Quantization of the null-surface formulation of general relativity

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(Received 17 October 1996)

We define and discuss various quantum operators that describe the geometry of spacetime in quantum general relativity. These are obtained by combining the null-surface formulation of general relativity, recently developed, with asymptotic quantization. One of the operators defined describes a ''fuzzy'' quantum light cone structure. Others, denoted ''spacetime-point operators,'' characterize geometrically defined physical points. We discuss the interpretation of these operators. This seems to suggest a picture of quantum spacetime as made of ''fuzzy'' physical points. We derive the commutation algebra of the quantum spacetime-point operators in the linearization around flat space. [S0556-2821(97)02414-4]

PACS number(s): 04.60.Ds, 04.20.Gz, 04.20.Ha

I. INTRODUCTION

The problem of finding and understanding the relationship between quantum theory and gravitation is an extremely difficult one (that has defied solution for close to 70 years) and is simultaneously such a profound problem that it has attracted a great deal of attention. Its resolution could easily be a major stepping stone to a more complete understanding of our physical world. The difficulties, however, are such that we might well need radical changes in our views or completely new ideas before the problem can be solved (see, for instance, [1]). Even if this is the case, this does not mean that we should necessarily abandon the exploration of more traditional approaches, since even if they fail, they could indicate possible directions to explore in the search for the unification of gravity with quantum theory.

In this paper we present an approach to this issue which, although based on many of the standard ideas, differs from other approaches in several substantial ways. In some sense our formulation lies between the conventional and nonconventional approaches [2].

The first issue we discuss is our view towards classical general relativity (GR). At the classical level, a clear distinction can be made between GR and other field theories. Only in GR does the geometry play a *dynamical* role. Though often noted, this distinction has been re-emphasized in a recent series of papers by presenting GR as a theory of characteristic *hypersurfaces* [3–5] rather than as a theory of the metric *field*. From this point of view the spacetime metric and associated connection are derived concepts: the basic variables are families of three-surfaces and a scalar function (a conformal factor) from which a metric can be derived. The surfaces are automatically the characteristic surfaces of the metric and the metric automatically satisfies the Einstein field equations. This reformulation of GR has been referred to as the null surface formulation (NSF) of GR. It appears that no other physically relevant field theory can be stated as such a theory of surfaces.

Here, we study the quantization of the linearized version of this approach. From this quantization of the NSF, we appear to be led to new ideas and results on the form a quantum theory of gravity might take. The new view essentially says that the null surfaces become operators that obey commutation relations. Furthermore, since there is a prescription for locating points of spacetime using foliations by families of null surfaces, the spacetime points themselves become operators.

Roughly speaking, our formalism is a union between the Ashtekar asymptotic quantization [6] of the gravitational field and the NSF. In our formalism, the free Bondi data at future null infinity $\mathcal{I}^+$ play a very important role. They enter as a source in the NSF field equations. Thus, for each data set, the solution to our classical equations represents a regular radiative spacetime. On the other hand, the formalism developed by Ashtekar gives a kinematic quantization of the radiative degrees of freedom of the gravitational field at $\mathcal{I}^+$. By promoting the classical Bondi data to quantum operators and introducing a Fock space of asymptotic states (modulo technical difficulties addressed in detail by Ashtekar), one is left with the ''in'' (or ''out'') states of quantum theory. What is missing in the Ashtekar approach is the dynamical part of the quantum theory, which would relate the asymptotic states to the geometry of the interior of spacetime.

In this paper we adopt Ashtekar's asymptotic quantization in its simplest form (avoiding infrared issues) by promoting the free Bondi data to quantum operators. The solutions to the classical NSF equations determine families of null surfaces in terms of these free data. It follows that in the ''quantum theory'' the null surfaces become operator functions of the operator data. Furthermore, since the spacetime points are themselves determined by the intersections of the null surfaces (and are expressible in terms of the surfaces), they can also be thought of as operator functions of the data, with implied nontrivial commutation relations. We emphasize that we neither give equal time commutation relations nor use a Hamiltonian to obtain the ''evolution'' of the operators: ap-

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propriate commutation relations for the operator data are given on $\mathcal{I}^+$, and the information about the dynamics (i.e., the full spacetime) is implicitly determined by the NSF equations. We emphasize that we are not discussing a field theory on spacetime; our variables are not fields, they are surfaces composed of spacetime points. The surfaces and associated points become the operators.

We point out that there is no Hamiltonian for a Schrödinger evolution; rather the operator ''evolution'' is given by the NSF equations. The formalism is most closely tied to a Heisenberg representation.

In Sec. II we will review some relevant aspects of the NSF of GR. [Note that we use signature $(+,-,-,-)$.] In Sec. III we discuss what happens when we implement the Ashtekar quantization procedure. In Sec. IV we summarize our main results and discuss possible meanings and ramifications of these ideas. An outline of our results and a more detailed discussion of their physical interpretation have appeared in [7]. We relegate many of the technical details, which can become complicated, to appendixes. In Appendix A, as an example, we apply our methods to the quantization of the Maxwell theory, obtaining the standard quantization in the Coulomb gauge.

II. NULL-SURFACE FORMULATION OF GR

In this section we review a new formulation, the null-surface formulation of classical general relativity [3–5,8], where the emphasis has been shifted away from more standard type of field variable (metric, connection, holonomy, curvature, etc.) to, instead, families of three-dimensional surfaces on a four-manifold M^4. (These surfaces eventually turn out to be the characteristic surfaces of a metric.) On the sphere bundle over M^4, topologically $M^4 \times S^2$, with no further structure, there are given differential equations for the determination of these surfaces. From the surfaces themselves, by differentiation and algebraic manipulation, a (conformal) metric tensor can be obtained. These surfaces, which play the role of the basic geometric quantities, are then automatically the characteristic surfaces of this conformal metric. Furthermore, the equations allow for a choice of conformal factor that makes the conformal metric into a metric which automatically satisfies the vacuum Einstein equations. In other words the vacuum Einstein equations are formulated as equations for families of surfaces and a single (scalar) conformal factor. All geometric quantities, the metric, the connection, spin coefficients, Weyl and Ricci tensors, can be expressed in terms of the surfaces and the conformal factor. In our present discussion we will be mainly concerned with these characteristic surfaces (i.e., the conformal structure), though of course in the full theory the conformal factor plays an essential role.

Since the details of the differential equations are relatively complicated [3–5,8] and we do not need them for the present work, we will adopt the following strategy. We will assume that the differential equations for the surfaces (and conformal factor) have been solved explicitly and then attempt to understand the meaning of the solutions and what can be derived from them.

First of all, we have the explicit expression for the conformal factor $\Omega = \Omega(x^a, \zeta, [\text{data}])$, where the x^a are some local coordinates on the manifold M^4, while the ζ is a complex stereographic coordinate on the sphere S^2, and [data] is the Bondi shear $[\sigma, \overline{\sigma}]$. We will have little further use here for $\Omega(x^a, \zeta)$. Of fundamental importance to us are the families of surfaces given as solutions to our equations, with specific free data. They take the form

$$Z(x^a, \zeta, [\text{data}]) = u = \text{const.} \tag{1}$$

For *fixed values* of (u, ζ) the above is a single function of the four coordinates x^a and thus describes a particular three-surface. As the value of u varies (*for fixed* ζ) we have a one-parameter foliation (of a local region) by the surfaces. The ζ then labels a sphere's worth of these foliations, i.e., a sphere's worth of surfaces passes through each spacetime point. Assuming that the Z satisfies the NSF differential equations, one can then, in a simple and straightforward fashion, *obtain a conformal metric in terms of* Z [3]. Symbolically, we thus have

$$g_{ab}(x^a, [\text{data}]) = g_{ab}[Z(x^a, \zeta, [\text{data}])],$$

where g_{ab} is a conformal metric, undetermined by an overall conformal factor. Note that while Z is a function of ζ, the metric g_{ab} is independent of ζ.

The details of this construction are not of importance here. What is important is that automatically there is a (conformal) metric, $g^{ab}(x^a)$ such that

$$g^{ab} Z_{,a} Z_{,b} = 0 \tag{2}$$

for all ζ; i.e., the surfaces $Z = \text{const}$, are characteristic surfaces of this metric. For simplicity, we can then choose (in a natural fashion) a special member of the conformal class yielding an explicit metric in terms of Z. [The ''naturalness'' arises from the fact that a simple function (see below) of Z is an affine parameter for this special member of the conformal class.] We emphasize that all conformal information about the spacetime is contained in knowledge of $Z(x^a, \zeta)$.

For each fixed value of ζ the level surfaces of Z describe a foliation by (null) surfaces: Treating Z simply as a sphere's worth of scalar functions on M^4, we can construct other scalar functions by differentiating Z several times in both the ζ and $\overline{\zeta}$ directions and then holding ζ constant afterwards.[1] Particularly useful to us are the two first derivatives and the mixed second derivative. Together with the $Z(x^a, \zeta)$, these are the four functions:

$$u = Z(x^a, \zeta, [\text{data}]), \tag{3a}$$

$$\omega = \eth Z(x^a, \zeta, [\text{data}]), \tag{3b}$$

$$\overline{\omega} = \overline{\eth} Z(x^a, \zeta, [\text{data}]), \tag{3c}$$

$$R = \eth\overline{\eth} Z(x^a, \zeta, [\text{data}]), \tag{3d}$$

[1] Note that differentiating Z with respect to ζ is equivalent to finding the intersections of adjacent null surfaces. For a detailed discussion, see [4].

where ð and ð̄ are (essentially) the ζ and $\overline{\zeta}$ derivatives [9]. With the definitions

$$\theta^i = (\theta^0, \theta^+, \theta^-, \theta^1) \equiv (u, \omega, \overline{\omega}, R),$$

we have

$$\theta^i = \theta^i(x^a, \zeta, [\text{data}]). \tag{4}$$

These four scalar functions (parametrized by ζ) have a simple geometric meaning.

(1) $\theta^0 = u = Z(x^a, \zeta, [\text{data}]) = \text{const}$, for fixed ζ, describes a null surface. Changing u leads to a one-parameter foliation of M^4 by null surfaces.

(2) $\theta^+ = \omega = \eth Z(x^a, \zeta, [\text{data}]) = \text{const}$ and $\theta^- = \overline{\omega} = \bar{\eth} Z(x^a, \zeta, [\text{data}]) = \text{const}$ choose a null geodesic on that surface.

(3) $\theta^1 = R = \eth\bar{\eth} Z(x^a, \zeta, [\text{data}])$ parametrizes points on that null geodesic. (In fact R is an affine parameter along the null geodesics for the special member of our conformal class mentioned earlier.) The four θ^i, for fixed ζ, thus locate spacetime points. They define a sphere's worth of null coordinate systems, and Eq. (4) gives the coordinate transformation between the θ^i and x^a for each fixed ζ.

Since $Z(x^a, \zeta, [\text{data}])$ contains all the conformal information of the spacetime, so do the θ^i.

An important conceptual issue is that Eq. (4) can, in principle, be (locally) algebraically inverted into the form

$$x^a = x^a(u, \omega, R; \zeta; [\text{data}]). \tag{5}$$

Since Eq. (5) is equivalent to Eq. (3), it too contains the full information about the solutions to the conformal Einstein equations; i.e., from knowledge of Eq. (5), a metric conformal to an Einstein metric can be obtained analytically [5]. The information about the conformal Einstein space is coded into the functional dependence on the data.

The information about the conformal structure of spacetime, originally encoded in Z, can now be extracted in an alternate manner from Eq. (5); a manner that is, at the moment, of direct interest to us. If values of the (u, ω, ζ) are chosen arbitrarily but kept constant and R is allowed to vary, Eq. (5) is the description of a *null geodesic* of the spacetime. The five-dimensional space of null geodesics is coordinatized by the (u, ω, ζ), with (ω, ζ) complex, while R parametrizes the individual geodesics. The conformal structure is determined by the knowledge of all null geodesics through each spacetime point, and the dependence of these on the [data] encodes the particular spacetime. Note the dual role Eqs. (4) and (5) play; Eq. (4) describes null surfaces, its null geodesics, and points on the geodesics in terms of some ''standard'' coordinates x^a, while Eq. (5) describes, in parametric form, all the null geodesics of the space. Though at first they appeared to describe the coordinate transformations between some null coordinates and an arbitrary set of coordinates x^a, they now have a coordinate-independent meaning. We return to Eq. (5) later.

Asymptotically flat vacuum spacetimes

Before we proceed further, we make the specialization from a description of any (local) Einstein spacetime to the study of asymptotically flat vacuum spacetimes. In this case the geometrical meanings of the various quantities become clearer. We begin with the fact that null infinity $\mathcal{I}^+$ exists. It can be coordinatized by a Bondi coordinate system,

$$(u, \zeta, \overline{\zeta}), \tag{6}$$

with u the Bondi retarded time, and $(\zeta, \overline{\zeta}) \in S^2$ labeling the null generators of $\mathcal{I}^+$. With this notation we can give a precise meaning to the null surfaces described by $u = Z(x^a, \zeta, [\text{data}])$; they are the past null cones of the points $(u, \zeta, \overline{\zeta})$ of $\mathcal{I}^+$. With this meaning to Z we have a dual interpretation of $Z(x^a, \zeta) = u$, namely, if the spacetime point x^a is held constant but the $(\zeta, \overline{\zeta})$ is varied over S^2, we obtain a two-surface (topologically S^2) on $\mathcal{I}^+$, the so-called light cone cut of $\mathcal{I}^+$, defined as the intersection of the future light cone of the point x^a with $\mathcal{I}^+$. It consists of all points of $\mathcal{I}^+$ reached by null geodesics from x^a. Z is then referred to as the light cone cut function.

We have a geometric interpretation, not only of $Z(x^a, \zeta, [\text{data}])$, but also of $\omega = \eth Z(x^a, \zeta, [\text{data}])$ and $R = \eth\bar{\eth} Z(x^a, \zeta, [\text{data}])$. ω is the ''stereographic angle'' that the light cone cuts make with the Bondi $u = \text{const}$ cuts [i.e., it labels the backward direction of the null geodesics from the point $(u, \zeta) \in \mathcal{I}^+$ to x^a]. R is a measure of the curvature of the cut, and thus a measure of the ''affine distance'' from $\mathcal{I}^+$ to x^a along the null geodesic.

The four functions $\theta^i(x^a, \zeta, [\text{data}])$, which are defined geometrically on $\mathcal{I}^+$, describe the interior of the spacetime. They can be inverted [see Eq. (5)], leading to

$$x^a = x^a(\theta^i; \zeta; [\text{data}]), \tag{7}$$

which gives the location of spacetime points in terms of (geometrical) information on $\mathcal{I}^+$, namely, the θ^i.

