

BRIEF REPORTS

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Chaos and rotating black holes with halos

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The occurrence of chaos for test particles moving around a slowly rotating black hole with a dipolar halo is studied using Poincaré sections. We find a novel effect: particles with angular momentum opposite to the black hole rotation have larger chaotic regions in phase space than particles initially moving in the same direction. [S0556-2821(97)03724-7]

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The main paradigm in the study of the motion of stars in a galaxy is a model of a central bulge surrounded by a halo [1]. In this context, for the particular axially symmetric case, arose the celebrated Hénon-Heiles model [2] whose study has been the source of inspiration of many researches on chaotic behavior [3]. The underlying theory in this case is the usual Newtonian gravitation that for large masses and velocities is known to be less appropriate than Einsteinian general relativity. In the latter case the Newtonian potential is replaced by the spacetime metric and Newton motion equations by geodesics. This change of dynamics can produce dramatic effects, for instance, test particles moving in the presence of systems of masses that are integrable in Newtonian theory are chaotic in general relativity; examples are the fixed two body problem [4,5], and particles moving in a monopolar center of attraction surrounded by a dipolar halo [6]. Also gravitational waves, a nonexistent phenomenon in the Newtonian realm, can produce an irregular motion of test particles orbiting around a static black hole [7,8]. Another distinctive feature of general relativity is the dragging of inertial frames due to mass rotation. This fact is observed, for instance, in the impressive differences of the geodesic motion in Schwarzschild and Kerr geometries [9].

In this Brief Report we study the effects of rotation in the motion of a particle orbiting around a slowly rotating black hole surrounded by a dipolar halo. The nonrotating case is chaotic, so we shall study mainly the change in the chaotic behavior due to the rotation of the center of attraction. A typical situation is represented by a galaxy with a rapid rotating center surrounded by a distant massive halo, ring or other shell-like distributions of matter. We study the motion of particles moving between the center and the halo whose first contribution is dipolar. This contribution is always

present whenever the halo does not possess reflection symmetry with respect to the black hole equatorial plane.

The metric that represents the superposition of a Kerr black hole and a dipole along the rotation axis is a stationary axially symmetric spacetime. The vacuum Einstein equations for this class of spacetimes is an integrable system of equations that is closely related to the principal σ model [10]. Techniques to actually find the solutions are Bäcklund transformations and the inverse scattering method, also a third method constructed with elements of the previous two is the “vesture method,” all these methods are closely related [10]. The general metric that represents the nonlinear superposition of a Kerr solution with a Weyl solution, in particular, with a multipolar expansion can be found by using the “inverse scattering method” [11]. We find

$$ds^2 = g_{tt}(r, z)dt^2 + 2g_{t\phi}(r, z)dtd\phi + g_{\phi\phi}(r, z)d\phi^2 + f(r, z) \times (dz^2 + dr^2), \quad (1)$$

where

$$g_{tt} = -\{e^{-2Dpuv}[e^{-4Dpu}(e^{-8Dpv}(p+1)^2(u^2-1) + e^{-8Dp}(p-1)^2(u^2-1) - 2e^{-4Dp(v+1)}(u^2-v^2)q^2) + e^{-4Dp(1+v)}(e^{-8Dpu}+1)(v^2-1)q^2]\}/F,$$

$$g_{t\phi} = 2pqe^{-2Dp(u+v+1)}\{e^{-4Dpu}[e^{-4Dpv}(p+1)(u-v) \times (u+1) + e^{-4Dp}(p-1)(u+v)(u-1)(v+1)] - e^{-4Dpv}(p+1)(u+v)(u+1)(v-1) - e^{-4Dp}(p-1) \times (u-v)(u-1)(v-1)\}/F,$$

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$$\begin{aligned}
g_{\varphi\varphi} &= [(g_{t\varphi})^2 - (mr)^2]/g_{tt}, \\
f &= m^2 \{ e^{-4Dp[(v+1)u-v+1]} [e^{-4Dpuv}(p-1)^2(u-1)^2 \\
&\quad + e^{-4Dp[(v+1)u+v-1]}(v+1)^2q^2] \\
&\quad + e^{-4Dp[(v+1)u+v-1]-4Dpuv}(p+1)^2(u+1)^2 \\
&\quad - 2e^{-2Dp[(v+1)u+v-1]-2Dp[(v+1)u-v+1]-4Dpuv} \\
&\quad \times (u^2-v^2)q^2 + e^{-8Dpuv}(v-1)^2q^2 \} / H, \\
F &\equiv e^{-4Dpu} [e^{-8Dpv}(p+1)^2(u+1)^2 + e^{-8Dp}(p-1)^2 \\
&\quad \times (u-1)^2 - 2e^{-4Dp(v+1)}(u^2-v^2)q^2], \\
H &\equiv 4 \exp\{-2Dp[(v+1)u+v-1] - 2Dp[(v+1)u-v+1] \\
&\quad - 6Dpuv - (u^2-1)(v^2-1)D^2p^2\}. \quad (2)
\end{aligned}$$

