

Quantum theory in curved spacetime using the Wigner function

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A nonperturbative approach to quantum theory in curved spacetime and to quantum gravity, based on a generalization of the Wigner equation, is proposed. Our definition for a Wigner equation differs somewhat from what has otherwise been proposed—it being an extension of methods from Yang-Mills theory in flat spacetimes. It is an exact equation, equivalent to the Heisenberg equations of motion. The approach makes different approximation schemes possible; e.g., it is in principle possible to perform a systematic calculation of the quantum effects order by order. The method is illustrated with some simple examples and applications. A calculation of the trace of the renormalized energy-momentum tensor is done, and the conformal anomaly is thereby related to nonconservation of a current in $d=2$ dimensions and a relationship between a vector and an axial-vector current in $d=4$ dimensions. The corresponding “hydrodynamic equations” governing the evolution of macroscopic quantities are derived by taking appropriate moments. The emphasis is put on the spin- $\frac{1}{2}$ case, but it is shown how to extend to arbitrary spins. Gravity is treated first in the Palatini formalism, which is not very tractable, and then more successfully in the Ashtekar formalism, where the constraints lead to infinite order differential equations for the Wigner functions. [S0556-2821(97)03414-0]

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INTRODUCTION

One of the greatest problems we are facing in physics today is gravitation: as is well known we do not possess a well-behaved theory of quantum gravity. Even semiclassical gravity, i.e., quantum field theory in curved spacetime, where the gravitational field is kept as a classical background, is not that well behaved. It seems that perturbation theory is just not applicable to problems involving gravitation; hence, new, nonperturbative approaches have to be found. In this paper, inspired by recent results in QCD (especially the study of quark-gluon plasmas), we propose a nonperturbative approach based on the Wigner operator. The equation governing the behavior of this operator is completely equivalent to the Heisenberg equations of motions for the fields. This Wigner equation is the quantum analogue of the classical Boltzmann or Vlasov equations [1–3] and thus allows for a better intuitive understanding of the problems, since it possesses a well-known classical analogue. Start with a flat spacetime manifold and a Dirac spinor field $\psi(x)$ on it. Instead of just looking at $\psi(x)$ or its Fourier transform $\tilde{\psi}(k)$ we consider a “distribution” $W(x,p)$ which is a functional of ψ and depends explicitly on both x and p . This “distribution,” the Wigner operator, is, however, a nonpositive operator, which is why we put the term “distribution” in quotation marks. The explicit form is given in the next section; it is essentially a kind of Fourier transform of the density matrix. The Dirac equation for ψ gives rise to an integro-differential equation for W , which is readily generalized to curved spacetimes. This allows us to attack the problem of fermions in curved spacetime from another angle than usual quantum field theoretical techniques and, furthermore, allow us to write down other approximation schemes. When the background is classical, i.e., when dealing with quantum field theory in curved spacetime, the integrations can be performed and one ends up with an infinite order differential equation.

The next step is naturally to write down a similar equation governing the gravitational fields. Since fermions causes torsion [4,5], it would at first sight seem convenient to introduce two Wigner operators $\Gamma_{\mu\nu}^{ab;cd}$ and $L_{\mu\nu}^{ab}$ for the spinor connection ω_{μ}^{ab} and the vierbein e_{μ}^a , respectively. These Wigner functions will not, however, be covariant, thus introducing an unwanted gauge dependency. In the Ashtekar formalism instead, we can introduce two covariant Wigner functions: one for the SU(2) connection A_i^a and one for its conjugate E_i^a . In both cases, however, the equations become highly complicated and we will not be able to say much about them. We will basically, in this paper, consider the background geometry as fixed (semiclassical quantum gravity). Let us note that such an approach should find many applications: Hawking radiation, the early universe, etc. We will not, however, deal so much with these applications in this paper.

This is not the first time the powerful Wigner-equation methods have been suggested as a framework for quantum fields in curved spacetime. Some earlier work by Calzetta *et al.* and Kandrup [6] uses either Riemann normal coordinates or a mode-decomposition (which only works for static spacetimes), they also show the connection between this kinetic approach and the usual path-integral or Green’s function-based approach; the Wigner function can be expressed as a Fourier transform of a two point function with respect to the middle point between the two points (the precise relationship is given in the next section). In general, neither the Fourier transform nor the concept of a middle point makes sense in a general curved background, at least not in a coordinate-independent manner. Therefore, one could attempt to use Riemann normal coordinates in order to be able to define $x-x'$ for x,x' points on the manifold, or one could make use of a mode decomposition of the solutions to the equations of motion of a free field to generalize the Fourier transform. Neither of these two approaches are

completely general, although they do represent useful calculational short-cuts. Winter [7] has proposed an approach similar to the one put forward here in which one integrates along curves (geodesics, to be precise). But his approach needs the solution of the geodesic as well as the geodesic deviation equations, a daunting task in general. This again essentially restricts the usefulness of his method to Riemann normal coordinates or to other approximation schemes (first quantum corrections to the classical kinetic equation). The solution of the geodesic equation is not needed for the approach presented here, even though we use parallel transport along geodesics at an intermediate stage. This parallel transport along geodesics is, however, not in general possible, since two arbitrary points on a manifold need not be connected by such a curve. In a general background, only points within a normal neighborhood will be connected by a (then unique) geodesic. In this respect the method put forward here is not more general than what has already been done. What is new, however, is the derivation of the kinetic equations and some of the results which we can draw from them.

Halliwell, Gell-Mann, and Hartle, and also Hu *et al.* have shown that the Wigner function appears naturally in a quantum theory of histories [8] such as, for instance, quantum cosmology; it is the only setting in which the Wigner function of the gravitational degrees of freedom has been studied. It is quite natural to extend the Wigner function to the Wheeler-DeWitt equation using the analogy with the nonrelativistic Schrödinger equation, and is then a very useful tool for studying precisely the enigmatic quantum to classical transition. In this paper, however, we will attempt another approach setting up Wigner functions for the gravitational fields in analogy with what has been done in Yang-Mills theory earlier.

It should finally be mentioned that Fonarev too, has extended the Wigner function technique from flat spacetime to a curved background [9]. Similar to the approach proposed here, he uses the structure of phase space as a cotangent bundle; our result, however, is much more closely related to the analogy with Yang-Mills theory. He furthermore emphasizes the case of the scalar field, whereas this paper mostly studies spinor fields, with some comments on the gravitational field. The derivation of the conformal anomaly and of the ‘hydrodynamic equations’ are also new.

Most of the calculations are done in $d=4$ dimensions for concreteness but are valid for all d . In some particular cases the case $d=2$ is also considered.

THE METHOD

We start out by considering the spacetime geometry as being fixed; i.e., we begin by studying quantum field theory (QFT) in curved spacetime. Our main source of inspiration are the papers by Vasak, Gyulassy, and Elze [10], in which the Wigner operator formalism is derived for Dirac fields interacting with Yang-Mills fields. We will give a brief introduction to their results here, as this will indicate how to generalize to curved spacetime.

Consider a second quantized Dirac field $\psi(x)$. The Wigner operator is defined to be [2,10]

$$\hat{W}(x,p) = \int e^{-iy \cdot p} \bar{\psi}\left(x + \frac{1}{2}y\right) \otimes \psi\left(x - \frac{1}{2}y\right) \frac{d^4y}{(2\pi)^4}; \quad (1)$$

note that \hat{W} takes values in the Clifford algebra, a point we will use later. This definition can be made gauge invariant by replacing

$$\psi\left(x - \frac{1}{2}y\right) = e^{-i(1/2)y \cdot \partial} \psi(x) \quad (2)$$

with the gauge-covariant generalization

$$\psi\left(x - \frac{1}{2}y\right) \equiv e^{-i(1/2)y \cdot \nabla} \psi, \quad (3)$$

where ∇ is the (gauge) covariant derivative. Writing $x_{\pm} \equiv x \pm \frac{1}{2}y$ we can write

$$\hat{W}(x,p) = \int e^{-ip \cdot y} \bar{\psi}(x_+) U(x_+, x) \otimes U(x, x_-) \psi(x_-) \frac{d^4y}{(2\pi)^4}, \quad (4)$$

where

$$U(b,a) \equiv P \exp\left(-ig \int_a^b A_{\mu}(z) dz^{\mu}\right), \quad (5)$$

where A_{μ} denotes the Yang-Mills field, and the path of integration is a straight line, $z(s) = a + (b-a)s$, which goes from a to b as s goes from 0 to 1. The Dirac equation then implies the following equation for \hat{W} :

$$\left[m - \gamma^{\mu} \left(p_{\mu} + \frac{1}{2}i\nabla_{\mu}\right)\right] \hat{W} = \frac{1}{2}ig \gamma^{\mu} \hat{X}_{\mu} \hat{W}, \quad (6)$$

where \hat{X}_{μ} is an integral operator involving the field strength tensor,

$$\hat{X}_{\mu} \hat{W} \equiv -\frac{\partial}{\partial p_{\nu}} \left[\int_0^1 \left(1 - \frac{1}{2}s\right) e^{-[i(1-s)/2]\Delta} F_{\mu\nu} \hat{W} ds \right. \\ \left. + \hat{W} \int_0^1 \frac{1-s}{2} e^{(1/2)is\Delta} F_{\mu\nu} ds \right], \quad (7)$$

where we have introduced the *triangle operator*

$$\Delta \equiv \frac{\partial}{\partial p} \cdot \nabla. \quad (8)$$

It is understood that the derivative with respect to momentum always acts on the Wigner distribution *alone* and *not* on any of the other terms.

One should also mention the relationship to the Green's functions. If $G(x,x')$ denotes the Green's function $G(x,x') = \langle \bar{\psi}(x) \psi(x') \rangle$ then the Wigner function can be rewritten as

$$W(x,p) = \langle \hat{W}(x,p) \rangle \propto \int e^{-iy \cdot p} G\left(x - \frac{1}{2}y, x + \frac{1}{2}y\right) d^4y,$$

i.e., as a Fourier transform of the two point function with respect to the distance between the two points. This shows the relationship with the more familiar approach to quantum theory using Green's functions.

The curvature two-form $F_{\mu\nu}$ appears through the holonomy, i.e., through reexpressing the derivative of the parallel transporter in the principal bundle $U(b,a)$ in terms of an integration along a closed curve. The above equation actually holds for A_μ^k being an operator, but for a classical background one can pull the field strength tensor out of the integral over y to obtain

$$\hat{X}_\mu = \pi^{-1/2} \frac{\partial}{\partial p_\nu} F_{\mu\nu} \left[j_0 \left(\frac{1}{2} \Delta \right) - i j_1 \left(\frac{1}{2} \Delta \right) \right], \quad (9)$$

where j_0, j_1 are the usual spherical Bessel functions $j_0 \sim J_{1/2}, j_1 \sim J_{3/2}$. We will now analyze this equation and the steps leading to it.

First the obvious translations: the gauge field is replaced by the spinor connection $\omega_\mu = \omega_\mu^{ab} \sigma_{ab}$, whereby the gauge covariant derivative becomes the spinor covariant derivative and the field strength tensor becomes the Riemann-Christoffel curvature tensor $R_{\mu\nu}^{ab}$. Let us next consider the path of integration. In flat space we integrate along the straight line from $x_- = x - \frac{1}{2}y$ to $x_+ = x + \frac{1}{2}y$; i.e., we go in the direction $\pm y$ from x for a ‘‘period of time’’ $\frac{1}{2}\|y\|$. The curved spacetime analogue is now obvious: y is a tangent vector, and we integrate along the (unique) geodesic with tangent $\pm y$ at x , the length of the curve segment at each side of x is $\frac{1}{2}\|y\|$. Note that this simplifies things a lot: *a priori* we would have expected y to lie in the manifold, but now we see that it really lies in the tangent space. Hence for each given point x in the manifold we integrate over all possible tangents y lying in the tangent space at x , $T_x M$, which is a flat space. This means that we keep the measure $d^4 y / (2\pi)^4$. To summarize, in flat spacetime x, y, p all belong to the same space, whereas in curved spacetime x is in the manifold, y in the tangent space and, p in the cotangent space. The dot product $y \cdot p$ is then just the pairing between $T_x M$ and $T_x^* M$. What we have done is to make extensive use of the intimate relationship between Yang-Mills theory as a theory of connections on a principal bundle and general relativity as related to connections on the frame bundle [5,11–13].