Linearization of the NSF

With this asymptotically flat point of view, we now consider the linearization of the null-surface formulation of the Einstein equations. The coordinates used here and subsequently are the standard Cartesian coordinates x^a of the background Minkowski spacetime. We will make extensive use of this later. In this case the conformal factor can be taken as one: i.e.,

$$\Omega(x^a, \zeta, [\text{data}]) = 1, \tag{8}$$

and the differential equation for Z becomes

$$\eth^2\bar{\eth}^2 Z = \eth^2 \overline{\sigma}_R(x^a, \zeta) + \bar{\eth}^2 \sigma_R(x^a, \zeta) \equiv D(x^a, \zeta, [\sigma]). \tag{9}$$

The data are given by a complex-valued spin-weight-2 function on $\mathcal{I}^+$, namely, $\sigma(u, \zeta)$ [and its complex conjugate $\overline{\sigma(u, \zeta)}$] which can be given freely. The data are then restricted to the Minkowski light cone cut $S^2(x^a)$, described by (see [10])

$$u(\zeta)=Z_0(x^a,\zeta)\equiv x^a \ell_a(\zeta),$$

$$\ell(\zeta)\cdot\ell(\zeta)\equiv \eta_{ab}\ell^a(\zeta)\ell^b(\zeta)=\ell_a(\zeta)\ell^a(\zeta)=0, \quad (10)$$

where $\ell^a=(1/\sqrt{2})(1,(\zeta+\overline{\zeta})/(1+\zeta\overline{\zeta}),-i(\zeta-\overline{\zeta})/(1+\zeta\overline{\zeta}),(-1+\zeta\overline{\zeta})/(1+\zeta\overline{\zeta}))$ satisfies $\eth^2\ell^a=\bar{\eth}^2\ell^a=0$ and $Z_0(x^a,\zeta)=Z(x^a,\zeta,[0])$, i.e., Z_0 is the Minkowski light cone cut function satisfying Eq. (9) with zero data. Note that the components of ℓ^a (and hence Z_0) are simple combinations of the first four spherical harmonics. Equation (10), in turn, leads to the restriction, to the light cone cuts, of the data $\sigma(u,\zeta)$: i.e.,

$$\sigma_R(x^a,\zeta)=\sigma(Z_0(x^a,\zeta),\zeta). \quad (11)$$

(Note that σ_R can be viewed in two different ways. It is the pullback or restriction of σ to a cut of $\mathcal{I}^+$ labeled by the spacetime points x^a, but it can also be directly viewed as a function on the sphere bundle over spacetime.) Equations (8) and (9) are equivalent to the linearized vacuum Einstein equations. The general regular solution to Eq. (9) is obtained as the sum of a particular solution plus the general solution Z_0 to the homogeneous equation: i.e.,

$$Z(x^a,\zeta,[\text{data}])=Z_0(x^a,\zeta)+\int_{S^2}G(\zeta,\eta)D(x^a,\eta,[\sigma])dS^2_\eta, \quad (12)$$

where dS^2_η is the volume element on the two-sphere and $G(\zeta,\eta)$ is a simple Green's function of the operator $\eth^2\bar{\eth}^2$, given by

$$G(\zeta,\eta)=\frac{1}{4\pi}\ell(\zeta)\cdot\ell(\eta)\ \ln[\ell(\zeta)\cdot\ell(\eta)]. \quad (13)$$

We want to point out and emphasize an important aspect of the solution (12). $Z_0(x^a,\zeta)$ consists only of combinations of $l=0,1$ spherical harmonics; the second term (the particular solution) has been chosen so that its spherical harmonic expansion contains no $l=0,1$ harmonics. One could have chosen other particular solutions *with* $l=0$ or $l=1$ harmonics having as coefficients four arbitrary functions of the x^a. These four functions would constitute an arbitrary gauge transformation in the linear theory. Our choice for them to vanish is equivalent to a particular gauge choice [12]. The implied gauge is the equivalent of the Coulomb gauge of Maxwell theory, namely, for $g^{ab}=\eta^{ab}+h^{ab}$, we have $h^{0a}=0$. The analogous gauge choice for Maxwell theory is described in Appendix A 3.

For later use, Eq. (12) can be rewritten as

$$\begin{aligned}Z(x^a,\zeta,[\sigma])&=Z_0(x^a,\zeta)+\int_{S^2}[\bar{\eth}^2_\eta G(\zeta,\eta)\sigma_R(x^a,\eta)\\&\quad+\eth^2_\eta G(\zeta,\eta)\overline{\sigma_R}(x^a,\eta)]d^2S_\eta\\&\equiv Z_0(x^a,\zeta)+Z_1(x^a,\zeta,[\sigma]). \quad (14)\end{aligned}$$

This expression is obtained from Eq. (12) by using properties of the Green's function $G(\zeta,\eta)$ (see Appendix C), and from the assumption that σ_R is a regular function on the sphere. By differentiation (with respect to ζ) of Eq. (12) it is a simple matter to construct the full set of θ^i, i.e., Eqs. (3a)–(3d), and invert them explicitly to obtain Eq. (5). The explicit linearized inversion is given in Sec. III D.

Simply for completeness, we mention that the full (exact) set of Einstein equations are a generalization of Eqs. (8) and (9); Eq. (8) for the conformal factor becomes more complicated, while Eq. (9), the equation for Z, retains the same form; it has an additional, rather complicated, term added to the right-hand side that does depend on the Ω [11].

III. QUANTIZATION OF LINEARIZED GR

In the previous section we described how the classical data on $\mathcal{I}^+$ can be used to reconstruct various geometrical structures in the interior of the spacetime: null surfaces, null geodesics, and the locations of spacetime points in a given local chart. In this section, by analogy, we begin with an asymptotic quantum theory at $\mathcal{I}^+$ [6], and subsequently extend it into the interior of the spacetime. We implement this idea by constructing quantum operators corresponding to the various geometrical entities described in the previous section. We finally compute various physically interesting commutation relations obtained from the free-field commutation relations on the data at $\mathcal{I}^+$.

While most of our calculations are formal, all quantities (in the linearized case) can be defined rigorously on the asymptotic Fock space. Alternatively, we can think of all quantities as abstract operators subject to nontrivial commutation relations.

In the first subsection we briefly introduce the asymptotic quantum theory (done in detail for the free Maxwell field in Appendix A 1), essentially the quantization of the characteristic free data at $\mathcal{I}^+$, and describe the construction of the asymptotic Fock space (the details are given in Appendix A 2). We describe Ashtekar's asymptotic quantization [6], differing only in notational details. In addition, we ignore infrared sectors.

The remaining subsections contain the construction of the new quantum operators. Since all of them have a functional dependence on the data through the Z function, our first result is the quantization of the null surfaces, in Sec. III B. The commutator for the Z function at two different points is, then, of fundamental importance to the remainder of the section, in which we construct the quantum analogues of the various geometrical quantities (Sec. III C) and quantum spacetime points (Sec. III D).

A. Asymptotic quantum theory

As is well known, the radiative degrees of freedom of the gravitational field are specified by the characteristic initial data on $\mathcal{I}^+$. The space of characteristic initial data is a phase space coordinatized by either the Bondi shear $\sigma_{ab}(u,\zeta)=\sigma(u,\zeta)\overline{m}_a\overline{m}_b+\overline{\sigma}(u,\zeta)m_am_b$, or the complex Bondi news $N_{ab}=N(u,\zeta)\overline{m}_a\overline{m}_b+\overline{N}(u,\zeta)m_am_b$. The complex Bondi shear $\sigma(u,\zeta)$ serves as a potential for the complex Bondi news $N=\partial\sigma/\partial u$. The action of the symplectic form on two-vectors (infinitesimal news scalars) δN_1 and δN_2 tangent to the phase space of characteristic data is [6]

$$\Omega(\delta N_1,\delta N_2)=\frac{1}{2\pi}\int\int_{\mathcal{I}^+}du\ dS^2\ du'\ dS^{2'}\delta^2(\zeta-\zeta')\times\Delta(u-u')[\delta N_1(u,\zeta)\overline{\delta N_2}(u',\zeta')-\delta N_2(u,\zeta)\overline{\delta N_1}(u',\zeta')], \tag{15}$$

where $\Delta(u)=\frac{1}{2}\,\mathrm{sgn}(u)$ is the skew-symmetric antiderivative of $\delta(u)$, so that $\delta(u)=\partial\Delta(u)/\partial u$ (as distributions); and $du\ dS^2=-2idu\wedge d\zeta\wedge d\bar{\zeta}/(1+\zeta\bar{\zeta})^2$ is the volume element on $\mathcal{I}^+$. Note that this phase space is analogous to the phase space for source-free Maxwell theory, with σ,N playing the roles of A,E, respectively. Thus the asymptotic aspects of the quantization are identical to the construction detailed in Appendix A.

Consider the space $\mathcal{S}$ of C^∞ spin-weight-2 complex scalar fields N on $\mathcal{I}^+$, all of whose components in a (u,ζ) chart and all their derivatives fall off faster than $1/|u|^n$ for any n, for large values of $|u|$. On the positive frequency (with respect to u) subspace $\mathcal{S}^+$ of news functions, one can introduce a Hermitian inner product analogous to Eq. (A28). One can then Cauchy complete this space to obtain the one particle Hilbert space, on which one constructs the asymptotic Fock space of the characteristic data for the radiative modes of GR. In a fashion analogous to that for the free Maxwell field, one then constructs operator-valued distributions corresponding to the Bondi news, and the Bondi shear $\sigma(u,\zeta)$. These operator-valued distributions satisfy [6] the formal commutation relations given by

$$[\hat{\sigma}(u,\zeta),\hat{\bar{\sigma}}(u',\zeta')]=-2\pi i\hbar\Delta(u-u')\delta^2(\zeta-\zeta')\hat{1}, \tag{16}$$

where $\delta^2(\zeta-\zeta')$ has spin weight 2 in ζ and -2 in ζ', and is defined such that $\int_{S^2}\delta^2(\zeta-\zeta')f(\zeta')dS'^2=f(\zeta)$ for all spin weight $+2$ functions f (see [9] for the treatment of δ functions in the context of spin-s spherical harmonics). These are the fundamental commutation relations for the data on $\mathcal{I}^+$. Since all the other operators are constructed via their functional dependence on the data, these commutation relations are critical to obtaining the commutation relations between the interesting geometrical operators.

B. Quantum hypersurfaces

We now present a construction that extends the quantization available at $\mathcal{I}^+$ into the interior of the spacetime. In a rather nonstandard fashion, we proceed to the quantization of hypersurfaces and spacetime points, instead of the more traditional approach of quantizing the metric fields or connections. This construction is based on the null-surface formulation of GR and a (classical) *dynamical prescription* to specify a location in the interior manifold. In Sec. II, we had two dynamical prescriptions, with different meanings: Eq. (4), $\theta^i=\theta^i(x^a,\zeta,[\sigma])$, which for given x^a,ζ, and σ define four null-geodesic quantities; or Eq. (5), $x^a=x^a(\theta^i,\zeta,[\sigma])$, which for given values of θ^i and ζ (fixed σ) locates an interior spacetime point.

Both alternatives require the explicit expression for the function $Z(x^a,\zeta,[\sigma])$, obtained in Sec. II [Eq. (14)]. Z can be viewed as describing null hypersurfaces of the spacetime by setting $Z(x^a,\zeta,[\sigma])=\text{const}$. Therefore, we will first develop the formal quantization of $Z(x^a,\zeta,[\sigma])$, without attempting to give it a meaning immediately.

We define the operator $\hat{Z}$ by simple substitution, in Eq. (14), of the classical variables σ with their quantum analogues $\hat{\sigma}$: i.e.,

$$\hat{Z}(x^a,\zeta)\equiv Z(x^a,\zeta,[\hat{\sigma}])=Z_0(x^a,\zeta)\hat{1}+\int_{S^2}\{\bar{\eth}^2_\eta G(\zeta,\eta)\hat{\sigma}(Z_0(x^a,\eta),\eta)+\eth^2_\eta G(\zeta,\eta)\hat{\bar{\sigma}}(Z_0(x^a,\eta),\eta)\}d^2S_\eta. \tag{17}$$

The operator $\hat{Z}$ is manifestly linear in the free data $\hat{\sigma}$. The free-data commutation relation (16) implies the following integral representation of the commutation relations for $\hat{Z}$:

$$[\hat{Z},\hat{Z}']\equiv[\hat{Z}(x^a,\zeta),\hat{Z}(x'^a,\zeta')]=-2\pi i\hbar\int_{S^2}[\bar{\eth}^2_\eta G(\zeta,\eta)\eth^2_\eta G(\zeta',\eta)+\eth^2_\eta G(\zeta,\eta)\bar{\eth}^2_\eta G(\zeta',\eta)]\Delta[y\cdot\ell(\eta)]d^2S_\eta\hat{1}, \tag{18}$$

where we use the notation $v\cdot w\equiv v^a\eta_{ab}w^b$ for vectors v^a and w^a on Minkowski space, $y^a\equiv x^a-x'^a$, and the vector ℓ^a was introduced in Eq. (10).

The commutator $[\hat{Z},\hat{Z}']$ is symmetric under interchange of only ζ with ζ' and antisymmetric under interchange of only x^a with x'^a. The latter antisymmetry implies that $[\hat{Z},\hat{Z}]$ and its ζ derivatives vanish identically, a property that has important consequences in the following two subsections.

The evaluation of the commutator (18) in closed form is a cumbersome calculation. In the case of *timelike* y^a the closed-form commutator is

$$[\hat{Z},\hat{Z}']=-2\pi i\hbar[\ell\cdot\ell'\ln(\ell\cdot\ell')+\tfrac{1}{3}-\tfrac{1}{6}\ell\cdot\ell']\times\Delta(x^0-x'^0)\hat{1}, \tag{19}$$

where x^0 and x'^0 are the time components of x^a and x'^a, respectively. The calculation follows essentially the same steps as in the analogous case of Maxwell fields, which we include in Appendix A 4. This calculation is considerably simpler than the case of spacelike separation because, in the timelike case, the step function $\Delta(y\cdot\ell)$ takes a constant value on the sphere ($+\frac{1}{2}$ if y^a is future pointing, or $-\frac{1}{2}$ if y^a is past pointing). If the step function changes sign on the sphere, as in the spacelike case, there is a nonvanishing line integration on the boundary where the sign change takes place. This line integral becomes lengthy and cumbersome (though straightforward) to evaluate (see Appendix A 4 for a very similar calculation in the case of Maxwell fields). Though this calculation has not yet been completed, it is not clear that the closed form will shed light on the discussion that follows.

In the remainder of this section, we turn our attention to the interpretations of two of the several alternate quantum descriptions which arise from the fact that Z is quantized.