The coordinates (r, ϕ, z) are dimensionless and have the range of the usual cylindrical coordinates. They are related to u and v by: $z = uv$ and $r = (u^2 - 1)^{1/2}(1 - v^2)^{1/2}$, $u \geq 1$ and $-1 \leq v \leq -1$. Our units are such that $c = G = 1$; D represents the dipole strength and a the rotation parameter, $q = a/m$ and $p^2 + q^2 = 1$. The coordinate transformation $t' = t + 2a\varphi$, $u = R/m - 1$, $v = \cos\vartheta$, $\varphi' = \varphi$ reduces Eq. (2) with $D = 0$ to the Kerr solution in the usual Boyer-Lindquist coordinates [12].

To study the slow rotation case is better to use the metric obtained by keeping the first order terms in the rotation parameter a in the exact metric (2). This approximation, for the parameters and range of coordinates used, will not produce a significant information loss; we shall come back to this point later. We find, for $g_{\mu\nu} = g_{\mu\nu}^0 + ag_{\mu\nu}^1$,

$$\begin{aligned}
g_{tt} &= -\frac{u-1}{u+1} \exp(-2Duv), \\
g_{t\phi} &= \frac{a}{u+1} \{ (u+v)(1-v) \exp[-2D(1-u-v)] \\
&\quad + (u-v)(1+v) \exp[-2D(1+u-v)] \}, \\
g_{\phi\phi} &= m^2(1-v^2)(1+u)^2 \exp(2Duv), \\
f &= m^2 \frac{(u+1)^2}{u^2-v^2} \exp\{D[(u^2-1)(v^2-1)D + 2uv - 4v + 4]\}. \quad (3)
\end{aligned}$$

The geodesic equations for the metric (1) can be cast as

$$\dot{t} = g^{tb} E_b, \quad \dot{\phi} = g^{\phi b} E_b, \quad (4)$$

$$\ddot{r} = -\frac{1}{2f} [g_{,r}^{ab} E_a E_b + f_{,r} (r^2 - \dot{z}^2) + 2f_{,z} \dot{r} \dot{z}], \quad (5)$$

$$\ddot{z} = -\frac{1}{2f} [g_{,z}^{ab} E_a E_b + f_{,z} (\dot{z}^2 - r^2) + 2f_{,r} \dot{r} \dot{z}], \quad (6)$$

where the dots denote derivation with respect to s and the indices a and b take the values (t, ϕ) , g^{ab} stands for the inverse of g_{ab} . $E_t = -E$ and $E_\phi = L$ are integration con-

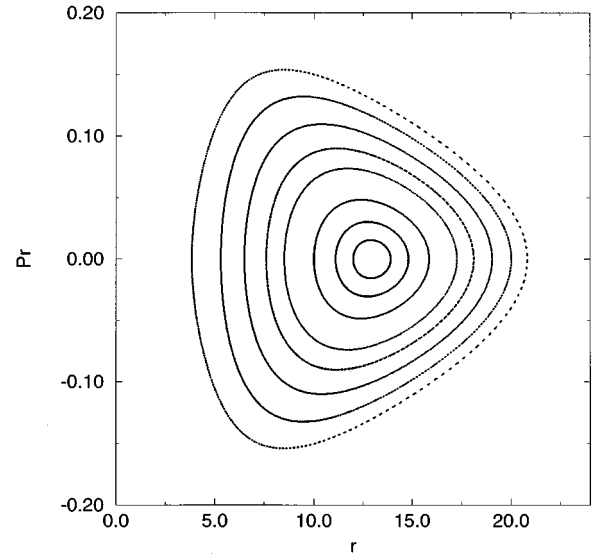


FIG. 1. Poincaré section of test particles moving with angular momentum $L = -3.75$ in an approximate Kerr geometry with rotation parameter $a = 0.01$ (counter rotation) and mass $m = 1$. This is a typical section of an integrable system.

stants; E and L are the test particle energy and angular momentum, respectively. The set (4)–(6) admits a third integration constant

$$E_3 = g^{ab} E_a E_b + f(\dot{r}^2 + \dot{z}^2) = -1. \quad (7)$$

Thus to have complete integrability we need one more independent constant of integration. In the case of pure Kerr solution ($D = 0$) we have a fourth constant due to the existence of a Killing tensor and for the nonrotating case we have another constant related to a third Killing vector associated to spherical symmetry [9].

