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Bogomol'nyi-type mass formulas for a class of nonrotating black holes

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In the presence of a Killing symmetry, various self-gravitating field theories with massless scalars (moduli) and vector fields reduce to σ models, effectively coupled to three-dimensional gravity. We argue that this particular structure of the Einstein-matter equations gives rise to quadratic relations between the asymptotic flux integrals and the area and surface gravity (Hawking temperature) of the horizon. The method is first illustrated for the Einstein-Maxwell system. A derivation of the mass formula is then also presented for the Einstein-Maxwell-dilaton-axion model, which is relevant to the bosonic sector of heterotic string theory. [S0556-2821(97)01114-4]

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I. INTRODUCTION

It has been known for a long time that the Einstein-Hilbert action in the presence of a Killing field k^μ, say, describes a two-dimensional σ model effectively coupled to three-dimensional gravity [1]. The target manifold of the σ model is the symmetric space SL(2)/SO(2), which is parametrized in terms of two gravitational scalars (the norm of k^μ and its twist potential).

The Einstein-Maxwell (EM) system with a Killing symmetry reveals a similar structure, where now the σ model comprises the gravitational scalars and two additional electromagnetic potentials [2]. Again, the target manifold is a symmetric space G/H. If the dimensional reduction is performed with respect to a timelike Killing field one finds $G/H = \mathrm{SU}(2,1)/S(\mathrm{U}(1,1)\times \mathrm{U}(1))$, whereas $G/H = \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(2)\times \mathrm{U}(1))$ if k^μ is spacelike. It is this particular property of the EM equations which gives rise to the Ernst potentials [3], the Mazur identity [4] and, in the presence of a second Killing field, to the total integrability of the field equations [5]. Moreover, it is most likely that the black hole uniqueness theorem itself owes its existence to the σ-model structure (see, e.g., [6]).

Obvious generalizations of the EM system are self-gravitating field theories with massless scalars and Abelian vector fields. Considering scalar fields with symmetric target space $\overline{G}/\overline{H}$, Breitenlohner *et al.* [7] were able to classify those models for which the dimensional reduction yields again a σ model with symmetric space G/H. Hence, these models admit a symmetry group which is large enough to comprise *all* scalar fields arising on the effective level within *one* coset space. In terms of a representation Φ of G/H, the field equations assume the form

$$\boldsymbol{R}^{(p)} = \mathrm{Tr}\{J \otimes J\}, \quad d*J = 0. \tag{1.1}$$

Here $\boldsymbol{R}^{(p)}$ denotes the Ricci tensor with respect to the projection metric $\boldsymbol{p}$, and J is the σ-model current,

$$\boldsymbol{p} \equiv V\boldsymbol{g} + k \otimes k, \quad J \equiv \tfrac{1}{2}\Phi^{-1} d\Phi, \tag{1.2}$$

where $\boldsymbol{g}$ is the spacetime metric and $V \equiv -g_{\mu\nu}k^\mu k^\nu$.

For the vacuum and the EM equations the explicit parametrization of the matrix Φ in terms of the target space coordinates (Ernst potentials) resulted from the work of Ehlers [1], Ernst [3], Geroch [8], Kinnersley and co-workers [9,10], Neugebauer and Kramer [2], and others. Only recently, Gal'tsov and Kechkin were able to find the generalized Ernst potentials and the corresponding σ-model representation for the Einstein-Maxwell-dilaton-axion (EMDA) equations [11]. The EMDA model is relevant to $N=4$ supergravity and to the bosonic sector of four-dimensional heterotic string theory. In fact, this system provides the simplest (nontrivial) example of the models classified by Breitenlohner *et al.* [7]. The relevant coset turns out to be $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(1,1)$, where the fact that SO(2,3) is locally isomorphic to $\mathrm{Sp}(4,\mathbb{R})$ is of crucial importance to the explicit representation of Φ [12].

The matrix J comprises $\dim(G)$ algebraically independent current one-forms j_i, say. However, since the target manifold is a symmetric space, only $\dim(G/H)$ of the conservation laws $d*j_i=0$ are independent. By virtue of the Killing symmetry, each conserved current gives rise to a *closed* two-form $\Omega_i \equiv *(k\wedge j_i)$. Integrating these two-forms over a spacelike hypersurface (which intersects the horizon and extends to infinity), Stokes' theorem yields a set of relations between the asymptotic flux integrals, the corresponding horizon quantities, and the values of the σ-model fields (potentials) at the horizon. In this way one obtains, for instance, the Smarr formula [13] for stationary EM black hole configurations.

As one is only dealing with $\dim(G/H)$ independent equations of the form $d\Omega_i=0$, one might expect that Stokes' theorem yields as many relations between the charges and the horizon values of the potentials. This is, however, not the case. In fact, the situation is better: Although there are $\dim(H)$ conservation laws which can be obtained from the remaining ones, *all* currents j_i are algebraically independent. For this reason, Stokes' theorem yields $\dim(G)$ nonredundant relations of the Smarr-type when applied to the two-forms Ω_i. The entire set of relations may then be used to eliminate the unknown horizon values of the σ-model scalars. In this way one ends up with a relation which involves only the total charges and the corresponding horizon quantities. For both the EM and the EMDA system we shall prove that all stationary black hole configurations with nonrotating Killing horizon satisfy

$$M_H^2 = M^2 + N^2 + D^2 + A^2 - Q^2 - P^2, \qquad (1.3)$$

where the right-hand side (RHS) comprises the asymptotic flux integrals, i.e., the total mass, the Newman-Unti-Tamburino (NUT) charge, the dilaton and axion charges, and the electric and magnetic charges, respectively. The quantity M_H is the Komar integral over the horizon, $M_H = -(8\pi)^{-1}\int_H * dk$. The left-hand side (LHS) of the above relation can, therefore, be expressed in terms of the area of the horizon $\mathcal{A}$, and its surface gravity κ or, equivalently, its Hawking temperature T_H:

$$M_H = \frac{1}{4\pi}\kappa\mathcal{A} = \frac{1}{2}T_H\mathcal{A}. \qquad (1.4)$$

The ''extreme'' Reissner-Nordström solution is well known to satisfy the bound $0 = M^2 - Q^2 - P^2$. The corresponding Bogomol'nyi-Prasad-Sommerfield (BPS) bound for the EMDA system, $0 = M^2 + D^2 + A^2 - Q^2 - P^2$, was obtained by Clément and Gal'tsov [14], constructing the null geodesics of the target space. Discussing the asymptotic behavior of target space geodesics for spherically symmetric configurations, Breitenlohner *et al.* obtained Eq. (1.3) with unspecified LHS [7]. In fact, many of the *spherically symmetric* black hole solutions with scalar and vector fields (see, e.g., [15–17]) are known to satisfy Eq. (1.3), where the LHS is expressed in terms of the horizon radius (see also [18] and references therein). Using the generalized first law of black hole thermodynamics, Gibbons *et al.* [19] were recently able to derive Eq. (1.3) for spherically symmetric solutions with an arbitrary number of vector and moduli fields.

In the present paper we establish Eq. (1.3) for arbitrary soliton ($M_H \equiv 0$) and stationary, nonrotating black hole solutions of the EM and EMDA equations. Our derivation is neither restricted to spherical symmetry, nor do we require the configurations to be static. The crucial observation is that the coset structure gives rise to a set of Smarr-type formulas which is sufficiently large to derive the desired relation. Since the EMDA σ model does not reduce to the EM σ model for vanishing dilaton and axion fields [14], we derive Eq. (1.3) separately for the two cases.

Although the recipe is simple, it is a rather unpleasant task to write out the current matrix J for a given representation Φ. We think that it should be possible to obtain the formula (1.3) even without having an explicit representation of the matrix Φ at hand. We, therefore, conjecture that relations similar to Eq. (1.3) hold for all models which reduce to the form (1.1) in the presence of a stationary Killing symmetry.

The paper is organized as follows: We start with a simple example: the static, purely electric EM system. In this case, the conserved currents are derived ''from scratch,'' that is, without making use of the σ-model structure (see also [6] for this approach). The third section is devoted to the general stationary EM equations. We recall the dimensional reduction and use the coset structure to construct all conserved currents and closed two-forms. Integrating the latter over a spacelike hypersurface will provide us with a set of generalized Smarr formulas, which we then use to compute the horizon potentials and to derive Eq. (1.3). In the fourth section the procedure is repeated for the EMDA system, where we take advantage of the coset representation found by Gal'tsov and Kechkin [11]. Since we prefer to use the exterior calculus, some computational rules for differential forms are compiled in the Appendix.