C. Quantum light cone cuts and associated geometric quantities

Consider Eqs. (3). At the classical level, they define four geometric quantities associated with null surfaces (see Sec. II). In the linearization, they are explicitly given by

$$u = Z_0 + Z_1 = x^a \ell_a + Z_1(x^a,\zeta,[\sigma]), \tag{20}$$

$$\omega = \eth Z_0 + \eth Z_1 = x^a m_a + \eth Z_1(x^a,\zeta,[\sigma]), \tag{21}$$

$$\overline{\omega} = \bar{\eth} Z_0 + \bar{\eth} Z_1 = x^a \overline{m_a} + \bar{\eth} Z_1(x^a,\zeta,[\sigma]), \tag{22}$$

$$R = \eth\bar{\eth} Z_0 + \eth\bar{\eth} Z_1 = x^a(n_a - \ell_a) + \eth\bar{\eth} Z_1(x^a,\zeta,[\sigma]), \tag{23}$$

where $m_a \equiv \eth \ell_a$, $\overline{m_a} \equiv \bar{\eth} l_a$, and $n_a \equiv \eth\bar{\eth}\ell_a + \ell_a$, or

$$\theta^i = x^a \lambda^i_a(\zeta) + \theta^i_1(x^a,\zeta,[\sigma]), \tag{24}$$

where $\theta^i_1(x^a,\zeta,[\sigma]) \equiv (Z_1, \bar{\eth} Z_1, \eth Z_1, \eth\bar{\eth} Z_1)$ and $\lambda^i_a(\zeta) \equiv (\ell_a, m_a, \overline{m_a}, n_a - \ell_a)$. For future reference we recall that the four vectors $\ell_a, m_a, \overline{m_a}$, and n_a satisfy $\ell_a n^a = -m_a \overline{m^a} = 1$, while the remaining scalar products among any two of them are zero. Furthermore, $n^a + \ell^a = \sqrt{2}\,\delta^a_0$.

We now define a set of quantum operators:

$$\hat{\theta}^i(x^a,\zeta) \equiv \theta^i(x^a,\zeta,[\hat{\sigma}]). \tag{25}$$

Explicit expressions of these in terms of the data can be obtained from Eqs. (24) and (14): namely,

$$\hat{u} \equiv x^a \ell_a(\zeta) \quad \hat{1} + Z_1(x^a,\zeta,[\hat{\sigma}]), \tag{26}$$

$$\hat{\omega} \equiv x^a m_a(\zeta) \quad \hat{1} + \eth Z_1(x^a,\zeta,[\hat{\sigma}]), \tag{27}$$

$$\hat{\overline{\omega}} \equiv x^a \overline{m_a}(\zeta) \quad \hat{1} + \bar{\eth} Z_1(x^a,\zeta,[\hat{\sigma}]), \tag{28}$$

$$\hat{R} \equiv x^a[n_a(\zeta) - \ell_a(\zeta)] \quad \hat{1} + \eth\bar{\eth} Z_1(x^a,\zeta,[\hat{\sigma}]). \tag{29}$$

They are manifestly linear in $\hat{\sigma}$.

$\hat{\theta}^i(x^a,\zeta)$ constitute a set of four quantum operators depending on (x^a,ζ). Therefore, in this picture, the interior points x^a are considered as c numbers, whereas $\hat{\theta}^i$, the geometric structures at $\mathcal{I}^+$, are quantum variables, subject to possible fluctuations.

The commutator $[\hat{u},\hat{u}'] \equiv [\hat{u}(x^a,\zeta),\hat{u}(x'^a,\zeta')]$ is simply $[\hat{Z},\hat{Z}']$, obtained earlier; i.e., Eq. (18). The other commutators $[\hat{\theta}^i,\hat{\theta}'^j] \equiv [\hat{\theta}^i(x^a,\zeta),\hat{\theta}^j(x'^a,\zeta')]$ can be obtained by differentiation of $[\hat{Z},\hat{Z}']$:

$$[\hat{u},\hat{u}'] = [\hat{Z},\hat{Z}'],$$

$$[\hat{u},\hat{\omega}'] = \eth'[\hat{Z},\hat{Z}'],$$

$$[\hat{u},\hat{\overline{\omega'}}] = \bar{\eth}'[\hat{Z},\hat{Z}'],$$

$$[\hat{u},\hat{R}'] = \eth'\bar{\eth}'[\hat{Z},\hat{Z}'],$$

$$[\hat{\omega},\hat{\omega}'] = \eth\eth'[\hat{Z},\hat{Z}'],$$

$$[\hat{\omega},\hat{\overline{\omega'}}] = \eth\bar{\eth}'[\hat{Z},\hat{Z}'],$$

$$[\hat{\omega},\hat{R}'] = \eth\eth'\bar{\eth}'[\hat{Z},\hat{Z}'],$$

$$[\hat{\overline{\omega}},\hat{\overline{\omega'}}] = \bar{\eth}\bar{\eth}'[\hat{Z},\hat{Z}'],$$

$$[\hat{\overline{\omega}},\hat{R}'] = \bar{\eth}\eth'\bar{\eth}'[\hat{Z},\hat{Z}'],$$

$$[\hat{R},\hat{R}'] = \eth\bar{\eth}\eth'\bar{\eth}'[\hat{Z},\hat{Z}']. \tag{30}$$

It can be inferred from Eq. (18) that these commutators are, generically, nonvanishing functions of x^a, x'^a, ζ, and ζ' (the closed forms are lengthy and complicated). The immediate consequence of the nonvanishing of the commutators is that the four geometric operators θ^i do not have a complete set of common eigenstates. Furthermore, since a generic state is not an eigenstate of any of the four operators, in a generic state, all four geometric quantities will fail to have well-defined values. In this sense, the light cone cut (u), its curvature (R), and the angle of emittance (ω) of the null geodesics at $\mathcal{I}^+$ are "fuzzy."

D. Quantum spacetime points

We now consider the "dual" picture, which arises from the inversion (5). Classically, the x^a represent an interior spacetime point which can be reached from $\mathcal{I}^+$ by specifying the values of (i) the observation point (u,ζ) at $\mathcal{I}^+$, (ii) the angle ω of the null geodesic emitted inwardly from (u,ζ), aimed at x^a, and (iii) the focusing distance R along the null geodesic (u,ζ,ω) at which the point x^a is located. The linearized version of Eq. (5) can be obtained from Eq. (24) in the form

$$x^a(\theta^k,\zeta,[\sigma]) = \lambda^a_i(\zeta)\,\theta^i - \lambda^a_i(\zeta)\,\theta^i_1(\lambda^a_j(\zeta)\,\theta^j,\zeta,[\sigma]), \tag{31}$$

where by $\lambda^a_i(\zeta)$ we denote the inverse matrix to $\lambda^i_a(\zeta)$, namely, $\lambda^a_j(\zeta)\lambda^i_a(\zeta) = \delta^i_j$, explicitly given by

$$\lambda^a_i(\zeta) = (\lambda^a_0, \lambda^a_+, \lambda^a_-, \lambda^a_1) = (n^a + \ell^a, -\overline{m^a}, -m^a, \ell^a). \tag{32}$$

We now define the operators associated with the spacetime points x^a as

$$\hat{x}^a(\theta^i,\zeta) \equiv x^a(\theta^i,\zeta,[\hat{\sigma}])$$
$$= \lambda^a_i(\zeta)\,\theta^i \hat{1} - \lambda^a_i(\zeta)\,\theta^i_1(\lambda^a_j(\zeta)\,\theta^j,\zeta,[\hat{\sigma}]) \tag{33}$$

and obtain a quantized description of the interior spacetime points x^a. Now the surface quantities θ^i remain c numbers. $\hat{x}^a(\theta^i,\zeta)$ constitute a set of four operators dependent on the six parameters (θ^i,ζ).

Since the spacetime-point operators $\hat{x}^a$ are functions of the fundamental operators $\hat{\sigma}$, they also are subject to commutation relations $[\hat{x}^a,\hat{x}'^b]\equiv[\hat{x}^a(\theta^i,\zeta),\hat{x}^b(\theta'^i,\zeta')]$ which can be derived from $[\hat{\sigma},\widehat{\overline{\sigma'}}]$. The commutators are

$$[\hat{x}^a,\hat{x}'^b]=\lambda_i^a\lambda_j'^b[\theta_1^i(\lambda_k^c\theta^k,\zeta,[\hat{\sigma}]),\theta_1^j(\lambda_k'^c\theta'^k,\zeta',[\hat{\sigma}'])], \tag{34}$$

where the commutators $[\theta_1^i(\lambda_k^c\theta^k,\zeta,[\hat{\sigma}]),\theta_1^j(\lambda_k'^c\theta'^k,\zeta',[\hat{\sigma}'])]$ are found from Eq. (18) by using Eq. (30). Explicitly,

$$\begin{aligned}[\hat{x}^a,\hat{x}'^b]=&\,2\delta_0^a\delta_0^b[\hat{Z},\hat{Z}']+\sqrt{2}\delta_0^a\ell'^b\eth'\bar{\eth}'[\hat{Z},\hat{Z}']-\sqrt{2}\delta_0^a\overline{m'}^b\eth'[\hat{Z},\hat{Z}']-\sqrt{2}\delta_0^a m'^b\bar{\eth}'[\hat{Z},\hat{Z}']+\sqrt{2}\ell^a\delta_0^b\eth\bar{\eth}[\hat{Z},\hat{Z}']\\
&+\ell^a\ell'^b\eth\bar{\eth}\eth'\bar{\eth}'[\hat{Z},\hat{Z}']-\ell^a\overline{m'}^b\eth\bar{\eth}\eth'[\hat{Z},\hat{Z}']-\ell^a m'^b\eth\bar{\eth}\bar{\eth}'[\hat{Z},\hat{Z}']-\sqrt{2}\overline{m^a}\delta_0^b\eth[\hat{Z},\hat{Z}']-\overline{m^a}\ell'^b\eth\eth'\bar{\eth}'[\hat{Z},\hat{Z}']\\
&+\overline{m^a}\overline{m'}^b\eth\eth'[\hat{Z},\hat{Z}']+\overline{m^a}m'^b\eth\bar{\eth}'[\hat{Z},\hat{Z}']-\sqrt{2}m^a\delta_0^b\bar{\eth}[\hat{Z},\hat{Z}']-m^a\ell'^b\bar{\eth}\eth'\bar{\eth}'[\hat{Z},\hat{Z}']+m^a\overline{m'}^b\bar{\eth}\eth'[\hat{Z},\hat{Z}']\\
&+m^a m'^b\bar{\eth}\bar{\eth}'[\hat{Z},\hat{Z}'],\end{aligned} \tag{35}$$

where $[\hat{Z},\hat{Z}']$ is given by Eq. (18) evaluated at $x^a=\lambda_k^a\theta^k=u(n^a+\ell^a)+R\ell^a-\omega\overline{m^a}-\bar{\omega}m^a$ and $x'^a=\lambda_k'^a\theta^k=u'(n'^a+\ell'^a)+R'\ell'^a-\omega'\overline{m'^a}-\overline{\omega'}m'^a$. We have thus obtained nontrivial commutators for operators which correspond to the coordinates of spacetime points. A series of conceptual issues arise from the existence of the nontrivial commutators. In this quantum picture, we would like to define the notion of spacetime point. Classically, a spacetime point can be specified by giving a 4-tuple of numbers, the values of the coordinates x^a on a four manifold. In the quantum description, however, an operator $\hat{x}^a$ (fixed a) takes a well-defined value only when acting on an eigenstate and a set of operators (all $\hat{x}^a$) have a complete set of simultaneous eigenstates if and only if all pairs mutually commute. Let us explore what kind of an analogue of a spacetime point we can construct.

Let us fix the values of the classical parameters $(\theta^i;\zeta)$. Classically, these define the spacetime point whose coordinates are $x_{\rm cl}^a=x^a(\theta^i;\zeta)$ [see Eq. (5)]. An important question at this juncture is whether the set of four operators $\hat{x}^a(\theta^i;\zeta)$ form a commuting set. It can be checked by inspection, setting $\theta'^k=\theta^k$ and $\zeta'=\zeta$ in Eq. (35), that all four operators $\hat{x}^a(\theta^i;\zeta)$ do commute with one another, as a consequence of the vanishing of $[\hat{Z},\hat{Z}]$ and all its ζ derivatives [see the discussion after Eq. (18)]. Therefore, we can define the quantum analogue of a spacetime point as a common eigenstate of the four coordinates $\hat{x}^a$. Let us denote this eigenstate by $|x^a_{\theta^i;\zeta}\rangle$. Now note that the eigenvalues of the operators $\hat{x}^a(\theta^i;\zeta)$, which are denoted by $x^a_{\theta^i;\zeta}$, can in general take a wide range of values and need not be equal to $x_{\rm cl}^a$. Thus, in any state of quantum gravity, there is a ''probability of finding'' the spacetime point defined by $(\theta^i;\zeta)$ at values other than the classical value $x_{\rm cl}^a=x^a(\theta^i;\zeta)$.

Next let us consider whether all spacetime points can simultaneously be assigned values. This would require that the right-hand side of Eq. (35) vanish identically. However, if $\theta'^k\neq\theta^k$ and $\zeta'\neq\zeta$ the commutator between two separate spacetime-point operators $\hat{x}^a$ and $\hat{x}'^b$ is generically nonvanishing. Thus there are no common eigenstates of all the distinct spacetime points, and as a consequence, we have no candidate for a quantum analogue of the spacetime manifold. Another way to see this is that in a common eigenstate of a particular spacetime-point set of operators, only that one point in the manifold is well defined, while the rest of the manifold becomes ''fuzzed'' out. In our second quantum picture, then, the interior spacetime is lost as a distinct classical manifold.

On the technical side, the commutators (35) display a singular behavior at the points $\zeta=\zeta'$, which makes the exploration of the ideas in the preceding paragraph a complicated task. Removal of the S^2's degrees of freedom from the commutators has been tried by means of double integration on the sphere, with the unsatisfactory result that the commutators (35) *vanish* upon integration.

IV. REMARKS

In this final section we will summarize our results and discuss their relevance to the issues of quantum spacetime.