We will follow customary notation and denote flat space indices by Roman letters from the beginning of the alphabet, and curved spacetime ones by Greek letters. In curved spacetime, the Clifford algebra relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ implies that the Dirac matrices are in general x dependent; we introduce the vierbein as the transformation connecting them to the usual flat (constant) Dirac matrices [4,5]:

$$\gamma_\mu = e_\mu^a \gamma_a. \quad (10)$$

The connection between ‘‘curved’’ and ‘‘flat’’ indices is established by this vierbein. The covariant derivative then reads [5]

$$\nabla_\mu = \partial_\mu + \omega_\mu^{ab} \sigma_{ab}, \quad (11)$$

with $\sigma_{ab} = \frac{1}{2}i[\gamma_a, \gamma_b]$ the generators of the Lorentz algebra $\text{so}(3,1)$ in the spinor representation, and is the analogue of the generator of the gauge algebra in the Yang-Mills case. The spin connection ω_μ^{ab} is related to the Christoffel symbol through

$$\omega_\mu^{ab} = \eta^{ca} e_c^\nu e_\rho^b \Gamma_{\mu\nu}^\rho - \eta^{ca} e_\rho^b \partial_\mu e_c^\rho, \quad (12)$$

and it is this quantity which is the analogue of the Yang-Mills potential A_μ^a . Even though we deal with fermions, we will in general assume that no torsion is present, i.e., $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$. This is a valid assumption since torsion does not propagate and since we only want to deal with fields in a given classical background.

With these comments the Wigner operator equation now reads

$$\left[m + \gamma^\mu \left(e_\mu^a p_a + \frac{1}{2} i \nabla_\mu \right) \right] \hat{W} = -\frac{1}{2} \kappa \gamma^a \hat{X}_a \hat{W}, \quad (13)$$

where now

$$\hat{X}_a \hat{W} \equiv \frac{1}{\sqrt{\pi}} e_a^\mu \frac{\partial}{\partial p_\nu} \left[j_0 \left(\frac{1}{2} \Delta \right) - i j_1 \left(\frac{1}{2} \Delta \right) \right] R_{\mu\nu}^{bc} \{ \sigma_{bc}, \hat{W} \}, \quad (14)$$

with the triangle operator given by

$$\Delta \equiv e_a^\mu \frac{\partial}{\partial p_a} \nabla_\mu. \quad (15)$$

The curly brackets denote an anticommutator. Both the curvature two-form and the Wigner operator are Clifford algebra valued, thus the Clifford algebra and not only $\text{so}(3,1)$ appears here as the analogue of the internal symmetry group of Yang-Mills theory. One should remember that $\text{so}(3,1) \simeq \text{spin}(3,1) \subseteq C(3,1)$, so we have in a sense ‘‘extended’’ the gauge algebra from the Lorentz algebra to the full Clifford algebra, with the price, of course, that we then no longer have a Lie algebra but a Clifford algebra instead. Also note that $\text{so}(3,1) \simeq \text{spin}(3,1)$ acts naturally on the Clifford algebra $C(3,1)$ through $x \mapsto [x, \sigma^{ab}]$; it is under this Lie algebra that the Wigner function transforms.

Here we have just made the obvious translations from Yang-Mills theory to minimally coupled Dirac fermions in a curved background. Nonminimally coupled fermions, i.e., including a coupling to torsion, should be dealt with by making the substitution $R_{\mu\nu}^{ab} \sigma_{ab} \rightarrow R_{\mu\nu}^{ab} \sigma_{ab} + S_{\mu\nu}^\rho \nabla_\rho$, where $S_{\mu\nu}^\rho$ is the torsion (the antisymmetric part of the Christoffel symbol). This is because the curvature only enters through the commutator $[\nabla_\mu, \nabla_\nu] = R_{\mu\nu}^{ab} \sigma_{ab}$ (via the holonomy) and this gets modified to the above mentioned linear combination of curvature and torsion when the latter is present [5]. In a similar fashion one could introduce gauge degrees of freedom, coupling to a Yang-Mills field A_μ^k by adding $ig A_\mu^k T_k$ to ∇_μ , where T_k are the generators of the gauged Lie algebra. We would then have to add $ig F_{\mu\nu}^k T_k$ to the right-hand side of the commutator $[\nabla_\mu, \nabla_\nu]$. The resulting Wigner function and its equation of motion would then also be gauge covariant.

Covariance is ensured by noting that ψ transforms as $\psi \rightarrow \mathcal{U} \psi$, where \mathcal{U} is the transformation matrix ($\mathcal{U} = \exp[i\alpha_{ab}(x)\sigma^{ab}]$ for the purely gravitational case and $\mathcal{U} = \exp[i\alpha_{ab}(x)\sigma^{ab} + ig\alpha^k(x)T_k]$ with a coupling to a Yang-Mills field), and that ∇_μ transforms covariantly (adjoint representation), $\nabla_\mu \rightarrow \mathcal{U} \nabla_\mu \mathcal{U}^{-1}$; one then sees $\hat{W} \rightarrow \mathcal{U} \hat{W} \mathcal{U}^{-1}$.

The Wigner equation is then an infinite-order differential equation. It is this equation which is the subject of this study.

CLIFFORD DECOMPOSITION

The Wigner operator takes values in the Clifford algebra, since it is given as a product of two spinors. Hence it can be decomposed. In $d=4$ a basis for the Clifford algebra is given by $1, \gamma_5, \gamma_a, \gamma_a \gamma_5, \sigma_{ab}$, as is well known. We thus write

$$\hat{W} = \mathcal{S} + \mathcal{P}\gamma_5 + \mathcal{V}^i \gamma_i + \mathcal{A}^a \gamma_a \gamma_5 + \mathcal{T}^{ab} \sigma_{ab}, \quad (16)$$

where then, of course, \mathcal{S}, \mathcal{P} are a scalar and pseudoscalar, $\mathcal{V}^i, \mathcal{A}^a$, are a vector and axial vector, and \mathcal{T}^{ab} an antisymmetric tensor. A similar splitting up of the operators can be made using the properties of the Dirac matrices. The left-hand side can be written

$$\left[m + \gamma^d \left(p_d + \frac{1}{2} i e_d^\mu \partial_\mu - \frac{1}{2} \omega_\mu^{bc} e_b^\mu \eta_{cd} \right) + 2i \epsilon_{abcd}^a e_a^\mu \omega_\mu^{bc} \gamma^5 \gamma^d \right] \hat{W} \equiv [m + \gamma^d A_d + \gamma_5 \gamma^d B_d] \hat{W} \quad (17)$$

having used

$$\gamma^d \sigma^{bc} = i(\eta^{ac} \eta^{bd} - \eta^{ab} \eta^{dc}) \gamma_a + 4 \epsilon^{dcb a} \gamma_5 \gamma_a,$$

which follows from the definition of σ_{ab} and the standard trace formulas for the Dirac matrices, see, e.g., Itzykson and Zuber [14]. We will write the right-hand side as

$$\hat{J}^{abc} \gamma_a \{ \sigma_{bc}, \hat{W} \}, \quad (18)$$

where \hat{J}^{abc} then contains all the curvature information and no Clifford algebra information (i.e., it is proportional to 1). With this notation we get

$$\begin{aligned} m\mathcal{S} - A_a \mathcal{V}^a + B_a \mathcal{A}^a &= -4 \epsilon_{abcd} \hat{J}^{abc} \mathcal{A}^d, \\ m\mathcal{P} + A_a \mathcal{A}^a - B_a \mathcal{V}^a &= -\eta_{ac} \eta_{bd} \hat{J}^{abc} \mathcal{A}^d, \\ m\mathcal{V}^d - A^d \mathcal{S} + B^d \mathcal{P} - i(\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) A^e \mathcal{T}^fg + 4 \epsilon_{efg}^d B^e \mathcal{T}^fg \\ &= i \eta_{ab} \delta_c^d \hat{J}^{abc} \mathcal{S} - 2 \epsilon_{abc}^d \hat{J}^{abc} \mathcal{P} + 4 \delta_a^d \eta_{bf} \eta_{ce} \hat{J}^{abc} \mathcal{T}^{ef}, \\ m\mathcal{A}^d + A^d \mathcal{P} - B^d \mathcal{S} + i(\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) B^e \mathcal{T}^fg - 4 \epsilon_{efg}^d A^e \mathcal{T}^fg \\ &= -4 \epsilon_{abc}^d \hat{J}^{abc} \mathcal{S} + 8i(\eta_{ab} \delta_c^d - \eta_{ac} \delta_b^d) \hat{J}^{abc} \mathcal{P}, \\ m\mathcal{T}^{ef} + \frac{1}{2} i A^{[e} \mathcal{V}^{f]} - \frac{1}{2} i B^{[e} \mathcal{A}^{f]} \\ &= 12 \epsilon_{abc}^g \epsilon_{dg}^{ef} \hat{J}^{abc} \mathcal{T}^{dg} - (\eta_{ac} \epsilon_{db}^{ef} - \eta_{da} \epsilon_{cb}^{ef}) \hat{J}^{abc} \mathcal{A}^d, \end{aligned}$$

where we have used the symmetry properties of \hat{J}^{abc} , i.e., $\hat{J}^{abc} = -\hat{J}^{acb}$, to remove some terms involving η 's. One will often be able to assume $\mathcal{P} = \mathcal{A}^a = 0$, in which case the equations can be simplified a little bit.

The equations involve not only the curvature two-form but also its dual, since

$$\epsilon_{abcd} \hat{J}^{abc} = -\frac{1}{2} i \kappa e_a^\mu (j_0 - i j_1) \frac{\partial}{\partial p_\nu} (*R_{\mu\nu}^{ab}) \eta_{bd}, \quad (19)$$

with

$$*R_\mu^{ab} := \frac{1}{2} \epsilon_{abcd} R_{\mu\nu}^{cd}. \quad (20)$$

One knows that in Yang-Mills theory self-dual and anti-self-dual curvature two-forms play an important role (they correspond to instantons [5,11–13]), therefore it is encouraging that the dual of the curvature two-form also appears in this gravitational setting.

One should also note that the combination $\eta_{ab} \delta_c^d \hat{J}^{abc}$ is proportional to the Ricci tensor $\eta_{ab} \delta_c^d \hat{J}^{abc} = \frac{1}{2} \pi^{-1/2} \kappa (\partial/\partial p_\nu) R_\nu^d (j_0 - i j_1)$, which implies that this combination vanishes for the vacuum solution of general relativity, i.e., in Ricci-flat spacetimes $R_{\mu\nu} = R_\mu^d \eta_d^a e_\nu^a = 0$. This implies that in many backgrounds of real physical interest, the right-hand sides of these coupled equations simplify.

Another important consequence of this Clifford algebra decomposition concerns the classical limit. In this limit one would expect the Wigner function, considered as a 4×4 matrix to be diagonal:

$$\hat{W} = \begin{pmatrix} f_1 & & & \\ & f_2 & & \\ & & g_1 & \\ & & & g_2 \end{pmatrix} \equiv \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix} \equiv \mathcal{S} 1 + \mathcal{V}^0 \gamma_0.$$

This leads to the set of coupled equations

$$m\mathcal{S} - A_0 \mathcal{V}^0 = 0, \quad (21)$$

$$B_0 \mathcal{V}^0 = 0, \quad (22)$$

$$m\mathcal{V}^0 - A^0 \mathcal{S} = i \eta_{ab} \delta_c^0 \hat{J}^{abc} \mathcal{S}, \quad (23)$$

$$B^d \mathcal{S} = 4 \epsilon_{abc}^d \hat{J}^{abc} \mathcal{S}, \quad (24)$$

where, as we have seen, the right-hand side in the third of these vanish if the spacetime is Ricci-flat, i.e., a solution to the vacuum Einstein equations.

