Global vortex and black cosmic string

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We study global vortices coupled to (2+1)-dimensional gravity with a negative cosmological constant. We found nonsingular vortex solutions in ϕ^4 theory with a broken U(1) symmetry, of which the spacetimes do not involve a physical curvature singularity. When the magnitude of a negative cosmological constant is larger than a critical value at a given symmetry breaking scale, the spacetime structure is a regular hyperbola; however, it becomes a charged black hole when the magnitude of the cosmological constant is less than the critical value. We explain through a duality transformation the reason why a static global vortex which is electrically neutral forms a black hole with electric charge. Under the present experimental bound of the cosmological constant, implications for cosmology as a straight black cosmic string are also discussed in comparison with a global U(1) cosmic string in the spacetime of the zero cosmological constant. [S0556-2821(97)02324-2]

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I. INTRODUCTION

Einstein gravity in 2+1 dimensions [(2+1)D] has no local degrees of freedom and the matter coupled to gravity changes only the global structure of spacetime outside sources [1]. Subsequently anti–de Sitter solutions in three-dimensional gravity were analyzed in the 1980's [2]; however, it took many years thenceforth to find out the black hole structure among those solutions of the negative cosmological constant [3].

One of the reasons why conic solutions formed by point particles in (2+1)D have attracted attention is that they stand for the asymptotic space of cylindrically symmetric local cosmic strings which are extended solitonic objects [4]. In this context, an intriguing question in 2+1 dimensions with the negative cosmological constant is whether or not one can find the structure of black cosmic strings. In relation to the stability of such stringlike objects in (3+1)D, the topological vortex solution in (2+1)D is the first candidate. There has been another subject in 3+1 dimensions: The study of black holes, particularly charged black holes, formed by solitons, e.g., monopoles, Skyrmions, etc., has been an interesting subject [5]. Since the negative cosmological constant amounts to the term of the energy proportional to the area of the spatial manifold ($\sim r^2$), one can easily guess that static extended objects carrying a long-range tail are important. The simplest candidate in 2+1 dimensions may be the global U(1) vortex of which energy diverges logarithmically in flat spacetime, which is a viable cosmic string candidate in cos-

mology [4,6,7]. Here, if we recall the fact that a physical curvature singularity in global string spacetime is unavoidable in the case of a zero cosmological constant, we can add another question as to whether we can find regular global cosmic strings in anti–de Sitter spacetime or black cosmic strings with no divergent curvature [8].

In this paper, we will consider a complex scalar ϕ^4 model in (2+1)-dimensional anti–de Sitter spacetime and look for global vortex solutions. There are cylindrically symmetric global U(1) vortex solutions connecting smoothly the symmetric local maximum at the origin and a broken vacuum point at spatial infinity. The spacetimes formed by these strings are regular hyperbola with a deficit angle, extremal black hole, and charged black hole as the magnitude of the cosmological constant decreases. The curvature of these solutions is not divergent everywhere even for charged black holes. So it is contrary to the zero cosmological constant case, where the U(1) global string admits no globally well-behaved solution.

This paper is organized as follows. We begin in Sec. II by establishing explicitly the relation that the (2+1)D spinless black hole solutions in Ref. [3] are part of general anti–de Sitter space solutions in Ref. [2]. In Sec. III, we introduce the model and obtain the global U(1) vortex solutions. Possible geodesics of massless and massive test particles are also given. In Sec. IV, the connection between the topological charge and the electric charge of a black hole is illustrated by use of duality transformations. In Sec. V, questions about the physical relevance of a charged black cosmic string in 3+1 dimensions are addressed. We conclude in Sec. VI with a brief discussion.

II. BLACK HOLE AS AN ANTI–de SITTER SOLUTION

In this section, let us recapitulate what the (2+1)-dimensional Schwarzschild black hole solution is among a

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series of anti–de Sitter solutions of which all static metrics can be characterized in terms of one complex function. Under a conformal gauge, the static metric compatible with static objects is parametrized by

$$ds^2 = \Phi^2(z, \bar{z})dt^2 - b(z, \bar{z})dz d\bar{z}, \qquad (2.1)$$

where $z \equiv x + iy = Re^{i\Theta}$. For n massive spinless point particles located at positions $z = z_a$, $a = 1, 2, \ldots, n$, each with mass m_a, the cosmological constant Λ is obtained by solving the time-time component of the Einstein equations[1]

$$\Lambda = -\frac{2}{b} \partial_z \partial_{\bar{z}} \ln b \sum_{a=1}^{n} |z_a - z|^{8Gm_a}. \qquad (2.2)$$

The space-space components give two independent equations: One is for the spatial trace [9],

$$\Lambda = -\frac{2}{\Phi b} \partial_z \partial_{\bar{z}} \Phi, \qquad (2.3)$$

and the other is for the traceless part,

$$\partial_z \left(\frac{1}{b} \partial_z \Phi \right) = 0. \qquad (2.4)$$

As obtained in Ref. [2], the general anti–de Sitter solution of Eq. (2.2), Eq. (2.3), and Eq. (2.4) is

$$b = \frac{\varepsilon}{|\Lambda| V(z) \bar{V}(\bar{z}) \sinh^2 \sqrt{\varepsilon}(\zeta - \zeta_0)}, \qquad (2.5)$$

$$\Phi = \sqrt{\varepsilon} \coth \sqrt{\varepsilon}(\zeta - \zeta_0), \qquad (2.6)$$

where $V(z)$ $[\bar{V}(\bar{z})]$ is an arbitrary [anti]holomorphic function and ζ is a real variable defined by

$$\zeta \equiv \frac{1}{2} \left[\int^z \frac{dw}{V(w)} + \int^{\bar{z}} \frac{d\bar{w}}{\bar{V}(\bar{w})} \right]. \qquad (2.7)$$

ε is a real positive integration constant for $\Lambda > 0$ and is an arbitrary nonzero real constant for $\Lambda < 0$.