By combining Ashtekar's asymptotic quantization of the gravitational field with the null-surface formulation of GR we have (in the linear version) constructed certain nonstandard operators on the quantum state space. The classical variables (to which these operators correspond) are not, in any conventional sense, the usual or standard field variables: they are families of point sets, specifically, families of three-dimensional surfaces. Though the surfaces are described by functions, it is the surfaces themselves which are fundamentally important, not the numerical values associated with them. Thus, it is not important if the functions that describe the surfaces are ''large,'' or ''small,'' or even whether they ''vanish.'' From knowledge of these surfaces, all null geodesics, light cones, and the conformal structure of a spacetime can be constructed. By analyzing the intersections of these surfaces one could even ''pick out'' or choose spacetime points [4]. It is possible to even think of these surfaces as being the primitive elements of the theory with the spacetime points and light cones as derived concepts. One thus sees that the associated operators are not, in any obvious fashion, standard field operators. Instead, we have operators that correspond to null surfaces, null geodesics, and field ''points.'' The novelty of this approach to quantum gravity lies in this feature. It appears to be saying that it is the spacetime itself, i.e., the manifold structure, that is undergoing the quantization process and not, as in the more standard approaches, some metric or connection field.

More specifically, the first and most important of our operators is $\hat{Z}(x^a,\zeta)$, defined in Eq. (17). The classical analogue $Z(x^a,\zeta)$ determines the characteristic surfaces in the NSF. In the ''presumed'' quantum theory, only the average position of the surfaces (whatever interpretation one might give to that) is determined for any given quantum state, by the expectation value of the operator. The ''observed'' position can be predicted only probabilistically.

The other operators of the set $\hat{\theta}^i$, i.e., Eqs. (27)–(29), for asymptotically flat spaces, correspond to simple classical geometric objects, angles at $\mathcal{I}^+$ labeling null geodesics (directions of sight) and curvatures of light cone cuts (focus distances) at $\mathcal{I}^+$. Once again, as quantum operators they are nonconventional; nevertheless, ''observed'' values are probabilistically determined.

The third, and perhaps most interesting, family of operators is given by the ''spacetime-point'' operators $\hat{x}^a(\theta^i;\zeta)$, defined in Eq. (33). Let us discuss an aspect of their classical physical meaning. In order to fix ideas physically, imagine that we wish to describe a gravitational phenomenon localized in a certain spacetime region $\mathcal{R}$, which we consider to be small. Consider the classical quantities $x^a(\theta^i;\zeta) = x^a(u,\omega,R;\zeta;[\text{data}])$. The three independent variables u,ζ determine a point on future (null) infinity $\mathcal{I}^+$. Recall that ζ coordinates the celestial sphere, and u the Bondi time. One may think of u,ζ as labeling asymptotic observers. Imagine that these observers look into the region $\mathcal{R}$. Each of them can vary the direction of sight, labeled by the independent variable ω. Finally, using a focusing distance labeled by the variable R, each of them can determine the distance to a point in $\mathcal{R}$. Thus, the set $(u,\omega,R;\zeta)$ determines the locations of observers and the direction of sight and focus distance of their observations, looking into $\mathcal{R}$ from a surrounding region. Now, since the trajectories of light rays are determined by the gravitational field, the actual point x^a seen by the observer at (u,ζ) looking at a distance R in the direction ω depends on the gravitational field. For a given spacetime, the quantities $x^a(u,\omega,R;\zeta;[\text{data}])$ determine this point.

It is a rather remarkable fact that these quantities, $x^a = x^a(u,\omega,R;\zeta;[\text{data}])$, specify the conformal spacetime geometry uniquely. Let us describe them in slightly more detail before returning to the quantum case. Consider the six-dimensional ''observation space'' defined by the three coordinates (u,ζ) of an observer's position on $\mathcal{I}^+$, the two angles of observation, ω, and the focus distance R. On this observation space consider a four-parameter family of two-dimensional surfaces, topologically S^2, each two-surface will be referred to as a leaf and the leaves foliate the observation space. Our equations $x^a = x^a(u,\omega,R;\zeta)$ are precisely of this form, i.e., each spacetime point x^a is equivalent to a leaf. [Notice furthermore, that it is the family of leaves that defines the spacetime points geometrically even if we change the gauge arbitrarily to $y^a = f^a(x^b)$.] Physically, this amounts to saying that a spacetime point can be viewed as the collection of points in observation space, i.e., locations, directions of sight, focus-distances, from which surrounding observers see it. Remarkably, this foliation by the equivalence classes of points in the observation space that ''see'' the same spacetime point is equivalent to giving the conformal pseudo-Riemannian geometry [8].

In the quantum domain, it is worth asking what validity this picture might have even when the spacetime geometry undergoes ''quantum fluctuations.'' The equations that define the leaves become operator equations, i.e., $\hat{x}^a = x^a(u,\omega,R;\zeta;[\widehat{\text{data}}])$. Now imagine that we are in the realm of quantum gravity. Then it is difficult to imagine how we could identify points physically inside $\mathcal{R}$. However, the construction partially survives. The ''observation space'' remains classical and hence we still have a family of observers surrounding $\mathcal{R}$ and looking in; specifically, the observers' locations, their directions of sight, and focus distances are still labeled by the classical parameters $(u,\omega,R;\zeta)$. What changes is that for a fixed quantum state, we will not have a sharply defined value for the operator $\hat{x}^a$ (the leaf)—except when it is in an eigenstate—but only a probability distribution of values. We are thus led to associate a ''fuzzy'' nature to quantum spacetime points by this asymptotic construction. Note thus that the question of whether two observations $(u,\omega,R;\zeta)$ and $(u',\omega',R';\zeta')$ ''see'' the same point can only be determined probabilistically.

As we just mentioned, there are equivalence classes (topological two-spheres) of observation points, i.e., points in the six-dimensional observation space, which correspond to the same *spacetime* point. In the quantum theory, we could raise the following question: Are there sets of observation points which are equivalent in the above sense, i.e., define the ''same'' $\hat{x}^a$? While we have no conclusive answers yet, there are possible directions in which to explore this question. For example, we could consider a collection of observation points to be ''equivalent'' if the corresponding spacetime-point operators mutually commute. Weaker alternatives would be to look for sets of $(u,\omega,R;\zeta)$ such that the $\hat{x}^a = x^a(u,\omega,R;\zeta;[\widehat{\text{data}}])$ possess *some* common eigenstates with the same eigenvalues, or the same expectation values in some quantum states. These are only some of the questions that remain to be thought about and explored.

Finally, the algebraic structure of the ''quantum spacetime'' defined in this way is characterized by the commutation relations between the spacetime-point operators. These are given in Eq. (35). We suspect that some relevant physical or mathematical result is hidden in these relations; but we have not been able, so far, to get to a fully convincing understanding of them. Two ideas may be relevant in this context. First, as the classical dynamics of a particle is fully determined by its gravitational interactions, one is tempted to speculate that its quantum properties can be derived from quantum geometry as well and, therefore, might be hidden in Eq. (35). Second, the commutation relations (35) could be relevant to the present efforts towards understanding quantum spacetime in terms of noncommutative geometry [13]. In that context, the commutative algebra of smooth functions over the manifolds is replaced by some noncommutative algebra, but it is difficult to find guidelines for guessing this noncommutative algebra. The commutation relations (35) define a noncommutative algebra that, *if* the Planck constant goes to zero, is equivalent to the commutative algebra of smooth functions over the manifold. Notice that this noncommutative algebraic structure is not assumed here, rather, it is *derived* from quantum general relativity. We leave the analysis of these suggestions for future investigations.

Notice that the picture of quantum gravity presented here is very far from conventional local quantum field theory where one assumes that physical points and the spacetime manifold are well defined to start with. It is, therefore, also very far from any approach to quantum gravity based on conventional quantum field theoretical ideas.

ACKNOWLEDGMENTS

We would like to thank Abhay Ashtekar for helpful discussions. This research was supported in part by the National Science Foundation under Grant Nos. PHY 89-04035 and PHY 92-05109.

APPENDIX A: ASHTEKAR'S ASYMPTOTIC QUANTIZATION OF THE FREE MAXWELL FIELD AND APPLICATIONS

In the main text, we are interested in the asymptotic quantization of linearized GR. Since the asymptotic phase spaces of GR and the free Maxwell theory are very similar, in this appendix we describe the asymptotic quantization of the free Maxwell field. The quantization follows very closely the usual construction of the Maxwell Fock space for initial data on a Cauchy surface [14]. Our aim here is to derive the standard covariant commutation relations between the Maxwell tensor in the interior at two different spacetime points, from the commutation relations on the asymptotic fields, which themselves are represented on a Hilbert space. Our description of linearized GR in the main text is completely analogous to this. (In fact, in the absence of IR sectors, we simply make the substitution $A \leftrightarrow \sigma$ and $E \leftrightarrow N$.)

The material in the following Secs. **1** and **2** is quite well known and is simply Ashtekar's asymptotic quantization of the Maxwell field and GR [6]. We present it here for the sake of completeness. We differ from [6] in one notable detail, namely, the definitions of the distributional field operators (A21) and (A22). Finally, in Secs. **3** and **4** we construct, respectively, an integral representation and then the closed form of the covariant commutation relations for the Maxwell field.

1. Phase space and algebra of observables

Let γ_a denote the connection field in the interior of Minkowski space. The Maxwell tensor is then obtained by $F_{ab}=2\nabla_{[a}\gamma_{b]}$. On $\mathcal{I}^+$, with null generators n^a, we define $A_a := \underleftarrow{\gamma_a}$ as the restriction of γ_a to $\mathcal{I}^+$, and $E_a := \partial A_a/\partial u = \pounds_n A_a$, the electric field on $\mathcal{I}^+$.

The space of solutions to Maxwell's equations is a linear phase space Γ, and we can introduce as coordinates on Γ the electric fields $E_a(u,\zeta)$ on $\mathcal{I}^+$. Note that $E_a(u,\zeta)$ is a gauge invariant quantity, and it is normal to the null generators of $\mathcal{I}^+$, namely, $E_a(u,\zeta)n^a=0$. Thus $E_a(u,\zeta)$ is completely defined by the complex scalar $E(u,\zeta)=-m^aE_a(u,\zeta)$; i.e., $E_a(u,\zeta)=E\overline{m}_a+\overline{E}m_a$.

For the purposes of easing later calculations, let us introduce some new notation [15]. Let $\alpha,\beta,\ldots$ be infinite-dimensional abstract indices on Γ which take values in the continuous set $(u,\zeta)\in\mathcal{I}^+$. Thus, $\{(\delta/\delta E(u,\zeta))^\alpha,(\delta/\delta\overline{E(u,\zeta)})^\alpha\}$ $[\{\mathbb{d}_\alpha E(u,\zeta),\mathbb{d}_\alpha\overline{E(u,\zeta)}\}]$ is a complex vector (covector) coordinate basis on Γ (since Γ is a linear space, we do not make a distinction betwen Γ and its tangent space at a point). Thus, for example, a complex scalar field on $\mathcal{I}^+$ is a vector V^α in Γ, with "components" $(V(u,\zeta),\overline{V(u,\zeta)})$. In the index notation we have introduced, a vector is represented by $V^\alpha=\int_{\mathcal{I}^+}du\ dS^2\{V(u,\zeta)(\delta/\delta E(u,\zeta))^\alpha+\overline{V(u,\zeta)}(\delta/\delta\overline{E(u,\zeta)})^\alpha\}$. We follow the abstract index "summation" convention, which, in our case, since the index takes a continuum of values, leads to an integral. The action of a covector $W_\alpha=\int_{\mathcal{I}^+}du\ dS^2[W(u,\zeta)\mathbb{d}_\alpha E(u,\zeta)+\overline{W(u,\zeta)}\mathbb{d}_\alpha\overline{E(u,\zeta)}]$ on a vector V^α is given by

$$W_\alpha V^\alpha=\int_{\mathcal{I}^+}du\ dS^2[V(u,\zeta)W(u,\zeta)+\overline{V(u,\zeta)}\,\overline{W(u,\zeta)}]. \tag{A1}$$

In this notation, the symplectic structure on the phase space [6] is given by

$$\Omega_{\alpha\beta}=\frac{1}{2\pi}\int\int_{\mathcal{I}^+}dudS^2du'\ dS^{2'}\delta^2(\zeta-\zeta')\times\Delta(u-u')\mathbb{d}_\alpha E(u,\zeta)\wedge\mathbb{d}_\beta\overline{E(u',\zeta')}, \tag{A2}$$

where $\Delta(u)=\frac{1}{2}\,\mathrm{sgn}(u)$ is the skew-symmetric antiderivative of $\delta(u)$, so that $\delta(u)=\partial\Delta(u)/\partial u$. Note that the symplectic structure is a *constant* real two-form on Γ, and its action $\Omega(V,W)$ on two vectors V^α and W^α is given by

$$\Omega_{\alpha\beta}V^\alpha W^\beta=\frac{1}{2\pi}\int\int_{\mathcal{I}^+}dudS^2du'dS^{2'}\delta^2(\zeta-\zeta')\Delta(u-u')\times[V(u,\zeta)\overline{W(u',\zeta')}-\overline{V(u',\zeta')}W(u,\zeta)]. \tag{A3}$$

There are two other naturally defined constant tensors on Γ which are useful. Since the electric fields on $\mathcal{I}^+$ are orthogonal to the null generators of $\mathcal{I}^+$, the (degenerate) metric on $\mathcal{I}^+$ defines a nondegenerate metric tensor on Γ itself:

$$\begin{aligned}Q_{\alpha\beta}&=\int\int_{\mathcal{I}^+}du\ dS^2du'dS^{2'}\delta^2(\zeta-\zeta')\\&\quad\times\delta(u-u')[\mathbb{d}_\alpha E(u,\zeta)\mathbb{d}_\beta\overline{E(u',\zeta')}\\&\quad+\mathbb{d}_\alpha\overline{E(u,\zeta)}\mathbb{d}_\beta E(u',\zeta')]\\&=2\int_{\mathcal{I}^+}dudS^2\ \mathbb{d}_{(\alpha}E(u,\zeta)\ \mathbb{d}_{\beta)}\overline{E(u,\zeta)},\end{aligned} \tag{A4}$$

whose action on two vectors V^α and W^α is given by

$$Q_{\alpha\beta}V^\alpha W^\beta=\int_{\mathcal{I}^+}dudS^2[V(u,\zeta)\overline{W(u,\zeta)}+\overline{V(u,\zeta)}W(u,\zeta)]. \tag{A5}$$

Next, consider the linear operator corresponding to the u derivative of fields on $\mathcal{I}^+$: $\dot{V}(u,\zeta)\equiv\partial V(u,\zeta)/\partial u$. This is a (1,1) tensor, defined by

$$T^\alpha{}_\beta V^\beta:=\dot{V}^\alpha\equiv\int_{\mathcal{I}^+}dudS^2\{\dot{V}(u,\zeta)(\delta/\delta E(u,\zeta))^\alpha+\dot{\overline{V(u,\zeta)}}(\delta/\delta\overline{E(u,\zeta)})^\alpha\}. \tag{A6}$$

It is straightforward to check that the u derivative operator satisfying Eq. (A6) can be written as

$$T^{\alpha}{}_{\beta}=\int\int_{\mathcal{I}^{+}}dudS^{2}du'dS^{2'}\;\delta^{2}(\zeta-\zeta')\frac{\partial}{\partial u}\delta(u-u')\times\left[\left(\frac{\delta}{\delta E(u,\zeta)}\right)^{\alpha}\mathbb{d}_{\beta}E(u',\zeta')+\left(\frac{\delta}{\delta\overline{E}(u,\zeta)}\right)^{\alpha}\mathbb{d}_{\beta}\overline{E}(u',\zeta')\right]. \quad \text{(A7)}$$

In relation to the analogy with the linearized NSF of GR, we are interested in considering the connections as characteristic free data on $\mathcal{I}^{+}$, rather than the electric fields. The connections are now determined, with respect to the electric fields, as the corresponding elements $A^{\alpha}=\int_{\mathcal{I}^{+}}dudS^{2}\{A(u,\zeta)(\delta/\delta E(u,\zeta))^{\alpha}+\overline{A(u,\zeta)}(\delta/\delta\overline{E(u,\zeta)})^{\alpha}\}$ of Γ such that

$$E^{\alpha}=T^{\alpha}{}_{\beta}A^{\beta}. \quad \text{(A8)}$$

Defined in this way, the connections are completely determined by a single complex scalar field $A(u,\zeta)$. This single complex scalar is related to the standard real A_{a} (introduced earlier) by $A=-m^{a}A_{a}$ and represents the two degrees of freedom of the Maxwell fields. In order to stay away from infrared sectors, the remaining component of A_{a} is chosen to vanish, namely, $A_{a}n^{a}=0$ (in this gauge, the Maxwell connection is equivalently represented by either A_{a} or A). Note that $T^{\alpha}{}_{\beta}$ is degenerate, since it annihilates fields which do not depend on u; thus, it has no unique inverse. However, the ambiguity in defining A^{α} by Eq. (A8) is precisely the remaining gauge freedom, that of an additive field which depends only on ζ.