The second of these equations implies $\mathcal{V}_0 = 0$, which then, from the first, yields $m = 0$, hence we are left with the pair of equations

$$A_0 \mathcal{S} = -i \eta_{ab} \delta_c^0 \hat{J}^{abc} \mathcal{S}, \quad (25)$$

$$B^d \mathcal{S} = 4 \epsilon_{abc}^d \hat{J}^{abc} \mathcal{S}. \quad (26)$$

Thus in a Ricci-flat spacetime, the only quantum corrections enter through the second of these, as the first become $A_0 \mathcal{S} = 0$, i.e., the quantum corrections will depend only on the dual of the curvature two-form. The first equation then implies that \mathcal{S} is an eigenstate of the differential operator $e_0^\mu \partial_\mu + i \omega_\mu^a e_\mu^a$ with eigenvalue $-2ip_0$, whereas the second equation determines the dependency on the other components of the momentum. We will return to the case $\hat{W} = \mathcal{S}$ in later sections. Let us note in passing that the extra term

$-2ip_0$ added to the Dirac operator can be interpreted as a coupling to an Abelian connection, or as corresponding to a Dirac operator in five dimensions (as in the families index theorem [12,13]).

For later use, we also give the Clifford decomposition in $d=2$ dimensions. Here γ^a corresponds to σ_1, σ_2 (the Pauli matrices) and both γ_5 and σ_{ab} correspond to $\sigma_3 = -i\sigma_1\sigma_2 = -\frac{1}{2}i[\sigma_1, \sigma_2]$. The covariant derivative can be written as $\nabla_\mu = \partial_\mu$, since there is no spinor connection in $d=2$ [13]. With this (Latin indices going from 1 to 2) the Wigner equation reads

$$\left[m - \sigma^i \left(p_i + \frac{1}{2} i \partial_i \right) \right] \hat{W} = -\frac{1}{4} i \kappa \sigma^j e_i^\alpha \hat{J} R_{\alpha\beta} \left\{ \sigma_3, \frac{\partial}{\partial p_\beta} \hat{W} \right\}. \quad (27)$$

With the decomposition

$$\hat{W} = W_0 1 + W_i \sigma^i + W_3 \sigma^3 \quad (28)$$

we then arrive at

$$m W_0 - \left(p_i + \frac{1}{2} i \partial_i \right) W^i = -\frac{1}{2} i \kappa \epsilon^{ij} e_i^\alpha R_{\alpha\beta} \hat{J} \frac{\partial}{\partial p_\beta} W_j, \quad (29)$$

$$\begin{aligned} m W_i - \epsilon_{ij} W^j - \left(p_i + \frac{1}{2} i \partial_i \right) W_0 \\ = -\frac{1}{2} i \kappa \left(e_i^\alpha R_{\alpha\beta} \hat{J} \frac{\partial}{\partial p_\beta} W_3 + \epsilon_{ij} e^{j\alpha} R_{\alpha\beta} \hat{J} \frac{\partial}{\partial p_\beta} W_0 \right), \end{aligned} \quad (30)$$

$$m W_3 - i \epsilon_{ij} \left(p^i + \frac{1}{2} i \partial^i \right) W^j = -\frac{1}{4} i \kappa \eta^{ij} e_i^\alpha R_{\alpha\beta} \hat{J} \frac{\partial}{\partial p_\beta} W_j. \quad (31)$$

We will return to this set of equations in the section concerning the trace of the energy momentum tensor in $d=2$ and $d=4$. For now let us just note that the set of coupled equations in $d=2$ can be re-expressed as an equation for W_i alone by using the first and third to express W_0, W_3 as some operators acting on W_i . Explicitly

$$W_0 = -\frac{i\kappa}{2m} \epsilon^{ij} e_i^\alpha R_{\alpha\beta} \hat{J} \frac{\partial W_j}{\partial p_\beta} + \frac{1}{m} \left(p_i + \frac{1}{2} i \partial_i \right) W^i, \quad (32)$$

$$W_3 = -\frac{i\kappa}{4m} \eta^{ij} e_i^\alpha R_{\alpha\beta} \hat{J} \frac{\partial W_j}{\partial p_\beta} + \frac{1}{m} \epsilon_{ij} \left(p^i + \frac{1}{2} i \partial^i \right) W^j, \quad (33)$$

leading to

$$\begin{aligned} (m^2 \eta_{ij} - m \epsilon_{ij}) W^j - \frac{1}{m} \left(p_i + \frac{1}{2} i \partial_i \right) \left(p_j + \frac{1}{2} i \partial_j \right) W^j \\ + \frac{1}{2} i \kappa \epsilon^{kj} \left(p_i + \frac{1}{2} i \partial_i \right) \left(e_k^\alpha R_{\alpha\beta} \hat{J} \frac{\partial W^j}{\partial p_\beta} \right) \\ = -\frac{1}{8} \kappa^2 \eta^{kj} e_i^\alpha e_k^\gamma R_{\alpha\beta} R_{\gamma\delta} \hat{J}^2 \frac{\partial^2 W^j}{\partial p_\beta \partial p_\delta} \\ - \frac{1}{4} \kappa^2 \epsilon_i^j \epsilon^{kl} e_j^\alpha e_l^\gamma R_{\alpha\beta} R_{\gamma\delta} \hat{J}^2 \frac{\partial^2 W_k}{\partial p_\beta \partial p_\delta} \\ + \frac{1}{2} \kappa \epsilon_{kj} e_i^\alpha R_{\alpha\beta} \left(p^k + \frac{1}{2} i \partial^k \right) \hat{J} \frac{\partial W^j}{\partial p_\beta} \\ - \frac{1}{2} i \kappa \epsilon_i^j e_j^\alpha R_{\alpha\beta} \left(p_k + \frac{1}{2} i \partial_k \right) \hat{J} \frac{\partial W^k}{\partial p_\beta}. \end{aligned} \quad (34)$$

Remembering $\hat{J} = j_0(\frac{1}{2}\Delta) - ij_1(\frac{1}{2}\Delta)$ and that Δ is Hermitian, the Hermitian and anti-Hermitian parts of this equation must be satisfied separately, which then leads to two coupled equations which W_i has to satisfy.

EXAMPLE: COVARIANTLY CONSTANT CURVATURE

When the curvature tensor is covariantly constant, $\nabla_\lambda R_{\mu\nu}^{ab} = 0$, we can take the square of the equation for \hat{W} ,

$$\left(m - \gamma_a A^a - \gamma_5 \gamma_a B^a + \frac{1}{4} i \kappa e_a^\mu \gamma^a R_{\mu\nu}^{bc} \left\{ \sigma_{bc}, \frac{\partial}{\partial p_\nu} \right\} \right) \hat{W} = 0$$

to obtain

$$\begin{aligned} 0 = [m^2 - A^2 - B^2 - 2\gamma_5 A \cdot B] \hat{W} \\ + \frac{\kappa^2}{16} e_a^\mu e_d^\rho \gamma^a R_{\mu\nu}^{bc} R_{\rho\sigma}^{ef} \frac{\partial^2}{\partial p_\nu \partial p_\sigma} \{ \sigma_{bc}, \gamma^d \{ \sigma_{ef}, \hat{W} \} \} \\ - \frac{1}{4} i \kappa \left\{ \gamma_a A^a, e_d^\mu \gamma^d R_{\mu\nu}^{ef} \frac{\partial}{\partial p_\nu} \{ \sigma_{ef}, \hat{W} \} \right\} \\ - \frac{1}{4} i \kappa \left\{ \gamma_5 \gamma_a B^a, e_d^\mu \gamma^d R_{\mu\nu}^{ef} \frac{\partial}{\partial p_\nu} \{ \sigma_{ef}, \hat{W} \} \right\}. \end{aligned} \quad (35)$$

We can split this into two equations by taking the Hermitian and anti-Hermitian parts out. For the Hermitian part we then get the constraint equation generalizing the mass-shell condition

$$\begin{aligned} 0 = (m^2 - A^2 - B^2 - 2A \cdot B \gamma_5) \hat{W} \\ + \frac{\kappa^2}{16} e_a^\mu e_d^\rho \gamma^a R_{\mu\nu}^{bc} R_{\rho\sigma}^{ef} \frac{\partial^2}{\partial p_\nu \partial p_\sigma} \{ \sigma_{bc}, \gamma^d \{ \sigma_{ef}, \hat{W} \} \}, \end{aligned} \quad (36)$$

whereas the anti-Hermitian part becomes the kinetic equation proper (as in the Yang-Mills case [10])

$$0 = \left\{ \gamma_a A^a + \gamma_5 \gamma_a B^a, e_d^\mu \gamma^d R_{\mu\nu}^{ef} \frac{\partial}{\partial p_\nu} \{ \sigma_{ef}, \hat{W} \} \right\}. \quad (37)$$

The mass-shell condition, Eq. (36), can also be seen as a momentum diffusion equation of Fokker-Planck type [1,2], but the important thing to note is that it is *not* on its own a kinetic equation specifying the state of the matter fields. Comparing this with a standard classical kinetic equation for a phase space distribution function \mathcal{F} [1],

$$\frac{\partial}{\partial p_\nu}(a_\nu \mathcal{F}) + \frac{1}{2} \frac{\partial^2}{\partial p_\mu \partial p_\nu}(b_{\mu\nu} \mathcal{F}) = f(p, x) \mathcal{F}, \quad (38)$$

we see that a_ν , the dynamical friction in our case is

$$a_\nu = 0, \quad (39)$$

while $b_{\mu\nu}$, the momentum diffusion coefficient, is (simply take $\hat{W} = S$ to get the nonspin terms)

$$b_{\nu\sigma} = \frac{\kappa^2}{4} e_a^\mu e_d^\rho (\eta^{gf} \eta^{de} - \eta^{ge} \eta^{df}) (\eta_{gc} \delta_b^a - \eta_{gb} \delta_c^a) R_{\mu\nu}^{bc} R_{\rho\sigma}^{ef} + \text{spin terms} \quad (40)$$

and $f(p, x)$, a kind of momentum force term, finally reads

$$f(p, x) = -(m^2 - A^2 - B^2) + \text{Clifford algebra terms} \quad (41)$$

and is thus an operator (remember, $A \sim p + \partial + \omega$).

Thus in this particularly simple case, we see that the mass-shell condition can be interpreted as a diffusion equation in momentum space. Now, this momentum diffusion is provided by the quantum and curvature corrections to the mass, i.e., the difference between m^2 and p^2 , it is thus a pure quantum phenomenon. This is a kinetic interpretation of the uncertainty principle and of the renormalization of the mass. It is intuitively pleasing to see that the momentum diffusion coefficient is given simply by the curvature, and is thus a purely geometric object, even though its source is purely quantum. For a curvature which is *not* covariantly constant, the mass-shell condition contains higher derivatives with respect to the momentum, such terms do not have a direct classical interpretation and therefore represent pure quantum effects.

The main difference between Eq. (37) and the result for Yang-Mills theory is the appearance of the anticommutator. As we have seen earlier, this anticommutator can be traced back to \hat{W} being Clifford algebra valued and that the generators of the Lorentz algebra for Dirac fermions is precisely σ_{ab} which is an element of the Clifford algebra. Remember that the elements $\gamma_a \gamma_b$ of any Clifford algebra $C(r, s)$ generate the Lie algebra $\text{spin}(r, s)$ which is isomorphic to $\text{so}(r, s)$. For $r=3, s=1$ this is the Lorentz algebra in $d=3+1$ dimensions, while for $r=s=1$ it is the corresponding algebra in $d=1+1$ dimensions [11]. It is this interrelationship between the Clifford algebra and the Lorentz algebra (which is then the gauged algebra for gravitational systems) which accounts for the anticommutator.

