Global view of kinks in 1+1 gravity

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Following Finkelstein and Misner, kinks are nontrivial field configurations of a field theory, and different kink numbers correspond to different disconnected components of the space of allowed field configurations for a given topology of the base manifold. In a theory of gravity, nonvanishing kink numbers are associated with a twisted causal structure. In two dimensions this means, more specifically, that the light cone tilts around (nontrivially) when going along a noncontractible non-self-intersecting loop on spacetime. One purpose of this paper is to construct the maximal extensions of kink spacetimes using Penrose diagrams. This will yield surprising insights into their geometry but also allow us to give generalizations of some well-known examples such as the bare kink and the Misner torus. However, even for an arbitrary 2D metric with a Killing field we can construct continuous one-parameter families of inequivalent kinks. This result has already interesting implications in the flat or de Sitter case, but it applies, e.g., also to generalized dilaton gravity solutions. Finally, several coordinate systems for these newly obtained kinks are discussed. [S0556-2821(97)05624-5]

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I. INTRODUCTION

About 40 years ago, Finkelstein and Misner considered integer-valued quantities that are conserved during time evolution as they are protected by a topological index [1]. These quantities, which may be used to characterize a field configuration of an appropriate field theory, were named ‘‘kinks’’ thereafter [2]. The idea of kinks is simple and by now standard: Suppose you are dealing with a field theory where the fields take values in a space Ω of nontrivial topology (such as, e.g., in a σ model; in gravity this nontriviality results from the required signature of the metric). Now consider the map Φ from a $t=$const hypersurface Σ into Ω given by the initial values of the field(s). If Σ has nontrivial topology (possibly due to boundary conditions imposed on the fields on an originally trivial space), it may well happen that there is more than one homotopy class in $H(\Sigma,\Omega)$. In this case the initial data, respectively Φ, single out some element $h \in H(\Sigma,\Omega)$. As time evolution is a smooth deformation of the map Φ, it will not move Φ out of its original homotopy class h. Thus, h is a conserved quantity and for $h\neq 0$ (0 denoting the trivial homotopy class defined by the constant map) the field configuration is said to have a kink (characterized by h).

In 3+1 gravity on a spacetime $\Sigma\times\mathbb{R}$ with Σ being a 3-sphere, one has $H(\Sigma,\Omega)=\pi_3(\Omega)=\mathbb{Z}$, the group of all integers [1,3]. The situation is unchanged if $\Sigma=\mathbb{R}^3$ and one requires spacetime to be (appropriately) Minkowskian asymptotically. Such spacetimes are characterized by a kink number $k\in\mathbb{Z}$ therefore. In subsequent works then it has been shown that all spacetimes with $k\neq 0$ have a ‘‘twisting light-cone structure’’ and gravitational kinks were in part viewed as ‘‘black holes without curvature singularities’’ (cf., e.g., [4–6]). This connection of homotopical considerations with those concerning the causal structure becomes most transparent for (1+1)-dimensional spacetimes, which, as often, may serve as a suitable laboratory to improve one’s understanding of the role of kinks in gravitational theories [7]. Here $\det g \equiv g_{00}g_{11}-g_{01}^2\neq 0$ separates $\mathbb{R}^3$ (the space of real symmetric matrices) into three regions characterized by the signatures $(++)$, $(--)$, and $(+-)$, respectively. The latter of these regions is Ω. With $\Sigma=S^1$ one obtains $H(\Sigma,\Omega)=\pi_1(\Omega)=\mathbb{Z}$, so that there again is a winding number k characterizing kinks (cf. Fig. 1 of [8] for a nice illustration). On the other hand, given an explicit kink metric, such as, e.g., the ‘‘bare kink’’ 2) [7,9],

$$g=-\cos 2x\,dt^2-2\sin 2x\,dt\,dx+\cos 2x\,dx^2,$$

it is easily verified that the light-cone turns upside down k times when going from $x=0$ to $x=k\pi$ along a ($t=$const) line Σ [cf. Fig. 1(a)]. Each such half-turn of the light cone clearly defines a noncontractible loop in Ω, which may serve as generator of $\pi_1(\Omega)$.[1]

In the literature 1+1 kink metrics have often been written down in explicit coordinates only [6–8]. Their kink nature is then usually shown by studying the behavior of the light

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[1]Here we always considered spacetimes $\mathcal{M}$ of the form $\mathcal{M}=\Sigma\times\mathbb{R}$ with $\Sigma=S^3,\mathbb{R}^3,S^1$, or $\mathbb{R}^1$, all of which are parallelizable. In the case of a more general, not parallelizable spacetime manifold $\mathcal{M}$ the metric is a section of a nontrivial bundle and the above homotopical considerations have to be modified accordingly.

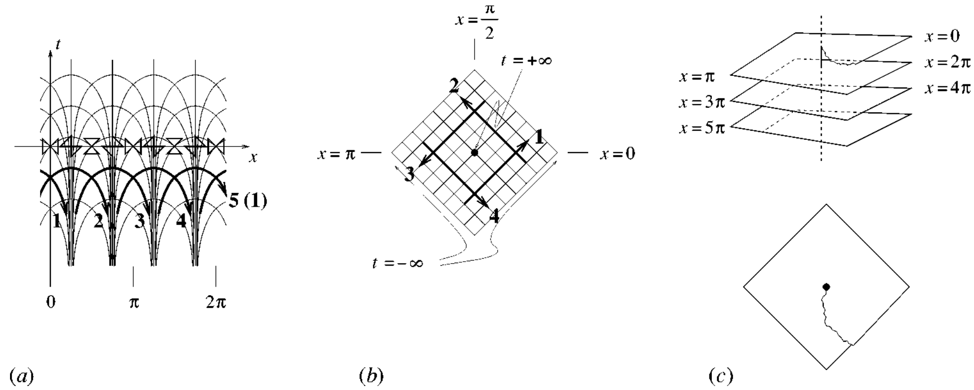

FIG. 1. The kink metric (2) and its conformal derivates [such as, e.g., Eq. (9)]: (a) shows the light-cone structure and the null extremals in the original x,t coordinates. In a Penrose diagram of the corresponding coordinate patch [(b), (c)] they play the role of polar coordinates, x being the angle and t some radial coordinate which goes $\to +\infty$ near the origin (for the extended Penrose diagram cf. Fig. 2). The 2-kink manifold (b) is obtained from (a), respectively, (c), by applying the identification $x \sim x+2\pi$, thereby mapping, e.g., the null extremal **5** onto **1**.

cone as sketched briefly in the example above. Consequently, one rarely finds any global analysis of the resulting spacetimes (the papers of [10] being a positive exception, where the kink number of the boundary components of a 3+1 spacetime $\mathcal{M}$ is shown to be directly related to the Euler characteristic of $\mathcal{M}$). In the present paper we want to fill this gap at least partially. Moreover, our analysis will lead to a systematic way of constructing new kink spacetimes. In particular it will enable us to construct, e.g., a one-parameter family of distinct kinks of given kink number for *any* given 1+1 metric with a local Killing symmetry. Kinks with $\Sigma = \mathbb{R}$ instead of $\Sigma = S^1$ may also be obtained, provided only that the metric allows for appropriate asymptotic regions (say, asymptotically flat or de Sitter); however, in these cases the kink number will turn out not to be an intrinsic property of the spacetime but rather a feature of the chosen coordinates.