The three tensors $\Omega_{\alpha\beta},Q_{\alpha\beta},T^{\alpha}{}_{\beta}$ on Γ are not all independent. In order to derive the relation between them, note first that the inverse $\Omega^{\alpha\beta}$ of the symplectic structure, defined by $\Omega^{\alpha\beta}\Omega_{\beta\gamma}=\mathbb{1}^{\alpha}{}_{\gamma}$, is given by

$$\Omega^{\alpha\beta}=4\pi\int\int_{\mathcal{I}^{+}}dudS^{2}du'dS^{2'}\delta^{2}(\zeta-\zeta')\frac{\partial}{\partial u}\delta(u-u')\times\left(\frac{\delta}{\delta E(u,\zeta)}\right)^{[\alpha}\left(\frac{\delta}{\delta\overline{E}(u',\zeta')}\right)^{\beta]} \quad \text{(A9)}$$

and that the inverse of the metric (A4) is given by

$$Q^{\alpha\beta}=2\int_{\mathcal{I}^{+}}dudS^{2}\left(\frac{\delta}{\delta E(u,\zeta)}\right)^{(\alpha}\left(\frac{\delta}{\delta\overline{E}(u,\zeta)}\right)^{\beta)}. \quad \text{(A10)}$$

Now, combining Eqs. (A7), (A9), and (A10), a short calculation shows that

$$\Omega^{\alpha\beta}=2\pi T^{\alpha}{}_{\gamma}Q^{\beta\gamma}. \quad \text{(A11)}$$

This relationship will be useful later for defining distributional operators corresponding to the connections.

We now want to construct the Poisson bracket algebra of elementary functions on the phase space, which are to be represented in the quantum theory by quantum operators. Since the phase space is a linear space, it will be most convenient to consider the space of all (sufficiently smooth) linear functions on Γ, together with the constant functions. This space can be parametrized in the following manner. Let $S\subset\Gamma$ be the space of complex covector test fields on $\mathcal{I}^{+}$. Let $V^{\alpha}\in S$, and define a function $\mathcal{F}_{V}$ on Γ, whose value, evaluated at the point $E^{\alpha}\in\Gamma$, is given by

$$\mathcal{F}_{V}[E]:=\Omega_{\alpha\beta}E^{\alpha}V^{\beta}. \quad \text{(A12)}$$

This is a linear function on Γ. Its gradient is given by $\nabla_{\alpha}\mathcal{F}_{V}=\Omega_{\alpha\beta}V^{\beta}$. The Poisson brackets between any two such functions is

$$\{\mathcal{F}_{V}[E],\mathcal{F}_{W}[E]\}\equiv\Omega^{\alpha\beta}\nabla_{\alpha}\mathcal{F}_{V}\nabla_{\beta}\mathcal{F}_{W}=-\Omega_{\alpha\beta}V^{\alpha}W^{\beta}, \quad \text{(A13)}$$

where $\Omega^{\alpha\beta}$ is the inverse of the symplectic structure, defined in Eq. (A9). Since the function on the right-hand side of Eq. (A13) is independent of E^{α}, the algebra is closed under Poisson brackets. This defines the algebra of elementary classical functions.

From the linear functions (A12), the classical distributional electric fields can be obtained via

$$E^{\alpha}=-\Omega^{\alpha\beta}\frac{\delta}{\delta V^{\beta}}\mathcal{F}_{V}[E]=-2\pi T^{\alpha}{}_{\gamma}Q^{\beta\gamma}\frac{\delta}{\delta V^{\beta}}\mathcal{F}_{V}[E], \quad \text{(A14)}$$

where we have used Eq. (A11). Comparing Eq. (A14) with Eq. (A8), and making the same gauge choice for the connection as before, we see that the distributional connection field is given by

$$A^{\alpha}=-2\pi Q^{\beta\alpha}\frac{\delta}{\delta V^{\beta}}\mathcal{F}_{V}[E]. \quad \text{(A15)}$$

From (A13) and the definition (A15) of the classical distributional connection field on $\mathcal{I}^{+}$, we can obtain the fundamental Poisson brackets between two connections:

$$\begin{aligned}\{A^{\alpha},A^{\beta}\}&=4\pi^{2}Q^{\gamma\alpha}Q^{\delta\beta}\frac{\delta}{\delta V^{\gamma}}\frac{\delta}{\delta W^{\delta}}\{\mathcal{F}_{V}[E],\mathcal{F}_{W}[E]\}\\&=4\pi^{2}Q^{\gamma\alpha}Q^{\delta\beta}\Omega_{\delta\gamma}\\&=-4\pi\int\int_{\mathcal{I}^{+}}dudS^{2}du'dS^{2'}\delta^{2}(\zeta-\zeta')\\&\quad\times\Delta(u-u')\left(\frac{\delta}{\delta E(u,\zeta)}\right)^{[\alpha}\left(\frac{\delta}{\delta\overline{E}(u',\zeta')}\right)^{\beta]}.\end{aligned} \quad \text{(A16)}$$

On the other hand, in terms of components we have

$$\{A^\alpha,A^\beta\}=\int\int_{\mathcal{I}^+}du dS^2 du' dS^{2'}\{A(u,\zeta),A(u',\zeta')\}\times\left(\frac{\delta}{\delta E(u,\zeta)}\right)^\alpha\left(\frac{\delta}{\delta E(u',\zeta')}\right)^\beta+\{\overline{A}(u,\zeta),\overline{A}(u',\zeta')\}\times\left(\frac{\delta}{\delta\overline{E}(u,\zeta)}\right)^\alpha\left(\frac{\delta}{\delta\overline{E}(u',\zeta')}\right)^\beta+2\{A(u,\zeta),\overline{A}(u',\zeta')\}\times\left(\frac{\delta}{\delta E(u,\zeta)}\right)^{[\alpha}\left(\frac{\delta}{\delta\overline{E}(u',\zeta')}\right)^{\beta]}. \quad \text{(A17)}$$

By comparing Eqs. (A16) and (A17) (or more directly), we obtain

$$\{A(u,\zeta),\overline{A}(u',\zeta')\}=-2\pi\Delta(u-u')\delta^2(\zeta-\zeta') \quad \text{(A18a)}$$

and

$$\{A(u,\zeta),A(u',\zeta')\}=\{\overline{A}(u,\zeta),\overline{A}(u',\zeta')\}=0. \quad \text{(A18b)}$$

These are the fundamental distributional Poisson brackets on the data on $\mathcal{I}^+$.

Let us summarize what we have done so far. First, we have shown that the linear space of free data of the Maxwell field can be parametrized by the characteristic data $A(u,\zeta)$ on $\mathcal{I}^+$. The data satisfy the Poisson brackets relations (A18). From the characteristic data $A(u,\zeta)$, we can obtain the Maxwell fields in the interior of the spacetime (see Sec. A 3), and their corresponding Poisson brackets. From the point of view of quantization, the characteristic data are not convenient elementary observables, since they correspond to distributions on Γ and cannot be directly represented on a Hilbert space as bounded self-adjoint operators. However, since the phase space is linear, we introduced the space of linear functionals on Γ in a particularly convenient way, as the space of smeared electric fields $\mathcal{F}_V[E]$ [Eq. (A12)]. These smeared fields satisfy the elementary Poisson brackets relations (A13). From the smeared electric fields $\mathcal{F}_V[E]$, the characteristic data $A(u,\zeta)$ can be reobtained by the functional derivative with respect to the test fields, via Eq. (A15).

Now in the quantum theory, the elementary operator algebra that one works with corresponds to the Poisson brackets algebra of the smeared fields. Following Ashtekar [6], in the next subsection we construct a representation of this algebra on an asymptotic Fock space. We are primarily interested in the distributional connections on $\mathcal{I}^+$, and these can be obtained from the smeared electric field operators via the quantum analogue of Eq. (A15). The distributional operators corresponding to the Maxwell fields in the interior can be constructed from the distributional connections by analogy with the classical construction (Sec. A 3).

Hence, to begin with, let us construct the algebra of elementary operators which we wish to represent in the quantum theory. We want to construct operators $\hat{E}(V)$ corresponding to the classical functions $\mathcal{F}_V[E]$. These smeared field operators are defined to satisfy the standard commutation relations corresponding to the Poisson brackets (A13)

$$[\hat{E}(V),\hat{E}(W)]=i\hbar\widehat{\{\mathcal{F}_V[E],\mathcal{F}_W[E]\}}=-i\hbar\Omega_{\alpha\beta}V^\alpha W^\beta\hat{1}. \quad \text{(A19)}$$

As we noted, we are primarily interested in operator-valued distributions corresponding to the electric fields and the connections at a point on $\mathcal{I}^+$. Thus, in analogy with the classical fields [see Eq. (A14)], let us define an operator-valued distributional electric field $\hat{E}^\alpha\equiv(\hat{E}(u,\zeta),\widehat{\overline{E}}(u,\zeta))$ by

$$\hat{E}^\alpha=-\Omega^{\alpha\beta}\frac{\delta}{\delta V^\beta}\hat{E}(V). \quad \text{(A20)}$$

By contracting Eq. (A20) with $\Omega_{\gamma\alpha}V^\gamma$, one can see that the smeared field operators are obtained from the distributional operators in the same manner as the linear functions are smeared with the test fields:

$$\hat{E}(V)=:\Omega_{\alpha\beta}\hat{E}^\alpha V^\beta. \quad \text{(A21)}$$

[Compare Eq. (A21) with Eq. (A12).]

Similarly, in analogy with Eq. (A15), we define an operator-valued distribution corresponding to the connection fields as follows:

$$\hat{A}^\alpha=-2\pi Q^{\beta\alpha}\frac{\delta}{\delta V^\beta}\hat{E}(V). \quad \text{(A22)}$$

Using this definition and the commutator (A19), we compute the commutator between the connection operators

$$[\hat{A}^\alpha,\hat{A}^\beta]=4\pi^2\, Q^{\gamma\alpha}Q^{\delta\beta}\frac{\delta}{\delta V^\gamma}\frac{\delta}{\delta W^\delta}[\hat{E}(V),\hat{E}(W)]=4\pi^2 i\hbar\hat{1}Q^{\gamma\alpha}Q^{\delta\beta}\Omega_{\delta\gamma}. \quad \text{(A23)}$$

Evaluating the components of $Q^{\gamma\alpha}Q^{\delta\beta}\Omega_{\delta\gamma}$, as in the classical case, finally leads to

$$[\hat{A}(u,\zeta),\widehat{\overline{A}}(u',\zeta')]=-2\pi i\hbar\Delta(u-u')\delta^2(\zeta-\zeta')\hat{1} \quad \text{(A24a)}$$

and

$$[\hat{A}(u,\zeta),\hat{A}(u',\zeta')]=[\widehat{\overline{A}}(u,\zeta),\widehat{\overline{A}}(u',\zeta')]=0. \quad \text{(A24b)}$$

These are the fundamental free-data commutators [16], on which the commutators for the interior fields are based. They will be of use in Sec. A 4. In the following subsection we describe the space of states on which the operators act.

2. Asymptotic Fock space

We are going to construct the (standard) antiholomorphic representation [6] of the free data on $\mathcal{I}^+$ for the free Maxwell field. Let $S\subset\Gamma$ be the Schwartz space of complex spin-1 test fields on $\mathcal{I}^+$. For any $V(u,\zeta)\in S$, define the Fourier transform

$$\mathcal{V}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}du V(u)e^{i\omega u}, \tag{A25}$$

and the positive frequency part of V

$$^{+}V(u)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}d\omega\mathcal{V}(\omega)e^{-i\omega u}, \tag{A26}$$

where the dependence on $(\zeta,\overline{\zeta})$ is understood. On S (or S^{+}), the symplectic structure (A2) can be expressed as

$$\Omega_{\alpha\beta}=\frac{1}{2\pi}\int d^2S\int_{-\infty}^{\infty}d\omega\frac{1}{i\omega}\mathrm{d}_{\alpha}\mathcal{E}(\omega,\zeta)\wedge\mathrm{d}_{\beta}\overline{\mathcal{E}(}-\omega,\zeta). \tag{A27}$$

On S^{+}, define a Hermitian inner product

$$\langle\ ^{+}V\mid{}^{+}W\ \rangle:=-\frac{i}{\hbar}\Omega(\overline{^{+}V},{}^{+}W)$$
$$=\frac{1}{2\pi\hbar}\int d^2S\int_{0}^{\infty}\frac{d\omega}{\omega}\overline{\mathcal{V}(\omega,\zeta)}\mathcal{W}(\omega,\zeta). \tag{A28}$$

By inspection, this is positive definite. Note that it can be written in the more familiar form [17]:

$$\langle\ ^{+}V\mid{}^{+}W\ \rangle:=\mu(V,W)-\frac{i}{2\hbar}\Omega(V,W), \tag{A29}$$

where $\mu(V,W)=(1/\hbar)Im\ \Omega(\overline{^{+}V},{}^{+}W)$ is a real inner product on S.