In the extreme classical limit $\hat{W} = S$ and the kinetic equation reduces to

$$0 = 2i(\eta_{af} \eta_{de} - \eta_{ae} \eta_{df}) A^a e^{\mu d} R_{\mu\nu}^{ef} \frac{\partial}{\partial p_\nu} \mathcal{S} - \epsilon_{ef}^{gh} (\eta_{ab} \eta_{dg} - \eta_{ag} \eta_{dh}) B^a e^{\mu d} R_{\mu\nu}^{ef} \frac{\partial}{\partial p_\nu} \mathcal{S}. \quad (42)$$

This can be seen as an equation involving only dynamical friction and not momentum diffusion. Writing it in the classical form

$$f_1(p, x) \mathcal{F} = \frac{\partial}{\partial p_\nu}(a_\nu \mathcal{F}), \quad (43)$$

where the ‘‘force term’’ f_1 contains the operator Δ , we see that the dynamical friction is given by

$$a_\nu = (\eta_{af} \eta_{de} - \eta_{ae} \eta_{df}) \left(p^a - \frac{1}{2} \omega_\rho^{ba} \right) e^{\mu d} R_{\mu\nu}^{ef} - \epsilon_{ef}^{gh} \epsilon_{qr}^{pa} e_p^\rho e^{\mu d} \omega_\rho^{qr} R_{\mu\nu}^{ef} \quad (44)$$

It is very interesting to note that the resulting equations for \hat{W} splits into two set of equations, one containing only momentum diffusion and the other only dynamical friction. In both cases the sources for the processes are given by quantum effects, i.e., vanish in the classical limit if one ignores spin effects. Furthermore, the coefficients $a_\mu, b_{\mu\nu}$ are in both cases given by the curvature and are thus geometrical quantities as one would expect intuitively.

Again, in the general case we would get higher derivatives of momentum, the ‘‘diffusion’’ equation containing all even powers of $\partial/\partial p$, and the ‘‘friction’’ equation containing all the odd powers. These extra terms are pure quantum effects and have no direct classical interpretation.

To round off the discussion we also give the splitting of the Wigner equation in $d=2$. As we have seen, in this case we can make do with an equation for W_i alone. The Hermitian part of this equation is seen to be, in the case of covariantly constant curvature (corresponding to taking $\hat{J} \equiv 1$),

$$\begin{aligned} & (m^2 \eta_{ij} - m \epsilon_{ij}) W^j - \left(p_i p_j + \frac{1}{2} i (p_i \partial_j + p_j \partial_i) - \frac{1}{4} \partial_i \partial_j \right) W^j \\ &= -\frac{1}{8} \kappa^2 \eta^{kj} e_i^\alpha e_k^\gamma R_{\alpha\beta} R_{\gamma\delta} \frac{\partial^2 W^j}{\partial p_\beta \partial p_\delta} \\ & \quad - \frac{1}{4} \epsilon_i^j \epsilon^{kl} e_j^\alpha e_l^\gamma R_{\alpha\beta} R_{\gamma\delta} \frac{\partial^2 W_k}{\partial p_\beta \partial p_\delta} \\ & \quad + \frac{1}{2} \kappa \epsilon_{kj} e_i^\alpha R_{\alpha\beta} \left(p^k + \frac{1}{2} i \partial^k \right) \frac{\partial W^j}{\partial p_\beta} \end{aligned} \quad (45)$$

for the Hermitian part, and

$$\epsilon^{kj} \left(p_i + \frac{1}{2} i \partial_i \right) e_k^\alpha R_{\alpha\beta} \frac{\partial W_j}{\partial p_\beta} = \epsilon_i^j e_j^\alpha R_{\alpha\beta} \left(p^k + \frac{1}{2} i \partial^k \right) \frac{\partial W_j}{\partial p_\beta} \quad (46)$$

for the anti-Hermitian part. Once again, we can interpret the Hermitian part as a momentum diffusion equation. This time we also have a dynamical friction contribution

$$\frac{1}{2} \frac{\partial^2}{\partial p_\beta \partial p_\delta} (b_{\beta\delta}^{kl} W_l) + \frac{\partial}{\partial p_\beta} (a_\beta^{kl} W_l) = f^{kl}(x, p) W_l.$$

The diffusion coefficient is

$$b_{\beta\delta}^{ik} = \frac{1}{2} \kappa^2 \left(\epsilon^{ij} \epsilon^{kl} e_j^\alpha e_l^\gamma + \frac{1}{2} \eta^{jk} \eta^{il} e_i^\alpha e_j^\gamma \right) R_{\alpha\beta} R_{\gamma\delta}, \quad (47)$$

while the dynamical friction becomes

$$a_\beta^{kl} = -\frac{1}{2} \kappa \epsilon^{il} \eta^{jk} e_j^\alpha R_{\alpha\beta} \left(p_i + \frac{1}{2} i \partial_i \right), \quad (48)$$

which is then a differential operator in this case. The force term is again related to the mass-shell constraint, although in a somewhat more complicated form:

$$f^{kl} = m^2 \eta^{kl} - m \epsilon^{kl} - p^k p^l - \frac{1}{2} i (p^k \partial^l - p^l \partial^k) + \frac{1}{4} \partial^k \partial^l. \quad (49)$$

The anti-Hermitian part is once more a pure dynamical friction equation (it would have a momentum diffusion term too, if the curvature was not covariantly constant), but there is no source term. Thus there is quite a big difference between the results in $d=2$ and $d=4$ dimensions, a difference which cannot simply be referred to the different Lorentz character of the ‘‘distribution’’ in the two cases (in $d=2$ we considered a vector, and in $d=4$ a scalar), but is very much due to the difference in the structure of the two Clifford algebras and the fact that there is no spin connection in $d=2$. This is an important caveat.

THE Δ -EXPANSION: QUANTUM CORRECTIONS

The case of covariantly constant curvature corresponds, as we have seen, to a kind of classical limit involving only a few quantum corrections. In this section we will commence a more systematic study of quantum corrections. This is done by noting that Δ appears multiplied by \hbar , and that Planck’s constant only enters in this combination. An expansion in Δ is therefore related to an expansion in \hbar . Thus, one can calculate quantum corrections by expanding the spherical Bessel functions in \hat{X} to a given order. To do this we need the standard formulas

$$j_0(z) = \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!},$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} = \sum_{n=1}^{\infty} (-1)^n \frac{2n}{(2n+1)!} z^{2n-1}.$$

Now, Δ is a Hermitian operator and the Bessel functions only appear in the combination $j_0 - i j_1$. Consequently, it is rather simple to make a separation into Hermitian and anti-Hermitian parts of the Wigner equation. One of these will then only contain odd powers of Δ and the other only even powers.

Due to the complicated nature of the operators appearing in the Wigner equation, we cannot simply take the square and compare with the classical kinetic equations. Instead we

have to make do with the equations on the ‘‘Dirac form’’ (in contrast to the ‘‘Klein-Gordon form’’ resulting from taking squares), which do not have a direct classical analogue. The interpretation will therefore not be as precise as one would perhaps have wanted.

Writing

$$W(x, p) = \sum_{n=0}^{\infty} \hbar^n W^{(n)}(x, p) \quad (50)$$

and remembering $z = \frac{1}{2} \hbar \Delta$, we can collect terms with the same power of Planck’s constant. Doing this we get

$$\begin{aligned} & \left[m + e_a^\mu \gamma^a \left(p_\mu + \frac{1}{2} i \nabla_\mu \right) \right] W^{(0)} \\ &= -\frac{1}{2} i \kappa e_a^\mu \gamma^a R_{\mu\nu}^{bc} \left\{ \frac{\partial}{\partial p_\nu} W^{(0)}, \sigma^{bc} \right\} \end{aligned} \quad (51)$$

for the ‘‘classical’’ contribution \hbar^0 which is recognized as the same as for the case of covariantly constant curvature treated in the previous section, whereas the first correction satisfies

$$\begin{aligned} & \left[m + e_a^\mu \gamma^a \left(p_\mu + \frac{1}{2} i \nabla_\mu \right) \right] W^{(1)} \\ &= -\frac{1}{2} i \kappa e_a^\mu \gamma^a R_{\mu\nu}^{bc} \left\{ \frac{\partial}{\partial p_\nu} W^{(1)}, \sigma_{bc} \right\} \\ &+ \frac{1}{6} \kappa e_a^\mu \nabla_\rho (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^2}{\partial p_\nu \partial p_\rho} W^{(0)}, \sigma_{bc} \right\}, \end{aligned} \quad (52)$$

while the second order contribution satisfies (remembering that the vierbein is covariantly constant)

$$\begin{aligned} & \left[m + e_a^\mu \gamma^a \left(p_\mu + \frac{1}{2} i \nabla_\mu \right) \right] W^{(2)} \\ &= -\frac{1}{2} i \kappa e_a^\mu \gamma^a R_{\mu\nu}^{bc} \left\{ \frac{\partial}{\partial p_\nu} W^{(2)}, \sigma_{bc} \right\} \\ &+ \frac{1}{6} \kappa e_a^\mu \nabla_\rho (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^2}{\partial p_\nu \partial p_\rho} W^{(1)}, \sigma_{bc} \right\} \\ &+ \frac{1}{48} i \kappa e_a^\mu \nabla_\rho \nabla_\sigma (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^3}{\partial p_\nu \partial p_\rho \partial p_\sigma} W^{(0)}, \sigma_{bc} \right\}. \end{aligned} \quad (53)$$

In general we will have a recursive scheme

$$\begin{aligned} & \left[m + e_a^\mu \gamma^a \left(p_\mu + \frac{1}{2} i \nabla_\mu \right) \right] W^{(n)} \\ &+ \frac{1}{2} i \kappa e_a^\mu \gamma^a R_{\mu\nu}^{bc} \left\{ \frac{\partial}{\partial p_\nu} W^{(n)}, \sigma_{bc} \right\} \\ &= \text{terms involving only } W^{(k)} \text{ with } k < n. \end{aligned} \quad (54)$$

The terms on the right-hand side will involve more and more momentum derivatives of the Wigner functions $W^{(k)}$ and similarly higher and higher order covariant derivatives of the

vierbeins and curvature two-forms. Higher order derivatives of curvature can in general be considered as related to fluctuations of the geometry, thus the higher order Wigner functions are determined by the fluctuations of spacetime, moreover, they couple to higher order derivatives with respect to the momentum of the lower order Wigner functions, terms which, therefore, do not have a classical interpretation (only first and second order momentum derivatives appear in classical kinetic equations). This shows how the expansion in Δ is closely related to pure quantum effects with no classical analogue. One cannot, however, guarantee that this expansion is equivalent to the standard loop expansion in quantum field theory. In general there will be no such simple relationship.

One should also take notice of the fact that the momentum derivative operator is symmetric in its indices; this implies that only the symmetric part of $\nabla_\rho \cdot \cdot \nabla_\sigma$ acting on the curvature will contribute. From the commutator relation we get

$$\nabla_{(\mu} \nabla_{\nu)} = \nabla_\mu \nabla_\nu - R_{\mu\nu}^{ab} \sigma_{ab}, \quad (55)$$

whence it follows that the right-hand side of the recursive scheme not only includes higher and higher order derivatives of the curvature but also higher and higher powers of it.