While there is no problem in just writing down kink metrics [cf. e.g., Eq. (2)], the more interesting cases are certainly those where the metric fulfills some extra conditions. As an example, many flat or de Sitter kink metrics (also in 3+1 dimensions) have been studied [6–8,11]. At first sight, however, these solutions seem to be in conflict with a well-established approach: Any maximally extended multiply connected spacetime should occur as a factor space of the simply connected *universal covering* solution, but neither from Minkowski space nor from the universal covering of de Sitter space could kink solutions be obtained in this way [12,13]. This apparent paradox is resolved by noting that — even when requiring simple connectedness — there is no unique maximal (analytic) extension of a manifold (such a warning has, e.g., already been expressed in footnote 15 of [14]): Take, for instance, Minkowski space, cut out a point, and construct the universal covering of this punctured plane, which now winds around the removed point infinitely often in new layers [cf. e.g., Fig. 5(d)]. Clearly, this solution is no longer geodesically complete at the removed point but nevertheless maximally extended, since adding the point again would yield a conical singularity. Identifying different layers of this manifold, maximally extended $(k \neq 2)$-kinks can be obtained; only the corresponding 2-kink is extendible, since insertion of the point restores Minkowski space.

Certainly, these latter kink solutions (and in fact all flat kinks or kinks of constant curvature) are never geodesically complete. On the other hand, there are plenty of complete kinks.[2] In the case that they have a Killing symmetry, a complete classification has been provided in [13]. In Sec. II we will show by the example of the bare kink how searching rigorously for maximal extensions and applying the factorization method of [13] allows us not only to shed some light on the geometry of these kinks but even to derive a whole bunch of related new ones. The same concepts — when applied to ''incomplete universal coverings'' such as the one mentioned above — will also yield more interesting examples of incomplete kinks. This is demonstrated for the flat case in Sec. III and generalized to arbitrary metrics with a Killing field in Sec. IV. In Sec. IV (as well as at the end of Sec. II and in Appendix A) we will also provide explicit coordinate representations for the newly obtained kink metrics.

Much of the interest in spacetimes with nontrivial kink number centers around such spacetimes which are locally solutions to the field equations of some gravity model, i.e., of some appropriate gravity action. Since, e.g., all solutions of generalized dilaton gravity models have a Killing field [16–18], the scheme of the present paper allows for the construction of kink spacetimes for any of these models. But also conversely, for any given metric with a (local) Killing symmetry and hence also for all the kink metrics it gives rise to (cf. Secs. II and IV), there is some gravity action for which the metric solves the corresponding field equations [18]. Let us thus briefly recollect some results about those metrics. Here it is advisable to use a nonconformal gauge for the metric, in contrast to what is useful on other occasions such

[2] A simple example is obtained from a metric of the form (1) with one triple zero of $h(r)$ and $h(r) \sim r^{n \leq 2}$ for $r \to \pm\infty$ [e.g., $h(r) = r - \arctan r$]. As shown in [13,15] the maximal extensions of this metric are geodesically complete kinks of arbitrary kink number. If $h(r) \sim r^{n>2}$, these kinks are no longer geodesically complete but nevertheless inextendible, since the curvature diverges at the incomplete boundaries then (e.g., the solutions **G4** and **R2** in [13,15]).

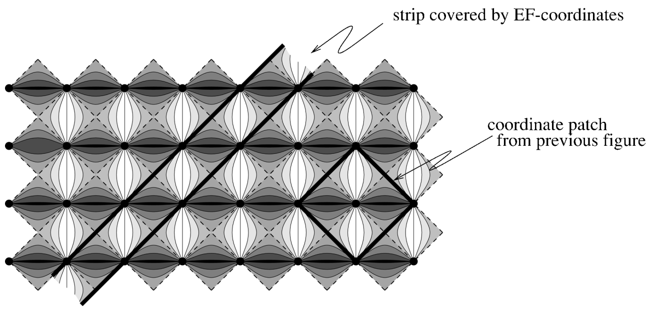

FIG. 2. A maximal extension of the bare kink, to be infinitely extended. The shading corresponds to the curvature ranging over $-4 \leqslant R \leqslant 4$, the thin $R=\text{const}$ lines being at the same time Killing trajectories for the field ∂_t. The coordinate patch (2) (with the identification $x \sim x+2\pi$) is shown, and also the patch covered by EF coordinates (1) with $h(r)=\cos 2r$. The universal covering would be obtained, if at every solid circle (indicating points at an infinite distance) the manifold were continued into a new overlapping layer as sketched in Fig. 1(c).

as in string theory, where the action is invariant under rescalings of the metric by a conformal factor:[3] Any two-dimensional (2D) metric with a (local) Killing field may be represented locally in the generalized Eddington-Finkelstein (EF) form

$$g = 2 dr dv + h(r) dv^2 \tag{1}$$

for some function h (cf., e.g., [15]). The Killing field in these coordinates is clearly ∂_v, its length squared equals h, and the curvature scalar is $R=h''$. For a given metric the function h in Eq. (1) is generically unique up to an equivalence relation $h(r) \sim b^2 h(r/b+a)$, $a,b=\text{const}$. Only for Minkowski and de Sitter space is this not quite true, because they have more than one Killing field (in fact three). For instance, Minkowski space can be described by $h^{\text{Mink}}(r)=ar+b$, where different choices of $h^{\text{Mink}}(r)$ may correspond to qualitatively different Killing fields ∂_v: $h^{\text{Mink}}(r)=b$ implies that ∂_v generates translations (timelike, null, or spacelike, according to sgn b), whereas for h^{Mink} linear in r ($a \neq 0$) the vector field ∂_v generates boosts.[4] Likewise, (anti–)de Sitter space of curvature R is described by $h^{\text{de S}}(r)=(R/2)(r+a)^2+b$, and again there are three qualitatively different Killing fields ∂_v according to sgn b (cf., e.g., [13,15]).

If the metric g is analytic, furthermore, then the function h is characteristic for the whole spacetime and there is a unique, analytic, simply connected extension where the boundary (to be defined properly) is either complete or a curvature singularity (called ''global'' in [13]; without such a requirement the extension is not unique, as mentioned before). An exposition of simple rules of how to obtain this extension and its Penrose diagram from a given function h may be found in [15]. The multiply connected global solutions can be obtained by factoring these universal coverings by discrete symmetry groups or, equivalently, by cutting out some region of the universal covering and gluing appropriately [13]. Whereas the first approach is favorable for a concise classification, the second one is more straightforward and shall be employed here.