Let us take the Cauchy completion of S^{+} under this inner product; denote it $\mathcal{H}=\overline{S^{+}}$. As we will see, $\mathcal{H}$ is the one-particle Hilbert space. The inner product (A28) defines a Hermitian metric on $\mathcal{H}$:

$$G_{\overline{\alpha}\alpha}\overline{^{+}V}^{\overline{\alpha}+}W^{\alpha}:=\langle\ ^{+}V\mid{}^{+}W\ \rangle. \tag{A30}$$

The introduction of this metric will be useful in what follows.

Consider the space $\mathcal{F}=\oplus_{n=1}^{\infty}\otimes_S\mathcal{H}^n\oplus\mathbb{C}$, where $\otimes_S$ stands for the symmetric tensor product. This consists of kets of the form $|\mathbf{T}\ \rangle$:

$$|\mathbf{T}\ \rangle=|\ T_0,T_1^{\alpha_1},\ldots,T_n^{\alpha_1\cdots\alpha_n},\ldots\ \rangle, \tag{A31}$$

where $T_0\in\mathbb{C}$ and $T_n^{\alpha_1\cdots\alpha_n}=T_n^{(\alpha_1\cdots\alpha_n)}\in\otimes_S\mathcal{H}^n=\mathcal{H}\otimes_S\cdots(n\ \text{times})\otimes_S\mathcal{H}$ is an element of the symmetric tensor product of n copies of $\mathcal{H}$.[2]

On this space of states $\mathcal{F}$ there is the inner product obtained by extending the inner product on $\mathcal{H}$:

$$\langle\mathbf{T}|\mathbf{W}\rangle=\overline{\mathbf{T}}_0\mathbf{W}_0+\sum_{\mathbf{n=1}}^{\infty}G_{\overline{\alpha}_1\alpha_1}\cdots G_{\overline{\alpha}_n\alpha_n}\overline{\mathbf{T}}_{\mathbf{n}}^{\overline{\alpha}_1\cdots\overline{\alpha}_n}\mathbf{W}_{\mathbf{n}}^{\alpha_1\cdots\alpha_n}. \tag{A33}$$

The Cauchy completion of $\mathcal{F}$ defines the desired asymptotic Fock space.

Now that we have the space of states, let us define the creation and annihilation operators $\hat{c}(^{+}V)$ and $\hat{a}(^{+}V)$, respectively. Given an element $^{+}V\in S^{+}$, define

$$\hat{c}(^{+}V)\circ|\ \mathbf{T}\ \rangle:=|\ 0,{}^{+}V^{\alpha}T_0,\ldots,\sqrt{n+1}\,{}^{+}V^{(\alpha}T_n^{\alpha_1\cdots\alpha_n)},\ldots \tag{A34}$$

and

$$\hat{a}(^{+}V)\circ|\ \mathbf{T}\ \rangle:$$
$$=|\ G_{\overline{\alpha}\alpha}\overline{^{+}V}^{\overline{\alpha}}T_1^{\alpha},\ldots,\sqrt{n}G_{\overline{\alpha}\alpha_n}\overline{^{+}V}^{\overline{\alpha}}T_n^{\alpha_1\cdots\alpha_n},\ldots\ \rangle. \tag{A35}$$

Using Eq. (A30), a straightforward calculation shows that these operators satisfy the commutation relations

$$[\hat{a}(^{+}V_1),\hat{c}(^{+}V_2)]=\langle\ ^{+}V_1\mid{}^{+}V_2\ \rangle\hat{1}, \tag{A36}$$

all other commutators vanishing. One can show that these operators are Hermitian adjoints of each other, i.e.,

$$\hat{a}^{\dagger}(^{+}V)=\hat{c}(^{+}V). \tag{A37}$$

In this representation, let us define the smeared electric field operators

$$\hat{E}(V):=\hbar[\hat{c}(^{+}V)+\hat{a}(^{+}V)]. \tag{A38}$$

From the commutator (A36), we easily see that

$$[\hat{E}(V),\hat{E}(W)]=2i\hbar^2\mathrm{Im}\langle\ ^{+}V\mid{}^{+}W\ \rangle\hat{1}, \tag{A39}$$

where we have used the form of the inner product (A29). It follows that the operators we have defined above in Eq. (A38) satisfy the desired commutation relations (A19). Furthermore, from the Hermiticity relations (A37) between the creation and annihilation operators, we see that the electric field operators (A38) are themselves Hermitian.

Thus, we have constructed a Hermitian representation of the smeared electric field operators defined in Appendix A 1. From these, via Eq. (A22) we can obtain the distributional connection operators $\hat{A}(u,\zeta)$, which satisfy the commutation relations (A24). Recall that the connections $A(u,\zeta)$ on $\mathcal{I}^{+}$ serve as data for the Maxwell fields in the interior of the spacetime. In the next subsection, we will use the commutation relations (A24) between the distributional connection field operators on $\mathcal{I}^{+}$ to compute the commutation relations between the field operators in the interior of the spacetime.

3. Integral representations of the covariant commutation relations

The fields in the interior of the spacetime can be reconstructed from knowledge of the fields at $\mathcal{I}^{+}$. The following is

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[2]The antiholomorphic representation can be easily constructed. For example, the one-particle state is represented by

$$\psi_{T_1}[Z]:=\langle\ Z\mid T\ \rangle=G_{\overline{\alpha}\alpha}\overline{Z}^{\overline{\alpha}}T^{\alpha}, \tag{A32}$$

where $Z^{\alpha}:={}^{+}E^{\alpha}$.

a reconstruction of the Maxwell fields based on the null-surface formulation of the background Minkowski spacetime [18].

In Minkowski space, the intersection of the future light cone of an interior point x^a with $\mathcal{I}^+$ is a topological sphere $S^2(x^a)$, denoted as the light cone cut of x^a. In coordinates (u,ζ), the light cone cut of a fixed point x^a is a two-surface $u=u(x^a,\zeta)$ in $\mathcal{I}^+$ given by

$$u=Z_0(x^a,\zeta)=x^a\ell_a(\zeta)=x^a\eta_{ab}\ell^b, \tag{A40}$$

where $\ell^b(\zeta)$ is a constant null vector in Minkowski space [see (Eq. 10)]. At any fixed point, $\ell^b(\zeta)$ defines the null cone by varying ζ. Z_0 denotes the Z function for Minkowski space.

If the (otherwise free) data $A(u,\zeta)$ is restricted to the light cone cut of a particular point x^a, it defines a function of six variables denoted $A_R(x^a,\zeta)\equiv A(Z_0(x^a,\zeta),\zeta)$. By giving asymptotic data A_R on a light cone cut, the Maxwell field and connection at the interior point x^a can be found, essentially by differentiation, from the knowledge of a real nonlocal superpotential $F(x^a,\zeta)$ which satisfies the following differential equation on the sphere:

$$\eth\bar{\eth}F=\eth\overline{A_R}(x^a,\zeta)+\bar{\eth}A_R(x^a,\zeta)\equiv D_M(x^a,\zeta)[A]. \tag{A41}$$

Regular solutions to this equation from given data can be found in integral form

$$F(x^a,\zeta;\,[A])=\int_{S^2}dS^2_\eta G_M(\zeta,\eta)D_M(x^a,\eta)[A], \tag{A42}$$

where $dS^2_\eta=-2i\,d\eta\wedge d\bar{\eta}/(1+\eta\bar{\eta})^2$ is the area form for S^2, $[A]$ indicates the functional dependence of the solution on the free data, and $G_M(\zeta,\eta)$ is a known Green's function [19], given by

$$G_M(\zeta,\eta)=\frac{1}{4\pi}\ln\left(\frac{(\zeta-\eta)(\bar{\zeta}-\bar{\eta})}{(1+\zeta\bar{\zeta})(1+\eta\bar{\eta})}\right)=\frac{1}{4\pi}\ln[\ell^a(\zeta)\ell_a(\eta)]. \tag{A43}$$

Note that any function of only x^a can be added to Eq. (A42) to obtain another solution of Eq. (A41) with the same data. This gauge freedom of the solutions to Eq. (A41) is equivalent to leaving free the $l=0$ term in their spherical-harmonic expansion. The Green's function (A43) has the property that if F is given by Eq. (A42), then $\int_{S^2}F=0$, hence F has no $l=0$ term in an expansion in spherical harmonics. Equation (A42) thus gives an integral representation of the superpotential $F(x^a,\zeta)$ in the $l=0$ gauge.

In a general gauge, the Maxwell connection $\gamma_a(x^c)$ is related to $F(x^a,\zeta)$ by

$$\ell^a(\zeta)\nabla_a F(x^c,\zeta)=\ell^a(\zeta)\gamma_a(x^c). \tag{A44}$$

By differentiation of Eq. (A44) with respect to ζ and by algebraic procedures we can reconstruct $\gamma_a(x^c)$, and $F_{ab}=2\nabla_{[a}\gamma_{b]}$. Explicitly,

$$\gamma_a=\gamma_i\lambda^i_a \tag{A45}$$

and

$$F_{ab}=2\lambda^i_{[b}\nabla_{a]}\gamma_i, \tag{A46}$$

where, by definition,

$$\gamma_1\equiv\ell^a\nabla_a F, \tag{A47}$$

$$\gamma_+\equiv-m^a\nabla_a F-\ell^a\nabla_a\eth F=-\eth(\ell^a\nabla_a F), \tag{A48}$$

$$\gamma_-\equiv-\overline{m^a}\nabla_a F-\ell^a\nabla_a\bar{\eth}F=-\bar{\eth}(\ell^a\nabla_a F), \tag{A49}$$

$$\gamma_0\equiv+(n^a-\ell^a)\nabla_a F+m^a\nabla_a\bar{\eth}F+\ell^a\nabla_a\eth\bar{\eth}F=\eth\bar{\eth}(\ell^a\nabla_a F), \tag{A50}$$

and $\lambda^i_a(\zeta)\equiv(\ell_a,m_a,\overline{m_a},n_a-\ell_a)$, $i=0,+,-,1$. If $F(x^a,\zeta)$ satisfies Eq. (A41), then $F_{ab}(x^c)$ is automatically a solution to the Maxwell equations $\nabla^a F_{ab}=0$. It is worth noticing that our $l=0$ gauge implies that the connection γ_a is fixed in the Coulomb gauge, being explicitly given by

$$\gamma_a=\int_{S^2}dS^2_\eta\{\dot{\bar{A}}(x^b\ell_b(\eta),\eta)m_a(\eta)+\dot{A}(x^b\ell_b(\eta),\eta)\overline{m_a}(\eta)\}. \tag{A51}$$

We return to the gauge issue at the end of this subsection.

In the quantization, we define the operators simply by replacing the classical variables with their quantum versions, i.e.,[3]

$$\hat{F}(x^a,\zeta;\,[A])=\int_{S^2}dS^2_\eta G_M(\zeta,\eta)D_M(x^a,\eta)[\hat{A}]. \tag{A52}$$

In the following, we find integral representations of the commutation relations of $\hat{F}$, $\hat{\gamma}_a$, and $\hat{F}_{ab}$ at different values of their arguments. These follow from the fundamental commutators (A24) for the free data.

Using the notation $[\hat{F},\hat{F}']\equiv[\hat{F}(x^a,\zeta),\hat{F}(x'^a,\zeta')]$, from Eqs. (A52) and (A24) we obtain

$$\begin{aligned}[\hat{F},\hat{F}']=&-2\int\int_{S^2}[\eth_\eta G_M(\zeta,\eta)\;\bar{\eth}_{\eta'}G_M(\zeta',\eta')\\&+\bar{\eth}_\eta G_M(\zeta,\eta)\;\eth_{\eta'}G_M(\zeta',\eta')]\\&\times\{\hat{A}(x^a\ell_a(\eta),\eta),\hat{\bar{A}}(x'^a\ell_a(\eta'),\eta')\}d^2S_\eta d^2S_{\eta'}\\=&4\pi i\hbar\int_{S^2}[\eth_\eta G_M(\zeta,\eta)\bar{\eth}_\eta G_M(\zeta',\eta)\\&+\bar{\eth}_\eta G_M(\zeta,\eta)\eth_\eta G_M(\zeta',\eta)]\Delta[y\cdot\ell(\eta)]d^2S_\eta\hat{1},\end{aligned} \tag{A53}$$

[3]We assume, as appears to be done for evolution from Cauchy data, that the operators corresponding to the restrictions of the connections on $\mathcal{I}^+$ to the cuts of x^a exist on the Fock space.

where $y\cdot\ell(\eta)\equiv y^a\ell_a(\eta)$ and $y^a\equiv x^a-x'^a$. To obtain this result we used the explicit expression of the Green's function, Eq. (A43), and the method for the evaluation of integrals on the sphere described in [20]. With $m_a\equiv\eth\ell_a$ [see the definition of the null tetrad $(\ell^a,m^a,\overline{m^a},n^a)$, Eq. (23)], Eq. (A53) takes the compact form

$$[\hat{F},\hat{F}']=8\pi i\hbar\int_{S^2}\frac{\ell^a(\zeta)\ell^b(\zeta')m_{(a}\overline{m_{b)}}}{\ell(\zeta)\cdot\ell\ell(\zeta')\cdot\ell}\Delta(y\cdot\ell)d^2S\hat{1}. \tag{A54}$$

Here and in the following, we omit the explicit dependence on a dummy variable, such as the integration variable in Eq. (A54).