If we introduce the operator \hat{Y} by

$$\hat{Y}W := \left[m + e_a^\mu \gamma^a \left(p_\mu + \frac{1}{2} i \nabla_\mu \right) + \frac{1}{2} i \kappa e_a^\mu \gamma^a R_{\mu\nu}^{bc} \left\{ \frac{\partial}{\partial p_\nu} \cdot, \sigma_{bc} \right\} \right] W, \quad (56)$$

then we can write the recursive scheme as

$$\hat{Y}W^{(n)} = F^{(n)}, \quad (57)$$

where $F^{(0)}=0$ and $F^{(n)}$ depends on $W^{(k)}$, $k < n$. Thus, on the formal level,

$$W^{(n)}(x, p) = \hat{Y}^{-1} F^{(n)}(x, p), \quad n \geq 1. \quad (58)$$

Written out explicitly, the first corrections then read

$$W^{(1)} = \frac{1}{6} \kappa \hat{Y}^{-1} e_a^\mu \nabla_\rho (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\nu \partial p_\rho}, \sigma_{bc} \right\}, \quad (59)$$

$$W^{(2)} = \frac{1}{6} \kappa \hat{Y}^{-1} e_a^\mu \nabla_\rho (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^2 W^{(1)}}{\partial p_\nu \partial p_\rho}, \sigma_{bc} \right\} + \frac{i\kappa}{48} \hat{Y}^{-1} e_a^\mu \nabla_\rho \nabla_\sigma (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^3 W^{(0)}}{\partial p_\nu \partial p_\rho \partial p_\sigma}, \sigma_{bc} \right\}. \quad (60)$$

Defining

$$H_{\rho\nu}^{bc} := e_a^\mu \nabla_\rho (\gamma^a R_{\mu\nu}^{bc}), \quad (61)$$

we can rewrite this as

$$W^{(1)} = \frac{1}{6} \kappa \hat{Y}^{-1} H_{\rho\nu}^{bc} \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\rho \partial p_\nu}, \sigma_{bc} \right\}, \quad (62)$$

$$W^{(2)} = \left(\frac{1}{6} \kappa \right)^2 \hat{Y}^{-1} H_{\rho\nu}^{bc} \left\{ \frac{\partial^2}{\partial p_\rho \partial p_\nu} \hat{Y}^{-1} \times H_{\kappa\epsilon}^{gh} \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\kappa \partial p_\epsilon}, \sigma_{gh} \right\}, \sigma_{bc} \right\} + \frac{i\kappa}{48} \hat{Y}^{-1} \nabla_\rho H_{\sigma\nu}^{bc} \left\{ \frac{\partial^3 W^{(0)}}{\partial p_\nu \partial p_\rho \partial p_\sigma}, \sigma_{bc} \right\}, \quad (63)$$

and so on.

Let us Clifford decompose $\hat{Y}W$. This is straightforward and we get

$$mS - A^a \mathcal{V}_a + B^a \mathcal{A}_a + 2i\kappa \epsilon_{bcd}^a e_a^\mu R_{\mu\nu}^{bc} \frac{\partial}{\partial p_\nu} \mathcal{A}^d$$

for the scalar part,

$$m\mathcal{P} + A^a \mathcal{A}_a - B^a \mathcal{V}_a + \frac{1}{2} i \kappa \delta_c^a \eta_{bd} e_a^\mu R_{\mu\nu}^{bc} \frac{\partial}{\partial p_\nu} \mathcal{A}^d$$

for the pseudoscalar contribution,

$$m\mathcal{V}^d - A^d \mathcal{S} + B^d \mathcal{P} - i(\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) A^e \mathcal{T}^g + 4\epsilon_{efg}^d B^e \mathcal{T}^g - \frac{1}{2} i \kappa e_a^\mu R_{\mu\nu}^{bc} \frac{\partial}{\partial p_\nu} [-i \delta_b^a \delta_c^d \mathcal{S} - 2\epsilon_{bc}^{ad} \mathcal{P} + 4\eta^{ad} \mathcal{T}_{bc}]$$

for the vector part, while the axial vector contribution turns out to be

$$m\mathcal{A}^d + A^d \mathcal{P} - B^d \mathcal{S} + i(\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) B^e \mathcal{T}^g - 4\epsilon_{efg}^d A^e \mathcal{T}^g - \frac{1}{2} i \kappa e_a^\mu R_{\mu\nu}^{bc} \frac{\partial}{\partial p_\nu} [-4\epsilon_{bc}^{da} \mathcal{S} + 8i(\delta_b^a \delta_c^d - \delta_c^a \delta_b^d) \mathcal{P}],$$

and finally

$$m\mathcal{T}^{ef} + \frac{1}{2} i A^{[e} \mathcal{V}^{f]} - \frac{1}{2} i B^{[e} \mathcal{A}^{f]} - \frac{1}{2} i \kappa e_a^\mu R_{\mu\nu}^{bc} \frac{\partial}{\partial p_\nu} [12\epsilon_{bc}^{ag} \epsilon_{dg}^{ef} \mathcal{T}^{dg} + (\delta_c^a \epsilon_{db}^{ef} - \delta_d^a \epsilon_{cb}^{ef}) \mathcal{A}^d]$$

for the tensor part. On the right-hand side of the recursion relation we have terms of the form

$$\frac{1}{6} \kappa e_a^\mu \nabla_\rho (\gamma^a R_{\mu\nu}^{bc}) \left\{ \frac{\partial^2}{\partial p_\nu \partial p_\rho} W, \sigma_{bc} \right\}.$$

Now, here the covariant derivative is to act on the two form $R_{\mu\nu}^{bc}$, i.e.,

$$\nabla_\rho R_{\mu\nu}^{bc} = \partial_\rho R_{\mu\nu}^{bc} + \Gamma_{\rho\mu}^\lambda R_{\lambda\nu}^{bc} + \Gamma_{\rho\nu}^\lambda R_{\lambda\mu}^{bc},$$

and hence ∇_ρ does not involve any Clifford algebra elements [σ_{bc} is only the generator of $\text{so}(3,1)$ in the spin- $\frac{1}{2}$ representation]. Therefore we can move the (constant) Dirac matrix γ^a outside the covariant derivation. A Clifford decomposition of this term is thus straightforward, and we obtain

$$S: -\frac{2}{3} \kappa e_a^\mu \epsilon_{bcd}^a (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{A}^d}{\partial p_\nu \partial p_\rho},$$

$$P: -\frac{1}{6}\kappa e_a^\mu \eta_{bd} \delta_c^a (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{A}^d}{\partial p_\nu \partial p_\rho},$$

$$\begin{aligned} V: & \frac{1}{6} i \kappa \delta_b^a \delta_c^d e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{S}}{\partial p_\nu \partial p_\rho} - \frac{1}{3} \kappa \epsilon_{bc}^{ad} e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{P}}{\partial p_\nu \partial p_\rho} \\ & + \frac{2}{3} \kappa \eta^{ad} \eta_{bf} \eta_{ce} e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2}{\partial p_\nu \partial p_\rho} T^{ef} \\ A: & -\frac{2}{3} \epsilon_{bc}^{da} e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{S}}{\partial p_\nu \partial p_\rho} + \frac{4}{3} i \kappa (\delta_v^a \delta_c^d - \delta_c^a \delta_b^d) \\ & \times (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{P}}{\partial p_\nu \partial p_\rho}, \end{aligned}$$

$$\begin{aligned} T: & 2 \kappa e_a^\mu \epsilon_{bc}^{ag} \epsilon_{dg}^{ef} (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2}{\partial p_\nu \partial p_\rho} T^{dg} - \frac{1}{6} \kappa (\delta_c^a \epsilon_{db}^{ef} - \delta_d^a \epsilon_{cb}^{ef}) \\ & \times (\nabla_\rho R_{\mu\nu}^{bc}) \frac{\partial^2 \mathcal{A}^d}{\partial p_\nu \partial p_\rho}. \end{aligned}$$

To have a look at a solution we can take the extreme case

$$W^{(0)}(x, p) = S_0(x, p) = \mathcal{N} e^{-\alpha^{\mu\nu}(x) p_\mu p_\nu + \beta^\mu(x) p_\mu + \gamma(x)}. \quad (64)$$

As we have seen earlier, this is only possible provided $m=0$, in which case S_0 has to satisfy the coupled set of equations

$$-A^d S_0 + \frac{1}{2} \kappa e_a^\mu R_{\mu\nu}^{ad} \frac{\partial}{\partial p_\nu} S_0 = 0, \quad (65)$$

$$-B^d S_0 + 2i \kappa e_a^\mu R_{\mu\nu}^{bc} \epsilon_{bc}^{da} \frac{\partial}{\partial p_\nu} S_0 = 0. \quad (66)$$

The second of these yields, upon insertion of the explicit form for S_0 ,

$$\omega_\mu^{bc} = -\kappa R_{\mu\nu}^{bc} (\alpha^{\nu\rho} p_\rho + \beta^\nu), \quad (67)$$

which implies $R_{\mu\nu}^{ab} \alpha^{\nu\rho} \equiv 0$. Inserting this into the first we arrive at

$$p_d + \frac{1}{2} i e_d^\mu (-\partial_\mu \alpha^{\nu\rho} p_\nu p_\rho + \partial_\mu \beta^\nu p_\nu + \partial_\mu \gamma) = 0, \quad (68)$$

which is only possible if

$$\alpha^{\mu\nu} = 0, \quad (69)$$

$$\beta^\nu = 2ix^\nu + \text{const}, \quad (70)$$

$$\gamma = \text{const}. \quad (71)$$

Thus S_0 is of the so-called Jüttner form [2]:

$$\exp\{\beta(x)[\mu(x) - p^\mu U_\mu(x)]\},$$

where β is the inverse temperature, μ the Gibbs energy, and U_μ the four velocity, i.e., $\beta^\mu = \beta U^\mu$, $\gamma = \beta\mu$. Let us now

make a similar ansatz for the first quantum correction, i.e., $W^{(1)} \equiv S_1 = \mathcal{N}' \exp(-\bar{\alpha}^{\mu\nu} p_\mu p_\nu + \bar{\beta}^\mu p_\mu + \bar{\gamma})$. The equations then reduce to

$$-A^d S_1 + \frac{1}{2} \kappa e_a^\mu R_{\mu\nu}^{ad} \frac{\partial}{\partial p_\nu} S_1 = \frac{1}{6} i \kappa e_a^\mu (\nabla_\rho R_{\mu\nu}^{ad}) \frac{\partial^2}{\partial p_\nu \partial p_\rho} S_0, \quad (72)$$

$$\begin{aligned} -B^d S_1 + 2i \kappa e_a^\mu R_{\mu\nu}^{bc} \epsilon_{bc}^{da} \frac{\partial}{\partial p_\nu} S_1 = & -\frac{2}{3} \kappa \epsilon_{bc}^{da} e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \\ & \times \frac{\partial^2}{\partial p_\nu \partial p_\rho} S_0. \end{aligned} \quad (73)$$

Then, evaluating the differentiations with respect to the momenta and collecting powers of these, we arrive at the following set of conditions:

$$\partial_\mu \bar{\alpha}^{\nu\rho} = 0, \quad (74)$$

$$p_d - \frac{1}{2} i e_d^\mu (\partial_\mu \bar{\beta}^\nu) p_\nu + \frac{1}{2} \kappa e_a^\mu R_{\mu\nu}^{ad} \bar{\alpha}^{\nu\rho} p_\rho = 0, \quad (75)$$

$$e_d^\mu \partial_\mu \bar{\gamma} = 0, \quad (76)$$

$$\frac{1}{2} e_a^\mu (\omega_\mu^{bc} \delta_b^a \eta_{cd} + \kappa R_{\mu\nu}^{ad}) S_1 = \frac{1}{6} i \kappa e_a^\mu (\nabla_\rho R_{\mu\nu}^{ad}) \alpha^{\rho\nu} S_0 \quad (77)$$

from the first of the equations for S_1 , while the second give us

$$\begin{aligned} -i \epsilon_{bcd} e_a^\mu \omega_\mu^{bc} + i \kappa e_a^\mu R_{\mu\nu}^{bc} \epsilon_{bc}^{da} (\bar{\beta}^\nu - \bar{\alpha}^{\nu\rho} p_\rho) S_1 \\ = \frac{1}{3} \kappa \epsilon_{bc}^{da} e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \alpha^{\nu\rho} S_0. \end{aligned} \quad (78)$$

Combining these we get the following set of conditions:

$$\partial_\mu \bar{\alpha}^{\nu\rho} = 0,$$

$$R_{\mu\nu}^{bc} \bar{\alpha}^{\nu\rho} = 0,$$

$$i e_a^\mu (\partial_\mu \bar{\beta}^\nu) p_\nu - \kappa e_a^\mu R_{\mu\nu}^{ac} \eta_{cd} \bar{\alpha}^{\nu\rho} p_\rho = 2p_d,$$

$$(\epsilon_{bcd} e_a^\mu \omega_\mu^{bc} - \kappa e_a^\mu R_{\mu\nu}^{bc} \epsilon_{bc}^{da} \bar{\beta}^\nu) S_1 = \frac{1}{3} \kappa \epsilon_{bc}^{da} e_a^\mu (\nabla_\rho R_{\mu\nu}^{bc}) \alpha^{\nu\rho} S_0.$$

In a general spacetime the first two of these will give $\bar{\alpha}^{\mu\nu} \equiv 0$ as for the lowest order term. Hence, the solution S_1 will once more be of the Jüttner form, albeit with a much more complicated expression for $\beta(x) U^\mu(x) = \bar{\beta}^\mu$.