II. BARE KINK, ITS PENROSE DIAGRAM, AND GENERALIZATIONS

An in some sense prototypical example of a kink metric is the ''bare kink'' [7,9]

$$g = -\cos 2x\, dt^2 - 2 \sin 2x\, dt\, dx + \cos 2x\, dx^2. \tag{2}$$

Its null extremals are calculated easily to

$$\frac{dt}{dx} = -\tan\left(x \pm \frac{\pi}{4}\right), \tag{3}$$

and it is thus clear that the light cone tilts with increasing x [cf. Fig. 1(a)]. In order to obtain a Penrose diagram it is advisable to interpret x and t as polar coordinates,

$$\tilde{x} = e^{-t} \cos x, \quad \tilde{t} = e^{-t} \sin x, \tag{4}$$

which brings Eq. (2) immediately into conformally flat form:

$$g = \frac{d\tilde{t}^2 - d\tilde{x}^2}{\tilde{t}^2 + \tilde{x}^2}. \tag{5}$$

Note that when applying Eq. (4) and thus also in the Penrose diagram Fig. (1(b) we have tacitly assumed that the identification $x \sim x+2\pi$ is made. If this is not desired, then there will occur overlapping layers as displayed in Fig. 1(c). Obviously any kink number k can be obtained from this manifold by imposing the identification $x \sim x+k\pi$. While for even kink numbers this amounts to identifying overlapping layers in Fig. 1(c), odd kink numbers involve a point reflection (inverting space and time).

This patch is, however, still incomplete. The central point $t=+\infty$ is at an infinite affine distance, but the null infinities at $t \rightarrow -\infty$ are incomplete. A maximal extension can be obtained using Eddington-Finkelstein coordinates and following the recipe of [15],[5] or, as is even simpler in the present

[3]The use of *nonconformal* gauges proves to be especially powerful in the presence of a Killing field. As such, gauges closely related to Eq. (1) below have been used with success in the literature [16,19,20]. In particular we use this opportunity to gratefully acknowledge here the influence of W. Kummer on our work. Bringing to our attention the success of nonconformal gauge conditions in a 2D gravity model [20] was essential for our interest in two-dimensional gravity theories, culminating finally in a series of papers on this subject.

[4]In this respect the latter EF coordinates, e.g., $g=2drdv+rdv^2$, resemble the Rindler coordinates [21] $g=x^2dt^2-dx^2$ [substitute $r=x^2/4$, $v=2(t-\ln x)$], where also ∂_t generates boosts. However, the EF coordinates cover a considerably larger portion of Minkowski space [cf. Fig. 8(b)].

[5]Substituting $x=r+\pi/2$, $t=\ln|\cos r-\sin r|-v$ into Eq. (2), g is easily brought into the form (1) with $h(r)=\cos 2r$. This means, according to [15], that the building block (as defined there) is infinite, periodic, with nondegenerate horizons, and the maximal extension (if identifying overlapping layers) is the chessboardlike arrangement of Fig. 2.

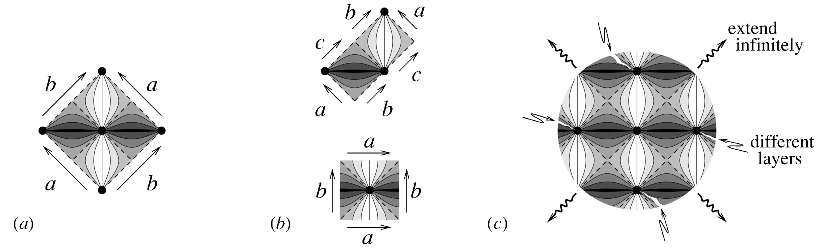

FIG. 3. Some further maximal extensions of the bare kink. (a) yields a torus with two holes, whereas (b) shows two equivalent constructions of a torus with one hole. It is also possible to obtain a globally cylindrical extension of the original kink, by repeating the construction of the universal covering; i.e., at any complete point except for the very first one the manifold has to be extended into different overlapping layers (c).

context, by applying the transformation $\tilde{x}\pm\tilde{t}=\tan(\hat{x}\pm\hat{t})$, which yields[6]

$$g=\frac{d\hat{t}^2-d\hat{x}^2}{\sin^2\hat{t}+\sin^2\hat{x}}. \tag{6}$$

The scalar curvature for this metric,

$$R=4\frac{\cos 2\hat{t}-\cos 2\hat{x}}{-2+\cos 2\hat{t}+\cos 2\hat{x}}, \tag{7}$$

ranges between $R=-4$ at $\hat{t}=n\pi$ (white regions in Fig. 2) and $R=4$ at $\hat{x}=n\pi$ (dark shaded regions). At $\hat{t},\hat{x}=n\pi$ the metric becomes singular, and these points (solid circles in Fig. 2, where the Killing trajectories meet) are at an infinite distance. Thus to obtain the universal covering the overlapping sectors after surrounding those points should *not* be identified [cf. Fig. 1(c)]. Disregarding this multilayered structure (after all, the 2-kink is single-layered), Eq. (6) provides a global chart for the bare kink.

What can be learned from this representation, taking into account the method of [13]? First, even when starting from a cylindrical kink, the maximal extension need not be a cylinder. Already the manifold displayed in Fig. 2 provides a counterexample, but many more may be found: For instance, gluing together opposite faces of the diamond-shaped coordinate patch yields a torus with two holes, Fig. 3(a), and likewise Fig. 3(b) yields a torus with one hole. On the other hand, proper topological cylinders will always be found among the covering solutions: Just start from the original punctured diamond but then proceed in the same way as if constructing the universal covering; i.e., whenever surrounding a complete point (solid circle) start with a new overlapping layer [Fig. 3(c)]. Topologically, this infinitely branching extension cannot be distinguished from a cylinder, though, admittedly, the causal structure near the exterior ''frazzled'' boundary will be rather involved. And many more cylindrical kinks can be obtained in a similar manner: Start from *any* noncontractible closed ribbon like A or C in Fig. 4 (A giving rise to the kink described previously), respectively any open ribbon covering (part of) the manifold like D with its short edges identified. Whenever the overall number of tilts of the light cone when going along the ribbon does not vanish, this is a kink manifold (cf. the Introduction) and proceeding as before a maximal cylindrical extension may be constructed. It is straightforward to give an explicit atlas for these manifolds, using EF coordinates (1) for straight segments and, e.g., Eq. (2) for the bent ones.