II. A SIMPLE EXAMPLE

As a motivation we consider the static, purely electric Einstein-Maxwell (EM) equations. In this case, the field strength two-form, $F = dA$, can be expressed in terms of the stationary Killing field (one-form) k and the electric one-form E: $F = (k/V) \wedge E$, where $V \equiv -k_\mu k^\mu \equiv -\langle k,k\rangle$. Staticity implies that the twist of the Killing field vanishes and, therefore, $d(k/V) = 0$ [see Eq. (A5)]. Hence, the Bianchi identity, $dF = 0$, and the Maxwell equation, $d*F = 0$, become

$$dE = 0, \quad d^\dagger\left(\frac{E}{V}\right) = 0, \qquad (2.1)$$

respectively, where $d^\dagger = *d*$ denotes the coderivative operator. [Here we have used Eq. (A4) for $\alpha = E/V$.] In addition, we consider the (00) component of Einstein's equations, $\boldsymbol{R}(k,k) = 8\pi\boldsymbol{T}(k,k) = \langle E,E\rangle$. In the static case, Eq. (A12) reduces to the Poisson equation, $d^\dagger(dV/V) = -2\boldsymbol{R}(k,k)/V$. Introducing the potential ϕ, $d\phi = E$, and using the formula $d^\dagger(f\alpha) = f d^\dagger\alpha - \langle df,\alpha\rangle$ (for arbitrary functions f and one-forms α), Eq. (2.1) implies $(1/V)\langle E,E\rangle = -d^\dagger(\phi E/V)$. Hence, both the Maxwell and the Poisson equation assume the form of current conservation laws:

$$d^\dagger j_Q = 0, \quad j_Q \equiv \frac{d\phi}{V}, \qquad (2.2)$$

$$d^\dagger j_M = 0, \quad j_M \equiv -\frac{1}{2}\frac{dV}{V} + \phi\frac{d\phi}{V}. \qquad (2.3)$$

In the presence of the Killing field k, every conserved one-form j, gives rise to a closed two-form, $\Omega \equiv *(k\wedge j)$. As $d\Omega$ vanishes, Stokes' theorem, $\int_{\partial\Sigma}$ (two-form) $= \int_\Sigma$ d(two-form), implies

$$\int_{S^2_\infty} *(k\wedge j) = \int_H *(k\wedge j), \qquad (2.4)$$

where the integral on the RHS extends over the topological two-sphere $H = \mathcal{H}\cap\Sigma$, $\mathcal{H}$ and Σ being the horizon and a spacelike hypersurface, respectively. In order to apply this formula to the above currents, we have to express $*(k\wedge j_Q)$ and $*(k\wedge j_M)$ in terms of the two-forms $F, *F$ and $*dk$. This is immediately achieved by using the static, purely electric identities $-(k\wedge dV/V) = dk$ and $(k\wedge E/V) = F$ [see Eq. (A5)]. The closed two-forms corresponding to the currents defined in Eqs. (2.2) and (2.3) are

$$*(k\wedge j_Q) = *F \quad \text{and} \quad *(k\wedge j_M) = \tfrac{1}{2}*dk + \phi *F, \qquad (2.5)$$

respectively.

Defining the horizon quantities $M_H \equiv -(1/8\pi)\int_H *dk$ and $Q_H \equiv -(1/4\pi)\int_H *F$, and using the Komar expression $M = -(1/8\pi)\int_\infty *dk$ for the total mass of a stationary spacetime, as well as the corresponding expression for the total charge, $Q=-(1/4\pi)\int_\infty *F$, we immediately find from Eqs. (2.4) and (2.5)

$$Q=Q_H, \quad M=M_H+\phi_H Q_H, \tag{2.6}$$

which implies the Smarr formula, $M=M_H+\phi_H Q$. We also recall that, for a Killing horizon $\mathcal{H}$ with null generator Killing field k, we have $M_H=(1/4\pi)\kappa\mathcal{A}$, where κ and $\mathcal{A}$ are, respectively, the surface gravity and the area of the horizon (at time Σ). Here we have adopted the gauge $\phi_\infty=0$ and used the fact that the electric potential assumes a constant value on the Killing horizon, ϕ_H, say. We also recall that asymptotic flatness and the Killing property of the horizon imply $V_\infty=1$ and $V_H=0$, respectively. As a consequence of the above relations (i.e., the Smarr formula), the horizon value of the electric potential is determined by the total mass M, the total charge Q, and the horizon quantities κ and $\mathcal{A}$,

$$\phi_H=\frac{1}{Q}\left(M-\frac{1}{4\pi}\kappa\mathcal{A}\right). \tag{2.7}$$

Until now we have only used Stokes' theorem and the fact that the field equations assume the form of differential conservation laws. One may wonder if there exist additional conserved currents which can also be expressed in terms of the one-forms dV/V and $d\phi/V$. Although the conservation laws for these currents will give rise to redundant equations on the differential level, they may, nevertheless, provide us with new information after integration. This is due to the fact that the coefficients in front of the one-forms dV/V and $d\phi/V$ can be pulled out of the boundary integrals, provided that they depend only on the potentials V and ϕ, and assume, therefore, constant values on H and S^2_∞. In this way one obtains combinations of M, Q, M_H, and Q_H which are independent of the relations (2.6) derived from the field equations. In fact, it is immediately verified from Eqs. (2.2) and (2.3) that

$$d^\dagger j_3=0, \quad j_3\equiv(V+\phi^2)\frac{d\phi}{V}-\phi\frac{dV}{V}. \tag{2.8}$$

[Use $d^\dagger(f\alpha)=fd^\dagger\alpha-\langle df,\alpha\rangle$ (for arbitrary functions f and one-forms α) to show that j_3 is conserved.] We can, therefore, apply Stokes' theorem (2.4) to the new closed two-form obtained from j_3,

$$*(k\wedge j_3)=(V+\phi^2)*F+\phi*dk. \tag{2.9}$$

As the potentials assume constant values on the horizon and at infinity, we immediately find $Q=\phi_H^2 Q_H+2\phi_H M_H =-\phi_H^2 Q+2\phi_H M$, where we have also used Eqs. (2.6) in the second step. Now using the expression (2.7) for ϕ_H gives $Q^2=(M-M_H)(M+M_H)$ and hence

$$M^2=\left(\frac{1}{4\pi}\kappa\mathcal{A}\right)^2+Q^2 \quad \text{i.e.,}\ \ T_H=\frac{2}{\mathcal{A}}\sqrt{M^2-Q^2}, \tag{2.10}$$

where $T_H=(1/2\pi)\kappa$ is the Hawking temperature.

The relation between the charges and the horizon quantities following from Eq. (2.8) was already derived by Israel in 1967 for a nondegenerate Killing horizon, $\kappa\neq 0$ [20]. [The above derivation does not require that the horizon contains its bifurcation surface, implying that Eq. (2.10) also holds in the degenerate case.] In fact, Israel and other authors used quadratic relations of the above kind to conclude that the electric potential depends only on the gravitational potential, $\phi=\phi(V)$. This important result opened the way for the extension of the vacuum Israel theorem [20] to static electrovac black hole spacetimes [21,22].

The existence of the additional conserved current (2.8) is not accidental: In the presence of a Killing field, the EM equations form a nonlinear σ model (effectively coupled to three-dimensional gravity) with a symmetric target space G/H [2] (see the next section). The isometries of the target space imply that, in addition to the $\dim(G/H)$ field equations, there exists an extra set of $\dim(H)$ conserved currents. In the static, purely electric case under consideration one ends up with the two equations (2.2) and (2.3) for V and ϕ, respectively, and the additional conserved current j_3, given in Eq. (2.8). (The *full* EM system comprises four plus four conserved currents; the truncation $U=0$, $\psi=0$ is, in this case, compatible with the coset representation. Here U and ψ are the twist and the magnetic potential, respectively, to be defined in the following section.)

III. THE STATIONARY EINSTEIN-MAXWELL SYSTEM

In the previous section we have restricted ourselves to the static, purely electric case. We shall now construct the complete set of conserved currents for the general stationary EM equations. We do so by taking advantage of the σ-model structure of the EM equations in the presence of a Killing field. The eight conserved currents give rise to eight closed two-forms which will be integrated over a spacelike hypersurface. The resulting Smarr formulas are finally used to obtain the desired quadratic relation (1.3) between the flux integrals and the quantity $M_H=(1/4\pi)\kappa\mathcal{A}$.

A. Dimensional reduction

We start by briefly recalling some basic facts concerning the dimensional reduction of the Maxwell and the Einstein equations in the presence of a (stationary) Killing field [8] (see also [23,24, or 6]). Throughout this paper we use the symbols k,V, and ω for the stationary Killing field (one-form), its norm, and its twist one-form, respectively:

$$V\equiv-\langle k,k\rangle, \quad \omega\equiv\tfrac{1}{2}*(k\wedge dk). \tag{3.1}$$

In the presence of a Killing field, the Bianchi identity, $dF=0$, and the Maxwell equation, $d*F=0$, give rise to two (local) scalar potentials ϕ and ψ, respectively: Since the Lie derivatives of F and $*F$ with respect to k vanish, one obtains (with $L_k=i_k\circ d+d\circ i_k$) the equations $d(i_kF)=0$ and $d(i_k*F)=0$, and hence

$$E\equiv-i_kF=d\phi, \quad B\equiv i_k*F=d\psi. \tag{3.2}$$

[Here and in the following $i_k\alpha$ denotes the interior derivative of the p-form α with respect to k, $(i_k\alpha)_{\mu_2\ldots\mu_p}\equiv k^\mu\alpha_{\mu\mu_2\ldots\mu_p}$; see also Eq. (A3).] By virtue of

Eq. (3.2), the electromagnetic two-form can be expressed in terms of k, $d\phi$, and $d\psi$ as follows:

$$F=\frac{k}{V}\wedge d\phi+*\left(\frac{k}{V}\wedge d\psi\right). \tag{3.3}$$

In the Appendix we show that each closed and invariant two-form gives rise to a local conservation law for a current one-form [see Eq. (A7)]. Applying this result to F and $*F$ brings the Maxwell equations in the form (A8),

$$d^{\dagger}\left(\frac{d\psi}{V}+2\phi\frac{\omega}{V^2}\right)=0, \quad d^{\dagger}\left(\frac{d\phi}{V}-2\psi\frac{\omega}{V^2}\right)=0. \tag{3.4}$$