The system (5) and (6) can be written as a four dimensional dynamical system in the variables $(r, z, P_r = \dot{r}, P_z = \dot{z})$. A convenient method to study qualitative aspects of this system is to compute the Poincaré sections through the plane $z = 0$. The intersection of the orbits with this plane will be studied in some detail for bounded motions. We shall numerically solve the system (4)–(6) and use the integral (7) to control the accumulated error along the integration; we shall return to this point later.

The Poincaré section for different initial conditions with energy $E = 0.965$ and angular momentum $L = -3.75$ (counter rotation) moving in an approximate Kerr geometry ($D = 0$) with rotation parameter $a = 0.01$ and mass $m = 1$ (this value for the mass will be kept unchanged from now on) are presented in Fig. 1. We have the typical section of an integrable motion, i.e., the sectioning of invariant tori, for integrability and KAM theory, see, for instance, [13].

We also studied the same case for direct rotation $L = 3.75$, as well as, the corresponding Schwarzschild limit $a = 0$, and $L = -3.75$. All these cases present Poincaré sections almost identical to Fig. 1. The section area for the Schwarzschild case is slightly smaller than the section area of Fig. 1 and the section area for particles in direct rotation in a Kerr geometry is even smaller. We have that counter rotation enlarges the

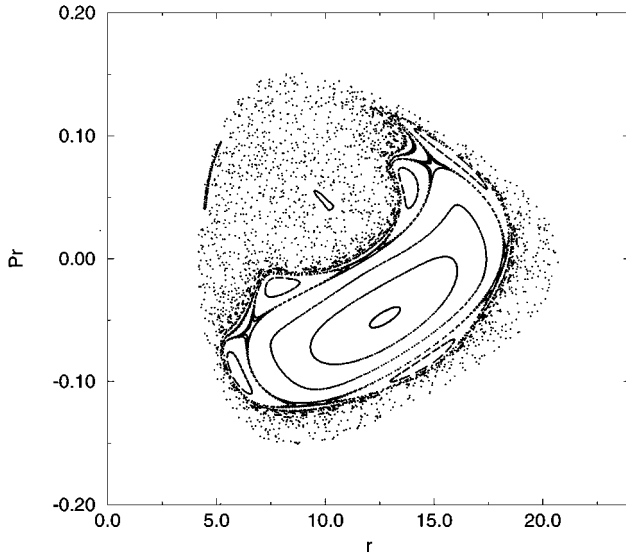


FIG. 2. Poincaré section for $D=0.0005$, $E=0.965$, $a=0$, $m=1$, and $L=\pm 3.75$. The two isolated islands around the points $(10, 0.05)$ and $(5, 0.075)$ are parts of the same torus.

area of the section and direct rotation shrinks it. This is clearly an effect of the dragging of inertial frames due to rotation.

The motion of test particles around a static black hole with a dipolar halo ($a=0$ and $D=0$) is chaotic and it was studied in some detail in [6] for a different energy shell. In Fig. 2 we show the Poincaré section for $D=0.0005$ and the same values of $E=0.965$, and $L=3.75$ as in Fig. 1. We find islands of integrability surrounded by chaotic motion. The two isolated islands around the points $(10, 0.05)$ and $(5, 0.075)$ are parts of the same torus. In the case studied in [6] they were closer.

Now we shall consider a particle moving around a slowly rotating attractive center surrounded by a dipolar halo for

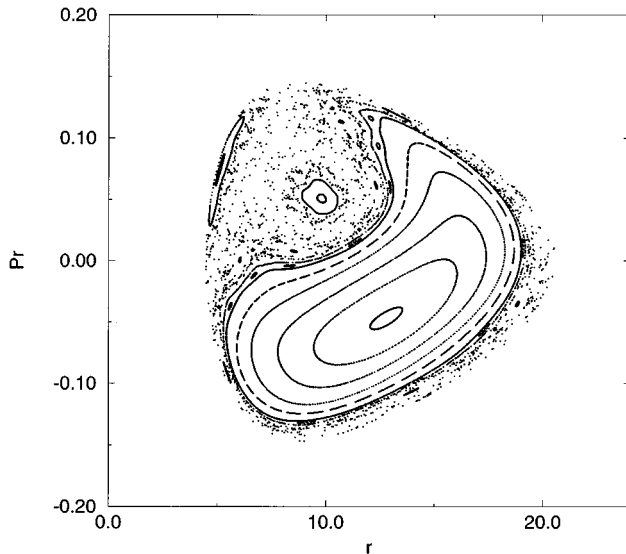


FIG. 3. Poincaré section for $D=0.0005$, $a=0.01$, $m=1$, $E=0.965$, and $L=3.75$ (direct rotation). The islands of stability are larger in this case than in the nonrotating case (cf. Fig. 2).