Second and perhaps even more surprising, a continuous family of locally equivalent but globally different kinks can be obtained. Note that (in the spirit of [13]) already the original k-kink (2) is obtained by identifying overlapping sectors of the universal covering (i.e., gluing $x\sim x+k\pi$). However, since the bare kink has a Killing symmetry [∂_t in coordinates (2)], one may apply a Killing transformation prior to the gluing. This amounts to an identification $(x,t)\sim(x+k\pi, t+\pi k\alpha)$ in Eq. (2); i.e., in Fig. 1(a) a vertical shift of length $k\pi\alpha$ is applied before gluing. Alternatively, one could as well substitute $t\to t+\alpha x$ into Eq. (2), while sticking to the original identification $(x,t)\sim(x+k\pi,t)$. The metric for these ''not-so-bare'' kinks can then be written as

$$g=-\cos 2x\,dt^2-2(\sin 2x+\alpha\cos 2x)dt\,dx \\ +[(1-\alpha^2)\cos 2x-2\alpha\sin 2x]dx^2. \tag{8}$$

For a proof that these solutions are not isometric for different α and for a geometrical characterization of this parameter cf. [13]. Of course, the same construction applies also to the more elaborate kinks discussed before (such as those obtained from C, D of Fig. 4), introducing a continuous param-

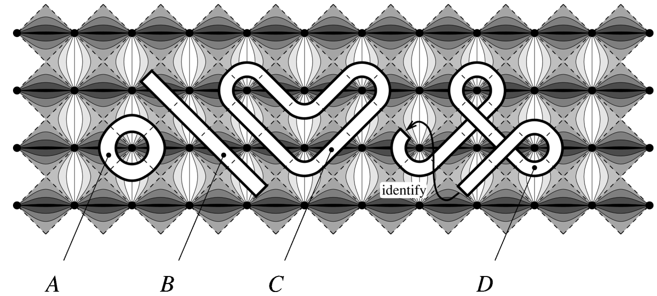

FIG. 4. Further manifolds derived from the bare kink. Whereas A and C are 2-kinks (cylindrical, if extended suitably), D with the short edges identified depicts a 4-kink. Identifying the opposite faces of strip B yields an incomplete ''Taub-NUT'' torus resembling in this respect the Misner torus, which is obtained from A by gluing the inner to the outer boundary. Similar incomplete tori may be constructed from C and D (again identifying inside and outside).

[6]A similar form of the metric has also been found in [22].

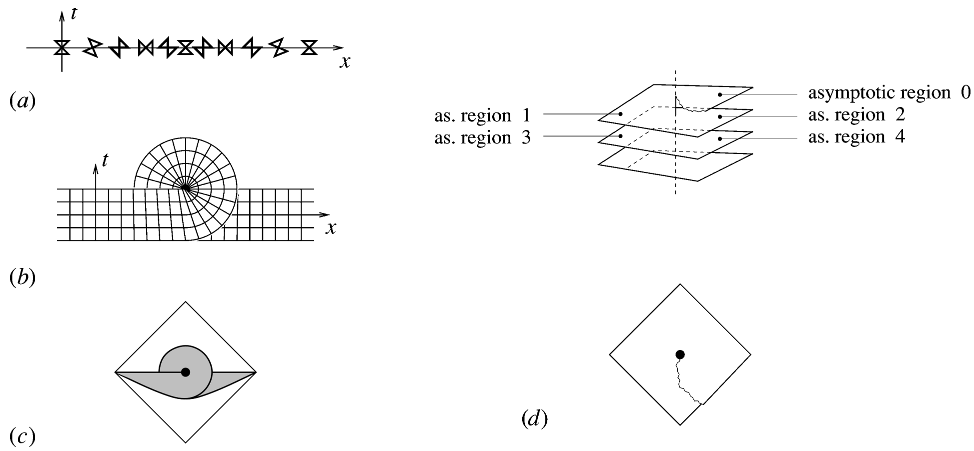

FIG. 5. Flat 2-kink of topology $\mathbb{R}\times\mathbb{R}$: (a) shows the light-cone structure in the original x,t coordinates, (b) an ''embedding'' (allowing for overlapping layers) of this patch into Minkowski space, and (c) the corresponding Penrose diagram. Its maximal extension, however, contains infinitely many asymptotic regions (d) and thus subsets of arbitrary kink number. Note that (d) represents an incomplete but inextendible simply connected flat manifold different from Minkowski space.

eter also there, and furthermore to noncylindrical extensions, where each generator of $\pi_1(\mathcal{M})$ acquires a parameter.

According to [13] the above constructions [i.e., gluing isometric sectors of the universal covering, thereby introducing a continuous parameter for each generator of $\pi_1(\mathcal{M})$] exhaust the complete manifolds obtained from the bare kink metric. However, there are further incomplete yet inextendible manifolds: One such example is the Misner torus (see the following paragraph) which could only be extended when abandoning the Hausdorff property. But also the constructions of Sec. IV can be applied [leading to Eq. (12) with $h(r)=\cos 2r$], in which case smoothness prohibits a further extension of the resulting kinks.

Finally, we shortly touch the case of another manifold which can be derived from the bare kink metric: the Misner torus [9,23]. It is obtained by also wrapping up periodically the coordinate t in Eq. (2), $t\sim t+\omega$ (besides $x\sim x+2\pi$), or equivalently from the annulus A shown in Fig. 4 by identifying the inner and outer boundary along the Killing trajectories (thin lines). This metric, although well behaved everywhere on a compact manifold, is incomplete near the Killing horizons (dashed lines), the singularity being of the Taub-Newman-Unti-Tamburino-(NUT-) type [23,24]. Given the extended bare-kink manifold, however, many more tori of that type can be obtained. For instance, gluing opposite faces of the strip B yields another torus with similar pathological completeness properties.[7] Explicit coordinates for the torus B are easily obtained from the EF coordinates $g=2drdv+\cos 2r dv^2$ by identifying $r\sim r+2\pi$ as well as $v\sim v+\omega$. But even more exotic specimens may be constructed, e.g., starting from C or D and gluing again the two boundaries. And certainly the parameter α from Eq. (8) can be introduced also here; together with ω this yields the two parameters for the two generators of $\pi_1(\text{torus})=\mathbb{Z}^2$.

[7]The tori obtained from A and B are indeed inequivalent: Whereas in B there are complete null extremals (the ones running alongside of the displayed strip), this is not the case in Misner's example A.

III. FLAT KINKS

Adding a simple conformal factor to the metric (2) yields quite a different example. The resulting metric

$$g=e^{-2t}(-\cos 2x dt^2-2\sin 2x dt dx+\cos 2x dx^2) \quad (9)$$

has the same tilting-light-cone structure as Eq. (2) and is thus also a kink metric. Calculating the curvature of this metric shows, however, that it is actually flat: $R\equiv 0$.

Again the transformation (4) clarifies the situation [as before we choose the 2-kink version of Eq. (9), i.e., identify $x\sim x+2\pi$]: This time it leads to

$$g=d\tilde{t}^2-d\tilde{x}^2, \quad (10)$$

and so this is nothing but flat Minkowski space in polar coordinates, the origin being removed. In contrast to Eq. (2) the metric (9) is incomplete at the origin; it has a hole which can (and thus should) be filled by inserting a point, leaving ordinary Minkowski space without any kink. In retrospect thus Eq. (9) is a rather blunt construction of a kink. In fact,

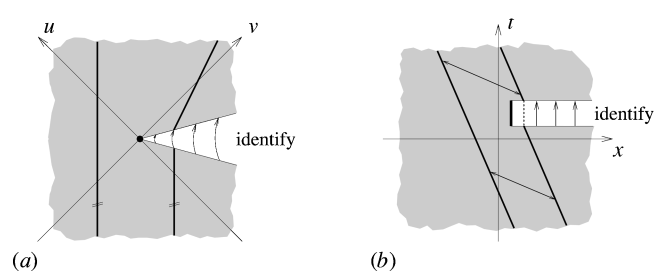

FIG. 6. (a) Minkowski kink with nontrivial holonomy. This space can be obtained by removing a wedge from flat Minkowski space and gluing together the corresponding boundary lines by a boost. As a result of this construction, two extremals which are parallel on one side of the origin are mutually boosted on the other side (cf. bold lines). Thus the holonomy is nontrivial (surrounding the origin yields a boosted frame), and at the origin there would occur a conical singularity. (b) Another Minkowski kink; it has trivial holonomy but the distance of parallels passing the hole changes (for a possible maximal extension cf. Fig. 12).