As for the reduction of the Einstein equations, the $\boldsymbol{R}(k,\cdot)$ components of the Ricci tensor are obtained from the general Killing field identity (A11) derived in the Appendix. Also using the expressions $8\pi*[k\wedge\boldsymbol{T}(k)]=-2d\phi\wedge d\psi$ and $8\pi\boldsymbol{T}(k,k)=\langle d\phi,d\phi\rangle+\langle d\psi,d\psi\rangle$ for the electromagnetic stress-energy tensor [where $T(k)_{\mu}\equiv T_{\mu\nu}k^{\nu}$], the general identities (A12) and (A13) yield

$$d^{\dagger}\left(\frac{dV}{V}\right)=4\frac{\langle\omega,\omega\rangle}{V^2}-2\frac{\langle d\phi,d\phi\rangle+\langle d\psi,d\psi\rangle}{V} \tag{3.5}$$

and

$$d\omega=-2d\phi\wedge d\psi \;\Rightarrow\; \omega=dU+\psi d\phi-\phi d\psi, \tag{3.6}$$

respectively, where U denotes the twist potential. We have already argued that the Maxwell equations for ϕ and ψ can be cast into the form of conservation laws (3.4). This is, in fact, also true for the Poisson equation (3.5): Using again the identity $d^{\dagger}(\omega/V^2)=0$, we have $d^{\dagger}(U\omega/V^2)=-\langle dU,\omega/V^2\rangle$ which, by virtue of Eq. (3.6), brings Eq. (3.5) into the form (3.8) below. We, therefore, end up with the following set of conserved currents, given in terms of the four potentials V, U, ϕ, and ψ:

$$d^{\dagger}j_N=d^{\dagger}\left(\frac{\omega}{V^2}\right)=d^{\dagger}\left(\frac{1}{V^2}(dU+\psi d\phi-\phi d\psi)\right)=0, \tag{3.7}$$

$$d^{\dagger}j_M=d^{\dagger}\left(-\frac{1}{2}\frac{dV}{V}+\psi\frac{B}{V}+\phi\frac{E}{V}-2U\frac{\omega}{V^2}\right)=0, \tag{3.8}$$

$$d^{\dagger}j_Q=d^{\dagger}\left(\frac{E}{V}-2\psi\frac{\omega}{V^2}\right)=0, \tag{3.9}$$

$$d^{\dagger}j_P=d^{\dagger}\left(\frac{B}{V}+2\phi\frac{\omega}{V^2}\right)=0. \tag{3.10}$$

In addition to these equations for the electromagnetic and the gravitational potentials, one has the Einstein equations for the projection metric $\boldsymbol{p}\equiv V\boldsymbol{g}+k\otimes k$ ($\boldsymbol{g}$ being the spacetime metric). These are readily obtained from Eq. (A14) and the fact that the electromagnetic stress-energy tensor satisfies $\boldsymbol{T}(X,Y)=(1/V)[\boldsymbol{T}(k,k)\boldsymbol{g}-(1/4\pi)(d\phi\otimes d\phi+d\psi\otimes d\psi)](X,Y)$ for vector fields X and Y orthogonal to k. The equation for the Ricci tensor of the projection metric $\boldsymbol{p}$ thus becomes

$$\boldsymbol{R}^{(p)}=\frac{1}{2V^2}(dV\otimes dV)+\frac{2}{V^2}(\omega\otimes\omega)-\frac{2}{V}(d\phi\otimes d\phi+d\psi\otimes d\psi). \tag{3.11}$$

It is well known, and of crucial importance to what follows, that the entire set of field equations (3.7)–(3.11) is obtained from the effective action (see, e.g., [24])

$$\mathcal{S}_{\text{eff}}=\int\left(-R^{(p)}+\frac{\langle dV,dV\rangle}{2V^2}+2\frac{\langle\omega,\omega\rangle}{V^2}-2\frac{\langle d\phi,d\phi\rangle+\langle d\psi,d\psi\rangle}{V}\right)\eta^{(p)}, \tag{3.12}$$

by considering variations with respect to the electromagnetic potentials ϕ,ψ, the gravitational potentials V,U, and the metric $\boldsymbol{p}$ [where $\omega=\omega(U,\phi,\psi)=dU+\psi d\phi-\phi d\psi$]. Here $R^{(p)}$ and $\eta^{(p)}$ denote the Ricci scalar and the volume three-form with respect to $\boldsymbol{p}$. Two comments may be helpful.

First, we note that $*j=-(k/V)\wedge\tilde{*}j$ for arbitrary one-forms j orthogonal to k, $\langle j,k\rangle=0$, (where $\tilde{*}$ denotes the Hodge dual with respect to the metric $\boldsymbol{p}$). The identity (A5), therefore, implies that the conservation laws $d\tilde{*}j=0$ obtained from the effective action (3.12) can also be written in the form $d*j=0$, that is, in the four-dimensional notation of Eqs. (3.7)–(3.10).

Second, the $\boldsymbol{R}(k,X)$ component of the Einstein equations is not obtained from the effective action (3.12) but has already been used in order to express the one-form ω in terms of the potentials U, ϕ, and ψ. The *systematic* Kaluza-Klein reduction of the Einstein-Hilbert action in the presence of a Killing field yields an effective action in terms of the gravitational potential V, the projection metric $\boldsymbol{p}$, and the bundle connection one-form γ, say. The equation for γ then implies the existence of the potential U. Substituting dU for $\tilde{*}d\gamma$ (by applying the Lagrange multiplier method) yields the ''partially on shell'' action (3.12).

B. Coset formulation

The action (3.12) describes a harmonic mapping into a four-dimensional target space, effectively coupled to three-dimensional gravity. Ernst [3] was able to parametrize the target space in terms of two complex potentials, $\mathcal{E}$ and Λ,

$$\mathcal{E}\equiv V-(\phi^2+\psi^2)+2iU, \quad \Lambda\equiv-\phi+i\psi. \tag{3.13}$$

In order to find the isometries of the target manifold, Neugebauer and Kramer [2] solved the corresponding Killing equations. This revealed the coset structure of the target space [4] and provided a parametrization of the latter in terms of the Ernst potentials [2,3]. (See also [25,5,26] for the complete integrability of the reduced system in the case of *two* Killing fields.) In the simplest case, that is for vacuum gravity, the coset space G/H is SU(1,1)/U(1), whereas $G/H=\mathrm{SU}(2,1)/S(\mathrm{U}(1,1)\times\mathrm{U}(1))$ for the Einstein-Maxwell equations with a timelike Killing field. [If the dimensional reduction is performed with respect to a spacelike Killing field, then $H=S(\mathrm{U}(2)\times\mathrm{U}(1))$.]

The explicit representation of the coset manifold in terms of the above Ernst potentials $\mathcal{E}$ and Λ is given by the Hermitian matrix

$$\Phi_{ab}=\eta_{ab}-2\overline{v_a}v_b, \tag{3.14}$$

where $\eta=\text{diag}(-1,+1,+1)$, and where v is the Kinnersley vector [9,10],

$$(v_0,v_1,v_2)=\frac{1}{2\sqrt{V}}(\mathcal{E}-1,\mathcal{E}+1,2\Lambda). \tag{3.15}$$

It is not hard to verify that, in terms of Φ, the effective action (3.12) assumes the manifestly SU(2,1)-invariant form

$$\mathcal{S}_{\text{eff}}=\int(-R^{(p)}+\text{Tr}\langle J,J\rangle)\eta^{(p)},\quad \text{with}\quad J=\tfrac{1}{2}\Phi^{-1}d\Phi. \tag{3.16}$$

The equations of motion following from the above action are the three-dimensional Einstein equations (obtained from variations with respect to $\boldsymbol{p}$) and the σ-model equations (obtained from variations with respect to Φ):

$$\boldsymbol{R}^{(p)}=\text{Tr}\{J\otimes J\},\quad d*J=0. \tag{3.17}$$

[Here we have again used the four-dimensional notation; see the comment below Eq. (3.12).] An important feature of the coset structure is the fact that it provides one with a set of differential equations which is *larger* than the original one: In addition to the $\dim[\text{SU}(2,1)/S(\text{U}(1,1)\times\text{U}(1))]=\text{four}$ equations (3.7)–(3.10), the above equation for the matrix current J comprises $\dim[S(\text{U}(1,1)\times\text{U}(1))]=\text{four}$ extra conservation laws. A straightforward (but rather unpleasant) computation gives the following explicit representation for J:

$$-J=\begin{pmatrix}-ij_N & j_M & 0\\ j_M & ij_N & j_Q-ij_P\\ 0 & -j_Q-ij_P & 0\end{pmatrix}+\begin{pmatrix}i(j_1+j_2) & ij_2 & -j_{34}\\ -ij_2 & i(j_1-j_2) & j_{34}\\ -\overline{j_{34}} & -\overline{j_{34}} & -2ij_1\end{pmatrix},$$

where j_N, j_M, j_Q, and j_P were given in Eqs. (3.7)–(3.10). The four additional currents j_1, j_2, and $j_{34}\equiv j_3+ij_4$ are linear combinations of the one-forms ω/V^2, dV/V, E/V, and B/V as well. Using Eqs. (3.7)–(3.10) to express the latter in terms of j_N,j_M,j_Q, and j_P, one finds

$$j_1=(\phi^2+\psi^2)j_N+(\psi j_Q-\phi j_P), \tag{3.18}$$

$$j_2=2Uj_M+(\phi^2+\psi^2-V)(\psi j_Q-\phi j_P)+\tfrac{1}{2}[1+4U^2+(\phi^2+\psi^2-V)(3\phi^2+3\psi^2-V)]j_N, \tag{3.19}$$

$$j_3=\phi j_M-2\left[-\phi U+\psi\left(\phi^2+\psi^2-\frac{V}{2}\right)\right]j_N-\tfrac{1}{2}(3\psi^2+\phi^2+1-V)j_Q+(\phi\psi-U)j_P, \tag{3.20}$$

$$j_4=-\psi j_M-2\left[\psi U+\phi\left(\phi^2+\psi^2-\frac{V}{2}\right)\right]j_N+\tfrac{1}{2}(3\phi^2+\psi^2+1-V)j_P-(\phi\psi+U)j_Q. \tag{3.21}$$

It is obvious from Eq. (3.17), and also easy to verify directly from Eqs. (3.7)–(3.10), that $d^\dagger j_1=d^\dagger j_2=d^\dagger j_3=d^\dagger j_4=0$. As an example, we obtain for the first current $d^\dagger j_1=-2\langle\phi E+\psi B,j_N\rangle-\langle B,j_Q\rangle+\langle E,j_P\rangle=0$. [Use the identity $d^\dagger(f\alpha)=fd^\dagger\alpha-\langle df,\alpha\rangle$ (for arbitrary functions f and one-forms α) to obtain this.]

