=\partial_z\psi$, $A_{\bar{z}}=\pm\partial_{\bar{z}}\psi$ (note that in principle one could choose two different functions ψ, but this is unnecessary, since only the sum enters the other equations). Now, one can directly verify that two classes of independent eigenfunctions $A\equiv(A_t,A_r,A_z,A_{\bar{z}})$ of Eqs. (14)–(16), satisfying the constraint, Eq. (13), can be put in the form

$$A^{(1)}\equiv\left(0,0,\frac{1}{\sqrt{2l(l+1)\omega}}\partial_z\Phi,\frac{-1}{\sqrt{2l(l+1)\omega}}\partial_{\bar{z}}\Phi\right), \tag{17}$$

$$A^{(2)}\equiv\left(0,\sqrt{\frac{l(l+1)}{2\omega^3}}\frac{\Phi}{r^2},\ \frac{g^{rr}}{\sqrt{2l(l+1)\omega^3}}\partial_z\partial_r\Phi,\ \frac{g^{rr}}{\sqrt{2l(l+1)\omega^3}}\partial_{\bar z}\partial_r\Phi\right), \tag{18}$$

where $\Phi(t,r,z,\overline{z})=e^{-i\omega t}f(r)Y_l^m(z,\overline{z})$ is a scalar field satisfying the equation

$$\Box\Phi=\frac{2g^{rr}}{r}\partial_r\Phi. \tag{19}$$

In the tortoise coordinate

$$r^*=r+\frac{r_+^2}{r_+-r_-}\ln(r-r_+)-\frac{r_-^2}{r_+-r_-}\ln(r-r_-),$$

$$r_\pm=M\pm\sqrt{M^2-Q^2},$$

the radial part of the field satisfies the ordinary differential equation

$$\left(\frac{d^2}{dr^{*2}}+\omega^2-V_l(r^*)\right)f(r^*)=0,\quad V_l(r^*)=\frac{l(l+1)}{r^2}g^{rr}, \tag{20}$$

with the normalization property

$$\int |f(r^*)|^2dr^*=1. \tag{21}$$

$r_+=r_h$ is the radius of the horizon. The above solutions form a set of orthonormal eigenfunctions with respect to the scalar product [30,31]

$$(A^{(1)},A^{(2)})=i\int g^{ij}(A_i^{*(1)}\partial_tA_j^{(2)}-\partial_tA_i^{*(1)}A_j^{(2)})g_{z\bar z}g_{rr}dzd\overline{z}dr. \tag{22}$$

Note that the more general expression for the scalar product has been specialized to our particular case.

III. ONE-LOOP CONTRIBUTION TO THE ENTROPY

The (leading) one-loop contribution to the entropy of the black hole due to the electromagnetic field can be easily computed using the brick wall method [2,17]. The computation is parallel to the original one given by 't Hooft. In the WKB approximation, the energy spectrum is given by

$$\oint k_l(r^*)dr^*=\oint k_l(r)g_{rr}dr=2\pi\hbar n,\quad n\in\mathbb{N}, \tag{23}$$

where

$$k_l(r^*)^2=\omega^2-V_l(r^*)=E^2-\frac{l(l+1)g^{rr}}{r^2}=k_l(r)^2. \tag{24}$$

Following 't Hooft, we consider the field in the region $r_h+\varepsilon<r<R$ and suppose it to satisfy Dirichlet boundary conditions. Then, the number of eigenstates with energy smaller than E read

$$\begin{aligned}\nu(E)&=\frac{1}{\pi}\sum_l(2l+1)n\\&=\sum_l(2l+1)\int_{r_h+\varepsilon}^R\sqrt{E^2-V_l(r)}g_{rr}dr\\&\sim\frac{1}{\pi}\int_0^{r^2E^2/g^{rr}+1/4}d\lambda\int_{r_h+\varepsilon}^R\sqrt{E^2-\frac{g^{rr}}{r^2}\left(\lambda-\frac{1}{4}\right)}g_{rr}dr,\end{aligned} \tag{25}$$

where we have put $\lambda=\hbar^2(l+1/2)^2$ and $\hbar^2\Sigma_l(2l+1)\to\int d\lambda$. The extreme of integration in the variable λ is due to the fact that $k_l(r)$ has to be positive. The integration in λ can be performed and so

