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General solution for self-gravitating spherical null dust

Aleksandar Miković*

Departamento de Fisica Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC, Facultad de Fisica, Burjassot-46100, Valencia, Spain

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We find the general solution of equations of motion for self-gravitating spherical null dust as a perturbative series in powers of the outgoing matter energy-momentum tensor, with the lowest order term being the Vaidya solution for the ingoing matter. This is done by representing the null-dust model as a 2D dilaton gravity theory, and by using a symmetry of a pure 2D dilaton gravity to fix the gauge. Quantization of this solution would provide an effective metric which includes the back-reaction for a more realistic black hole evaporation model than the evaporation models studied previously.

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Two-dimensional (2D) dilaton gravity theories have turned out to be very useful toy models of black hole formation and evaporation [1]. They are also relevant for four-dimensional (4D) black holes, since a spherically symmetric scalar field collapse can be described by a 2D dilaton gravity model

$$S_0=\frac{1}{2}\int d^2x\sqrt{-g}\left[e^{-2\Phi}[R+2(\nabla_\mu\Phi)^2+2e^{2\phi}] - \frac{G}{2}e^{-2\Phi}(\nabla_\mu f)^2\right], \tag{1}$$

where G is the Newton constant and the 4D line element ds_4 is related to the 2D line element ds by

$$ds_4^2=ds^2+e^{-2\Phi}d\Omega^2. \tag{2}$$

R is a 2D scalar curvature associated with a 2D metric $g_{\mu\nu}$, ∇_μ is the corresponding covariant derivative, Φ is the dilaton field, f is the scalar field, and $d\Omega$ is a two-sphere line element. Quantization of Eq. (1) would provide us with a semiclassical metric which would include the back reaction. However, the progress is hindered by the absence of explicit solutions of the classical equations of motion. In the case of 2D black holes described by the Callan-Giddings-Harvey-Strominger (CGHS) model [2], the analog of Eq. (1) is exactly solvable, and by quantizing the solution, one can obtain an effective semiclassical metric up to any finite order in matter loops [3–5]. Since the matter in the CGHS model is 2D conformally coupled, this motivates us to consider a modification of Eq. (1),

$$S=\frac{1}{2}\int d^2x\sqrt{-g}\left[e^{-2\Phi}[R+2(\nabla_\mu\Phi)^2+2e^{2\phi}] - \frac{G}{2}(\nabla_\mu f)^2\right], \tag{3}$$

so that f will obey a free-field equation of motion in the conformal gauge, and consequently the general solution could be found more easily. Action (3) describes the dynamics of spherically symmetric self-gravitating null dust (SSND). For purely ingoing matter, the solution of the equations of motion is given by the Vaidya metric [6]

$$ds^2=-\left(1-\frac{2m(v)}{r}\right)dv^2+2\,dv\,dr, \tag{4}$$

where $r=\exp(-\Phi)$, $G=1$, and

$$\frac{dm(v)}{dv}=T_{vv}(v)=\frac{1}{2}(df/dv)^2. \tag{5}$$

An analytic model of black hole evaporation based on the quantization of the Vaidya solution has been studied in [7], and a qualitative agreement with the numerical results of [8] has been found. However, in order to fully take into account the back reaction in the analytic approach, one needs the most general solution for which the outgoing matter is also present [9,7].

In order to find the general solution of the SSND model, we rewrite the action (3) as

$$S=\frac{1}{2}\int d^2x\sqrt{-\tilde g}\left[\tilde R\phi+V(\phi)-\frac{1}{2}(\tilde\nabla_\mu f)^2\right], \tag{6}$$

where $\phi=e^{-2\Phi}=r^2$, $\tilde g_{\mu\nu}=rg_{\mu\nu}$, and $V=2/r$. We do this in order to establish the connection with a generic 2D dilaton gravity model, which can be represented by the action of the form (6). For example, the CGHS model is given by $V=4\lambda^2$, where λ is a 2D cosmological constant. The equations of motion are given by

$$\nabla_\mu\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}V=-T_{\mu\nu}, \tag{7}$$

$$R+\frac{dV}{d\phi}=0,\quad \Box f=0, \tag{8}$$

where we have omitted the tildes, $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$ and

*On leave from Institute of Physics, P.O. Box 57, 11001 Belgrade, Yugoslavia. Electronic address: mikovic@lie.ific.uv.es

$$T_{\mu\nu}=\partial_\mu f\partial_\nu f-\frac{1}{2}g_{\mu\nu}(\nabla_\mu f)^2.$$