Here let us consider the simplest case that $V = z/c$ where c is a real constant so as to keep the single valuedness of ζ. When $\varepsilon > 0$, one can set $\varepsilon = 1$ without loss of any generality. In the radial coordinate, c is identified as $c = 1 - 4Gm$ and the metric in Eq. (2.1) becomes

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[1] Our equation in Eq. (2.2), which uses the action for the point particles,

$$S_{pp} = \sum_{a=1}^{n} m_a \int ds \sqrt{g^{\mu\nu}(x_a(s)) \frac{dx_\mu^a}{ds} \frac{dx_\nu^a}{ds}},$$

is different from the equation written as Eq. (2.7) in Ref. [2]. Specifically $m_a = \Phi(x)^2 m_a^{DJ}$. For the point sources, these two equations lead to the same solutions which were first obtained by the authors in Ref. [2]. However, there is a possibility that the solutions for the extended sources may include different curved spacetime. Y.K. would like to thank R. Jackiw for a discussion on this point.

$$ds^2 = \left(\frac{R^{(1-4Gm)} + R^{-(1-4Gm)}}{R^{(1-4Gm)} - R^{-(1-4Gm)}} \right)^2 dt^2$$
$$- \frac{4(1-4Gm)^2}{|\Lambda| R^2 (R^{(1-4Gm)} - R^{-(1-4Gm)})^2} (dR^2 + R^2 d\Theta^2), \qquad (2.8)$$

where m is the total mass of the point particle at $R = 0$. Introducing the new coordinates r and θ such as

$$r = \frac{2}{|\Lambda|^{1/2}} \frac{1}{|R^{(1-4Gm)} - R^{-(1-4Gm)}|} \quad \text{and} \quad \theta = (1 - 4Gm)\Theta, \qquad (2.9)$$

we can rewrite the metric in Eq. (2.8) as

$$ds^2 = (1 + |\Lambda| r^2)dt^2 - (1 + |\Lambda| r^2)^{-1} dr^2 - r^2 d\theta^2. \qquad (2.10)$$

Now we can easily identify the structure of manifold as a hyperbola with deficit angle $\delta = 8\pi Gm$ where $4Gm < 1$.

The above is the physical interpretation provided in Ref. [2]. Then, how about the solutions with negative ε? Or, equivalently, the $\varepsilon = -1$ case? In this case, the metric in Eq. (2.1) can be reexpressed as

$$ds^2 = \frac{1}{\tan^2(2c \ln R)} dt^2 - \frac{c^2}{|\Lambda| \sin^2(2c \ln R)} (d\ln R^2 + d\theta^2). \qquad (2.11)$$

One can easily notice that the coordinate R has unconventional ranges such that $\Phi(R)$ and $R^2 b(R)$ diverge at $R = \exp(k\pi/4c)$ (k is an integer). Because of the unconventional behavior of metric functions at $R = 0$, it seems rather difficult to pin the unknown constant c down by use of the point particle mass m under this coordinate system. To make the physics clear, let us do a coordinate transformation

$$r = \frac{c}{|\Lambda|^{1/2} \sin(2c \ln R)}. \qquad (2.12)$$

The result leads to the well-known $(2+1)$D (Schwarzschild-type) black hole solution with mass c^2 and negative cosmological constant Λ [3]:

$$ds^2 = (-c^2 + |\Lambda| r^2)dt^2 - (-c^2 + |\Lambda| r^2)^{-1} dr^2 - R^2 d\theta^2. \qquad (2.13)$$

As shown in Fig. 1, each range of r ($\exp(k\pi/4c) < r < \exp[(k+1)\pi/4c]$) covers the exterior region of the Bañados-Teitelboim-Zanelli (BTZ) solution.

One step extension may be the quadratic $V(z)$, i.e., $V = (z - z_1)(z - z_2)/\tilde{c}$, $(z_1 \neq z_2)$. Reexamining the above computation, we can easily conclude that this reproduces the same result again: (i) When $\varepsilon = 1$, it goes to anti–de Sitter space with point particle mass $4Gm = 1 - |\tilde{c}/(z_1 - z_2)|$, and (ii) when $\varepsilon = -1$, it goes to the BTZ black hole with the black hole mass $8GM = |\tilde{c}/(z_1 - z_2)|^2$. For more complicated examples, further study is needed.
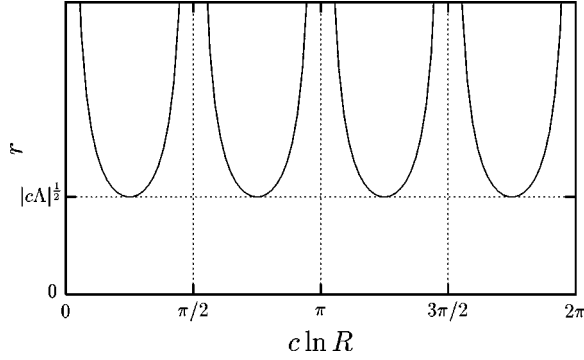
FIG. 1. The radial coordinate R under the conformal gauge vs r of the Schwarzschild metric.