To get more information about the recursive scheme presented in this section, we can find the Hermitian and anti-Hermitian parts of the squared equations (i.e., the Wigner equation in ‘‘Klein-Gordon form’’), as this is where we expect to see the analogy with classical kinetic theory most clearly.

We essentially calculated the square of \hat{Y} in the previous section. The square of $F^{(1)}$ is similarly

$$(F^{(1)})^2 = \frac{\kappa^2}{36} H_{\nu\rho}^{bc} \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\nu \partial p_\rho}, \sigma_{bc} \right\} H_{\kappa\epsilon}^{gh} \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\kappa \partial p_\epsilon}, \sigma_{gh} \right\}. \quad (79)$$

A Clifford decomposition of this is complicated by the γ^a part of $H_{\mu\nu}^{bc}$ and the quadratic appearance of $W^{(0)}$. We note that $(F^{(1)})^2$ is Hermitian, hence, it only contributes to the momentum diffusion equation generalizing the mass-shell constraint, whereas the kinetic equation proper (which comes from the anti-Hermitian part of \hat{Y}^2) does not get any contribution from lower order terms. The momentum diffusion equation then reads

$$\begin{aligned} & (m^2 - A^2 - B^2 - 2A \cdot B \gamma_5) W^{(1)} \\ & + \frac{\kappa^2}{16} e_a^\mu e_b^\nu \gamma^a R_{\mu\nu}^{bc} R_{\rho\sigma}^{ef} \frac{\partial^2}{\partial p_\nu \partial p_\sigma} \{ \sigma_{bc}, \gamma^d \{ \sigma_{ef}, W^{(1)} \} \} \\ & = \frac{\kappa^2}{36} H_{\nu\rho}^{bc} \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\nu \partial p_\rho}, \sigma_{bc} \right\} H_{\kappa\epsilon}^{gh} \left\{ \frac{\partial^2 W^{(0)}}{\partial p_\kappa \partial p_\epsilon}, \sigma_{gh} \right\}. \quad (80) \end{aligned}$$

At the next order $O(\hbar^2)$, however, there will be contributions from $W^{(0)}, W^{(1)}$ to the kinetic equation proper for $W^{(2)}$ too: namely,

$$\frac{1}{288} \kappa^2 \left\{ H_{\rho\nu}^{bc} \left\{ \frac{\partial^2 W^{(1)}}{\partial p_\nu \partial p_\rho}, \sigma_{bc} \right\}, (\nabla_\sigma H_{\kappa\epsilon}^{ef}) \left\{ \frac{\partial^3 W^{(0)}}{\partial p_\sigma \partial p_\kappa \partial p_\epsilon}, \sigma_{ef} \right\} \right\}.$$

In general, the momentum diffusion equation will contain only squares, whereas the kinetic equation will contain only cross products (always in the form of an anticommutator) of the lower order Wigner functions.

THE CONFORMAL ANOMALY IN $d=2$ AND $d=4$

Let us calculate the trace of $\langle T_{\mu\nu} \rangle$, denoted by $\langle T \rangle$. The nonvanishing of this quantity is the conformal anomaly when $m=0$ [4].

The general expression for $\langle T_{ab} \rangle$ in terms of the Wigner function is given by

$$\langle T_{ab}(x) \rangle = \text{Tr} \int_{T_x^* M} \gamma_a p_b \langle \hat{W} \rangle d^d p. \quad (81)$$

It follows that (in d dimensions, $W = \langle \hat{W} \rangle$)

$$\langle T \rangle := \langle T_{ab} \eta^{ab} \rangle = \text{Tr} \int_{T_x^* M} \gamma^a p_a W d^d p. \quad (82)$$

Now, from the equation for W ,

$$\left[m - \frac{1}{2} i \gamma^a (2i p_a + e_a^\mu \nabla_\mu) \right] W = \hat{X} W,$$

it follows that

$$\gamma^a p_a W = \left(\frac{1}{2} i \gamma^a e_a^\mu \nabla_\mu - m \right) W - \hat{X} W. \quad (83)$$

Hence,

$$\langle T \rangle = \text{Tr} \int_{T_x^* M} \left[\left(\frac{1}{2} i \gamma^a e_a^\mu \nabla_\mu - m \right) W - \hat{X} W \right] d^d p. \quad (84)$$

We also know

$$\hat{X} W = \frac{\partial}{\partial p_\nu} [W \times (\text{other terms})]; \quad (85)$$

i.e., that $\hat{X} W$ is a total derivative, and thus that its integral over the cotangent space at x vanish, leaving us with

$$\langle T \rangle = \text{Tr} \int_{T_x^* M} \left(\frac{1}{2} i \gamma^a e_a^\mu \nabla_\mu - m \right) W d^d p. \quad (86)$$

From this we see that $\langle T \rangle$ measures the failure of W to satisfy a Dirac equation with mass $2m$ (thus it is most interesting when $m=0$). This equation is valid for all d . To carry out the trace we need the Clifford decomposition, which is then d dependent.

For $d=2$ we get

$$\langle T(x) \rangle = \frac{1}{2} i \int_{T_x^* M} e_a^\mu \eta^{ab} \partial_\mu W_b d^2 p - m \int_{T_x^* M} W_0 d^2 p, \quad (87)$$

where we have written $W = W_0 1 + W_a \sigma^a + W_3 \sigma_3$ with $a, b = 1, 2$. Introducing the current density $\langle j_a(x) \rangle := \int W_a d^2 p$ and the number density $\langle n(x) \rangle := \int W_0 d^2 p$ we can write this in terms of these macroscopic quantities only as

$$\langle T(x) \rangle = \frac{1}{2} i e_a^\mu \eta^{ab} \langle \partial_\mu j_b(x) \rangle - m \langle n(x) \rangle. \quad (88)$$

For massless fields, the conformal anomaly is then related to the nonconservation of the current j_a , such that $\langle T \rangle \neq 0$ if and only if j_a is not conserved in expectation value.

In $d=4$ we similarly get

$$\begin{aligned} \langle T(x) \rangle &= \int_{T_x^* M} \left[\frac{1}{2} (i e_d^\mu \partial_\mu - \omega_\mu^{bc} e_b^\mu \eta_{cd}) \mathcal{V}^d \right. \\ &\quad \left. + 2i \epsilon_{bcd}^a e_a^\mu \omega_\mu^{bc} \mathcal{A}^d - m \mathcal{S} \right] d^4 p: \\ &= \frac{1}{2} (i e_d^\mu \partial_\mu - \omega_\mu^{bc} e_b^\mu \eta_{cd}) \langle j^d \rangle \\ &\quad + \epsilon_{bcd}^a e_a^\mu \omega_\mu^{bc} \langle k^d \rangle - m \langle n \rangle, \quad (89) \end{aligned}$$

where we have introduced the vector current $\langle j_a \rangle := \int \mathcal{V}_a d^4 p$, the axial current $\langle k_a \rangle := \int \mathcal{A}_a d^4 p$, and the number density $\langle n \rangle := \int \mathcal{S} d^4 p$. We see that in $d \neq 2$ the possible existence of a conformal anomaly or not is not as simply related to the question of the conservation of a current as in $d=2$.

The fact that an anomaly can be expressed in terms of the Wigner function integrated over the cotangent space suggests a closer relationship between W and anomalies, in particular with the spin complex [12,13]. The very nature of the Wigner function, or rather the entire Wigner-Weyl-Moyal formalism where operators are replaced by symbols on the cotangent bundle, makes the translation of analytical properties into geometrical or algebraic ones possible. Such a translation is at the heart of index theorems, and it has in fact

been shown that the Atiyah-Singer index theorem can be related to the classical limit of the Wigner-Weyl-Moyal (WWM) formalism for the ordinary Heisenberg algebra [15]. The way in which this generalized WWM formalism relates to index theorems is presently under study.

THE HYDRODYNAMIC EQUATIONS: MOMENTS

The previous section calculated the trace of the energy-momentum tensor from the Wigner equation, this is only one macroscopic quantity which one can define. Let us define the following moments of the Wigner function (for simplicity we suppress the $\langle \cdot \rangle$):

$$\begin{aligned} n(x) &:= \text{Tr} \int W d^4 p = \int S d^4 p, \\ U^a(x) &:= \text{Tr} \int p^a W d^4 p = \int p^a S d^4 p, \\ j_a(x) &:= \text{Tr} \int \gamma_a W d^4 p = \int \mathcal{V}_a d^4 p, \\ k_a(x) &:= \text{Tr} \int \gamma_5 \gamma_a W d^4 p = \int \mathcal{A}_a d^4 p, \\ T_{ab}(x) &:= \text{Tr} \int p_a \gamma_b W d^4 p = \int p_a \mathcal{V}_b d^4 p, \\ s_{ab}(x) &:= \text{Tr} \int \sigma_{ab} W d^4 p = \int \mathcal{T}_{ab} d^4 p, \end{aligned}$$

$$\tau_{ab}(x) := \text{Tr} \int p_a \gamma_5 \gamma_b W d^4 p = \int p_a \mathcal{A}_b d^4 p,$$

$$\chi(x) := \text{Tr} \int \gamma_5 W d^4 p = \int \mathcal{P} d^4 p,$$

$$\chi_a(x) := \text{Tr} \int p_a \gamma_5 W d^4 p = \int p_a \mathcal{P} d^4 p,$$

$$\lambda_{abc}(x) := \text{Tr} \int p_a \sigma_{bc} W d^4 p = \int p_a \mathcal{T}_{bc} d^4 p.$$

These are all the zeroth and first moments of the Wigner function in $d=4$ dimensions. Most of these quantities have a direct physical interpretation, $n(x)$ is the number or energy density, U^a is a momentum density, j_a a current, k_a and axial-vector current, T_{ab} the energy-momentum tensor, and s_{ab} the spin density, while τ_{ab} is a kind of ‘‘pseudoenergy-momentum tensor,’’ χ and χ_a are related to chirality, and λ_{abc} represents a spin-momentum interaction term. The equations of motion for these macroscopic quantities constitute the corresponding set of (quantum) hydrodynamic equations, which are derived by simply taking the zeroth and first moments of the Wigner equation in its Clifford decomposed form. Hence, each of the five equations in this Clifford decomposition gives rise to two equations for these moments. If we write

$$A_a = p_a + \frac{1}{2} i \hat{D}_a, \quad J_a^{bc} = \hat{J} e_a^\mu R_{\mu\nu}^{bc} \frac{\partial}{\partial p_\nu}, \quad (90)$$

we get the following ten equations:

$$mn - T - \frac{1}{2} i \hat{D}_a j^a + B_a k^a = 0, \quad (91)$$

$$m U_b - \int p_b p_a \mathcal{V}^a d^4 p + \frac{1}{2} i \hat{D}_a T_b^a + B_a \tau_b^a = \epsilon_{b'cd}^a \hat{J} e_a^\mu R_{\mu\nu}^{b'c} e_b^\nu k^d, \quad (92)$$

$$m \chi + \tau + \frac{1}{2} i \hat{D}_a k^a - B_a j^a = 0, \quad (93)$$

$$m \chi_b + \int p_a p_b \mathcal{A}^a d^4 p + \frac{1}{2} i \hat{D}_a \tau_b^a - B_a T_b^a = \eta_{b'd} \hat{J} e_a^\mu R_{\mu\nu}^{b'c} e_b^\nu k^d, \quad (94)$$

$$m j^d - U^d - \frac{1}{2} i \hat{D}^d n + B^d \chi - (\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) \left(\lambda^{efg} + \frac{1}{2} i \hat{D}^e s^{fg} \right) + 4 \epsilon_{efg}^d B^e s^{fg} = 0, \quad (95)$$