In the conformal gauge $d\tilde{s}^2=-e^{2\rho}dx^+dx^-$ one obtains

$$\partial_+\partial_-\phi=-\frac{1}{4}Ve^{2\rho}, \tag{9}$$

$$\partial_+^2\phi-2\partial_+\rho\partial_+\phi=-(\partial_+f)^2, \tag{10}$$

$$\partial_-^2\phi-2\partial_-\rho\partial_-\phi=-(\partial_-f)^2, \tag{11}$$

$$\partial_+\partial_-\rho=-\frac{1}{8}\frac{dV}{d\phi}e^{2\rho}, \tag{12}$$

and $\partial_+\partial_-f=0$. We want to solve the system (9)–(12) in analogy to the CGHS case, where a gauge $\rho=0$ can be chosen. This is possible since $\partial_+\partial_-\rho=0$ is a consequence of the equations of motion. In the SSND case ρ is not a free field, so that we have to find an appropriate modification. We will use as the starting point the free-field currents for the $f=0$ case [11]

$$j_1=\int\frac{d\phi}{2E+J(\phi)}, \tag{13}$$

$$\tilde{j}_2=\log(\nabla\phi)^2, \tag{14}$$

where $2E=(\nabla\phi)^2-J(\phi)$, $dJ/d\phi=V$, and E is a constant of motion. j's satisfy

$$\Box j_1=0,\quad \Box\tilde{j}_2+R=0. \tag{15}$$

By going into the conformal gauge one can see that the equations (15) imply that j_1 and $j_2=\tilde{j}_2-2\rho$ are free fields. In the SSND case, when $f=0$ one has $2j_1=r+2M\log|r/2M-1|$ and $j_2=\log|r-2M|-2\rho$, where $M=-E/4$ is the black hole mass. The Schwarzschild solution is obtained for

$$2j_1=\frac{1}{2}(v-u), \tag{16}$$

$$j_2=0. \tag{17}$$

The gauge choice (16) gives the familiar relation between r and u, v coordinates, while the gauge choice (17) is equivalent to

$$ds^2=-\frac{e^{2\rho}}{r}du\,dv=-(1-2M/r)du\,dv. \tag{18}$$

Note that the gauge choice $j_2=0$ gives a relation between ρ and ϕ, which is of the type we are looking for, and hence we will concentrate on it.

When the matter is added, the relations (15) change as

$$\Box j_1=\frac{\partial^2 j_1}{\partial E^2}T^{\mu\sigma}T^\nu_\mu\nabla_\sigma\phi\nabla_\nu\phi+\left(\frac{\partial j_1}{\partial E}-\frac{4}{(\nabla\phi)^4}\right)T_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi,$$

$$\Box\tilde{j}_2+R=-\frac{4}{(\nabla\phi)^4}[T^{\mu\sigma}T^\nu_\mu\nabla^\sigma\phi\nabla^\nu\phi+V(\phi)T_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi]+\frac{2}{(\nabla\phi)^2}T_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi, \tag{19}$$

so that j's are not free fields any more. The form of Eqs. (19) and our remark about j_2 suggest that we look for a free field of the form

$$j=j_2+X, \tag{20}$$

where X is to be determined from Eqs. (19). This gives

$$\partial_+\partial_-X=-\left[\frac{T_{++}}{(\partial_+\phi)^2}+\frac{T_{--}}{(\partial_-\phi)^2}\right]\partial_+\partial_-\phi. \tag{21}$$

Equation (21) is valid for any potential V, and can be solved as

$$X=X_0+\log\left|\frac{\partial_-\phi}{\partial_+\phi}\right|+2\int dx^-\frac{T_{--}}{\partial_-\phi}, \tag{22}$$

where X_0 is a free-field solution.

In the SSND case $\phi=r^2$, $x^+=v$, $x^-=u$, and we choose $X_0=0$, which fixes the gauge. It is convenient to use a generalized form of the Vaidya metric [10]

$$ds^2=-e^{2\psi}F\,dv^2+2e^\psi dr\,dv, \tag{23}$$

which can be related to the conformal form of the metric via $ds^2=-C(u,v)dudv$ and $C=e^{2\rho}/r$. Then the gauge choice $j=0$ is equivalent to

$$\psi=2\int du\frac{T_{uu}}{\partial_u r^2}, \tag{24}$$

where $T_{uu}=\frac{1}{2}T_{--}$. Note that we have not specified the limits of u integration in Eq. (24), which means that a constant of integration will occur. Only when this constant is specified, the gauge will be completely fixed.