It is now the turn of charged case. The metric compatible with such spinless static objects and with rotational symmetry is of a form

$$ds^2 = e^{2N(r)}B(r)dt^2 - \frac{1}{B(r)}dr^2 - r^2 d\theta^2. \quad (2.14)$$

Then the Einstein equations are

$$\frac{1}{r}\frac{dN}{dr} = \frac{8\pi G}{B}(T_t^t - T_r^r), \quad (2.15)$$

$$\frac{1}{r}\frac{dB}{dr} = 2|\Lambda| - 16\pi G T_t^t. \quad (2.16)$$

If the source is composed of the electrostatic field of a point charge q, the energy-momentum density is

$$T_t^t = T_r^r = \frac{e^{-2N(r)}}{2}E_i^2 = \frac{q^2}{2}\frac{e^{-2N(r)}}{r^2}. \quad (2.17)$$

Inserting Eq. (2.17) into the Einstein equations in Eq. (2.16) and Eq. (2.15), we have

$$N(r) = 0, \quad (2.18)$$

$$B(r) = |\Lambda|r^2 - 8\pi Gq^2 \ln r - 8GM, \quad (2.19)$$

where M is an undetermined mass parameter [of which the dimension is mass per unit length in (3+1)D]. When

$$M \geq \frac{\pi}{2}q^2\left(1 - \ln\frac{4\pi Gq^2}{|\Lambda|}\right),$$

a Reissner-Nordström-type black hole with two horizons is formed, and an extremal one is formed when the equality holds. For the Schwarzschild-type black hole case, there is no curvature singularity at any $r(r \geq 0)$. However, this charged black hole contains the curvature singularity at the origin, $R_t^t = 6|\Lambda| - 8\pi Gq^2/r^2$, due to the infinite self-energy of the point charge. It should be noted that the black hole charge is generated by the so-called logarithmically divergent energy term $\int_\epsilon^L dr r T_t^t \sim \ln L$ at large distance, but the ultraviolet singularity at the origin ($\epsilon \to 0$) is not essential for the formation of the charged black hole even though it gives a divergent curvature. Therefore, the global vortex attracts

our interest since it involves the long tail of energy density and does not involve the ultraviolet divergence at the vortex core of which the singularity is irrelevant to the black hole structures.

We have an explicit solution of the charged BTZ black hole, and so the question is whether there are two or more solutions under the conformal gauge in Eq. (2.1), of which one corresponds to the charged BTZ solution. Again, let us consider the spacetime geometry with a point particle of mass m and charge q sitting at the origin. The relevant Einstein equation is

$$\Lambda = -\frac{2}{b}\partial_z\partial_{\bar{z}}\ln b|z|^{8\pi Gm} - 16\pi G T_{emt}^t, \quad (2.20)$$

and the energy density distribution determined by Gauss' law, namely, $T_{emt}^t = (1/2b)(q^2/R^2)$. Again Eq. (2.20) with a negative cosmological constant reduces to a Liouville-type equation

$$2\partial_z\partial_{\bar{z}}\ln\left(\frac{b}{R^{-8Gm-4\pi Gq^2\ln R}}\right) = |\Lambda|b. \quad (2.21)$$

It is known that this equation is not integrable and there is no known exact solution of this equation, yet [10]. Once we try to do a coordinate transformation from Eq. (2.1) to Eq. (2.17) and Eq. (2.19), the reason why we cannot obtain the explicit form of the solution in the conformal metric is obvious though we could get it in the Schwarzschild metric. Despite the algebraic relation $R^2b(R) = r^2$, R cannot be expressed as a function of r in a closed form for charged objects:

$$r^2 = r_0^2 \exp\left\{\int^r \frac{d\ln R^2}{\sqrt{|\Lambda|R^2 - 4\pi Gq^2\ln R^2 - 8GM}}\right\}. \quad (2.22)$$

Only when the object is neutral ($q = 0$) is this integration done in a closed form. For a negative M, the integration range of r is not restricted and then we have the hyperbolic solution in Eq. (2.8), namely, a positive ϵ solution in Eq. (2.5). On the other hand, for a positive M, the integration range of r is larger than $M/|\Lambda|$ and it is the BTZ black hole solution outside horizon in Eq. (2.13), namely, a negative ϵ solution. Though we do not know the closed form of the metric function in conformal coordinates and do not determine the range of integration range of r explicitly, it is obvious that there are both types of solutions for the charged case ($q \neq 0$). When

$$M < \frac{\pi}{2}q^2\left(1 - \ln\frac{4\pi Gq^2}{|\Lambda|}\right),$$

we have a regular solution corresponding to the positive ϵ solution. When

$$M > \frac{\pi}{2}q^2\left(1 - \ln\frac{4\pi Gq^2}{|\Lambda|}\right),$$

the obtained metric describes the inside of inner horizon and outside of outer horizon of the charged black hole corre-

sponding to the negative ε. Note that the positivity of M is not a necessity for the charged black hole ($q \neq 0$). We will show it is indeed the case in global U(1) vortices.

In this section we have clarified a relation between the rotationally symmetric solutions in the metric under a conformal gauge and the static BTZ black hole solutions in the Schwarzschild-type metric. However, a similar construction like the above relation is not clear for more complicated solutions, e.g., multicenter solutions without rotational symmetry, solutions with the deficit angle equal to or bigger than 2π, and spinning black holes [11], yet.