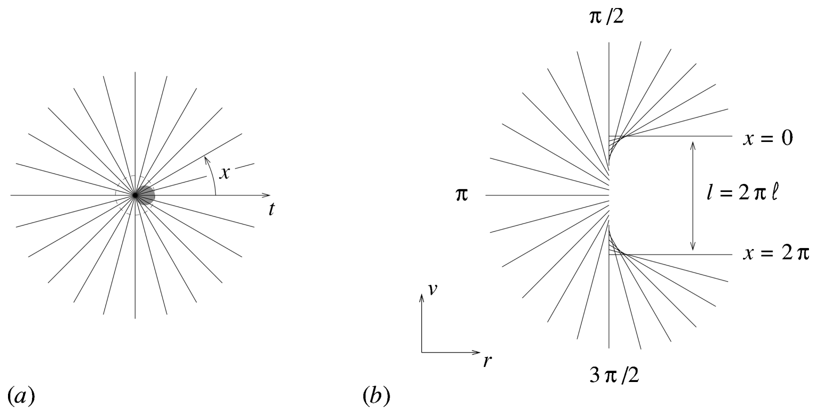

FIG. 7. The cutting procedure (11) for $n=1$ (2-kink). The patch (a) carries polar coordinates. The radii are mapped onto the corresponding lines in the diagram (b), where the EF metric (1) lives. The pullback of this metric is the desired kink metric. It is well defined everywhere in (a) except for the small shaded disk indicated there. While the transformation as such is discontinuous at $x=0,\ 2\pi,\ \ldots$, this does not matter for the resulting metric, since everything is v independent.

any nice no-kink manifold gives rise to many such kinks simply by cutting holes into it: Since the light cone tilts upsidedown twice when surrounding the hole, this is a 2-kink solution. Taking covering spaces arbitrary even kink numbers can be obtained (odd kink numbers may occur, if there is a point-reflection symmetry). Of course, these ($k \neq 2$)-kinks are no longer extendible, because at the inserted point there would occur a conical singularity (branch point) and the extension could not be smooth. Introducing polar coordinates (4) it is straightforward to write down explicit charts for these kinks (identifying $x \sim x + k\pi$ again). Nevertheless, this construction seems rather artificial.

Note that in a similar manner ''kinks'' of topology $\mathbb{R} \times \mathbb{R}$ may be obtained. This refers to coordinates x,t where the light cone tilts a couple of times when going from one asymptotic region, $x \to -\infty$, to the other, $x \to \infty$ [cf. Fig. 5(a) for kink number 2]. However, in 2D the maximal extension of such a kink usually has an infinite series of potential asymptotic regions [cf. Fig. 5(d)] and certainly one could as well choose coordinates which wrap around the origin more often before settling down in an asymptotic region. Thus ($\mathbb{R} \times \mathbb{R}$)-kinks of arbitrary kink number can be obtained (in contrast to the conclusion of [8]). On the other hand, this kink number merely characterizes the coordinates, not the (extended) spacetime itself, which is always of the form of Fig. 5(d) (or a cylindrical kink of sufficiently large kink number);[8] for these reasons we dismiss this topic and return to cylindrical kinks again.

It has been pointed out already that there are no geodesically complete flat kinks. The kinks (9), on the other hand, are at least inextendible (due to the conical singularity) except for the $k=2$ case, which in some sense reveals their construction (cutting a hole). However, in the presence of a Killing field — and in the case of Minkowski space there are three independent Killing fields — we can do more than just cutting out points or regions. This will allow us to construct continuous families of inequivalent flat kinks, which are inextendible even in the $k=2$ case. Moreover, the procedure allows for a straightforward generalization to any metric with a Killing field, as will be shown in Sec. IV.

Start from Minkowski space, but instead of only removing the origin cut out a whole wedge [cf. Fig. 6(a)]. By means of a Lorentz boost the two edges of the wedge can be mapped onto one another, and we use this boost to glue the remaining patch together (note that also the tangents at these edges must be mapped with the tangential map of this boost). Clearly such a space is everywhere flat (except at the origin, which is considered not to belong to the manifold) but it has nontrivial holonomy. For instance, two timelike extremals which are parallel ''before'' passing the origin at different sides will be mutually boosted afterwards [bold lines in Fig. 6(a)]. Also, because of the nontrivial holonomy, the origin can no longer be inserted [as was possible for the trivial Minkowski 2-kink (9)], since there would occur a conical singularity. Thus a continuous one-parameter family of 2-kinks is obtained, labeled by their holonomy (i.e., boost-parameter or angle of the removed wedge). Changing the sign of the boost parameter corresponds to *inserting* a wedge or, equivalently, to removing a ''timelike'' wedge. Of course, taking covering solutions (with an additional factoring by a point reflection for odd k) arbitrary kink numbers k can be obtained, and for a given k they are again characterized by the additional boost parameter.

Because of the nontrivial holonomy around the origin, there is certainly no global chart on the punctured plane where g takes the standard Minkowski form $g=2dudv$ [or Eq. (10)]. But there is not even a (globally smooth) conformal chart in this case. This may be seen as follows: A metric in conformal gauge is flat, iff the conformal factor equals a product of two functions of the light-cone coordinates u and v: $g=f_1(u)f_2(v)dudv$. But any such metric can be extended smoothly into the origin and is flat there. This contradicts the assumption of a nontrivial holonomy (which necessarily entails a conical singularity at the origin). Still, a smooth conformal coordinate system may be found on a cylinder which only misses an arbitrarily small square around the puncture. This chart will be provided in Appendix A; cf. Eq. (A1). We remark here only that in the limit of shrinking the square to the origin, the conformal factor given in Appendix A becomes distributional (nonsmooth even outside the origin).

In the above example we have chosen the boosts centered at the origin as Killing symmetry. However, Minkowski space also exhibits translation symmetries. An analogous construction can be applied in this case, too, with the following geometrical interpretation: Cut out a whole slit (in direction of the chosen translation), remove the strip on one side of the slit, and glue together the resulting faces [cf. Fig.

[8]This is a 2D artifact, however: In higher dimensions the asymptotic region is usually topologically sufficiently nontrivial to allow for ''genuine'' kink numbers *within* this region (if connected) or within its connected components, respectively.