The equations which determine r are Eqs. (9) and (10), while Eqs. (11) and (12) are the consistency conditions for the gauge choice, which are satisfied by construction. Equation (9) becomes

$$\partial_u\partial_v r^2=e^\psi\partial_u r, \tag{25}$$

while Eq. (10) gives

$$\partial_v^2 r^2-2\partial_v\rho\partial_v r^2=-(\partial_v f)^2, \tag{26}$$

where ρ is determined from the gauge choice as

$$2\rho=\log|-\partial_u r^2|+\psi. \tag{27}$$

Equation (11) follows from Eqs. (27) and (24), while Eq. (12) follows from Eqs. (27), (25), and (24). By integrating Eq. (25) with respect to u, we obtain

$$\partial_v r = \frac{1}{2}\left(1-\frac{2m(v)}{r}\right)+\frac{1}{2r}\int du(e^{\psi}-1)\partial_u r. \quad (28)$$

$m(v)$ is a constant of integration, and can be determined from Eq. (26). By inserting Eq. (28) into Eq. (26), and by using Eq. (27), we get

$$T_{vv}=\frac{dm}{dv}-\frac{1}{2}(1-e^{\psi}-2r\partial_v\psi)\partial_v r-\frac{1}{2}\partial_v\int du(e^{\psi}-1)\partial_u r. \quad (29)$$

When $\psi=0$, Eq. (28) yields $F=1-2m/r$ and Eq. (29) gives the relation (5), so that one recovers the Vaidya solution. When $\psi\neq 0$, the equations look difficult. However, their form is such that a perturbative solution in T_{uu} can be easily found. Let us introduce an expansion parameter ϵ by replacing T_{uu} with ϵT_{uu}. Then we will seek a solution in the form

$$r=r_0+\epsilon r_1+\epsilon^2 r_2+\cdots. \quad (30)$$

The expansion (30) then implies

$$\psi=\epsilon\psi_1+\epsilon^2\psi_2+\cdots, \quad (31)$$

where

$$\psi_1=\int du\,\frac{T_{uu}}{r_0\partial_u r_0},\quad \psi_2=-\int du\,T_{uu}\frac{\partial_u(r_1 r_0)}{(r_0\partial_u r_0)^2},\cdots. \quad (32)$$

In general one should also take

$$m(v)=m_0(v)+\epsilon m_1(v)+\epsilon^2 m_2(v)+\cdots, \quad (33)$$

although it is possible that the series in Eq. (33) gets truncated, like in the example we consider in this paper, where $m=\epsilon m_1$. By inserting the expansions (30), (31), and (33) into the equation for r (28), we get an infinite hierarchy of equations

$$\partial_v r_0=\frac{1}{2}\left(1-\frac{2m_0}{r_0}\right),$$

$$\partial_v r_1=\frac{m_0}{r_0^2}r_1-\frac{m_1}{r_0}+\frac{1}{2r_0}\int du\,\psi_1\partial_u r_0,$$

$$\partial_v r_2=\frac{m_0}{r_0^2}r_2-\frac{m_2}{r_0}-\frac{r_1}{r_0}\partial_v r_1+\frac{1}{2r_0}\int du\left[\left(\frac{1}{2}\psi_1^2+\psi_2\right)\partial_u r_0 + \psi_1\partial_u r_1\right],\ \ldots. \quad (34)$$

The system (34) can be solved, since for every n the equation for r_n does not involve r_k with $k>n$ and each equation is a first order linear differential equation for r_n, except for $n=0$, which is a nonlinear first order differential equation. Therefore by starting from $n=1$, one can write an explicit solution for any r_n in terms of r_0 and T_{uu}. At each step several integration constants $C(u)$ and $C(v)$ will arise (due to u and v integrations), and these constants can be determined from the constraint equation (29), exactly as in the $T_{uu}=0$ case. This can be done because Eq. (29) also decomposes into an infinite hierarchy of equations under the expansions (30), (31), and (33)

$$T_{vv}=dm_0/dv,$$

$$0=dm_1/dv+\frac{1}{2}(\psi_1+2r_0\partial_v\psi_1)\partial_v r_0-\frac{1}{2}\partial_v\int du\,\psi_1\partial_u r_0,$$

$$0=dm_2/dv+\frac{1}{2}\left(\psi_2+\frac{1}{2}\psi_1^2+2r_0\partial_v\psi_2+2r_1\partial_v\psi_1\right)\partial_v r_0 +\frac{1}{2}(\psi_1+2r_0\partial_v\psi_1)\partial_v r_1-\frac{1}{2}\partial_v\int du\left[\left(\psi_2+\frac{1}{2}\psi_1^2\right)\partial_u r_0 +\psi_1\partial_u r_1\right],\ \ldots. \quad (35)$$