III. GLOBAL VORTEX IN ANTI−de SITTER SPACE AS A REGULAR NEUTRAL VORTEX OR A CHARGED BLACK HOLE

Let us consider the anti−de Sitter spacetime in the presence of global vortices. The standard example is given by the action

$$S = \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G}(R+2\Lambda) + \frac{1}{2} g^{\mu\nu} \partial_\mu \overline{\phi} \partial_\nu \phi \right.$$
$$\left. - \frac{\lambda}{4}(\overline{\phi}\phi - v^2)^2 \right\}, \tag{3.1}$$

where $\phi(x)$ is a complex field. The ansatz for the static global vortices with rotational symmetry is

$$\phi = |\phi|(r) e^{in\theta}. \tag{3.2}$$

From the model given above in Eq. (3.1), the equation for a scalar field is

$$\frac{d^2|\phi|}{dr^2} + \left(\frac{dN}{dr} + \frac{1}{B} \frac{dB}{dr} + \frac{1}{r} \right) \frac{d|\phi|}{dr}$$
$$= \frac{1}{B} \left(\frac{n^2|\phi|}{r^2} + \lambda(|\phi|^2 - v^2)|\phi| \right). \tag{3.3}$$

The energy-momentum tensor for the Einstein equations includes the long-tail term ($\sim 1/r^2$):

$$T^t_t = \frac{1}{2} \left\{ B \left(\frac{d|\phi|}{dr} \right)^2 + \frac{n^2}{r^2}|\phi|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right\}, \tag{3.4}$$

$$T^r_r = \frac{1}{2} \left\{ -B \left(\frac{d|\phi|}{dr} \right)^2 + \frac{n^2}{r^2}|\phi|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right\}. \tag{3.5}$$

Substituting Eq. (3.4) and Eq. (3.5) into the Einstein equations in Eq. (2.15) and Eq. (2.16), we obtain the following equations:

$$\frac{1}{r} \frac{dN}{dr} = 8\pi G \left(\frac{d|\phi|}{dr} \right)^2, \tag{3.6}$$

$$\frac{1}{r} \frac{dB}{dr} = 2|\Lambda| - 8\pi G \left\{ B \left(\frac{d|\phi|}{dr} \right)^2 + \frac{n^2}{r^2}|\phi|^2 \right.$$
$$\left. + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right\}. \tag{3.7}$$

Then the metric functions $N(r)$ and $B(r)$ are expressed in terms of the scalar field:

$$N(r) = -8\pi G \int_r^\infty dr' r' \left(\frac{d|\phi|}{dr'} \right)^2, \tag{3.8}$$

$$B(r) = e^{-N(r)} \left\{ 2|\Lambda| \int_0^r dr' r' \right.$$
$$\times \exp \left[-8\pi G \int_{r'}^\infty dr'' r'' \left(\frac{d|\phi|}{dr''} \right)^2 \right]$$
$$- 8\pi G \int_0^r dr' r' \exp \left[-8\pi G \int_{r'}^\infty dr'' r'' \left(\frac{d|\phi|}{dr''} \right)^2 \right]$$
$$\times \left[\frac{n^2}{r'^2}|\phi|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right] + e^{N(0)} \right\}. \tag{3.9}$$

Here we choose a set of boundary conditions $B(0) = 1$ and $N(\infty) = 0$ according to the following reason: When we take the limit of both no matter ($T^\mu_\nu = 0$) and zero vacuum energy ($\Lambda = 0$), the spacetime reproduces Minkowski spacetime. Since a rescaling of the radial coordinate r leads to a flat cone with a deficit angle $2\pi[1 - \sqrt{B(0)}]$ in this limit, $B(0) = 1$ is an appropriate choice. If the coincidence of the proper time for the observer at spatial infinity is asked, then the temporal coordinate t selects $N(\infty) = 0$. In the context of a scalar field, the configuration of our interest is the solitonic one approaching its vacuum value at spatial infinity, i.e., $|\phi|(\infty) = v$. Now the one remaining boundary condition is about the scalar amplitude at the origin. If there is no coordinate and curvature singularity, single valuedness of scalar field forces $|\phi|(0) = 0$ for the vortex solution ($n \neq 0$). However, when we take into account the geometry with a curvature singularity or a black hole including the horizons, it is not necessary in general for the scalar field configuration to be nonsingular. Such singular solutions, so-called exotic black holes, where their scalar fields do not vanish at the origin, have been studied in (3+1)D [5]. However, it seems that there is a difference between the (3+1)D black holes and the BTZ solution: The mass accumulated at the core of the black hole induces a steep curvature change around its core and is crucial to make a black hole in (3+1)D curved spacetime, but the Schwarzschild-type BTZ solution does not have any signal of such an accumulation singularity and the divergent curvature at the origin of the Reissner-Norström-type BTZ solution in Eq. (2.19) is irrelevant to the black hole structure as explained previously. In this respect, an intriguing question is whether there is the global vortex solution interpolating smoothly $|\phi|(0) = 0$ and $|\phi|(\infty) = v$ even in asymptotically anti−de Sitter spacetime. As mentioned previously, the charged BTZ black hole made by the

electric point charge involves an unnecessary divergent curvature at the origin, and the regular extended objects, specifically the neutral global vortex, can have a chance to form a curvature-singularity-free charged BTZ black hole.