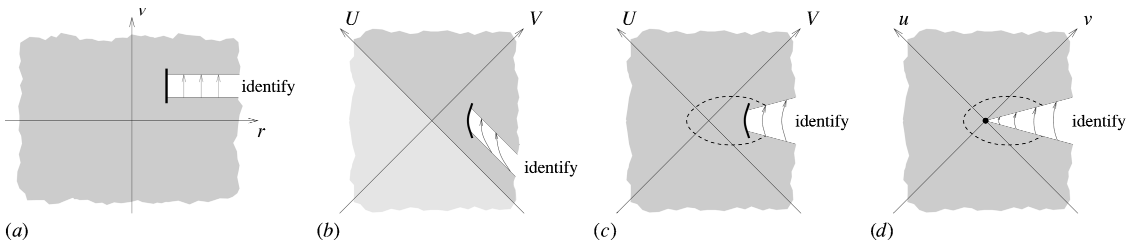

FIG. 8. Relationship between the kinks (12) and Fig. 6(a): The EF coordinates (a) with $h^{\rm Mink}(r) \propto r$ cover half of Minkowski space (b). Instead of removing a strip bounded by null lines, one can remove a wedge (c). On a cylindrical region outside the dashed ellipse this manifold coincides with the Fig. 6(a) ''boost'' kink (d).

6(b)]. This manifold has now trivial holonomy, but the metric distance of two generic parallels passing the hole changes. Thus the manifold is so badly distorted that it cannot be completed to ordinary Minkowski space, either. Also for this case smooth conformal coordinates on a cylindrical region can be given, Eq. (A6), but even analytic coordinates may be constructed, as will be demonstrated in the following section. However, in contrast to the former example this space can still be smoothly extended further, leaving behind only branch points. This is shown in detail in Appendix B; cf. Fig. 12.

IV. KINKS FROM ARBITRARY METRICS AND EXPLICIT COORDINATES

In the previous section several examples of flat kinks have been constructed. Clearly, since any 2D metric is locally conformally flat, some of these methods will be applicable also to nonflat metrics: For instance, to obtain a $2n$-kink from an arbitrary metric choose locally a conformal gauge, change to polar coordinates (4), and identify $x \sim x + 2\pi n$, leading to Eq. (9) with another conformal factor. Again, however, only the resulting $k \neq 2$-kinks (i.e., $n \neq 1$) will be inextendible. The more elaborate constructions of Fig. 6, on the other hand, may be transferred only if the metric in question has a Killing symmetry. (Note that this is, e.g., the case for practically all 2D vacuum gravity models; cf. [18]). Using, for instance, local EF coordinates (1), the construction of Fig. 6(b) can be applied literally, with (r,v) taking the role of (x,t), since the Killing field ∂_v generates vertical translations then.

Remarkably, it is even possible to obtain an explicitly analytic chart for such a kink (again at the cost of enlarging the hole slightly). Let (r,v) denote the original (EF) coordinates, (x,t) the new kink coordinates, and, without loss of generality, let the ''hole'' be centered at the origin, $r,v=0$. Then the desired transformation is

$$r = t \cos nx, \quad v = t \sin nx - \ell x, \quad n \in \mathbb{N}, \tag{11}$$

with, e.g., $0 \leq x < 2\pi$. If the term to the right in the expression for v were absent, this would merely be the formula for polar coordinates. The extra term accomplishes the ''cutting'' by producing a shift of length $l = 2\pi\ell$ in the v direction at each full turnaround of the ''angle'' variable x. This is illustrated in Fig. 7 for $\ell > 0$, $n = 1$.

In the left diagram, Fig. 7(a), x,t are polar coordinates (to be interpreted as cylinder coordinates for the kink spacetime, finally). The radii $x =$ const are mapped in a one-to-one fashion onto the rays in the right diagram, Fig. 7(b), where r,v are Cartesian coordinates (carrying the EF metric). Of course this transformation breaks down near the origin, as is seen by the intersecting rays in Fig. 7(b). If the corresponding region (eccentric shaded disk) is excluded from Fig. 7(a), e.g., by the stronger restriction $t > |\ell|/n$ (broken circle), then we are left with an ''annulus'' on which Eq. (11) is a local diffeomorphism. (The jump from $x = 2\pi$ to $x = 0$ does not raise any problems, since by its v homogeneity the EF metric g remains smooth.)

Inserting Eq. (11) into the EF metric (1) leads to the desired kink metric. Since the expressions are rather ugly, we will only write them down in terms of a zweibein. Note that via $g = 2e^+e^-$ the metric (1) can be obtained from $e^+ = dv, e^- = dr + \frac{1}{2}h(r)dv$. Transforming this zweibein into the new x,t coordinates yields

$$e^+ = \sin nx\, dt + [nt \cos nx - \ell]dx,$$

$$e^- = d[t \cos nx] + \frac{h(t \cos nx)}{2} e^+, \quad n \in \mathbb{N}. \tag{12}$$

As the expression (12) is 2π periodic in x, $g = 2e^+e^-$ pro-

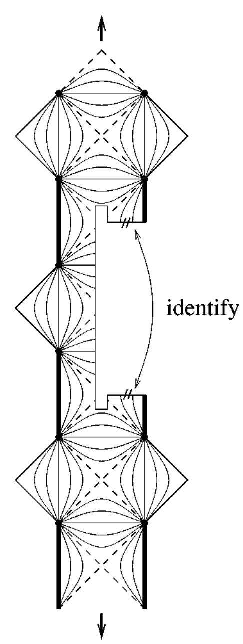

FIG. 9. Reissner-Nordström kink.

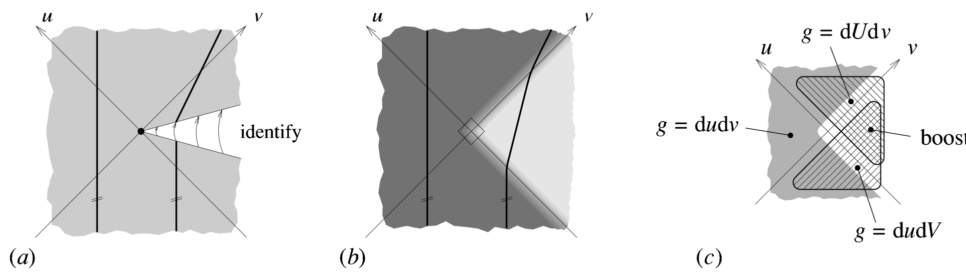

FIG. 10. Minkowski kink, with nontrivial holonomy. (a) shows once more the manifold Fig. 6(a). Instead of this cutting procedure it may as well be described by an explicit (nonanalytic) conformal chart, Eq. (A1), provided the puncture is enlarged to a small square. This is illustrated in (b), where the shading indicates the conformal factor (dark corresponding to a larger factor). (c) depicts some coordinate systems used in the text.

vides a metric on a cylindrical spacetime. The nondegeneracy of g is guaranteed (only) for values $t>|\ell|/n$,[9] since

$$e^{-}\wedge e^{+}=(nt-\ell\cos nx)dt\wedge dx. \qquad (13)$$

It is a $2n$-kink metric, i.e., moving along any circle $t=\mathrm{const}>|\ell|/n$, the light cone makes $2n$ half-turns or n full turns. This is evident by construction, but may be verified also explicitly: Obviously $e_{\mu}^{+}=(\sin nx,\ nt\cos nx-\ell)$ winds around the origin (0,0) n times when going along a $t=\mathrm{const}$ loop. Because of its linear independence, Eq. (13), e_{μ}^{-} is doomed to follow this tilting movement. This concludes the proof, since the null directions are determined precisely by $e^{\pm}$.