$$\begin{aligned}\nu(E)&\sim\frac{2}{3\pi}\int_{r_h+\varepsilon}^R\left(E^2+\frac{g^{rr}}{4r^2}\right)^{3/2}r^2g_{rr}^2dr\\&=\frac{2}{3\pi}\int_\varepsilon^{r_h}\frac{r_h^4}{x^2}\left(1+\frac{x}{r_h}\right)^4\left(E^2+\frac{x}{4r_h^3(1+x/r_h)^3}\right)^{3/2}dx\\&\quad+\frac{2}{3\pi}\int_{r_h}^{R-r_h}x^2\left(1+\frac{r_h}{x}\right)^4\\&\quad\times\left(E^2+\frac{1}{4x^2(1+r_h/x)^3}\right)^{3/2}dx\\&\sim\frac{2r_h^6E^3}{3\pi\varepsilon(r_h-r_-)^2}+\frac{VE^3}{6\pi}+\cdots,\end{aligned} \tag{26}$$

where in the latter expression only the leading divergences have been written down (V is the volume of the spherical box). The derivative of $\nu(E)$ in Eq. (26) represents the density of states with energy E.

Now for the partition function one easily gets

$$\begin{aligned}\ln Z&=-\sum_\nu\ln(1-e^{-\beta E_\nu})=\beta\int_0^\infty\frac{\nu(E)}{e^{\beta E}-1}dE\\&\sim\frac{\pi^2V}{90\beta^3}+\frac{2\pi^3r_h^6}{45\varepsilon\beta^3(r_h-r_-)^2}+\cdots,\end{aligned} \tag{27}$$

which agrees with a similar expression in Ref. [21] and reduces to the expression given in Refs. [2, 17] in the the limit $Q\to0$, that is $r_h=2M$, $r_-=0$. The first term on the right-hand side of the latter equation is the usual one proportional to the volume, while the second is a divergent contribution due to the presence of the horizon and is interpreted as a quantum contribution to the black hole entropy due to the matter field. Taking into account that we have two independent scalar fields both satisfying Eq. (19), we finally get

$$S_{\rm RN}=-(\beta\partial_\beta-1)\ln Z=\frac{16\pi^3 r_h^6}{45\varepsilon\beta^3(r_h-r_-)^2}. \qquad (28)$$

As already anticipated in the Introduction, the leading term in the one-loop contribution to the entropy due to the electromagnetic field is exactly twice the one due to the scalar field. In our derivation we do not obtain anomalous terms of the kind obtained for the Rindler case in Refs. [25–27]. In any case, as suggested in Ref. [27], such terms are nonphysical and have to be discharged.

All results of this section have a good limit for $Q\to 0$ and so they are valid also for the Schwarzschild black hole, with the simple substitution $r_h=2M,\ r_-=0$.

At the equilibrium temperature $T_H=r_h-r_-/4\pi r_h^2$ the entropy reads

$$S_{T=T_H}=\frac{\sqrt{M^2-Q^2}}{90\varepsilon}\sim\frac{1}{45}\left(\frac{r_h}{l}\right)^2, \quad l^2=\frac{4r_h^2\varepsilon}{r_h-r_-}, \qquad (29)$$

where the cutoff parameter ε has been expressed in terms of the proper distance l.

IV. CONCLUSION

We have written the Maxwell equations in the Reissner-Nordstöm background in terms of a couple of scalar fields satisfying a Klein-Gordon-like equation. As an application of such a nice result, Eq. (19), we have computed the first quantum correction to the black hole entropy due to the electromagnetic field, by using the semiclassical approximation. As it was expected, the leading term is exactly twice the one that one has for a scalar field.

It is not the aim of the present paper, but of course it would be interesting to go on in the approximation in order to analyze the possible existence of anomalous surface contributions similar to those which one has in the Rindler case [25–27]. Such an analysis is easily performed in Rindler, since in this case one knows the exact solutions of scalar field equations, while, for Schwarzshild, the solutions of Eq. (19) are not known exactly and so one has to find some useful approximation.

ACKNOWLEDGMENTS

It is a pleasure to thank V. Moretti, L. Vanzo, and S. Zerbini for many discussions and suggestions.

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