The question of whether or not there exist smooth vortex configurations is also intriguing in the context of the no-go theorem that this global U(1) scalar model cannot support a finite energy static regular vortex configuration in flat spacetime. Thus the global U(1) vortex carries logarithmically divergent energy. This symptom seems to appear in curved spacetime that the global U(1) vortex does not admit globally a well-behaved solution when $\Lambda=0$ [7]. The negative potential energy (the negative cosmological constant) comes in this model through the coupling of gravity although it does not have its own propagating degrees in (2+1)D. Therefore, one may expect the existence of a regular vortex configurations in anti–de Sitter spacetime, and we will show that it is indeed the case in the global U(1) model of our interest.

Near the origin, the power series solutions up to the leading term are

$$|\phi|(r)\sim\phi_0 r^n, \tag{3.10}$$

$$N(r)\sim N(0)+4\pi Gn\phi_0^2 r^{2n}, \tag{3.11}$$

$$B(r)\sim 1+(|\Lambda|-2\pi G\lambda v^4-8\pi G\phi_0^2\delta_{n1})r^2. \tag{3.12}$$

Since the right-hand side of Eq. (3.6) is positive definite, $N(r)$ is monotonically increasing everywhere. Equation (3.12) tells us that, when the scale of cosmological constant $|\Lambda|/\lambda v^2$ is smaller that the Planck scale $2\pi Gv^2$, then $B(r)$ starts to decrease near the origin.

Though we will take into account the geometry with the ''horizon'' and it may hinder the systematic expansion of the solution, let us attempt the power series solution up to the leading term for sufficiently large r:

$$|\phi|(r)\sim v-\frac{\phi_\infty}{r^2}, \tag{3.13}$$

$$N(r)\sim-\frac{8\pi G\phi_\infty^2}{r^4}, \tag{3.14}$$

$$B(r)\sim|\Lambda|r^2-8\pi Gv^2 n^2\ln r/r_c-8G\mathcal{M}+1+\mathcal{O}(1/r^2), \tag{3.15}$$

where $\mathcal{M}$ is the integration constant. Let us estimate the core mass $\mathcal{M}$ and the core size r_c of the global vortex. As a simple but valid approximation, let us assume

$$|\phi|(r)=\begin{cases}0 & \text{when } r<r_c,\\ v & \text{when } r\geqslant r_c,\end{cases} \tag{3.16}$$

and neglect the change of the metric function $N(r)$, i.e., $N\sim 0$. Substituting Eq. (3.16) into Eq. (3.9), we have

$$B(r)-|\Lambda|r^2-8\pi Gv^2 n^2\ln r-1$$

$$\approx 2\pi Gv^2(\lambda v^2 r_c^2-4n^2\ln r_c)$$

$$\geqslant 4\pi Gv^2 n^2(1-\ln 2n^2/\lambda v^2). \tag{3.17}$$

Here the minimum value in the second line of Eq. (3.17) is obtained when $r_c\sim n/\sqrt{\lambda}v$ and $\mathcal{M}\sim(\pi/2)v^2 n$. Crude as this approximation is, one can read the minimum point of $B(r)$:

$$r_m=\sqrt{\frac{4\pi Gv^2 n^2}{|\Lambda|}}, \tag{3.18}$$

which may be valid when the minimum position r_m is much larger than the core radius. The positivity of the core mass $\mathcal{M}$ after subtracting the logarithmic long tail is different from the global monopole in (3+1)D curved spacetime with a zero cosmological constant. The core mass of the global monopole is negative and the repulsive nature at the monopole core leads to the impossibility of the formation of a global monopole black hole even at the Planck scale [12].

Suppose that there exists a horizon, namely, the position r_H where the metric function $B(r)$ vanishes. At the horizon, the boundary conditions are from Eq. (3.3) and Eq. (3.7):

(i) $B_H=0,$

(ii) $$\left.\frac{d|\phi|}{dr}\right|_H=\frac{|\phi|_H\left[\dfrac{n^2}{r_H^2}+\lambda(|\phi|_H^2-v^2)\right]}{8\pi Gr_H\left[\dfrac{|\Lambda|}{4\pi G}-\left(\dfrac{n^2}{r_H^2}|\phi|_H^2+\dfrac{\lambda}{2}(|\phi|_H^2-v^2)^2\right)\right]}, \tag{3.19}$$

where $B_H=B(r_H)$ and $|\phi|_H=|\phi|(r_H)$. The behaviors of functions near the horizon are approximated by

$$|\phi|(r)\sim|\phi|_H+\left.\frac{d|\phi|}{dr}\right|_H(r-r_H)+\frac{1}{2}\left.\frac{d^2|\phi|}{dr^2}\right|_H(r-r_H)^2, \tag{3.20}$$

$$N(r)\sim N_H+N_{H1}(r-r_H)+N_{H2}(r-r_H)^2, \tag{3.21}$$

$$B(r) \sim B_{H1}(r - r_H) + B_{H2}(r - r_H)^2, \tag{3.22}$$

where $N_H = N(r_H)$, $d^2|\phi|/dr^2|_H$, N_{H1} N_{H2}, B_{H1}, and B_{H2} are expressed in terms of $|\phi|_H$ at r_H:

$$\frac{d^2|\phi|}{dr^2}\bigg|_H = \frac{|\phi|_H\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)^2\right]}{64\pi^2 G^2 r_H^2\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]^2}\left(-8\pi G\left\{\frac{n^2}{r_H^2}|\phi|_H^2 + \left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]\right.\right.$$

$$\left.\left.\times\left[1 + \frac{n^2}{r_H^2}\frac{1}{\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)^2\right]}\right]\right\} + \frac{3|\phi|_H^2\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)^2\right]^2}{2\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]} + \frac{1}{2}\left[\frac{n^2}{r_H^2} + \lambda(3|\phi|_H^2 - v^2)\right]\right), \tag{3.23}$$