As already pointed out, the function h in Eq. (12) is completely arbitrary and can be chosen to satisfy the equation of motion (EOM) of any desired gravity model [18]. For instance, if h is taken to be the function of de Sitter gravity, e.g., $h(r)=-1-r^2$, then we arrive exactly at the de Sitter-kinks of [11]. These solutions are perfectly smooth everywhere (in contrast to those of [7]) but still nontrivial (i.e., not extendible to the global de Sitter space).[10]

Another instructive example is obtained when taking $h(r)\propto r$. As discussed in the Introduction this yields flat space, but with the Killing field ∂_v describing boosts. Thus, when inserting $h(r)\propto r$ into Eq. (12) one would expect to recover the kinks of Fig. 6(a). This is only partially true, however: Substituting, e.g., $r=-UV/2$, $v=2\ln(V/2)$ into the metric $g=2drdv+rdv^2$ yields $g=2dUdV$ with $V>0$. Thus, the above EF coordinates underlying Fig. 8(a) cover only half of Minkowski space, namely, the dark shaded region of Fig. 8(b). Subsequently, one can certainly replace the removed strip by a wedge (of the same ''boost width'') as shown in Fig. 8(c). This patch coincides with the one of Fig. 6(a), displayed for comparison in Fig. 8(d) again, only on an annular (i.e., cylindrical) region remote from the origin (e.g., outside the dashed ellipse). This also shows once more that a not maximally extended kink can have quite different extensions [note that the patch, Fig. 8(c), has to be extended further, as shown in Appendix B, Fig. 12].

How exhaustive is the construction scheme described above? First of all, it has to be pointed out that we have only shown the simplest examples. In general *any* cylindrical *limited* covering[11] of the original spacetime or, worse, any factor space thereof with a nonzero number of tilts of the light cone may be addressed as kink (remember, e.g., the manifold D of Fig. 4). It is certainly tempting to assume that the maximally extended universal covering of any such kink is a branched covering of the unique ''global'' universal covering (as constructed in [15]), although we could not prove this assertion. Any inextendible kink solution could then be obtained as a factor space from the respective branched covering and a classification would amount to specifying the position of the branch points (up to symmetry transformations, of course) and finding the conjugacy classes of freely and properly discontinuously acting symmetry subgroups for this manifold [12,13]. For globally cylindrical kinks, furthermore, only the subgroups isomorphic to $\mathbb{Z}$ are relevant. However, even if this approach is feasible, it will sometimes obscure geometric features: Note that generically the presence of a branch point breaks the Killing symmetry [unless the Killing vector vanishes at that point, as is the case in Fig. 6(a)]. Consequently, when describing the kink of Fig. 6(b) as a factor space of its maximally extended universal covering (cf. Appendix B), the continuous parameter ''translation width'' does not emerge from a Killing symmetry during the factorization, but it is already encoded in the spacing of the branch points in the universal covering.

We conclude this section with two further constructions which have not been made explicit so far. First, if there are symmetries not generated by a Killing field, then further discrete parameters may be introduced. This is, e.g., the case for the Reissner-Nordström solution, which is an infinite periodic repetition of one patch: One could make a long vertical

[9]If preferred, one can of course reparametrize the t coordinate, e.g., $t\rightarrow|\ell|/n+e^t$, so as to obtain a metric defined for all (coordinate) values of t.

[10]Our present approach allows us to correct a small mistake in [11]: At that time it was believed that the spacetime described by Eq. (12) [with $h(r)=-1-r^2$] contains closed timelike curves. It is obvious from the present analysis that this is not the case. Quite on the contrary, for a cylindrical solution (12) there is *not a single* closed loop with a definite sign of ds^2. As a consequence there is no foliation of the cylinder into spacelike leaves $\Sigma\sim S^1$ (a Hamiltonian formulation, however, may still be defined, as demonstrated in [11,25]).

[11]In contrast to the familiar (unlimited) covering manifolds, a limited covering may have (ideal) boundaries which do not correspond to boundaries of the underlying base manifold. Especially, any open subset of a manifold is a limited covering.

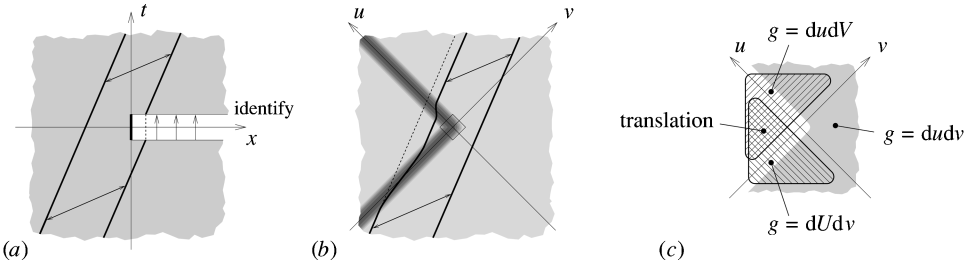

FIG. 11. A similar construction can be applied to the ''translational'' Minkowski kink of Fig. 6(b), leading to Eq. (A6).

slit through a number of patches, then remove a few patches on one side, and glue together again; cf. Fig. 9. The resulting kink is thus characterized by a Killing parameter *and* the patch number. Certainly, also this kink can be extended in a similar manner as the one of Fig. 6(b); cf. Appendix B. Second, let us note that such surgery is obviously not restricted to cylindrical solutions (i.e., one hole only), but within any solution one can cut any number of holes, each giving rise to a kink number k, a continuous parameter for the gluing (if there is a Killing symmetry), further ones for the relative position of the respective new hole, and perhaps some discrete parameters. And applying more advanced gluing techniques (such as making slits between branch points and sewing the overlapping layers crosswise) it is even possible to obtain surfaces of higher genus.[12]

V. CONCLUSION

For any given 2D metric we have constructed kink spacetimes (inextendible if $k\neq 2$). In the presence of a Killing symmetry (thus covering, e.g., all generalized dilaton gravity solutions), furthermore, there occurred actually a continuous one-parameter family of solutions for each kink number, which were inextendible even for $k=2$. A geometrical interpretation of this parameter has been provided in terms of holonomy, respectively parallel displacement. Although a complete classification of the maximal extensions of these kinks proves to be elusive (due to ambiguities in the extension process), the characteristic parameters have been pointed out. For cylindrical regions, furthermore, a conformal but nonanalytic as well as an analytic coordinate system is given. As a by-product we have also found generalizations of the bare kink and the Misner torus.