In order to find the commutator of $\hat{\gamma}_a$, we take two gradients $\nabla_a\nabla'_b$ in the spacetime arguments of Eq. (A54) and then contract with $\ell^a(\zeta)\ell^b(\zeta')$ since

$$\ell^a\ell'^b\nabla_a\nabla'_b[\hat{F},\hat{F}']=[\ell^a\nabla_a\hat{F},\ell'^b\nabla'_b\hat{F}']=[\ell^a\hat{\gamma}_a,\ell'^b\hat{\gamma}'_b]$$
$$=\ell^a\ell'^b[\hat{\gamma}_a,\hat{\gamma}'_b]. \tag{A55}$$

Using Eq. (A54), we have

$$\ell^a\ell'^b\nabla_a\nabla'_b[\hat{F},\hat{F}']$$
$$=-8\pi i\hbar\ell^a\ell'^b\int_{S^2}m_{(a}\overline{m_{b)}}\dot{\delta}(y\cdot\ell)d^2S\ \hat{1}, \tag{A56}$$

where we use the notation $\dot{f}(x)\equiv df(x)/dx$. Since the integral in the right-hand side of Eq. (A56) is not a function of (ζ,ζ'), then, from Eqs. (A55) and (A56), it follows that

$$[\hat{\gamma}_a,\hat{\gamma}'_b]=-8\pi i\hbar\int_{S^2}m_{(a}\overline{m_{b)}}\dot{\delta}(y\cdot\ell)d^2S\ \hat{1}. \tag{A57}$$

This is an integral representation of the commutator of $\hat{\gamma}_a$ at two different points, in the interior of the spacetime. The reason why it does not resemble the standard commutators for the Maxwell connection is that we have not made the standard gauge choice, namely, the Lorentz gauge, $\nabla^a\gamma_a=0$. Instead, by choosing the superpotential F as in Eq. (A42) we have picked the Coulomb gauge, i.e, $\nabla^a\gamma_a=0$ and $\gamma_0=0$. Interestingly, these gauge conditions are consistent with Eq. (A57). Namely, if the operators $\hat{\gamma}_a$ are constrained by $\nabla^a\hat{\gamma}_a=0$ and $\hat{\gamma}_0=0$, then it should also be true that $[\nabla^a\hat{\gamma}_a,\hat{\gamma}'_b]=0$ and $[\hat{\gamma}_0,\hat{\gamma}'_b]=0$. By taking a gradient ∇^a and observing that $m\cdot\ell=0$, it is straightforward to see that Eq. (A57) implies $[\nabla^a\hat{\gamma}_a,\hat{\gamma}'_b]=0$, whereas $[\hat{\gamma}_0,\hat{\gamma}'_b]=0$ holds trivially, since m_a has a vanishing timelike component [cf. Eq. (A86)].

4. Closed-form commutators

In this section we evaluate in closed form the integral representation of the commutator of the nonlocal potential $F(x^a,\zeta)$, Eq. (A53).

In the first place, we rewrite the integrand into two terms (by "flipping" an $\eth_\eta$ derivative from the Green's functions over to the step function Δ while keeping the so-called boundary terms):

$$(\eth_\eta G\bar{\eth}_\eta G'+\eth_\eta G'\bar{\eth}_\eta G)\Delta(y\cdot\ell)=\eth_\eta[(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G$$
$$+\overline{R}+\overline{R'})\Delta(y\cdot\ell)]-(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\overline{R}+\overline{R'})$$
$$\times y\cdot m\,\delta(y\cdot\ell), \tag{A58}$$

where $\overline{R}$ and $\overline{R'}$ are assumed to satisfy $\eth_\eta G=\overline{R}$ and $\eth_\eta G'=\overline{R'}$, respectively ($\overline{R}$ and $\overline{R'}$ are not unique). The integration variable on the sphere is η. The other parameters (y^a,ζ,ζ'), are fixed. With the integrand written in this way, the integral in Eq. (A53) splits into two terms:

$$[F,F']=-2\pi i\hbar\int_{S^2}\eth_\eta[(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\overline{R}+\overline{R'})$$
$$\times\Delta(y\cdot\hat{\ell})]d\hat{S}^2-\int_{S^2}(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\overline{R}+\overline{R'})$$
$$\times y\cdot\hat{m}\,\delta(y\cdot\hat{\ell})d^2S. \tag{A59}$$

The first term is a volume integral on the sphere which can be evaluated by a method that combines Stokes's theorem and the theorem of residues for a complex variable [20]. The second term in Eq. (A59) either vanishes (if $y\cdot\ell\neq 0$) or is a line integral, since the integrand has support only on the line defined by $y\cdot\ell=0$. These two distinctions correspond to y^a being timelike or spacelike, respectively.

For timelike future-pointing y^a the step function $\Delta(y\cdot\ell)$ takes the constant value $+1/2$. We will first evaluate Eq. (A53) in this case, and then extend the result to timelike past-pointing y^a by simply multiplying by an overall minus sign.

The commutator (A53) is reduced to

$$[F,F']=-\pi i\hbar\int_{S^2}\eth_\eta(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\overline{R}+\overline{R'})d^2S. \tag{A60}$$

From Eq. (A43) the following are obtained:

$$\overline{R(\zeta,\eta)}=\frac{1}{4\pi}\frac{\ell(\zeta)\cdot\ell(\eta)}{\ell(\zeta)\cdot m(\eta)}\{\ln[\ell(\zeta)\cdot\ell(\eta)]-1\} \tag{A61}$$

and

$$\bar{\eth}_\eta G'\equiv\bar{\eth}_\eta G(\zeta',\eta)=\frac{1}{4\pi}\frac{\ell(\zeta')\cdot\overline{m(\eta)}}{\ell(\zeta')\cdot\ell(\eta)}. \tag{A62}$$

Notice that G, $\overline{R}$, and $\bar{\eth}_\eta G'$ are singular at certain values of $(\eta,\overline{\eta})$. This implies that the integral in Eq. (A60) must be defined by a limiting process; the integral is performed on a domain $\mathcal{D}=S^2-\mathcal{B}$ that excludes small neighborhoods of the singular points, which are eventually shrunk to zero. By Stokes's theorem, the integral (A60) on $\mathcal{D}$ can be converted into contour integrations around the singular points. Furthermore, due to the theorem of residues (with an overall minus sign), the contour integrals can finally be evaluated by computing the residues at the simple poles inside $\mathcal{B}$ ($\mathcal{B}$ consists of a disjoint union of neighborhoods around singular points)

$$[F,F']=-2\pi i\hbar \oint_{\partial\mathcal{B}}(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\bar{R}+\bar{R}')\frac{-id\bar{\eta}}{1+\eta\bar{\eta}}$$
$$=8\pi^2 i\hbar\sum_k \operatorname{Res}\left(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\bar{R}+\bar{R}'\times\frac{1}{1+\eta\bar{\eta}}\right)\Bigg|_{\bar{\eta}=\bar{\eta}_k}. \tag{A63}$$

In the evaluation by residues, the variables $(\eta,\bar{\eta})$ are considered independent of each other, the singular points that affect the integration being those on the variable $\bar{\eta}$. We are thus interested in accounting for all the singular points $\bar{\eta}=\bar{\eta}_k$ which are simple poles, while the variable η is considered fixed, taking the limiting value $\eta=\eta_k$. Using the explicit expressions of the scalar products between ℓ^a and m^a [18],

$$\ell(\zeta)\cdot\ell(\eta)=\frac{(\eta-\zeta)(\bar{\eta}-\bar{\zeta})}{(1+\zeta\bar{\zeta})(1+\eta\bar{\eta})},$$
$$\ell(\zeta)\cdot m(\eta)=\frac{(\bar{\eta}-\bar{\zeta})(1+\zeta\bar{\eta})}{(1+\zeta\bar{\zeta})(1+\eta\bar{\eta})}, \tag{A64}$$

we see that the integrand in Eq. (A63) is singular at $\bar{\eta}=\bar{\zeta},\bar{\zeta}',-1/\zeta,-1/\zeta'$. These are simple poles. (The apparent pole at $\bar{\eta}=-1/\eta$ is ignored, since it does not affect the value of the integral.) A careful calculation gives only the nonzero residues

$$\operatorname{Res}\left(\frac{G\bar{\eth}_\eta G'}{1+\eta\bar{\eta}}\right)\Bigg|_{\bar{\eta}=\bar{\zeta}'}=\operatorname{Res}\left(\frac{G'\bar{\eth}_\eta G}{1+\eta\bar{\eta}}\right)\Bigg|_{\bar{\eta}=\bar{\zeta}}=\frac{1}{4\pi}\ln(\ell\cdot\ell') \tag{A65}$$

and

$$\operatorname{Res}\left(\frac{\bar{R}}{1+\eta\bar{\eta}}\right)\Bigg|_{\bar{\eta}=-1/\zeta}=\operatorname{Res}\left(\frac{\bar{R}'}{1+\eta\bar{\eta}}\right)\Bigg|_{\bar{\eta}=-1/\zeta'}=\frac{1}{4\pi}. \tag{A66}$$

Therefore, the commutator for the nonlocal potential F for future-pointing timelike separation y^a in closed form is

$$[F,F']=2\pi i\hbar(\ln[\ell\cdot\ell')+1]. \tag{A67}$$

Likewise, the commutator for the nonlocal potential F for past-pointing timelike separation y^a in closed form is

$$[F,F']=-2\pi i\hbar[\ln(\ell\cdot\ell')+1]. \tag{A68}$$

For spacelike separation y^a, the condition $y\cdot\ell=0$ defines a closed contour on the sphere. This has two immediate consequences. On one hand, the step function $\Delta(y\cdot\hat{\ell})$ changes sign across the contour, which implies that, in the evaluation by residues, there will be some likely cancellations, depending on whether the poles are all located on the same side or are scattered on both sides of the contour. On the other hand, there is a nonvanishing contour term that needs to be evaluated explicitly, in addition to the contribution of the residues.

We will first evaluate Eq. (A59) for spacelike separation of the form

$$y^a=(t,0,0,z). \tag{A69}$$

This has the considerable advantage of orienting the contour $y\cdot\ell(\eta)=0$ around the z axis; i.e., the contour is a horizontal circle on the sphere, not necessarily at the equator. Once we obtain the result, we will generalize it to an arbitrary spacelike y^a by means of a general three-dimensional rotation.

The first term in Eq. (A59) consists of a combination of the residues (A65) and (A66), with appropriate signs depending on whether the pole is above or below the contour. The step function Δ is negative above the contour. The second term in Eq. (A59) requires a cumbersome calculation, which we outline in the following.

Using standard spherical coordinates (θ,ϕ) on S^2, with $\theta=0$ at the north pole, the stereographic coordinates are given by $\eta=\cot(\theta/2)e^{i\phi}$, and the condition $y\cdot\ell(\eta)=0$ reads

$$t-z\cos\hat{\theta}=0, \tag{A70}$$

defining a circle at a latitude θ_0 given by $\cos\theta_0=t/z$. The second term in Eq. (A59) takes the form

$$\int_{S^2}(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\bar{R}+\bar{R}')y\cdot m\,\delta(y\cdot\ell)d^2S$$
$$=-\int_0^{2\pi}(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\bar{R}+\bar{R}')\frac{2\rho_0}{(1+\rho_0^2)}e^{-i\phi}d\phi, \tag{A71}$$

where we have used the notation $\rho_0\equiv\cot(\theta_0/2)$. Notice that ρ_0 increases from 0 at the south pole to ∞ at the north pole, taking the value of 1 at the equator. The line integral in Eq. (A71) can be written as the following contour integral around the unit circle in the complex plane:

$$\int_0^{2\pi}(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\bar{R}+\bar{R}')\frac{2\rho_0}{(1+\rho_0^2)}e^{-i\phi}d\phi$$
$$=\oint_{|v|=1}(G\bar{\eth}_\eta G'+G'\bar{\eth}_\eta G+\bar{R}+\bar{R}')\frac{2\rho_0}{(1+\rho_0^2)}\frac{dv}{iv^2}\equiv\mathbf{I}, \tag{A72}$$

where

$$G\bar{\eth}_\eta G'=\frac{1}{4\pi}\ln\left[\frac{(\rho_0 v-\zeta)(\rho_0/v-\bar{\zeta})}{(1+\rho_0^2)(1+\zeta\bar{\zeta})}\right]\frac{(1+\bar{\zeta}'\rho_0 v)}{(\rho_0/v-\bar{\zeta}')} \tag{A73}$$

and

$$\bar{R}=\frac{1}{4\pi}\frac{(\rho_0 v-\zeta)}{(1+\rho_0\zeta/v)}\left(\ln\left[\frac{(\rho_0 v-\zeta)(\rho_0/v-\bar{\zeta})}{(1+\rho_0^2)(1+\zeta\bar{\zeta})}\right]-1\right). \tag{A74}$$

With Eqs. (A73) and (A74), the contour integral **I** is explicitly

$$\mathbf{I}=\frac{\rho_0}{2\pi i(1+\rho_0^2)}\oint_{|v|=1}\ln\left[\frac{(\rho_0 v-\zeta)(\rho_0/v-\overline{\zeta})}{(1+\rho_0^2)(1+\zeta\overline{\zeta})}\right]\frac{(1+\rho_0^2)(1+\zeta\overline{\zeta'})}{(\rho_0-v\overline{\zeta'})(v+\rho_0\zeta)}-\frac{(\rho_0 v-\zeta)}{(v+\rho_0\zeta)v}$$
$$+\ln\left[\frac{(\rho_0 v-\zeta')(\rho_0/v-\overline{\zeta'})}{(1+\rho_0^2)(1+\zeta'\overline{\zeta'})}\right]\frac{(1+\rho_0^2)(1+\zeta'\overline{\zeta})}{(\rho_0-v\overline{\zeta})(v+\rho_0\zeta')}-\frac{(\rho_0 v-\zeta')}{(v+\rho_0\zeta')v}dv. \qquad \text{(A75)}$$

Technically, the contour integral **I**, Eq. (A75), cannot be evaluated by residues as it stands, because of the branch cut of the logarithm at $v=0$. One can rewrite the integrand as a rational function by introducing a parameter τ in the argument of the logarithm, and then differentiating with respect to τ, in the following fashion:

$$\mathbf{I}=\mathbf{J}(\tau=1)=\int_0^1\frac{d\mathbf{J}(\tau)}{d\tau}d\tau+\mathbf{J}(\tau=0), \qquad \text{(A76)}$$

where $\mathbf{J}(\tau)$ is a generalization of **I** defined by introducing τ, for convenience, as

$$\mathbf{J}(\tau)\equiv\frac{\rho_0}{2\pi i(1+\rho_0^2)}\oint_{|v|=1}\ln\left[\frac{(\tau\rho_0 v-\zeta)(\tau\rho_0/v-\overline{\zeta})}{(1+\rho_0^2)(1+\zeta\overline{\zeta})}\right]\frac{(1+\rho_0^2)(1+\zeta\overline{\zeta'})}{(\rho_0-v\overline{\zeta'})(v+\rho_0\zeta)}-\frac{(\rho_0 v-\zeta)}{(v+\rho_0\zeta)v}$$
$$+\ln\left[\frac{(\tau\rho_0 v-\zeta')(\tau\rho_0/v-\overline{\zeta'})}{(1+\rho_0^2)(1+\zeta'\overline{\zeta'})}\right]\frac{(1+\rho_0^2)(1+\zeta'\overline{\zeta})}{(\rho_0-v\overline{\zeta})(v+\rho_0\zeta')}-\frac{(\rho_0 v-\zeta')}{(v+\rho_0\zeta')v}dv. \qquad \text{(A77)}$$

Notice that Eq. (A77) is equal to Eq. (A72) if τ is set equal to 1. On the other hand, if τ is set equal to zero then the v dependence of the logarithm in the integrand of **J** disappears; consequently, the term $\mathbf{J}(\tau=0)$ in Eq. (A76) can be integrated by residues.