$$N_{H1} = \frac{1}{8\pi G r_H}\frac{|\phi|_H^2\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)\right]^2}{\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]^2}, \tag{3.24}$$

$$N_{H2} = \frac{|\phi|_H^2\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)^2\right]^2}{64\pi^2 G^2 r_H^2\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]^3}\left(-8\pi G\left\{\frac{n^2}{r_H^2}|\phi|_H^2 + \left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]\right.\right.$$

$$\left.\left.\times\left[\frac{1}{2} + \frac{n^2}{r_H^2}\frac{1}{\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)^2\right]}\right]\right\} + \frac{3|\phi|_H^2\left[\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)^2\right]^2}{2\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]} + \frac{1}{2}\left[\frac{n^2}{r_H^2} + \lambda(3|\phi|_H^2 - v^2)\right]\right), \tag{3.25}$$

$$B_{H1} = 8\pi G r_H\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right], \tag{3.26}$$

$$B_{H2} = 4\pi G\left[\frac{|\Lambda|}{4\pi G} + \left(\frac{n^2}{r_H^2} - \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right] - \frac{3|\phi|_H^2\left(\frac{n^2}{r_H^2} + \lambda(|\phi|_H^2 - v^2)\right)^2}{2\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_H^2}|\phi|_H^2 + \frac{\lambda}{2}(|\phi|_H^2 - v^2)^2\right)\right]}. \tag{3.27}$$

Noticing that the expansion coefficients depend only on $|\phi|_H$ and not on N_H in Eq. (3.21), one may suspect a possibility that there does not exist a smooth solution interpolating $|\phi|(0)=0$ and $|\phi|(\infty)=v$ with one horizon or two. However, the horizon r_H in the expressions of the coefficients is also an undetermined parameter, which is the position to which the boundary conditions are applied. Therefore, we can expect a difficulty in analyzing the solutions by using the numerical technique, e.g., the shooting method.

If $B(r)$ is a decreasing function near the origin, there must exist a minimum point r_m of the metric function: $B(r) \geq B_m = B(r_m)$. Let us do a series expansion of $B(r)$ about the minimum,

$$|\phi|(r) \approx |\phi|_m + \frac{d|\phi|}{dr}\bigg|_m (r-r_m) + \frac{1}{2}\frac{d^2|\phi|}{dr^2}\bigg|_m (r-r_m)^2, \tag{3.28}$$

$$N(r) \approx N_m + 8\pi G r_m \left(\frac{d|\phi|}{dr}\bigg|_m\right)^2 (r-r_m) + N_{m2}(r-r_m)^2, \tag{3.29}$$

$$B(r) \approx B_m + B_{m2}(r-r_m)^2, \tag{3.30}$$

where the coefficients $d^2|\phi|/dr^2|_m$, N_{m2}, B_m, and B_{m2} can easily be evaluated in terms of the scalar amplitude $|\phi|_m$ and its derivative $d|\phi|/dr|_m$ at the minimum point r_m:

$$\frac{d^2|\phi|}{dr^2}\bigg|_m = -\left[8\pi G r_m\left(\frac{d|\phi|}{dr}\bigg|_m\right)^2 + \frac{1}{r_m}\right]\frac{d|\phi|}{dr}\bigg|_m$$
$$+ |\phi|_m\left(\frac{d|\phi|}{dr}\bigg|_m\right)^2$$
$$\times \frac{\frac{n^2}{r_m^2} + \lambda(|\phi|_m^2 - v^2)}{\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_m^2}|\phi|_m^2 + \frac{\lambda}{2}(|\phi|_m^2 - v^2)^2\right)}, \tag{3.31}$$

$$N_{m2} = 8\pi G\left(\frac{d|\phi|}{dr}\bigg|_m\right)^2 \left\{-1 + \frac{1}{2}\frac{d|\phi|}{dr}\bigg|_m\right.$$
$$- 8\pi G r_m^2\left(\frac{d|\phi|}{dr}\bigg|_m\right)^2 + r_m|\phi|_m\frac{d|\phi|}{dr}\bigg|_m$$
$$\left. \times \frac{\frac{n^2}{r_m^2} + \lambda(|\phi|_m^2 - v^2)}{\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_m^2}|\phi|_m^2 + \frac{\lambda}{2}(|\phi|_m^2 - v^2)^2\right)}\right\}, \tag{3.32}$$

$$B_m = \frac{1}{\left(\frac{d|\phi|}{dr}\bigg|_m\right)^2}\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_m^2}|\phi|_m^2 + \frac{\lambda}{2}(|\phi|_m^2 - v^2)^2\right)\right], \tag{3.33}$$

$$B_{m2} = 8\pi G r_m\left\{-2|\phi|_m\frac{d|\phi|}{dr}\bigg|_m\left(\frac{n^2}{r_m^2} + \lambda(|\phi|_m^2 - v^2)\right)\right.$$
$$+ \frac{n^2|\phi|_m^2}{r_m^3} + \left(8\pi G r_m\left(\frac{d|\phi|}{dr}\bigg|_m\right)^2 + \frac{1}{r_m}\right)$$
$$\left.\times\left[\frac{|\Lambda|}{4\pi G} - \left(\frac{n^2}{r_m^2}|\phi|_m^2 + \frac{\lambda}{2}(|\phi|_m^2 - v^2)^2\right)\right]\right\}. \tag{3.34}$$

From now on, let us examine vortex solutions in detail for the cases of zero and negative cosmological constants separately.