ACKNOWLEDGMENTS

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APPENDIX A: CONFORMAL CHARTS FOR FLAT KINKS

In this appendix we provide conformal coordinate charts for the kink manifolds of Figs. 6(a) and 6(b). Let us start with the kink, Fig. 6(a) (cf. Fig. 10). As already proved in Sec. III, it is impossible to give a smooth conformal chart for the entire manifold. This problem can be circumvented, however, if we enlarge the puncture to a square: Let $f(x)$ be a smooth function which vanishes for $x<-1$ and equals 1 for $x>+1$. Then the metric

$$g=e^{-\omega f(-u)f(v)}du\,dv \tag{A1}$$

is flat outside the square $-1<u,v<1$ [cf. Fig. 10(b)], where the shading indicates the conformal factor]. Still, the holonomy is nontrivial, as is easily calculated, and it depends on the parameter ω in the exponent:[13] Let

$$F(x):=\int^x e^{-\omega f(z)}dz. \tag{A2}$$

Clearly $F(x)=x$ for $x<-1$, $F(x)=xe^{-\omega}+\text{const}$ for $x>1$, and for simplicity we assume that f is fine-tuned such as to make const vanish. Note that the metric (A1) is already in Minkowski form $g=du\,dv$ throughout the dark shaded part of Figs. 10(b) and 10(c). There are two possibilities to transform Eq. (A1) into Minkowski form also on the right-hand sector: Introducing $U(u):=-F(-u)$ instead of u does this job for the upper right half, $v>1$, and likewise replacing v by $V(v):=F(v)$ works for the lower right half, $u<-1$ [cf. Fig. 10(c)]. However, on the right-hand sector itself the two overlapping coordinate systems (U,v) and (u,V) disagree by a boost,

$$\begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} e^{-\omega} & 0 \\ 0 & e^{\omega} \end{pmatrix} \begin{pmatrix} u \\ V \end{pmatrix}, \tag{A3}$$

which proves the assertion.

In order to get the square cut out as small as possible (preferably pointlike), the interval where f ascends must be made narrower. Thus, in the limiting case, one could describe the entire solution — allowing for a distributional conformal factor — as

$$g=e^{-\omega\theta(-u)\theta(v)}du\,dv, \tag{A4}$$

where $\theta(x)$ denotes the Heaviside step function. The conformal factor then takes one constant value in the right-hand (respectively, any other) sector and another value every-

[12]This is also well known from complex analysis, where, e.g., the Riemann surface of the function $\sqrt{(z-a)(z-b)(z-c)(z-d)}$ is a torus with four branch points.

[13]However, even without calculation it is obvious that the conformal factor mimics the cutting procedure by giving less metric ''weight'' to the right-hand sector.

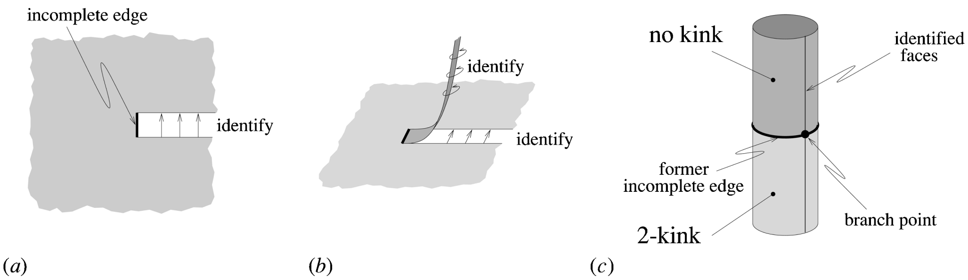

FIG. 12. Possible maximal extension of the ''translation'' kinks.

where else. A short exposition of distributional metrics and their usefulness may be found in [26].

For the Minkowski kink with trivial holonomy [Fig. 6(b) or Fig. 11(a)] an analogous chart can be found. The above reasoning would suggest to imitate the cutting by suitably ''diluting'' the metric on the strip in question. However, since the metric should remain nondegenerate, it seems wiser to turn the tables and *insert* a strip of the same width ω on the opposite side ($x<0$ in this example). This can be achieved by putting

$$g=[1+\omega f(-x)f'(t)]^2dt^2-dx^2, \tag{A5}$$

with f as before: Whereas on the right-hand side, $x>1$, Eq. (A5) is in Minkowski form, $g=dt^2-dx^2$, on the left-hand side, $x<-1$, this form can only be attained when substituting t by $T(t):=t+\omega f(t)$. But then crossing the strip $-1<t<1$ clearly increases the coordinate T by ω against the right-hand coordinate t, as desired. A conformal chart is obtained when splitting this vertical shift into two similar shifts of the null coordinates $u,v=t\mp x$ (the horizontal components canceling, but the vertical ones adding up):

$$g=[1+\omega f(u)f'(v)+\omega f(-v)f'(u)]dudv. \tag{A6}$$

This is illustrated in Fig. 11(b). Again, introducing $U(u):=u+\omega f(u)$ on the lower left half, $v<-1$, and $V(v):=v-\omega f(-v)$ on the upper left half, $u>1$, allows us to extend the Minkowski metric from the right-hand side into the left-hand sector; cf. Fig. 11(c). On the overlap the two coordinate systems are related by

$$\begin{pmatrix} U \\ v \end{pmatrix}=\begin{pmatrix} u+\omega \\ V+\omega \end{pmatrix}, \tag{A7}$$

i.e., by a shift of length ω into the t direction. [If the two terms in Eq. (A6) were given different (positive) weights ω_1, ω_2, with $\omega_1\omega_2=\omega^2$, then the corresponding translation direction would be boosted; sufficiently far from the origin, however, the resulting kink manifolds are still isometric.] Also here the limit of the shrinking interval can be taken, but this time it will involve a δ distribution, $f'(x)\to\theta'(x)=\delta(x)$. While this does not pose any problems for Eq. (A6), the metric (A5) looks rather undefined then, as it contains a term $\delta(t)^2$.

APPENDIX B: EXTENSION OF THE ''TRANSLATION'' KINK

Here we want to show how the flat kink of Fig. 6(b), respectively Fig. 11(a) can be extended further. A similar procedure may be applied to all kinks obtained in Sec. IV. In these examples a strip has been removed, leaving an incomplete edge [bold line in Fig. 12(a)]. However, one can simply extend the manifold beyond the edge into a new layer: The perhaps easiest way to look at this is to keep the previously removed strip attached to the rest, Fig. 12(b), and to sew together its faces, too. The result is then one cylinder Fig. 12(c), half of which is the original 2-kink solution, the other half having no kink. (In this example there are closed timelike curves in the latter half cylinder; they would not occur if we had chosen a spacelike translation.) The identified end points of the previously incomplete edge constitute a branch point which should be removed. Certainly, to obtain the universal covering one has now first to unwrap this cylinder (thus introducing infinitely many copies of the branch point) and subsequently also to unwrap the manifold around those branch points into new overlapping layers (thereby once again multiplying the number of branch points).

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