Equations (35) will also determine the $m_n(v)$, provided we fix the integration constants $\psi_n(v)$ which appear in Eqs. (32). These are related to the complete specification of the gauge, or equivalently, to the choice of the v coordinate, since a coordinate change $v=v(\tilde{v})$ in Eq. (23) gives

$$\tilde{\psi}=\psi+\log\frac{dv}{d\tilde{v}}. \quad (36)$$

At the end, one sets $\epsilon=1$ and writes the solution as

$$r=r_0+r_1+r_2+\cdots,\quad \psi=\psi_1+\psi_2+\cdots. \quad (37)$$

These general features can be nicely illustrated on the example of a white hole emiting a shock wave. This is simply a shock-wave Vaidya solution where the u and v coordinates are interchanged. The corresponding spacetime is described by the Penrose diagram of Fig. 1. In this case $T_{uu}\neq 0$, and one can find an exact solution for $r=r(u,v)$, so that

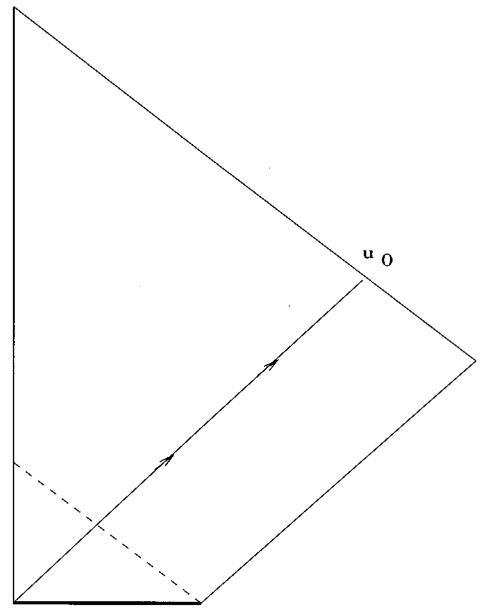

FIG. 1. Penrose diagram of a white hole emiting a shock wave.

the expansions (30) and (31) and the corresponding equations (34) and (35) can be checked. By taking $T_{uu}=M\delta(u-u_0)$, we get

$$ds^2=\begin{cases}-du^2-2\,du\,dr & u>u_0,\\ -(1-2M/r)du^2-2\,du\,dr & u<u_0.\end{cases} \tag{38}$$

This can be rewritten as

$$ds^2=\begin{cases}-dv^2+2\,dv\,dr & u>u_0,\\ -Fe^{2\psi}dv^2+2e^{\psi}dv\,dr & u<u_0,\end{cases} \tag{39}$$

where $F=1-2M/r$,

$$\psi=-\log\left|1-\frac{4M}{v-u_0}\right|, \tag{40}$$

and $r=1/2(v-u)$ for $u>u_0$, while for $u<u_0$

$$r+r_s\log|r/r_s-1|=\frac{1}{2}(v-u)+r_s\log|(v-u_0)/2r_s-1|, \tag{41}$$

where $r_s=2M$. The form of the solution, given by Eqs. (40) and (41), is such that

$$r=\sum_{n\geqslant 0} r_s^n\tilde{r}_n, \quad \psi=\sum_{n\geqslant 1} r_s^n\tilde{\psi}_n, \tag{42}$$

which is of the form (37). More exactly, one can show that $r_n=r_s^n\tilde{r}_n$ and $\psi_n=r_s^n\tilde{\psi}_n$ satisfy Eqs. (34) and (35) with $m=m_1=M$. Also, starting from $n=2$, nontrivial integration constants $\psi_n(v)$ appear.

In conclusion we can say that we have found a useful form of the general solution, given by the expansions (37) and Eqs. (34) and (35). The expansions (37) are clearly in powers of the outgoing energy-momentum tensor, and by truncating them at finite n we obtain an explicit perturbative solution. This form of the solution can be used to construct an effective quantum metric in the approximation of a finite number of matter loops via the method of quantization of the classical solution [3–5,7]. The corresponding construction is going to be more involved than in the 2D case [3–5] or the Vaidya case [7], since the relation between the metric and the dilaton (radius) is more complicated. For example, for the one-loop approximation one would take $r=r_0+r_1$ and $\psi=\psi_1$, with $T_{\mu\nu}$ replaced by $\langle T_{\mu\nu}\rangle$ evaluated in an appropriate quantum state. Note that in the one-loop case one does not have to truncate the expansions (37) at $n=1$. By including the higher-order terms, one extends the validity of the one-loop approximation to a smaller radius.

Our solution can also serve as a good starting point for finding approximate analytic solutions for the more realistic collapse described by the action (1).

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