A. $\Lambda = 0$

First of all, under the Schwarzschild-type metric in Eq. (2.14), we read the case of a zero cosmological constant. Consider a scalar configuration with the boundary condition $|\phi|(0)=0$, which is consistent with the single valuedness of it at the origin under the vortex ansatz in Eq. (3.2). Suppose that $B(r)$ is continuous and starts from $B(0)=1$. Then, near the origin, $B(r)$ is a decreasing function as given in Eq. (3.12) and, since the right-hand side of Eq. (3.7) is negative as far as $B(r)$ is positive, it is monotonically deceasing. Furthermore, since the second term of T_t^t in Eq. (3.4) is dominant for large r when $|\phi|$ approaches the vacuum expectation value v, the negativity of the coefficient of the logarithmic term in Eq. (3.15) tells us that $B(r)$ goes to negative infinity at spatial infinity if we keep forcing the boundary condition $|\phi|(\infty)=v$. It means that there should exist r_H such that $B(r_H)=0$. Obviously, the nonsingular global string solution connecting $|\phi|(0)=0$ and $|\phi|(\infty)=v$ can not be supported unless the asymptotic region of space constructed by the U(1) global string is not well behaved. Furthermore, the spacetime formed by these global strings includes a physical curvature singularity which is not removable through coordinate transformation [7]. In the viewpoint of a smooth solution, there still remains the possibility of the existence of a scalar configuration which has oscillatory behavior around the vacuum expectation value v in the asymptotic region of negative $B(r)$, because the sign change of $B(r)$ effectively flips up the potential term on the right-hand side of Eq. (3.3).

B. $\Lambda < 0$

When the cosmological constant is negative, $B(r)$ in Eq. (3.12) near the origin increases or decreases according as its rescaled magnitude of the cosmological constant $|\Lambda|/\lambda v^2$ whether it is larger or smaller than the ratio of the Planck scale and the symmetry-breaking scale Gv^2. Once $B(r)$ begins decreasing from $r=0$, sometimes it can form a horizon as expressed in Eq. (3.22). Here we study the vortex solutions for three cases such that they have zero (regular

vortex with asymptotic hyperbola), one (extremal black hole), and two (Reissner-Nordström-type black hole) horizon(s).

1. Regular neutral vortex

In the previous subsection, we studied the global vortex case of a zero cosmological constant. As expressed in Eq. (3.9), the gravitational correction due to the object contributes negatively to $B(r)$ and becomes dominant for large r and finally changes the sign of $B(r)$ for some r. The problem was started by the very coordinate singularity, that is, the zero point of $B(r)$, and finally turns out to have a physical singularity. Therefore, if the sign of $B(r)$ does not change for every r, there may probably be a regular global vortex solution interpolating $|\phi|(0)=0$ and $|\phi|(\infty)=v$ smoothly. In this viewpoint, the positive contribution of the negative cosmological constant in Eq. (3.9) is drastic. No matter how small the magnitude of the cosmological constant is, this term is dominant for a sufficiently large r since it is proportional to the area of space ($\sim r^2$) for any solution with a boundary condition $d|\phi|/dr|_{r\to\infty}=0$. When $|\Lambda|/v^2$ is larger than $8\pi G v^2$, the first term in Eq. (3.9) can be larger than the second term. Then, in accordance with another boundary condition $B(0)=1$, $B(r)$ can always be positive for every r. Once $B(r)$ is regular and has only a positive sign for every r, a regular global vortex solution connecting $|\phi|(0)=0$ and
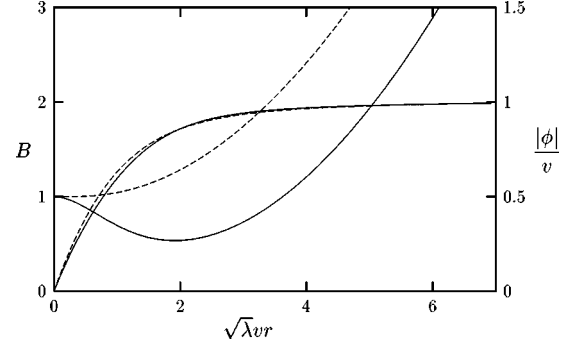

FIG. 2. Regular vortex configurations: $|\Lambda|/\lambda v^2=1.0$ and $8\pi G v^2=1.25$ for solid lines, and $|\Lambda|/\lambda v^2=0.1$ and $8\pi G v^2=1.15$ for dashed lines.

$|\phi|(\infty)=v$ smoothly exists, and so does $N(r)$ from Eq. (3.6). The shapes of solutions obtained numerically are shown in Fig. 2. The obtained regular neutral vortex solutions of the topological charge n are classified into two categories by the behavior of $B(r)$. When $|\Lambda|>2\pi G\lambda v^4-8\pi G\phi_0^2\delta_{n1}$ in Eq. (3.12), $B(r)$ is monotonically increasing as shown by the dashed lines in Fig. 2. When $|\Lambda|<2\pi G\lambda v^4-8\pi G\phi_0^2\delta_{n1}$ in Eq. (3.12), $B(r)$ must have a positive minimum for the regular solution as shown by the solid lines in Fig. 2.