$$\begin{aligned} m T_h^d - \int p_h p^d S d^4 p - \frac{1}{2} i \hat{D}^d U_h + B^d \chi_h - i (\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) \left(\int p_h p^e \mathcal{T}^{fg} d^4 p + \frac{1}{2} i \hat{D}^e \lambda_h^{fg} \right) + 4 \epsilon_{efg}^d B^e \lambda_h^{fg} \\ = -i \delta_c^d \hat{J} e_b^\mu R_{\mu\nu}^{bc} e_h^\nu n + 2 \epsilon_{bc}^{ad} \hat{J} e_a^\mu R_{\mu\nu}^{bc} e_h^\nu \chi - 4 \eta^{da} \eta_{bf} \eta_{ce} \hat{J} e_a^\mu R_{\mu\nu}^{bc} e_h^\nu s^{ef}, \end{aligned} \quad (96)$$

$$m k^d + \chi^d + \frac{1}{2} i \hat{D}^d \chi - B^d n + i (\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) B^e s^{fg} - 4 \epsilon_{efg}^h \left(\lambda^{efg} + \frac{1}{2} i \hat{D}^e s^{fg} \right) = 0, \quad (97)$$

$$\begin{aligned}
& m\tau_h^d + \int p_h p^d \mathcal{P} d^4 p + \frac{1}{2} i \hat{D}^d \chi_h - B^d U_h + i(\eta_{ef} \delta_g^d - \eta_{eg} \delta_f^d) B^e \lambda_h^{fg} - 4 \epsilon_{efg}^d \left(\int p_h p^e \mathcal{T}^g d^4 p + \frac{1}{2} i \hat{D}^e \lambda_h^{fg} \right) \\
& = 4 \epsilon_{bc}^{da} \hat{J} e_a^\mu R_{\mu\nu}^{bc} e_h^\nu n - 8i(\delta_b^a \delta_c^d - \delta_c^a \delta_b^d) \hat{J} e_a^\mu R_{\mu\nu}^{bc} e_h^\nu \chi, \tag{98}
\end{aligned}$$

$$m s^{ef} - \frac{1}{4} \hat{D}^{[e} j^{f]} + \frac{1}{2} i B^{[e} k^{f]} = 0, \tag{99}$$

$$m \lambda_h^{ef} + \frac{1}{2} i \int p_h p^{[e} \mathcal{V}^{f]} d^4 p - \frac{1}{4} \hat{D}^{[e} T_{|h}^{f]} - \frac{1}{2} i B^{[e} \tau_{|h}^{f]} = -12 \epsilon_{bc}^{ag} \epsilon_{dg}^{ef} \hat{J} e_a^\mu R_{\mu\nu}^{bc} e_h^\nu s^{dg} + (\delta_c^a \epsilon_{db}^{ef} - \delta_d^a \epsilon_{cb}^{ef}) \hat{J} e_a^\mu R_{\mu\nu}^{bc} e_h^\nu k^d, \tag{100}$$

where $T = T_a^a$, $\tau = \tau_a^a$. Some of these have direct physical interpretations, e.g., as we saw in the previous section Eq. (91) gives a kinetic expression for the conformal anomaly when $m=0$. In analogy with this, we will refer to the equation for τ , Eq. (93), as the pseudoconformal anomaly for lack of a better word. Another important equation is Eq. (95) which states that the momentum density and the mass times the current density (which is also the velocity density) are not identical; the difference between the two is an indication of heat flow. Also, Eq. (99) gives an expression for the spin density in terms of the velocity/current density j_a and the axial current k_a .

The quantities without a direct physical interpretation, such as, $\tau_{ab}, \chi, \chi_a, \lambda_{abc}$ can be eliminated from these equations. One should also note that we are not able to eliminate the second moments appearing in this set of equations. In general one will get an infinite hierarchy of moment equations. This can be truncated by brute force at any given stage resulting in a set of approximate hydrodynamic equations. The second moments have a physical interpretation in terms of viscous pressure, since

$$\Pi_h^d := \text{Tr} \int p_h p^d \hat{W} d^4 p = \int p_h p^d \mathcal{S} d^4 p \tag{101}$$

is the viscous pressure tensor [2]. The remaining second moments are then higher Clifford algebra analogues of this, lacking a straightforward classical interpretation.

ARBITRARY SPINS

Clearly the fermions are the most difficult to treat due to their Grassmannian nature and that is why we have chosen to consider such fields in great detail. Arbitrary spins are not that difficult. Consider a field equation for a field Φ of some spin s ,

$$(D_s + M_s)\Phi = gJ,$$

where D_s is some differential operator (first order whenever s is half integral and second order when s is an integer), M_s is a mass term including couplings to curvature such as ξR for $s=0$ and $R_{\mu\nu}$ for $s=1$, g is a coupling constant, and J is a source term. As we have seen above such an equation gives rise to an equation for the associated Wigner function W_Φ by the ‘‘minimal substitution’’

$$i \nabla_\mu \mapsto e_\mu^a p_a + \frac{1}{2} i \nabla_\mu \tag{102}$$

and the ‘‘source term transformation’’

$$gJ \mapsto \hat{X}^{(s)} W_\Phi, \tag{103}$$

where \hat{X} is some integrodifferential operator containing the source. For bosons with spin s , \hat{X} is a $4s$ tensor and W is a $2s$ tensor. For $s=1/2$ we have already seen that \hat{X} carries a Clifford algebra index (i.e., two spinor indices) and three Lorentz indices, two of which must be understood as coming from the Lie algebra $\text{so}(r, d-r)$. For $s=3/2$ we would get a Wigner function which had two more Lorentz indices, and \hat{X} would also have four extra Lorentz indices to account for the vector index on the Rarita-Schwinger field. It is clear that for $s \geq 1$ the notation quickly becomes cumbersome.

All Wigner functions are maps from the cotangent bundle into some algebra: for a scalar field the target space is simply \mathbb{C} , for spin- $\frac{1}{2}$ it is $C(r, d-r)$, for $s=1$ we get $(T^*M \otimes \mathfrak{g}) \otimes (T^*M \otimes \mathfrak{g}^\dagger)$ with \mathfrak{g} the internal gauge algebra, and so on. In general fermions will have Wigner functions with values in $C(r, d-r) \otimes (T^*M)^{\otimes (2s-1)}$ while bosonic fields will have Wigner functions taking their values in $(T^*M)^{\otimes 2s}$. Gauge degrees of freedom are handled by simply enlarging the target space by tensoring it with $\mathfrak{g} \otimes \mathfrak{g}^\dagger$.

A spin-0 field ϕ would then give rise to a Wigner equation of the form

$$\left[\left(i e_\mu^a p_a + \frac{1}{2} i \nabla_\mu \right)^2 + m^2 + \xi R \right] W_\phi = \hat{X}^{(0)} W_\phi, \tag{104}$$

while the Maxwell field A_μ gives rise to

$$\left(i e_\nu^a p_a - \frac{1}{2} \nabla_\nu \right)^2 W_{\mu\rho} + R_\mu^\nu W_{\nu\rho} = \hat{X}_{\mu\rho}^{(1)\nu\lambda} W_{\nu\lambda}, \tag{105}$$

and a Rarita-Schwinger field ψ_μ would give rise to

$$\frac{1}{2} \epsilon^{abcd} \gamma_5 \gamma_b e_c^\mu e_d^\nu \left(e_\mu^e p_e + \frac{1}{2} i \nabla_\mu \right) W_{\nu\rho} = \hat{X}_{\mu\rho}^{(3/2)\nu\lambda} W_{\nu\lambda}, \tag{106}$$

where we have omitted the A and ψ subscripts on the Wigner functions for the cases of $s=1, 3/2$ and suppressed the spinor indices in the latter case. Yang-Mills fields could be treated

analogously but will have more complicated kinetic equations due to the nonlinearity of their field equations.

Internal degrees of freedom is also treated easily. The fields will now be cross sections in some associated bundle, and the parallel transporter has to include not just the curvature effect but also the connection in the corresponding principal bundle (the gauge field). One simply replaces the covariant derivative with the appropriate gauge covariant derivative containing the gauge field and the metric connection. If the field transforms in the representation ρ of \mathfrak{g} then the Wigner function will transform in the representation $\bar{\rho} \otimes \rho$, if ρ is the fundamental representation, then this is the adjoint one.

THE GRAVITATIONAL FIELD: PALATINI FORMALISM

Next we write down the equations for the gravitational field. This can be viewed from two points: (1) as a means of calculating the back reaction of the quantum fields on the spacetime geometry (keeping the energy-momentum tensor fixed) or (2) as constituting full-fledged quantum gravity (giving a set of coupled equations).

We immediately face a problem: although the connection is the formal analogue of the Yang-Mills fields, we cannot take over the results by Elze, Gyulassy, and Vasak, as the Einstein equations are first order in the connection, while the Yang-Mills equations are second order. In the presence of torsion the field equations for gravity can be written

$$e_b^\nu R_{\mu\nu}^{ab} = \kappa \left(T_\mu^a + \frac{1}{2} e_\mu^a T \right), \quad (107)$$

$$S_{ab}^c = 2\kappa \left(C_{ab}^c - \frac{3}{7} \delta_a^c C_{db}^d + \frac{2}{7} \delta_b^c C_{ad}^d \right), \quad (108)$$

where T_μ^a is the energy-momentum tensor, T its trace, S_{bc}^a the torsion, and C_{bc}^a is given by

$$C_{ab}^c \equiv \frac{1}{4} i E \bar{\psi} \gamma^c \sigma_{ab} \psi, \quad (109)$$

with $E = \det e_\mu^a$, see Ramond [5]. We will have to introduce two Wigner operators, one for the vierbein and one for the spin connection:

$$\Gamma_{\mu\nu}^{ab;cd} \equiv \int e^{-ip \cdot y} U_+ \omega_\mu^{ab} \otimes U_- \omega_\nu^{cd} \frac{d^4 y}{(2\pi)^4}, \quad (110)$$

$$L_{\mu\nu}^{ab} \equiv \int e^{-ip \cdot y} U_+ e_\mu^a \otimes U_- e_\nu^b \frac{d^4 y}{(2\pi)^4}. \quad (111)$$

Unfortunately these are noncovariant, as the spin connection transforms in an affine way under local $\mathfrak{so}(3,1)$ transformations. This noncovariance will then introduce an unwanted dependency on coordinate choices (gauge dependency).

Finding the Wigner equations for these is straightforward (albeit tedious) and will not be done here. They will have the form

$$\left(e_\mu^a p_a + \frac{1}{2} i \nabla_\mu \right) \Gamma_{\nu\rho}^{ab;cd} - (\mu \leftrightarrow \nu) = \hat{X}_{\mu gh}^{(T)ab} \Gamma_{\nu\rho}^{gh;cd} - (\mu \leftrightarrow \nu), \quad (112)$$

$$\left(e_\mu^a p_a + \frac{1}{2} i \nabla_\mu \right) L_{\nu\rho}^{ab} - (\mu \leftrightarrow \nu) = \hat{X}_{\mu c}^{(C)a} L_{\nu\rho}^{cb} - (\mu \leftrightarrow \nu), \quad (113)$$

where $\hat{X}_\mu^{(T,C)}$ are integral operators containing the source terms. Expressing these sources in terms of the Wigner operator for the matter fields, e.g.,

$$T_{ab} \equiv \left\langle \text{Tr} \left(\int \gamma_a p_b \hat{W} d^4 p \right) \right\rangle, \quad (114)$$

would then lead to a set of coupled integrodifferential equations, constituting the full set of equations for quantum gravity. It must be remembered that for $g_{\mu\nu}$, a quantum field, we cannot take the curvature two-form out of the integral in the original expression for \hat{X} . Therefore, the equation for the matter Wigner function becomes much more complicated. In [10] the analogous situation in QCD is treated. We will not attempt this level of generality here, just note that the straightforward generalization from Yang-Mills fields to gravitational fields does not give tractable kinetic equations, at least not in the Palatini formalism.