C. Mass formulas

In order to apply Stokes' theorem (2.4) we use Eq. (A4), which shows that each conserved current j, $d^\dagger j=0$, gives rise to a closed two-form $*(k\wedge j)$, $d*(k\wedge j)=0$. Using Eqs. (3.3) and (3.9), and the identity (A5), $d(k/V)=2*(k\wedge\omega/V^2)$, we find, for instance,

$$*(k\wedge j_Q)=*\left(k\wedge\frac{E}{V}\right)-\psi d\left(\frac{k}{V}\right)=*F-d\left(\psi\frac{k}{V}\right).$$

In a similar way one derives the desired expressions for $*(k\wedge j_P)$ and $*(k\wedge j_M)$ [also taking advantage of the identity $*(k\wedge dV/V)=-*dk-2(k/V)\wedge\omega$]. The closed two-forms obtained from the conserved currents (3.7)–(3.10) become

$$*(k\wedge j_N)=*\left(k\wedge\frac{\omega}{V^2}\right)=\frac{1}{2}d\left(\frac{k}{V}\right), \tag{3.22}$$

$$*(k\wedge j_M)=\frac{1}{2}*dk+\psi F+\phi*F-d\left(U\frac{k}{V}\right), \tag{3.23}$$

$$*(k\wedge j_Q)=*F-d\left(\psi\frac{k}{V}\right), \tag{3.24}$$

$$*(k\wedge j_P)=F+d\left(\phi\frac{k}{V}\right). \tag{3.25}$$

Stokes' theorem (2.4) now yields a set of relations between the charges M,Q,P and the corresponding horizon quantities M_H,Q_H,P_H, defined by

$$M,M_H=-\frac{1}{8\pi}\int_{S^2_\infty,H}*dk,\quad Q,Q_H=-\frac{1}{4\pi}\int_{S^2_\infty,H}*F,$$

$$P,P_H=-\frac{1}{4\pi}\int_{S^2_\infty,H}F, \tag{3.26}$$

where, by definition, $M_H=(1/4\pi)\kappa\mathcal{A}$. For asymptotically flat solutions the NUT charge vanishes and the integrals over the exact two-forms do not contribute. In this case, we immediately obtain from Eqs. (3.23)–(3.25)

$$M=M_H+\phi_H Q_H+\psi_H P_H\,,Q=Q_H\,,P=P_H\,, \quad (3.27)$$

where we have used the fact that all potentials assume constant values on the horizon. We also recall that asymptotic flatness implies $V_\infty=1$, whereas, by the definition of a Killing horizon, $V_H=0$. Here and in the following we adopt a gauge for which all other potentials vanish in the asymptotic regime, $U_\infty=\phi_\infty=\psi_\infty=0$. (For static, regular configurations without horizon the above relations reduce to $M=Q=P=0$, which yields the well-known nonexistence theorem for self-gravitating Abelian *soliton* solutions [7].)

So far we have used the field equations to derive Eqs. (3.27), which imply the Smarr formula, $M=M_H+\phi_H Q+\psi_H P$. The interesting observation is that Stokes' formula for the *additional* conserved currents j_1–j_4 [given in Eqs. (3.18)–(3.21)] yields a set of *new* relations between the charges and the horizon quantities. Since the potentials assume constant values on the horizon, they can be pulled out of the integrals, which implies that the additional relations do not depend on the original ones [although, as already emphasized, the differential laws $d^\dagger j_i=0$ $(i=1,\ldots,4)$ do not contain new information]. In order to evaluate Stokes' theorem (2.4) for the additional four closed two-forms $*(k\wedge j_i)$, one uses

$$\int_\infty *(k\wedge j_N)=\int_H *(k\wedge j_N)=0,$$

$$\int_\infty *(k\wedge j_M)=\int_H *(k\wedge j_M)=-4\pi M,$$

$$\int_\infty *(k\wedge j_Q)=\int_H *(k\wedge j_Q)=-4\pi Q,$$

$$\int_\infty *(k\wedge j_P)=\int_H *(k\wedge j_P)=-4\pi P.$$

In this way we immediately obtain from Eqs. (3.18) and (3.19) the formulas

$$\phi_H P=\psi_H Q \quad \text{and} \quad U_H M=0, \quad (3.28)$$

respectively. Together with the Smarr formula (3.27), this enables one to solve for the horizon potentials in terms of the charges and M_H,

$$\phi_H=Q\frac{M-M_H}{Q^2+P^2}, \quad \psi_H=P\frac{M-M_H}{Q^2+P^2}, \quad U_H=0, \quad (3.29)$$

where $U_H=0$ reflects the fact that, for the moment, we have restricted ourselves to configurations with vanishing NUT charge. We may finally apply Stokes' theorem to either of the remaining equations (3.20) or (3.21). Using Eq. (3.20) we find

$$0=2\phi_H M-(3\psi_H^2+\phi_H^2+1)Q+2(\phi_H\psi_H-U_H)P. \quad (3.30)$$

Substituting the expressions (3.29) for the potentials into this equation eventually yields the desired formula, which involves only global charges and the horizon quantity M_H:

$$M^2=M_H^2+Q^2+P^2, \quad \text{with } M_H=\frac{1}{4\pi}\kappa\mathcal{A}=\tfrac{1}{2}T_H\mathcal{A}. \quad (3.31)$$

The derivation of Eq. (3.31) implies that this formula holds for every stationary, asymptotically flat black hole solution with nonrotating horizon, i.e., with Killing horizon generated by the stationary Killing field k. Considering the uniqueness theorem for the Reissner-Nordström metric, this is, of course, not surprising. However, the above derivation does, for instance, circumvent the staticity problem. Moreover, we have not required a nondegenerate horizon. Hence, the formula (3.31) also implies that the stationary, nonrotating solutions with vanishing surface gravity saturate the Bogomol'nyi bound $M^2=Q^2+P^2$, and vice versa [27,28] (provided, of course, that the horizon is connected). Before we derive a similar formula for the EMDA system, we also evaluate the above currents for configurations with nonvanishing NUT charge.

D. Mass formulas including NUT charge

The NUT charge N and its horizon counterpart N_H are defined by the boundary integrals

$$N,N_H=-\frac{1}{4\pi}\int_{S^2_\infty,H} *(k\wedge j_N)=-\frac{1}{8\pi}\int_{S^2_\infty,H} d\left(\frac{k}{V}\right). \quad (3.32)$$

Like the magnetic charge P,N is a topological quantity. (An illustration is provided by the Schwarzschild-NUT solution:

$$^{(4)}\boldsymbol{g}=-V(dt-2N\cos\vartheta d\varphi)^2+\frac{1}{V}dr^2+(r^2+N^2)d\Omega^2, \quad (3.33)$$

with

$$V(r)=\frac{r(r-2M)-N^2}{r^2+N^2}.$$

The stationary Killing one-form is $k=-V(dt-2N\cos\vartheta d\varphi)$. Hence, we have $d(k/V)=-2N\sin\vartheta d\vartheta\wedge d\varphi$ and $-(1/8\pi)\int d(k/V)=N$ for any two-sphere; in particular, $N=N_H$. Also using $*dk=-(r^2+N^2)(dV/dr)d\Omega+(dr\wedge\cdots)$, one finds

$$M,M_H=-\frac{1}{8\pi}\int_{S^2_\infty,H} *dk=\left[\frac{2N^2r+M(r^2-N^2)}{r^2+N^2}\right]_{\infty,r_H}.$$

As expected, the RHS yields M as $r\to\infty$ whereas, for $r=r_H=M+\sqrt{M^2+N^2}$, we obtain $M_H=\sqrt{M^2+N^2}$. For the Schwarzschild-NUT metric we, therefore, have

$$N=N_H, \quad M^2+N^2=M_H^2. \quad (3.34)$$

It will follow below, that this relation holds for any stationary, nonrotating vacuum black hole solution.)

Let us now return to the general stationary EM equations and evaluate Stokes' theorem for the closed two-forms (3.22)–(3.25) with nonvanishing NUT charge. Instead of Eqs. (3.27) we now obtain the slightly modified relations (in a gauge where $\phi_\infty=\psi_\infty=U_\infty=0$)

$$N=N_H, \quad Q=Q_H-2\psi_H N, \quad P=P_H+2\phi_H N, \tag{3.35}$$

and

$$M=M_H+\phi_H Q+\psi_H P-2U_H N. \tag{3.36}$$

Here we have already used the consequence $\phi_H Q_H+\psi_H P_H=\phi_H Q+\psi_H P$ of Eqs. (3.35) to obtain the Smarr formula (3.36) with NUT charge. Using the fact that the potentials assume constant values on the horizon, we can again evaluate Stokes' theorem for the remaining closed two-forms $*(k\wedge j_i)(i=1,\ldots,4)$, where now $-(1/4\pi)\int_\infty *(k\wedge j_N)=-(1/4\pi)\int_H *(k\wedge j_N)=N$. The expressions (3.18)–(3.21) then imply the relations (with $V_\infty=1$ and $V_H=0$)

$$0=(\phi_H^2+\psi_H^2)N+(\psi_H Q-\phi_H P),$$

$$N=2U_H M+(\phi_H^2+\psi_H^2)(\psi_H Q-\phi_H P)+\tfrac{1}{2}[1+4U_H^2+3(\phi_H^2+\psi_H^2)^2]N,$$

$$0=\phi_H M+2[\phi_H U_H-\psi_H(\phi_H^2+\psi_H^2)]N-\tfrac{1}{2}(3\psi_H^2+\phi_H^2+1)Q+(\phi_H\psi_H-U_H)P,$$

$$0=\psi_H M+2[\psi_H U_H+\phi_H(\phi_H^2+\psi_H^2)]N-\tfrac{1}{2}(3\phi_H^2+\psi_H^2+1)P+(\phi_H\psi_H+U_H)Q.$$

Adding together ϕ_H times the third and ψ_H times the fourth relation and using the Smarr formula (3.36) one finds $(\phi_H^2+\psi_H^2)(M+M_H)=\phi_H Q+\psi_H P$. In combination with the first of the above formulas, this enables one to solve for ϕ_H and ψ_H. Substituting the result into the Smarr formula then also yields U_H:

$$\phi_H=\frac{(M+M_H)Q-NP}{(M+M_H)^2+N^2}, \quad \psi_H=\frac{(M+M_H)P+NQ}{(M+M_H)^2+N^2}, \tag{3.37}$$

$$U_H=\frac{(M+M_H)(Q^2+P^2+M_H^2-M^2)-(M-M_H)N^2}{2N[(M+M_H)^2+N^2]}. \tag{3.38}$$

We may finally use these expressions for the horizon potentials in the second of the above formulas, which can also be written in the form $4U_H N(M+U_H N)=N^2-(\phi_H P-\psi_H Q)^2$. A short computation yields the desired relation between the total charges and the horizon quantity $M_H=(1/4\pi)\kappa\mathcal{A}$,

$$M^2+N^2=\left(\frac{1}{4\pi}\kappa\mathcal{A}\right)^2+Q^2+P^2, \tag{3.39}$$

which generalizes the previous result (3.31).