The derivative $d\mathbf{J}/d\tau$ is

$$\frac{d\mathbf{J}}{d\tau}=\frac{\rho_0^2}{2\pi i(1+\rho_0^2)}\oint_{|v|=1}\left(\frac{v}{(\tau\rho_0 v-\zeta)}+\frac{1}{(\tau\rho_0-\overline{\zeta}v)}\right)\frac{(1+\rho_0^2)(1+\zeta\overline{\zeta'})}{(\rho_0-v\overline{\zeta'})(v+\rho_0\zeta)}$$
$$+\left(\frac{v}{(\tau\rho_0 v-\zeta')}+\frac{1}{(\tau\rho_0-\overline{\zeta'}v)}\right)\frac{(1+\rho_0^2)(1+\zeta'\overline{\zeta})}{(\rho_0-v\overline{\zeta})(v+\rho_0\zeta')}dv. \qquad \text{(A78)}$$

The simple poles that are relevant to the evaluation of $d\mathbf{J}/d\tau$ as a function of τ are

$$v=\frac{\rho_0}{\overline{\zeta'}},\quad \frac{\rho_0}{\overline{\zeta}},\quad \frac{\zeta}{\tau\rho_0},\quad \frac{\zeta'}{\tau\rho_0},\quad \frac{\tau\rho_0}{\overline{\zeta}},$$
$$\frac{\tau\rho_0}{\overline{\zeta'}},\quad -\rho_0\zeta,\quad -\rho_0\zeta'. \qquad \text{(A79)}$$

Care must be taken to correctly account for the simple poles that are inside the unit circle at different values of τ.

After the evaluation by residues, $d\mathbf{J}/d\tau$ can be seen to be an explicit linear combination of terms of the form $1/(a+b\tau)$, which can be integrated in τ immediately as a logarithmic function. The procedure is rather lengthy but entirely straightforward.

In this way, we have given an outline of the main technical steps necessary to the evaluation of the second term in Eq. (A59). By combining the results obtained separately from the first and second terms in Eq. (A59), the final expression for the commutator of the nonlocal potential F at spacelike separation y^a of the form (A69) is obtained, which we present split into four different cases.

If ζ and ζ' are both above the contour, then

$$[F,F']=-2\pi i\hbar\left\{\ln(\ell\cdot\ell')-\ln\left[\frac{(\rho_0^2-\zeta\overline{\zeta'})(\rho_0^2-\zeta'\overline{\zeta})}{(1+\rho_0^2)^2\zeta\overline{\zeta}\zeta'\overline{\zeta'}}\right]+\frac{(1-\rho_0^2)}{(1+\rho_0^2)}\right\}. \qquad \text{(A80)}$$

If ζ is above and ζ' is below the contour, then

$$[F,F']=-2\pi i\hbar\left\{\ln\left[\frac{(1+\zeta'\overline{\zeta'})\zeta\overline{\zeta}}{(1+\zeta\overline{\zeta})}\right]+\frac{(1-\rho_0^2)}{(1+\rho_0^2)}\right\}. \qquad \text{(A81)}$$

If ζ is below and ζ' is above the contour, then

$$[F,F']=-2\pi i\hbar\left\{\ln\left[\frac{\zeta'\overline{\zeta'}(1+\zeta\overline{\zeta})}{(1+\zeta'\overline{\zeta'})}\right]+\frac{(1-\rho_0^2)}{(1+\rho_0^2)}\right\}. \tag{A82}$$

If ζ and ζ' are both below the contour, then

$$[F,F']=-2\pi i\hbar\left\{-\ln(\ell\cdot\ell')+\ln\left[\frac{(\rho_0^2-\zeta\overline{\zeta'})(\rho_0^2-\zeta'\overline{\zeta})}{(1+\rho_0^2)^2}\right]+\frac{(1-\rho_0^2)}{(1+\rho_0^2)}\right\}. \tag{A83}$$

The results (A80) and (A83) have a regular limit as the contour is shrunk to zero [unlike Eqs. (A81) and (A82), in which one of the points ζ or ζ' would disappear as the contour is shrunk to zero]. The contour is shrunk to zero by taking the limits $\rho_0\to 0$ (in which case the contour flies off the sphere at the south pole), and $\rho_0\to\infty$ (in which case the contour flies off the sphere at the north pole). The limiting values $\rho_0=0,\infty$ correspond to $t=+z,-z$, i.e., the null boundaries between the timelike and spacelike regions. Therefore, it is expected that Eq. (A80) has Eq. (A68) for a limit as $\rho_0\to 0$, whereas Eq. (A83) should have Eq. (A67) for a limit as $\rho_0\to\infty$. This is actually the case, as can be verified by inspection of Eqs. (A80) and (A83).

In order to generalize to an arbitrary spacelike separation y^a, in the following we rewrite the relevant quantities as invariants under general spatial rotations, keeping the time axis fixed.

We define the unit timelike vector

$$T^a\equiv(1,0,0,0), \tag{A84}$$

which is invariant under spatial rotations. We also have at our disposal the vectors ℓ^a and m^a given by

$$\ell^a=\frac{1}{\sqrt{2}}\left(1,\frac{\zeta+\overline{\zeta}}{1+\zeta\overline{\zeta}},-i\frac{\zeta-\overline{\zeta}}{1+\zeta\overline{\zeta}},\frac{-1+\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right), \tag{A85}$$

$$m^a\equiv\eth\ell^a=\frac{1}{\sqrt{2}}\left(0,\frac{1-\overline{\zeta}^2}{1+\zeta\overline{\zeta}},-i\frac{1+\overline{\zeta}^2}{1+\zeta\overline{\zeta}},\frac{2\overline{\zeta}}{1+\zeta\overline{\zeta}}\right). \tag{A86}$$

In terms of these vectors, the relevant quantities in Eqs. (A80)–(A83) take the form

$$t=y\cdot T,$$

$$z=\sqrt{(y\cdot T)^2-y\cdot y},$$

$$1+\zeta\overline{\zeta}=\frac{2z}{z-y\cdot T+\sqrt{2}y\cdot\ell},$$

$$\rho_0^2=\frac{z+y\cdot T}{z-y\cdot T},$$

$$\frac{(\rho_0^2-\zeta'\overline{\zeta})(\rho_0^2-\zeta\overline{\zeta'})}{(1+\rho_0^2)^2}=\frac{-y\cdot y\,\ell\cdot\ell'+2y\cdot\ell\, y\cdot\ell'}{(z-y\cdot T+\sqrt{2}y\cdot\ell)(z-y\cdot T+\sqrt{2}\,y\cdot\ell')}, \tag{A87}$$

where every scalar product is invariant with respect to spatial rotations. By substituting Eq. (A87) into Eqs. (A80)–(A83), the commutators are generalized to an arbitrary spacelike separation y^a.

APPENDIX B: EVALUATION OF THE NULL-SURFACE COMMUTATOR IN THE CASE OF TIMELIKE SEPARATION

In this appendix we evaluate Eq. (18) in closed form for a special range of the parameters y^a. As a first step, however, we rewrite Eq. (18) in the form

$$[\hat{Z},\hat{Z}']=-2\pi i\hbar\int_{S^2}\eth_\eta(V\Delta)-V\eth_\eta\Delta\, d^2S_\eta, \tag{B1}$$

where $V=V(\eta,\zeta,\zeta')$ is given by

$$V=\eth_\eta G'\bar{\eth}_\eta^2G+\eth_\eta G\bar{\eth}_\eta^2G'-G'\eth_\eta\bar{\eth}_\eta^2G-G\eth_\eta\bar{\eth}_\eta^2G'+Q'\eth_\eta^2\bar{\eth}_\eta^2G+Q\eth_\eta^2\bar{\eth}_\eta^2G'-R'\eth_\eta^3\bar{\eth}_\eta^2G-R\eth_\eta^3\bar{\eth}_\eta^2G' \tag{B2}$$

and the functions Q and R are (nonunique) first and second primitives of G, respectively, in the sense that $G=\eth_\eta Q$ and $G=\eth_\eta^2R$. A choice of the functions Q and R is given in Appendix C.

If y^a is timelike and future pointing, then $y\cdot\ell>0$ and thus $\Delta(y\cdot\ell)=+\frac{1}{2}$, constant on the sphere, whereas $\eth_\eta\Delta(y\cdot\ell)=\delta(y\cdot\ell)\eth_\eta(y\cdot\ell)=0$ everywhere on the sphere. Therefore, for this range of the parameters y^a the commutator reduces to

$$[\hat{Z},\hat{Z}']=-2\pi i\hbar\int_{S^2}\eth_\eta\left(\frac{1}{2}V\right)d^2S_\eta, \tag{B3}$$

which can be evaluated by residues (see [20]):

$$[\hat{Z},\hat{Z}']=4\pi\hbar\sum_j\oint_j\frac{1}{2}\frac{V}{(1+\eta\overline{\eta})}d\overline{\eta}=8\pi^2i\hbar\sum_j\operatorname{Res}\left(\frac{1}{2}\frac{V}{(1+\eta\overline{\eta})}\right)\bigg|_{\overline{\eta}=\overline{\eta}_j}. \tag{B4}$$

The poles $\overline{\eta_j}$ are $\overline{\zeta},\overline{\zeta'},(\zeta)^{-1}$, and $(\zeta')^{-1}$. This can be deduced by inspection of the explicit expression of V which is obtained from the information about the Green's function that we give in Appendix C. The evaluation of the residues at these poles is straightforward, and gives the final expression

$$[\hat{Z},\hat{Z}']=-2\pi i\hbar\left[\ell\cdot\ell'\ln(\ell\cdot\ell')-\frac{1}{6}\ell\cdot\ell'+\frac{1}{3}\right].$$

APPENDIX C: PROPERTIES OF THE GREEN'S FUNCTION

The function (13) gives solutions to the following differential equation for functions F of spin weight zero on the sphere, with given spin-weight-0 source J:

$$\eth^2\bar{\eth}^2 F = J. \tag{C1}$$

One of the properties of this Green's function is that, aside from possible distributional behaviors at $\zeta=\eta$, it is annihilated by application of the operation $\eth^4\bar{\eth}^2$ for all values of $\zeta\neq\eta$:

$$\bar{\eth}_\eta G(\zeta,\eta)=\frac{1}{4\pi}\ell(\zeta)\cdot\overline{m(\eta)}\{\ln[\ell(\zeta)\cdot\ell(\eta)]+1\},$$

$$\bar{\eth}^2_\eta G(\zeta,\eta)=\frac{1}{4\pi}\frac{[\ell(\zeta)\cdot\overline{m(\eta)}]^2}{\ell(\zeta)\cdot\ell(\eta)},$$

$$\eth_\eta\bar{\eth}^2_\eta G(\zeta,\eta)=\frac{1}{4\pi}\ell(\zeta)\cdot\overline{m(\eta)}\left(\frac{1}{\ell(\zeta)\cdot\ell(\eta)}-3\right),$$

$$\eth^2_\eta\bar{\eth}^2_\eta G(\zeta,\eta)=\frac{1}{2\pi}[3\ell(\zeta)\cdot\ell(\eta)-2],$$

$$\eth^3_\eta\bar{\eth}^2_\eta G(\zeta,\eta)=\frac{3}{2\pi}\ell(\zeta)\cdot m(\eta),$$

$$\eth^4_\eta\bar{\eth}^2_\eta G(\zeta,\eta)=0. \tag{C2}$$

This property allows for the rewriting of (18) in the form (B1) in Appendix B.

Another useful property of the Green's function is that, up to free constants of integration, its primitives $Q_{(n)}$ defined by $\eth^n_\eta Q_{(n)}=G$ can be found recursively. In general,

$$Q_{(n)}(\zeta,\eta)=\frac{H_{(n)}[\ell(\zeta)\cdot\ell(\eta)]}{[\ell(\zeta)\cdot m(\eta)]^n}, \tag{C3}$$

where $H_{(n)}(x)$ satisfies

$$\frac{d^n H_{(n)}}{dx^n}(x)=H_{(0)}(x)=\frac{1}{4\pi}x\ \ln x$$

or

$$\frac{dH_{(n)}}{dx}(x)=H_{(n-1)}(x). \tag{C4}$$

Equation (C4) can be solved by making the ansatz

$$H_{(n)}(x)=x^{n+1}(C_n\ln x-B_n). \tag{C5}$$

By imposing Eq. (C4) we find that the parameters C_n and B_n need to satisfy

$$C_{n-1}=(n+1)C_n,$$
$$B_{n-1}=(n+1)B_n-C_n, \tag{C6}$$

which are solved by

$$C_n=\frac{C_0}{(n+1)!},$$
$$B_n=\frac{C_0}{(n+1)!}\sum_{i=0}^{n-1}\frac{1}{(n+1-i)}, \tag{C7}$$

where $C_0=1/4\pi$. In this way, we have found a choice of the generic primitive of G to any desired order (note that the primitives are not unique).

Here we show explicitly the first and second primitives:

$$Q_{(1)}(\zeta,\eta)=\frac{1}{8\pi}\frac{[\ell(\zeta)\cdot\ell(\eta)]^2}{\ell(\zeta)\cdot m(\eta)}\{\ln[\ell(\zeta)\cdot\ell(\eta)]-\tfrac{1}{2}\}, \tag{C8}$$

$$Q_{(2)}(\zeta,\eta)=\frac{1}{24\pi}\frac{[\ell(\zeta)\cdot\ell(\eta)]^3}{[\ell(\zeta)\cdot m(\eta)]^2}\{\ln[\ell(\zeta)\cdot\ell(\eta)]-\tfrac{5}{6}\}. \tag{C9}$$

In Appendix B, we have used the notation $Q\equiv Q_{(1)}$ and $R\equiv Q_{(2)}$.

A third and essential property of the Green's function can be stated in terms of the solutions of Eq. (C1). A solution to Eq. (C1) can be found by

$$F_P=\int_{S^2}G(\zeta\eta)J(\eta)d^2S_\eta \tag{C10}$$

(any other solution can be found by adding to F_P a solution to the homogeneous equation $\eth^2\bar{\eth}^2F=0$). It can be shown that F_P has no $l=0,1$ terms in an expansion in spherical harmonics. Thus *the Green's function* (13) *provides a decomposition of a generic solution into its* $l=0,1$ *part and its* $l\geq 2$ *part*. This third property holds as a consequence of the *kernel exclusion property* of the Green's functions for $\eth^n$ acting on spin-weight-s functions; namely, they yield no spherical harmonics of order $l\in\{s,\ldots,s+n-1\}$ upon integration on the sphere against a given source [19].

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