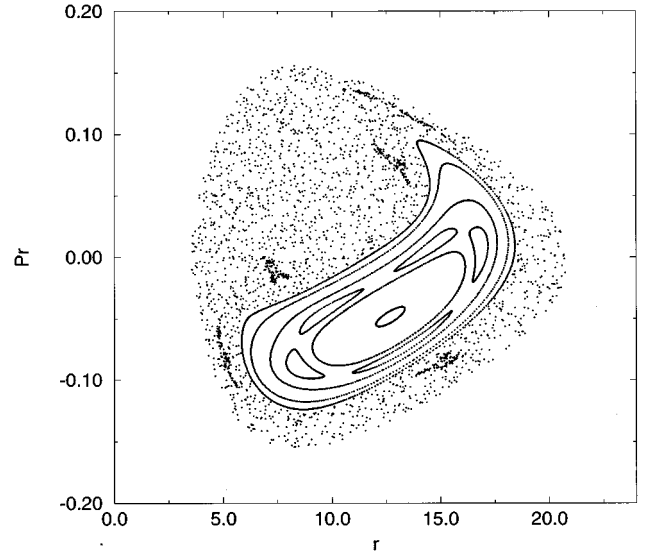


FIG. 4. Poincaré section with the same parameters of Fig. 3, except that now $L=-3.75$ (counter rotation). The chaotic region increases in a significant way, also the system of islands located around the points $(10, 0.05)$ and $(5, 0.075)$ in the two precedent cases has disappeared.

both direct rotation and counter rotation. In Fig. 3 we draw the Poincaré section for $D=0.0005$, $a=0.01$, $E=0.965$, and $L=3.75$ (direct rotation). We see that the islands of stability are larger in this case than in the nonrotating case (see Fig. 2); also we have new systems of small islands immersed in the chaotic region. We have that the chaotic region is smaller in this case than in the equivalent nonrotating one. It does seem that the direct rotation diminishes the effect of the dipolar strength as a chaos source. In Fig. 4 we present the section with the same parameters of Fig. 3, except that now we have counter rotation, $L=-3.75$. In Fig. 4 we observe that the chaotic region increases in a significant way and also that the islands located around the points $(10, 0.05)$ and $(5, 0.075)$ in Figs. 2 and 3 have disappeared in this scale. In other words, with $D=0$, the counter rotating motion of particles is more chaotic than the static case, the later being more chaotic than the direct rotation case. Also, we can say that the counter (direct) rotation reinforces (weakens) the strength of the dipole as a source of the chaotic motion (for $D=0$ we have an integrable system). This effect is a manifestation of the fact that particles moving in nonequatorial orbits of a rotating central body can suffer repulsive forces due to rotation. This fact allows in a Kerr geometry closed orbits of test particles moving on planes parallel to the equatorial plane [14], though these orbits are not stable. Although in the present paper we present results for particular values of the parameters involved, we did a rather extended numerical study that supports our conclusions, Figs. 3 and 4 being representative of this search.

For the values of the parameters $D=0.0005$, $a=0.01$, $E=0.965$, and $L=\pm 3.75$, we have that the particles move in the “box” $4.7 < r < 21$, $-5 < z < 9$. We take, as a measure of error the quantities,

$$\Delta g_{ab} = |(g_{ab}^{\text{ex}} - g_{ab}) / g_{ab}^{\text{ex}}|, \quad \Delta f = |(f^{\text{ex}} - f) / f^{\text{ex}}|, \quad (8)$$

where in these expressions the sum rule of repeated indices does not apply. $g_{\mu\nu}^{\text{ex}}$ and $g_{\mu\nu}$ refer to the solutions (2) and (3), respectively. We find that for the above mentioned range of coordinates and the values of parameters used in this paper the quantities defined in (8) are at most of the order of 10^{-6} . Also in this range, the error in the derivatives of the metric functions is even smaller (the metric functions are very smooth). We also want to mention that the Poincaré sections shown in this paper were computed from orbits with an accumulated error in the “energy” [cf. Eq. (7)] smaller than 10^{-10} .

We want to conclude with a discussion of a possible astrophysical implication of our main result: direct rotating particles are less chaotic than counter rotating ones. In a rotating center of gravitational attraction we can have structures formed by counter and direct rotating particles, thus our result favors larger life times of structures formed with direct rotating particles. A deeper discussion of this point will be presented elsewhere.

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