IV. THE EINSTEIN-MAXWELL-DILATON-AXION SYSTEM

Let us now consider the bosonic sector of four-dimensional heterotic string theory or, equivalently, $N=4$ supergravity with one vector field. Denoting the dilaton scalar field with S, the axion pseudoscalar field with κ, and the Abelian (Maxwell) vector field with A, the effective action can be cast into the form

$$\mathcal{S}=\frac{1}{16\pi}\int[-*R+2F\wedge*G+2dS\wedge*dS+\tfrac{1}{2}e^{4S}d\kappa\wedge*d\kappa], \tag{4.1}$$

where F is the field strength of the vector field. Here we have introduced the two-form G, which turns out to be very convenient in the following. For vanishing dilaton and axion fields we have $G=F$, whereas, in general, G is a combination of F and $*F$, involving the dilaton and the axion fields:

$$G\equiv e^{-2S}F-\kappa*F, \quad \text{where} \quad F=dA. \tag{4.2}$$

(Hence, $F\wedge*G=e^{-2S}F\wedge*F+\kappa F\wedge F$.) It is also worthwhile recalling that it is the boundary integral over $*G$ (rather than over $*F$) which is identified with the electric charge in the presence of a dilaton and an axion (see, e.g., [19] and Eq. (4.37) below).

A. Dimensional reduction

The dimensional reduction of the field equations in the presence of the stationary Killing field k can be performed along the same lines as for the EM system discussed in the previous section. The Bianchi identity, $dF=0$, and the "Maxwell" equation, $d*G=0$ (i.e., the variational equation with respect to A), give (locally) again rise to two scalar potentials, ϕ and ψ, say: Since $L_k F=0$ and $L_k*G=*L_k G=0$ one obtains (with $L_k=i_K\circ d+d\circ i_k$) the equations $d(i_k F)=0$ and $d(i_k*G)=0$, and, therefore,

$$d\phi=-i_k F, \quad d\psi=i_k*G. \tag{4.3}$$

Since both F and $*G$ are closed and invariant with respect to the Killing field k, we can apply the construction discussed in the Appendix [see Eq. (A7)] to obtain two conserved current one-forms:

$$d^\dagger j_P=0, \quad \text{where} \quad j_P=\frac{\hat{B}}{V}+2\phi\frac{\omega}{V^2}, \quad \hat{B}\equiv i_k*F, \tag{4.4}$$

$$d^\dagger j_Q=0, \quad \text{where} \quad j_Q=\frac{\hat{E}}{V}-2\psi\frac{\omega}{V^2}, \quad \hat{E}\equiv -i_k G. \tag{4.5}$$

It is easy to see that the one-forms $\hat{E}$ and $\hat{B}$ are linear combinations in $d\phi$ and $d\psi$:

$$\begin{pmatrix}\hat{E}\\ \hat{B}\end{pmatrix}=\mathcal{D}\begin{pmatrix}d\phi\\ d\psi\end{pmatrix}, \quad \text{with} \quad \mathcal{D}=\begin{pmatrix}e^{-2S}+\kappa^2 e^{2S} & \kappa e^{2S}\\ \kappa e^{2S} & e^{2S}\end{pmatrix}. \tag{4.6}$$

In terms of $\hat{E},\hat{B}$, the potentials and the Killing one-form, we also have

$$F=\frac{k}{V}\wedge d\phi+*\left(\frac{k}{V}\wedge\hat{B}\right), \quad G=\frac{k}{V}\wedge\hat{E}+*\left(\frac{k}{V}\wedge d\psi\right), \tag{4.7}$$

which generalize Eq. (3.3). It is worth recalling that the symmetric and symplectic matrix $\mathcal{D}$ is a special case of the matrix introduced in [7], parametrizing an arbitrary number of moduli fields (see also [19]). [For vanishing axion and dilaton fields we have $G \rightarrow F$, $\mathcal{D} \rightarrow \mathbb{1}$, $\hat{E} \rightarrow d\phi$, and $\hat{B} \rightarrow d\psi$, which shows that Eqs. (4.4) and (4.5) reduce to the ordinary Maxwell equations (3.4) in this case.]

The axion and dilaton equations are obtained from variations of the action (4.1) with respect to κ and S, respectively. One finds

$$d^{\dagger}(e^{4S}d\kappa) = -2\langle F, *F\rangle, \tag{4.8}$$

$$d^{\dagger}(dS - \tfrac{1}{2}e^{4S}\kappa d\kappa) = \langle F, G\rangle. \tag{4.9}$$

(Note that the variation with respect to the dilaton field first gives $d^{\dagger}dS + \frac{1}{2}e^{4S}\langle\kappa,\kappa\rangle = e^{-2S}\langle F,F\rangle$. Integrating by parts and using the axion equation (4.8) and the definition of G then yields Eq. (4.9). Also note that $\langle\alpha,\beta\rangle *1 \equiv \alpha \wedge *\beta$ for arbitrary forms of the same degree; hence $\langle F,G\rangle = \frac{1}{2}F_{\mu\nu}G^{\mu\nu}$.) Now using the ''Maxwell'' equations (4.4) and (4.5), the identity $d^{\dagger}(\omega/V^2) = 0$ and the formula $d^{\dagger}(f\alpha) = f d^{\dagger}\alpha - \langle df, \alpha\rangle$ (for arbitrary functions f and one-forms α), we can write the axion and dilaton equations (4.8) and (4.9) in the form of current conservation laws as well, since

$$\langle F, *F\rangle = -2d^{\dagger}\left(\phi\frac{\hat{B}}{V} + \phi^2\frac{\omega}{V^2}\right),$$

$$\langle F, G\rangle = d^{\dagger}\left(\phi\frac{\hat{E}}{V} - \psi\frac{\hat{B}}{V} - 2\phi\psi\frac{\omega}{V^2}\right).$$

It remains to consider the Einstein equations. In order to evaluate the Poisson equation (A12) and the twist equation (A13), we have to compute the Ricci one-form $\boldsymbol{R}(k)$. Since the kinetic terms of the axion and the dilaton do not contribute to $\boldsymbol{R}(k)$, we have $R(k)_\mu = [t_{\mu\nu} - \frac{1}{2}g_{\mu\nu}t^{\sigma}{}_{\sigma}]k^{\nu}$, where $t_{\mu\nu}$ is the stress-energy tensor of the vector field:

$$t_{\mu\nu} = \frac{1}{8\pi}[2F_{\mu\sigma}G_{\nu}{}^{\sigma} - g_{\mu\nu}\langle F,G\rangle]. \tag{4.10}$$

Contracting with k^{ν} and using the expressions (4.7), Eqs. (A12) and (A13) yield

$$d^{\dagger}\left(\frac{dV}{V}\right) = \frac{4}{V^2}\langle\omega,\omega\rangle - \frac{2}{V}(2\langle i_k F, i_k G\rangle + V\langle F,G\rangle), \tag{4.11}$$

and

$$d\omega = -2d\phi \wedge d\psi, \tag{4.12}$$

respectively. The twist equation (4.12) implies the existence of a twist potential U, such that $dU = \omega + \phi d\psi - \psi d\phi$. The Poisson equation (4.11), therefore, also assumes the form of a conservation law, since its RHS becomes

$$\frac{4}{V^2}\langle\omega,\omega\rangle - \frac{2}{V}(\langle d\phi,\hat{E}\rangle + \langle d\psi,\hat{B}\rangle)$$
$$= d^{\dagger}\left(2\phi\frac{\hat{E}}{V} + 2\psi\frac{\hat{B}}{V} - 4U\frac{\omega}{V^2}\right).$$

In conclusion, the field equations for the three pairs of scalar potentials (V,U), (ϕ,ψ) and (S,κ) can be cast into the form of six conservation laws for the following current one-forms (see [7] for the more general case of an arbitrary number of moduli fields):

$$d^{\dagger}j_N = d^{\dagger}\left(\frac{\omega}{V^2}\right) = 0, \quad \text{where} \quad \omega = dU + \psi d\phi - \phi d\psi, \tag{4.13}$$

$$d^{\dagger}j_M = d^{\dagger}\left(-\frac{1}{2}\frac{dV}{V} + \psi\frac{\hat{B}}{V} + \phi\frac{\hat{E}}{V} - 2U\frac{\omega}{V^2}\right) = 0, \tag{4.14}$$

$$d^{\dagger}j_Q = d^{\dagger}\left(\frac{\hat{E}}{V} - 2\psi\frac{\omega}{V^2}\right) = 0, \tag{4.15}$$

$$d^{\dagger}j_P = d^{\dagger}\left(\frac{\hat{B}}{V} + 2\phi\frac{\omega}{V^2}\right) = 0, \tag{4.16}$$

$$d^{\dagger}j_A = d^{\dagger}\left(e^{4S}d\kappa + 4\phi\frac{\hat{B}}{V} + 4\phi^2\frac{\omega}{V^2}\right) = 0, \tag{4.17}$$

$$d^{\dagger}j_D = d^{\dagger}\left(-2dS + e^{4S}\kappa d\kappa - 2\psi\frac{\hat{B}}{V} + 2\phi\frac{\hat{E}}{V} - 4\phi\psi\frac{\omega}{V^2}\right) = 0, \tag{4.18}$$

where $\hat{E}$ and $\hat{B}$ are defined in terms of S, κ, $d\phi$, and $d\psi$ by Eq. (4.6). It may be worth noticing that Eqs. (4.13)–(4.16) reduce to the corresponding Einstein and Maxwell equations (3.7)–(3.10) for vanishing dilaton and axion fields. However, for $\kappa = S = 0$, the *entire* set of equations (4.13)–(4.18) is not equivalent to Eqs. (3.7)–(3.10), since the dilaton and axion equations (4.17) and (4.18) impose additional restrictions to the vector field A. It is for this reason that the coset formulation to be discussed below does not reduce to the electrovac coset representation for $\kappa = S = 0$.