THE GRAVITATIONAL FIELD: ASHTEKAR VARIABLES

As Elze, Gyulassy, and Vasak [10] have developed the Wigner function technique for Yang-Mills fields, one would suspect that an approach similar to theirs can be fruitful if one uses the Ashtekar formulation of gravity. In this formulation [16] one has a complex $\text{SU}(2)$ connection $A = A_i^a dx^i \sigma_a$, where σ_a are the generators of \mathfrak{su}_2 and $i=1,2,3$ is a spatial index. This formalism is based on the fact that the Lorentz algebra $\mathfrak{so}(3,1)$ in four dimensions (and four dimensions only) is isomorphic to the complexification of \mathfrak{su}_2 . The Ashtekar formalism therefore makes use of two vital aspects of general relativity: the correct dimensionality and the correct signature of spacetime. Canonically conjugate to the connection, we have the ‘‘electric field’’ E_a^i , i.e.,

$$\{A_i^a(x), E_b^j(x')\}_{PB} = \delta_b^a \delta_i^j \delta(x, x'). \quad (115)$$

Introducing the field strength tensor F_{ij}^a , the constraints can be written as

$$D_i = E_a^j F_{ij}^a = 0,$$

$$H = \epsilon_c^{ab} E_a^i E_b^j F_{ij}^c = 0,$$

$$G_a = \nabla_j E_a^j = 0.$$

In exact analogy with the Yang-Mills case as presented in [10], we could define

$$\Gamma_{\mu\nu}^{ab}(x,p) = \int_{T_x M} \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} (e^{-(1/2)y \cdot \nabla} F_{\mu\rho}^a) \otimes (F_{\sigma\nu}^{b\dagger} e^{(1/2)y \cdot \nabla^\dagger}) g^{\rho\sigma},$$

with $F_{0i}^a = E_i^a$. Considering the Hamiltonian nature of the system, it is, however, more appropriate to introduce two slightly different Wigner functions (i.e., essentially splitting the above candidate into two); namely,

$$\Gamma_{ijkl}^{ab} = \int_{T_x \Sigma} \frac{d^3 y}{(2\pi)^3} e^{-ip \cdot y} (e^{-(1/2)y \cdot \nabla} F_{ij}^a) \otimes (F_{kl}^{b\dagger} e^{(1/2)y \cdot \nabla^\dagger}), \quad (116)$$

$$L_{ab}^{ij} = \int_{T_x \Sigma} \frac{d^3 y}{(2\pi)^3} e^{-ip \cdot y} (e^{-(1/2)y \cdot \nabla} E_a^i) \otimes (E_b^{j\dagger} e^{(1/2)y \cdot \nabla^\dagger}), \quad (117)$$

where we have used the global hyperbolicity of spacetime always assumed in a Hamiltonian formulation of gravity, $M \simeq \Sigma \times \mathbb{R}$, and, furthermore, made explicit use of the complex nature of the connection and the ‘‘electric field’’ (whence the daggers on the right-hand side).

The constraints can then be used to derive relationships between these two Wigner functions. The diffeomorphism constraint D_i can be rewritten

$$(E_a^i \otimes E_b^{j\dagger})(F_{ik}^a \otimes F_{jl}^{b\dagger}) = 0. \quad (118)$$

Noting that the Wigner functions $\Gamma_{ijkl}^{ab}, L_{ab}^{ij}$ are precisely the Weyl transforms of these two tensor products, we see that the quantum version of this constraint reads

$$\mathcal{D}_{kl} \equiv L_{ab}^{ij} * \Gamma_{ijkl}^{ab} = 0, \quad (119)$$

where $*$ denotes the twisted product (the noncommutative product induced on the set of phase-space functions by the non-Abelian product of operators),

$$f * g \sim f e^{(1/2)i\vec{\Delta}} g = f g + \frac{1}{2} i f \vec{\Delta} g \dots, \quad (120)$$

with $f \vec{\Delta} g := \{f, g\}$ being the Poisson bracket.

Similarly the Hamiltonian constraint H becomes

$$\mathcal{H} \equiv \epsilon_c^{ab} \epsilon_{c'}^{a'b'} L_{aa'}^{ii'} * L_{bb'}^{jj'} * \Gamma_{iji'j'}^{cc'} = 0. \quad (121)$$

One should note that while the classical constraints D_i, H are purely algebraic in F_{ij}^a, E_i^a , the quantum versions become infinite order differential equations in the corresponding Wigner functions—equivalently, they become $*$ algebraic, i.e., deformed. The last constraint, the Gauss one, however, gives rise to a differential equation for L_{ij}^{ab} , analogous to the equation for the Dirac-Wigner function (or more precisely, to the Yang-Mills Wigner function as derived by Elze, Gyulassy, and Vasak [10].

GROUP THEORETICAL ARGUMENTS

We can round off this discussion with a few comments about the algebraic structure behind the entire WWM for-

malism. The flat space Wigner function has a natural group theoretical interpretation [17,19]. Define the operator

$$\Pi(u, v) = \exp(iu\hat{p} - iv\hat{q}), \quad (122)$$

then $\Pi(u, v)$ forms a ray representation of the (Abelian) group of translations in phase space:

$$\Pi(u, v)\Pi(u', v') = e^{-i(uv' - v'u')}\Pi(u + u', v + v'). \quad (123)$$

This operator then gives the Weyl transformation, mapping an operator into a function on the classical phase space

$$A_W(u, v) = \text{Tr} \Pi(u, v) \hat{A}. \quad (124)$$

The Wigner function is the (symplectic) Fourier transform of the Weyl transform of the projection operator (for a pure state) $|\psi\rangle\langle\psi|$, i.e.,

$$W(x, p) = \int e^{i(up - vx)} \langle\psi|\Pi(u, v)|\psi\rangle du dv. \quad (125)$$

What we have done in this paper is to replace the Abelian group of translation on a flat phase space with the non-Abelian group of parallel transport on the curved phase space T^*M . This splits up into two parts, the parallel transport in the base manifold M , which is generated by the momentum operator, and the (Abelian) translations in the fiber T_x^*M , which is generated by \hat{q} .

In a previous work [18] I showed that one can generalize Π to a very large class of algebras, most notably finite dimensional Lie algebras, their corresponding loop and Kac-Moody algebras, as well as super Lie algebras and C^* -algebras. To put it in algebraic language then, what one does when going from flat space to curved space is to replace the usual Heisenberg algebra by the curved space analogue

$$[\hat{p}_\mu, \hat{p}_\nu] = -\hat{R}_{\mu\nu}, \quad [\hat{p}_\mu, \hat{q}^\nu] = -i\delta_\mu^\nu, \quad [\hat{q}^\mu, \hat{q}^\nu] = 0, \quad (126)$$

where $\hat{R}_{\mu\nu}$ is the curvature two-form (not the Ricci tensor).

The twisted product is given by

$$A_W * B_W := (\hat{A}\hat{B})_W = \text{Tr} \Pi(u, v) \hat{A} \hat{B} \quad (127)$$

and can be written in terms of a kernel as [18,19]

$$(f * g)(u, v) = \int K(u, v, u', v', u'', v'') f(u', v') g \times (u'', v'') du' dv' du'' dv'', \quad (128)$$

with

$$K(u, v, u', v', u'', v'') := \text{Tr} \Pi(u, v) \Pi(u', v') \Pi(u'', v''). \quad (129)$$

Thus the quantity Π is the essential ingredient in any generalized ‘‘Wigner-Weyl-Moyal formalism’’ or ‘‘symbol calculus’’ [18]. Hence what is needed in general is (i) a ‘‘symbol map’’ sending operators (typically pseudodifferential operators) into functions on the cotangent bundle and (ii) a map

connecting two fibers; this latter map is simply the symbol of the parallel translator. In principle one could have a different map in (ii), but the symbol of the parallel translator is the simplest choice.

The formula we have derived for W was based on the phase space of a classical *mechanical* system, namely, T^*M . The Wigner function took its values in $C(3,1) \otimes \Gamma(T^*M)$ then, where $C(3,1)$ is the Clifford algebra and $\Gamma(T^*M)$ denotes the set of cross sections of the cotangent bundle. We used the group of parallel transport and translations along the fibers to generalize the Wigner function from quantum mechanics in flat spacetime. One can proceed to quantum field theory by means of second quantization, although this is not usually easy to define in a curved background [4]. Locally this always makes sense, but there are often global obstructions. Another method, which is not very common, is to treat the field and its conjugate momenta as the fundamental phase space (which is then infinite dimensional). The Wigner function is then a functional of these fields. This is very similar to the way one treats quantum gravity in quantum cosmology, where one then considers the wave function of the universe. The Wigner function has already been extended to this situation. Let us consider a field theory in Hamiltonian formalism and denote the fields and their conjugate momenta by ϕ and π , respectively. These can be either bosonic or fermionic. We then want a functional \mathcal{W} such that

$$\langle A \rangle \equiv \int \mathcal{W}[\phi, \pi] A[\phi, \pi] \mathcal{D}\phi \mathcal{D}\pi, \quad (130)$$

which generalizes

$$\langle \hat{A} \rangle = \text{Tr} \rho \hat{A} = \int A_W(x, p) W(x, p) dx dp.$$

If we have free fields, then the phase space is flat, and the parallel transporter becomes simply

$$\Pi = \exp\left(i \int u \hat{\phi} - v \hat{\pi} dx\right). \quad (131)$$

From the quantum mechanical relation

$$W(u, v) = \text{Tr} \rho \Pi(u, v) = \langle \Pi(u, v) \rangle,$$

which is the symplectic Fourier transform of the Wigner function, we conclude

$$\mathcal{W}(u, v) := \langle \Pi(u, v) \rangle = \frac{1}{Z(0)} \int e^{iS} \Pi(u, v) \mathcal{D}\phi \mathcal{D}\pi. \quad (132)$$

Noticing the concrete form of Π we see that $\exp(iS)\Pi$ is equivalent to adding a source term (with the sources denoted by u and v , respectively), hence

$$\mathcal{W}(u, v) = \frac{Z(u, v)}{Z(0)}. \quad (133)$$

The proper Wigner functional is then a symplectic (functional) Fourier transform of this quantity. This makes it possible to interpret the partition function (with sources) as essentially the Wigner functional of the vacuum state (or, more generally, the vacuum to vacuum transition).

CONCLUSION

We have generalized the work by Elze, Gyulassy, and Vasak to gravitation and quantum fields in curved spacetime. Thereby we obtained a set of *exact* equations for QFT in curved spacetime and even quantum gravity, which allows us to make *nonperturbative* calculations in these cases. The major draw back is the complicated nature of the equations—especially in the full quantum gravity case. But on the other hand, we can develop a recursive scheme for these quantum corrections (the Δ expansion), and we can therefore avoid perturbation theory all together. One should also note that this approach is, on the one hand, intimately related to the algebraic structure of the space of quantum observables (the canonical commutator relations for the fields) and on the other, to the topological structure of spacetime (the Wigner function is, for Dirac fermions, a mapping from the cotangent bundle into the Clifford algebra, whereas for bosons it is a mapping from the cotangent bundle into the tensor algebra). It was this interplay that allowed us to find dynamical expressions for the conformal anomaly in $d=2$ and $d=4$.

We also saw that the kinetic equation satisfied by the Wigner function could be split up into two, one being the mass-shell constraint giving the quantum and curvature induced corrections to the mass, while the other was the kinetic equation proper. In certain classical-like situations these could be written as two Fokker-Planck equations, the one with no momentum diffusion the other with no dynamical friction. We derived expressions for these kinetic quantities. We also saw how quantum corrections modified this simple situation.

By taking appropriate moments of the Wigner equation we arrived at a set of coupled equations governing macroscopic quantities such as energy-momentum tensor and current and spin densities. These were the corresponding hydrodynamic equations. We saw that they, besides giving the kinetic interpretation of the conformal anomaly, also lead to an expression for the heat flow.

We attempted to handle first the gravitational degrees of freedom in the Palatini formalism, in which we had to introduce Wigner functions for the vierbein as well as the spin connection. These were not, however, covariant, and the resulting equations were too complicated. We then turned to the Ashtekar variables, where we could either introduce one Wigner function for $F_{\mu\nu}^a$ in the standard way or use a canonical description to split this into two, one for F_{ij}^a and the other for E_a^i . These were by construction covariant. The constraints induced conditions on these Wigner functions which were * algebraic, i.e., infinite order differential relations due to a quantum deformation of the product.

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