The remaining equations, which will not be used in the following, are the Einstein equations for the projection metric $\boldsymbol{p} = V\boldsymbol{g} + k \otimes k$. These are again obtained from the reduction formula (A14), using $R_{\mu\nu} = 8\pi t_{\mu\nu} + 2S_{\mu}S_{\nu} + \frac{1}{2}e^{4S}\kappa_{\mu}\kappa_{\nu}$. Also taking advantage of the expression (4.10) for $t_{\mu\nu}$, the Einstein equations for $\boldsymbol{p}$ become

$$\boldsymbol{R}^{(p)} = \frac{1}{2V^2}(dV \otimes dV) + \frac{2}{V^2}(\omega \otimes \omega) - \frac{2}{V}(d\phi \otimes \hat{E} + d\psi \otimes \hat{B})$$
$$+ 2dS \otimes dS + \tfrac{1}{2}e^{4S}d\kappa \otimes d\kappa, \tag{4.19}$$

which reduces to Eq. (3.11) for the EM case.

B. Coset representation

The entire set of field equations, i.e., the conservation laws (4.13)–(4.18) and the three-dimensional Einstein equations (4.19) can be obtained from variations of the effective action $\mathcal{S}_{\rm eff}$ with respect to the scalar fields V, U, ϕ, ψ, S, κ, and the projection metric $\boldsymbol{p}$, where

$$\mathcal{S}_{\rm eff}=\int\left[-R^{(p)}+\frac{\langle dV,dV\rangle}{2V^2}+2\frac{\langle\omega,\omega\rangle}{V^2}-2\frac{\langle d\phi,\hat{E}\rangle+\langle d\psi,\hat{B}\rangle}{V}+2\langle dS,dS\rangle+\tfrac{1}{2}e^{4S}\langle d\kappa,d\kappa\rangle\right]\eta^{(p)}. \tag{4.20}$$

[Recall that $\omega=dU-\phi d\psi+\psi d\phi$, and $\hat{E}$ and $\hat{B}$ are given in terms of the potentials by Eq. (4.6).] Combining the Maxwell potentials into a vector, $\underline{a}\equiv(\phi,\psi)^T$, and using the matrix $\mathcal{D}$ defined in Eq. (4.6), the effective action assumes the compact form

$$\mathcal{S}_{\rm eff}=\int\left[-R^{(p)}+\frac{\langle dV,dV\rangle}{2V^2}+2\frac{\langle\omega,\omega\rangle}{V^2}-2\frac{\langle d\underline{a}^T,\mathcal{D}d\underline{a}\rangle}{V}+\langle\mathcal{D}^{-1}d\mathcal{D},\mathcal{D}^{-1}d\mathcal{D}\rangle\right]\eta^{(p)}, \tag{4.21}$$

where the inner product in the last two terms also involves the matrix trace. In terms of $\underline{a}$ and the antisymmetric 2×2 tensor ε one has $\omega=\omega(U,\underline{a})=dU+\underline{a}^T\varepsilon^{-1}d\underline{a}$. The Eqs. (4.13)–(4.16) are obtained from variations with respect to the gravitational potentials V and U and the potential $\underline{a}$. Since $\mathcal{D}$ is symmetric and symplectic, the axion and the dilaton describe a nonlinear σ model with coset space $\overline{G}/\overline{H}=\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ (see, e.g., [29]). Hence, the variation of the above action with respect to $\mathcal{D}$ yields the axion and dilaton equations (4.17) and (4.18) *and* an additional equation. In fact, one easily finds the following additional conserved current:

$$d^\dagger j_{AD}\equiv d^\dagger\left(4\kappa dS+(1-\kappa^2e^{4S})d\kappa+4\psi\frac{\hat{E}}{V}-4\psi^2\frac{\omega}{V^2}\right)=0. \tag{4.22}$$

This formula is, of course, a consequence of the set of field equations (4.13)–(4.18), as can also be verified directly. However, its integrated version will provide us with an additional relation between the charges and the horizon quantities. The axion and dilaton equations (4.17), (4.18), and (4.22) assume the form $d^\dagger\mathcal{J}=0$, where

$$\mathcal{J}=\mathcal{D}^{-1}d\mathcal{D}+4(\underline{a}^T\otimes\varepsilon\underline{a})j_N+2[\underline{a}\otimes\underline{j}-\varepsilon^{-1}(\underline{a}\otimes\underline{j})^T\varepsilon], \tag{4.23}$$

and where we have introduced the notations

$$\mathcal{J}\equiv\begin{pmatrix} j_D & j_A\\ j_{AD} & -j_D\end{pmatrix},\quad \underline{j}\equiv\begin{pmatrix} j_Q\\ j_P\end{pmatrix}. \tag{4.24}$$

[In deriving Eq. (4.23) we have also used Eqs. (4.13), (4.15), and (4.16) to substitute the one-forms ω/V^2, $\hat{E}/V$, and $\hat{B}/V$ by the currents j_N, j_Q, and j_P.] Before we proceed, we recall some facts concerning the structure of the stationary EMDA system.

Since $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to SU(1,1), the axion and dilaton describe a nonlinear σ model with the same coset space as the vacuum Ernst system, SU(1,1)/SU(1). [In fact, using the complex target space coordinate $z=\kappa+ie^{-2S}$, the effective density $\mathrm{Tr}\langle\mathcal{D}^{-1}d\mathcal{D},\mathcal{D}^{-1}d\mathcal{D}\rangle$ becomes $\langle dz,d\overline{z}\rangle/(z-\overline{z})^2$, which is the same expression as one finds for the vacuum Ernst potential. For axion-dilaton gravity without vector fields, the Kähler metric on the target space is, therefore, generated by the potential $\ln(Ve^{-2S})$; see [29] for details.]

The action (4.20) obviously describes a harmonic mapping which is effectively coupled to three-dimensional gravity. This is indeed the case for an arbitrary number of self-gravitating Abelian vector fields coupled to massless scalar (moduli) fields which form a coset space $\overline{G}/\overline{H}$. Breitenlohner *et al.* [7] have given a classification of models which admit a sufficiently large symmetry group, such that the *entire* set of potentials, i.e., the moduli *and* the vector and gravitational potentials, form a coset space G/H.

The $\mathrm{SL}(2,\mathbb{R})$ axion-dilaton symmetry is still present in axion-dilaton gravity with an Abelian gauge field. Like in the EM case, the system also possesses an SO(1,2) symmetry, arising from the dimensional reduction with respect to the Abelian isometry group generated by the Killing field. Gal'tsov and Kechkin [11] have shown that the full symmetry group is, however, larger than $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(1,2)$. Indeed, the target space for dilaton-axion gravity with a U(1) vector field is the coset $\mathrm{SO}(2,3)/[\mathrm{SO}(2)\times\mathrm{SO}(1,2)]$ [12]. Using the fact that SO(2,3) is isomorphic to $\mathrm{Sp}(4,\mathbb{R})$, Gal'tsov and Kechkin [30] were also able to give a parametrization of the target space in terms of 4×4 (rather than 5×5) matrices. The relevant coset was shown to be $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(1,1)$, which implies that, in addition to the field equations (4.13)–(4.18), there exists a set of *four* additional conserved currents [one of which, j_{AD}, was already constructed above from the $\mathrm{SL}(2,\mathbb{R})$ symmetry].

The explicit representation of the target space in terms of the potentials (V,U), (ϕ,ψ), and (S,κ) is given by the symplectic 4×4 matrix Φ,

$$\Phi=\begin{pmatrix}\mathcal{P}^{-1} & \mathcal{P}^{-1}\mathcal{Q}\\ \mathcal{Q}\mathcal{P}^{-1} & \mathcal{P}+\mathcal{Q}\mathcal{P}^{-1}\mathcal{Q}\end{pmatrix}, \tag{4.25}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the 2×2 matrices

$$\mathcal{P}=e^{-2S}\begin{pmatrix} e^{2S}V-2\phi^2 & \sqrt{2}\phi\\ \sqrt{2}\phi & -1\end{pmatrix},$$

$$\mathcal{Q}=\begin{pmatrix}-2\phi(\psi+\kappa\phi)-2U & \sqrt{2}(\psi+\kappa\phi)\\ \sqrt{2}(\psi+\kappa\phi) & -\kappa\end{pmatrix}, \tag{4.26}$$

see, e.g., [31,14]. [Our potentials slightly differ from the ones used in [14]: The potential pairs (f,χ), (v,u), and (κ,ϕ) of [14] are our (V,U), $(-\sqrt{2}\phi,\sqrt{2}\psi)$, and (κ,S), respectively.] In terms of the matrix Φ the effective action (4.21) assumes the desired form

$$\mathcal{S}_{\rm eff}=\int[-R^{(p)}+\mathrm{Tr}\langle\Phi^{-1}d\Phi,\Phi^{-1}d\Phi\rangle]\eta^{(p)}, \tag{4.27}$$

where the trace-free matrix $\Phi^{-1}d\Phi$ comprises four 2×2 current matrices, three of which are algebraically independent. A lengthy computation yields the following explicit expressions for the latter in terms of the ten currents j_N, j_M, j_Q, j_P, j_A, j_D, j_{AD}, and j_1–j_3:

$$\mathcal{P}^{-1}d\mathcal{Q}\mathcal{P}^{-1}=-\begin{pmatrix} 2j_N & \sqrt{2}j_P \\ \sqrt{2}j_P & j_A \end{pmatrix}, \tag{4.28}$$

$$\mathcal{Q}\mathcal{P}^{-1}d\mathcal{Q}\mathcal{P}^{-1}+d\mathcal{P}\mathcal{P}^{-1}=\begin{pmatrix} -2j_M & -\sqrt{2}j_1 \\ \sqrt{2}j_Q & j_D \end{pmatrix}, \tag{4.29}$$

$$d\mathcal{Q}-d\mathcal{P}\mathcal{P}^{-1}\mathcal{Q}-\mathcal{Q}\mathcal{P}^{-1}d\mathcal{P}-\mathcal{Q}\mathcal{P}^{-1}d\mathcal{Q}\mathcal{P}^{-1}\mathcal{Q} =\begin{pmatrix} -2j_3 & \sqrt{2}j_2 \\ \sqrt{2}j_2 & -j_{AD} \end{pmatrix}. \tag{4.30}$$

The conservation laws for the currents $j_N, j_M, j_Q, j_P, j_A, j_D$ are identical with the field equations (4.13)–(4.18). The conserved current j_{AD}, arising from the dilaton-axion symmetry, was given in Eq. (4.22). The remaining additional conserved currents, j_1–j_3 can be expressed in terms of $j_N, j_M, \underline{j}=(j_Q, j_P)^T$ and the 2×2 matrix $\mathcal{D}^{-1}d\mathcal{D}$ as follows:

$$\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}=(\mathcal{D}^{-1}d\mathcal{D}+2\mathbb{1}j_M)\underline{a}+(V\mathcal{D}^{-1}-2U\varepsilon)(\underline{j}+2j_N\varepsilon\underline{a}), \tag{4.31}$$

$$j_3=\mathrm{Tr}\{\underline{a}^T\varepsilon[(\mathcal{D}^{-1}d\mathcal{D})\underline{a}+2V\mathcal{D}^{-1}(\underline{j}+2j_N\varepsilon\underline{a})]\}+4Uj_M +(V^2+4U^2)j_N. \tag{4.32}$$

C. Mass formulas

In order to apply Stokes' theorem (2.4), we have to compute the closed two-forms $*(k\wedge j)$ obtained from the ten conserved currents j_N, j_M, $\underline{j}$, $\mathcal{J}$, (j_1, j_2), and j_3, given in Eqs. (4.13)–(4.16), (4.23), (4.31), and (4.32), respectively [see Eq. (4.24) for the definitions of $\underline{j}$ and $\mathcal{J}$]. To this end, we first express the two-forms arising from the gravitational and the electromagnetic currents (4.13)–(4.16) in terms of the two-forms $*dk$, $d(k/V)$, $*G$, and F (which give rise to the mass, the NUT charge and the electric and magnetic charges, respectively). This is achieved in a similar way as in the EM case. One finds

$$*(k\wedge j_N)=*\left(k\wedge\frac{\omega}{V^2}\right)=\tfrac{1}{2}d\left(\frac{k}{V}\right), \tag{4.33}$$

$$*(k\wedge j_M)=\tfrac{1}{2}*dk+\psi F+\phi*G-d\left(U\frac{k}{V}\right), \tag{4.34}$$

$$*(k\wedge j_Q)=*G-d\left(\psi\frac{k}{V}\right), \tag{4.35}$$

$$*(k\wedge j_P)=F+d\left(\phi\frac{k}{V}\right). \tag{4.36}$$

[For vanishing axion and dilaton fields this reduces to the corresponding EM expressions (3.22)–(3.25), since then $G=F$.] The following integrals over S^2_∞ and $H=\Sigma\cap\mathcal{H}$ give the electric, magnetic, dilaton, and axion charges, and their counterparts, Q_H, P_H, D_H, and A_H defined on the horizon:

$$Q,Q_H=-\frac{1}{4\pi}\int_{S^2_\infty,H}*G,\quad P,P_H=-\frac{1}{4\pi}\int_{S^2_\infty,H}F, \tag{4.37}$$

$$D,D_H=-\frac{1}{8\pi}\int_{S^2_\infty,H}*(k\wedge j_D),$$

$$A,A_H=-\frac{1}{8\pi}\int_{S^2_\infty,H}*(k\wedge j_A). \tag{4.38}$$

Requiring that both S and κ remain finite on the horizon, we find from the general properties of Killing horizons that

$$D_H=0,\quad A_H=0. \tag{4.39}$$

We recall that the total mass M and the corresponding horizon quantity $M_H=(1/4\pi)\kappa\mathcal{A}$ are given by the Komar integrals over S^2_∞ and H. In a similar way one obtains the NUT charge N and its horizon counterpart N_H:

$$M,M_H=-\frac{1}{8\pi}\int_{S^2_\infty,H}*dk,\quad N,N_H=-\frac{1}{8\pi}\int_{S^2_\infty,H}d\left(\frac{k}{V}\right). \tag{4.40}$$

We may now apply Stokes' theorem (2.4) to the closed two-forms (4.33)–(4.36). Adopting a gauge for which the electromagnetic and the twist potentials vanish at infinity, $\phi_\infty=\psi_\infty=U_\infty=0$, we immediately obtain the relations

$$N=N_H,\quad Q=Q_H-2\psi_H N,\quad P=P_H+2\phi_H N, \tag{4.41}$$

and

$$M=M_H+\phi_H Q+\psi_H P-2U_H N, \tag{4.42}$$

where we have already used Eqs. (4.41) on the RHS of the Smarr formula (4.42), i.e., we have replaced $\phi_H Q_H+\psi_H P_H$ by $\phi_H Q+\psi_H P$.

The information from the remaining conservation laws is now extracted as follows: First, we choose $S_\infty=\kappa_\infty=0$. (This can by achieved by generalized Ehlers and Harrison transformations; see [32].) The currents j_A and j_{AD} then coincide at infinity and the definitions (4.38) of the dilaton and axion charges yield [with $\underline{a}_\infty=(\phi_\infty,\psi_\infty)^T=0$]

$$-\frac{1}{4\pi}\int_{S^2_\infty}*(k\wedge\mathcal{D}^{-1}d\mathcal{D})=2\begin{pmatrix} D & A \\ A & -D \end{pmatrix},$$

$$\int_H *(k\wedge\mathcal{D}^{-1}d\mathcal{D})=0. \tag{4.43}$$

(In the second integral we have used Eq. (4.39) and the fact that κ and S assume constant values on the horizon.) Since all potentials can be pulled out of the integrals, we are now able to evaluate Stokes' theorem for the remaining closed two-forms; $*(k\wedge\mathcal{J})$ and $*[k\wedge j_i](i=1,\ldots,3)]$, using

$$\int_\infty *(k\wedge j_N)=\int_H *(k\wedge j_N)=-4\pi N,$$

$$\int_\infty *(k\wedge j_M)=\int_H *(k\wedge j_M)=-4\pi M,$$

$$\int_\infty *(k\wedge \underline{j})=\int_H *(k\wedge \underline{j})=-4\pi (Q,P)^T.$$

From the closed matrix two-form $*(k\wedge\mathcal{J})$ [with $\mathcal{J}$ given in Eq. (4.23)] we obtain an expression for the dilaton charge D and two expressions for the axion charge A. Combining these gives

$$D=\phi_H(Q+N\psi_H)-\psi_H(P-N\phi_H),$$
$$A=\psi_H(Q+N\psi_H)+\phi_H(P-N\phi_H), \tag{4.44}$$

and

$$N(\phi_H^2+\psi_H^2)=\phi_H P-\psi_H Q. \tag{4.45}$$

Stokes' theorem for $*(k\wedge j_3)$ [with j_3 given in Eq. (4.32)] is easily evaluated, since the trace term gives no contribution at the horizon $[V_H=0,(\mathcal{D}^{-1}d\mathcal{D})_H=0]$ and also vanishes at infinity $(\underline{a}_\infty=0)$. We thus have, from Eq. (4.32),

$$N=4U_H(M+U_H N). \tag{4.46}$$

Finally, the evaluation of the two-forms $*(k\wedge j_1)$ and $*(k\wedge j_2)$ [with j_1 and j_2 given in Eq. (4.31)] yields

$$Q=2\phi_H(M+2U_H N)-2U_H P,$$
$$P=2\psi_H(M+2U_H N)+2U_H Q. \tag{4.47}$$

For vanishing NUT charge, Eq. (4.46) gives $U_H=0$. Otherwise, we can solve for U_H and use the result in Eqs. (4.47) to obtain the explicit expressions for the potentials ϕ_H and ψ_H in terms of the charges $M,\ N,\ Q$, and P. One finds

$$\phi_H=\frac{1}{2N}\left(\frac{NQ-MP}{\sqrt{M^2+N^2}}+P\right),\quad \psi_H=\frac{1}{2N}\left(\frac{NP+MQ}{\sqrt{M^2+N^2}}-Q\right),$$

$$U_H=\frac{1}{2N}(\sqrt{M^2+N^2}-M). \tag{4.48}$$

We may eventually use these formulas to eliminate the potentials from the relations (4.44) and (4.45). A short calculation gives $2D(M^2+N^2)=2NPQ+M(Q^2-P^2)$ and $2A(M^2+N^2)=2MPQ-N(Q^2-P^2)$. Hence,

$$2M_c D_c=(Q_c)^2, \tag{4.49}$$

where the complex charges $M_c,\ Q_c$, and D_c are defined by

$$M_c\equiv M+iN,\quad Q_c\equiv Q+iP,\quad D_c\equiv D+iA. \tag{4.50}$$

We have now exhausted all information from the additional conservation laws following from the coset structure. The only equation which has not been used yet is the Smarr formula (4.42). Substituting the horizon values (4.48) for the potentials into the Smarr formula (4.42), we obtain the following expression for M_H in terms of the charges:

$$M_H=\frac{2|M_c|^2-|Q_c|^2}{2|M_c|}. \tag{4.51}$$

Taking the square of this formula [and using Eq. (4.49) to eliminate the $|Q_c|^4$ term] finally yields the desired expression, $M_H^2=|D_c|^2+|M_c|^2-|Q_c|^2$, that is,

$$\left(\frac{1}{4\pi}\kappa\mathcal{A}\right)^2+Q^2+P^2=M^2+N^2+D^2+A^2. \tag{4.52}$$

For $M_H=0$, the above formulas have been obtained for various spherically symmetric BPS solutions of the EM and EMDA equations (see, e.g., [14,18], and references therein). For spherically symmetric configurations with nondegenerate horizons $(\kappa\neq 0)$, Eq. (4.52) was obtained by Breitenlohner *et al.* [7], where the term on the LHS was not specified. More recently, Gibbons *et al.* [19] were able to establish the *full* relation (4.52) for spherically symmetric solutions, using the generalized first law of black hole thermodynamics. A version of Eq. (4.52) which also includes the rotation parameter was derived by Gal'tsov and Kechkin [11] for the dilaton-axion-Kerr-NUT dyon solution. The latter was constructed from the Kerr-NUT metric, using the symmetries of the target space. The relation was also derived by applying generalized Ehlers and Harrison transformations to the seed Schwarzschild solution [18,32].

The above derivation shows that the generalization (4.52) of the Bogomol'nyi equation holds for arbitrary, stationary, asymptotically flat (or asymptotically NUT) solutions of the EM and EMDA equations. The non-negative term which transforms the inequality $M^2+N^2+D^2+A^2-Q^2-P^2\geqslant 0$ into an equality is found to be $[(1/4\pi)\kappa\mathcal{A}]^2$. Although we have established the above results by using explicit representations of the EM and EMDA cosets, we expect them to hold in the general case as well. More precisely, we conjecture that the Hawking temperature of all stationary, asymptotically flat (or asymptotically NUT) black holes with massless scalars and Abelian vector fields is given by

$$T_H=\frac{2}{\mathcal{A}}\sqrt{\sum (Q_S)^2-\sum (Q_V)^2}, \tag{4.53}$$

provided that the field equations assume the form (1.1), (1.2), and Φ is a map into a *symmetric space* G/H. Here Q_S and Q_V denote the charges of the scalars (including the gravitational ones) and the vector fields, respectively.

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APPENDIX

In this Appendix we recall some identities for Killing fields. Throughout, k will denote a timelike Killing field (one-form) with norm V and twist (one-form) ω:

$$V \equiv -\langle k,k\rangle, \quad \omega \equiv \tfrac{1}{2} *(k \wedge dk). \tag{A1}$$

The Lie derivative of an arbitrary p-form α with respect to a Killing field commutes with the Hodge dual, i.e.,

$$L_k * \alpha = * L_k \alpha, \quad \text{where } L_k = i_k \circ d + d \circ i_k. \tag{A2}$$

The operator i_k denotes the interior product (derivative) which assigns to α the $(p-1)$-form $(i_k\alpha)_{\mu_2 \ldots \mu_p} \equiv \alpha_{\mu\mu_2 \ldots \mu_p} k^\mu$. The latter is also obtained from the exterior product of the dual of α with k:

$$i_k \alpha = -*(k \wedge * \alpha), \quad i_k * \alpha = *(\alpha \wedge k), \tag{A3}$$

where the second identity is obtained from the first one by replacing α with its dual and using $*^2\alpha = -(-1)^p \alpha$. For an invariant one-form α, $L_k\alpha = 0$, Eqs. (A2) and (A3) imply $d*(k\wedge\alpha) = -d(i_k*\alpha) = i_k d*\alpha$, and hence

$$d*(k \wedge \alpha) = -(d^\dagger \alpha) * k \quad \text{if } L_k\alpha = 0, \tag{A4}$$

where $d^\dagger \equiv *d*$ is the coderivative operator. The above formula also provides one with a coordinate-invariant formulation of Stokes' theorem for stationary (but not necessarily static) spacetimes.

The Frobenius theorem, implying that the hypersurfaces of constant V are orthogonal to k if and only if the twist ω vanishes, is recovered from the identity

$$d\left(\frac{k}{V}\right) = 2*\left(k \wedge \frac{\omega}{V^2}\right) = -\frac{2}{V^2} i_k * \omega. \tag{A5}$$

[This is obtained from Eqs. (A1) and (A2), which yield $2 i_k * \omega = i_k(k \wedge dk) = -V dk - k \wedge dV$, since $i_k dk = -d i_k k = dV$.] Applying the exterior derivative to the above identity and using Eq. (A2) yields $0 = i_k d*(\omega/V^2)$ and thus [since $d*(\omega/V^2)$ is a four-form]

$$d^\dagger\left(\frac{\omega}{V^2}\right) = 0. \tag{A6}$$

The identities (A5) and (A6) also imply the following: Let Ω be a closed two-form which is invariant with respect to the isometry group generated by the Killing field k. Then there exist (locally) a function f and a (current) one-form j, such that

$$d^\dagger j = 0, \quad \text{where} \quad j = \frac{i_k * \Omega}{V} - 2f\frac{\omega}{V^2} \quad \text{and} \quad df = i_k\Omega. \tag{A7}$$

First, by virtue of Eq. (A2), $L_k\Omega = 0$ and $d\Omega = 0$ imply $d(i_k\Omega) = 0$ and thus the local existence of a potential f, such that $df = i_k\Omega$. Using this and the identity (A5) gives $d((k/V)\wedge\Omega) = -2\Omega \wedge i_k * \omega/V^2 = 2 df \wedge * \omega/V^2 = 2d(f*\omega/V^2)$, where we have taken advantage of the identity (A6) in the last step. Using Eq. (A3) in the first term proves the above formula.

As an application of Eq. (A7) one can write the Maxwell equations in the presence of a Killing field in the form of conservation laws for two current one-forms. The Bianchi identity and the Maxwell equation imply that both the electromagnetic two-form F and its dual $*F$ are closed. Moreover, assuming that F is stationary, $L_kF = 0$, Eq. (A2) implies that $*F$ is stationary as well, $L_k*F = 0$. Hence, we can either choose $\Omega = F$ or $\Omega = *F$ in Eq. (A7). Introducing the potentials ϕ and ψ, defined by $-d\phi = i_kF$ and $d\psi = i_k*F$, respectively, the stationary Maxwell equations become

$$d^\dagger j_P = 0, \quad \text{where } j_P = \frac{d\psi}{V} + 2\phi\frac{\omega}{V^2},$$

$$d^\dagger j_Q = 0, \quad \text{where } j_Q = \frac{d\phi}{V} - 2\psi\frac{\omega}{V^2}. \tag{A8}$$

In the presence of a (timelike) Killing field, the Ricci tensor can be reduced with respect to the projection metric $\boldsymbol{p} = V\boldsymbol{g} + k \otimes k$. The $\boldsymbol{R}(k,\cdot)$ components can also be obtained from the Ricci identity,

$$-\Delta k = d^\dagger dk = 2\boldsymbol{R}(k), \quad \text{with } R(k)_\mu \equiv R_{\mu\nu}k^\nu, \tag{A9}$$

by expressing the Laplacian of k in terms of V and ω. [For a Killing one-form one has $d^\dagger k = 0$ and, therefore, $-\Delta k = (d^\dagger d + dd^\dagger)k = d^\dagger dk$.] To this end, one uses Eq. (A5) in the form

$$*dk = i_k * \frac{dV}{V} - 2\frac{k}{V} \wedge \omega. \tag{A10}$$

Applying the exterior derivative to this identity and taking advantage of Eq. (A2) in the first and of Eq. (A5) in the second term gives

$$d*dk = -i_k d*\frac{dV}{V} + \frac{4}{V^2} i_k * \omega \wedge \omega + 2\frac{k}{V} \wedge d\omega.$$

Now using $i_k\omega = 0$, $i_k*1 = *k$, and the Ricci identity (A9) yields the desired result,

$$2\boldsymbol{R}(k) = -\Delta k = \left[d^\dagger\left(\frac{dV}{V}\right) - 4\frac{\langle\omega,\omega\rangle}{V^2}\right]k + 2*\left(\frac{k}{V}\wedge\omega\right). \tag{A11}$$

Since the last one-form is orthogonal to k, we immediately find

$$2\boldsymbol{R}(k,k) = -Vd^\dagger\left(\frac{dV}{V}\right) + 4\frac{\langle\omega,\omega\rangle}{V}, \tag{A12}$$

$$k\wedge \boldsymbol{R}(k) = -*d\omega, \tag{A13}$$

where the second equation implies $\boldsymbol{R}(k,X) = (1/V)(*d\omega)(k,X)$, for any vector field X orthogonal to k. Finally, we also recall the projection formula for the remaining components of the Ricci tensor [8] (see, e.g., [24]). For X and Y orthogonal to k one finds

$$\boldsymbol{R}(X,Y) = \frac{1}{V}\boldsymbol{R}(k,k)\boldsymbol{g}(X,Y) + \boldsymbol{R}^{(p)}(X,Y) - \frac{1}{2V^2}(dV\otimes dV + 4\omega\otimes\omega), \tag{A14}$$

where $\boldsymbol{R}^{(p)}$ denotes the Ricci tensor obtained from the metric $\boldsymbol{p} = V\boldsymbol{g} + k\otimes k$.

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