From Eq. (3.33), we can roughly estimate a criterion for positive B_m or equivalently for the regular solution:

$$\frac{|\Lambda|}{4\pi G}\underset{\sqrt{\lambda}vr_m\sim1,\ |\phi|_m\sim v}{>}\lambda\left\{\frac{n^2}{(\sqrt{\lambda}vr_m)^2}\left(\frac{|\phi|_m}{v}\right)^2+\frac{1}{2}\left[\left(\frac{|\phi|_m}{v}\right)^2-1\right]^2\right\}\sim O(\lambda). \tag{3.35}$$

2. Extremal black hole

The extremal black hole has to satisfy the boundary condition $dB/dr|_{r_H}=0$ in addition to $B(r_H)=0$. Suppose that there is an extremal black hole; the position of the horizon can exactly be expressed by $|\phi|_H$ only:

$$r_H=\begin{cases}\sqrt{\dfrac{4\pi Gn^2|\phi|_H^2}{|\Lambda|-2\pi G\lambda(|\phi|_H^2-v^2)^2}} & \text{from Eq. (3.9) or } B_{H1}=B_m=0,\\[20pt]\dfrac{n}{\sqrt{\lambda(v^2-|\phi|_H^2)}} & \text{from Eq. (3.3).}\end{cases} \tag{3.36}$$

Combining these two, we obtain an explicit value for $|\phi|_H$,

$$|\phi|_H=v\left(1-\frac{|\Lambda|}{2\pi G\lambda v^4}\right)^{1/4}<v, \tag{3.37}$$

and r_H,

$$r_H=\frac{n}{\sqrt{\lambda}v\left(1-\sqrt{1-\dfrac{|\Lambda|}{2\pi G\lambda v^4}}\right)^{1/2}}\approx\begin{cases}\dfrac{n}{\sqrt{\lambda}v} & \text{when } 2\pi Gv^2\approx\dfrac{|\Lambda|}{\lambda v^2},\\[20pt]\sqrt{\dfrac{4\pi G\lambda v^4}{|\Lambda|}}\dfrac{n}{\sqrt{\lambda}v} & \text{when } 2\pi Gv^2\gg\dfrac{|\Lambda|}{\lambda v^2}.\end{cases} \tag{3.38}$$
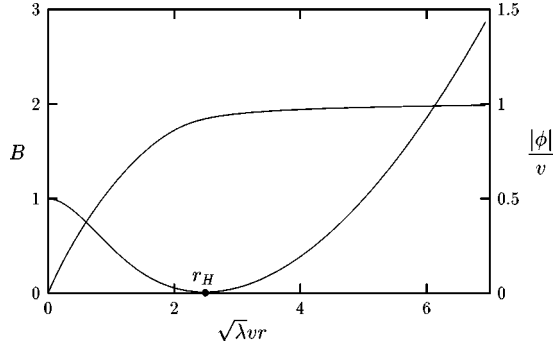
FIG. 3. Extremal black hole solution: $|\Lambda|/\lambda v^2 = 0.1$ and $8\pi G v^2 = 1.338$.

The second line in Eq. (3.38) coincides with the result in Eq. (3.18) based on a rough estimation. The complete determination of the position of the horizon r_H and the value of the scalar amplitude $|\phi|_H$ is an inevitable result since the leading terms of Eq. (3.3) and Eq. (3.7) lead to two algebraic equations in the case of the extremal black hole. Therefore, the series expansion of Eqs. (3.3), (3.6), and (3.7) in order of $(r - r_H)$ determines all coefficients of the power series solutions $|\phi|(r)$, $N(r)$, and $B(r)$ in closed form.

Before arguing for the existence of a global vortex solution constituting an extremal black hole, a comment should be made here. To guarantee a solution of Eq. (3.37) and Eq. (3.38) we must impose the following condition: $|\Lambda|/2\pi G \lambda v^4 < 1$, since this is nothing but the condition to distinguish the $n > 2$ solutions with a minimum at nonzero r from those with monotonically increasing $B(r)$ in Sec. III B 1. So it is not useful except for $n = 1$ solutions. Since $dB/dr|_H = 0$ at the horizon, B_{H1} in Eq. (3.22) and Eq. (3.26) must vanish at the horizon r_H, and then B_{H2} in Eq. (3.27) is automatically positive:

$$B_{H2} = B_{m2} = 8\pi G \lambda v^4 \sqrt{1 - \frac{|\Lambda|}{2\pi G \lambda v^4}}$$

$$\times \left(1 - \sqrt{1 - \frac{|\Lambda|}{2\pi G \lambda v^4}}\right) > 0. \qquad (3.39)$$

This implies a convex-down property of $B(r)$ at the horizon of the extremal black hole. In addition the positivity of the slope of the scalar amplitude gives a condition

$$\frac{|\Lambda|}{\lambda v^2} + \frac{1}{32\pi G v^2} < \frac{1}{2}, \qquad (3.40)$$

which is more restrictive than the previous one from Eq. (3.37) and Eq. (3.38). The power series expansion of higher order terms will make the condition more and more restrictive. Note that the data for our numerical solution in the caption of Fig. 3 are consistent with Eq. (3.40).

Now the remaining task is to find numerically a patch of solution connecting smoothly the boundary conditions $[|\phi|(0) = 0, \; B(0) = 1]$ and $[|\phi|_H = (3.37), \; B_H = dB/dr|_H = 0]$ by use of the shooting method for various

(ϕ_0, N_0)'s in Eq. (3.10) and Eq. (3.12), and the other patch from the horizon to spatial infinity $[|\phi|(\infty) = v, \; N(\infty) = 1]$. We obtain one for a model with $8\pi G v^2 = 1.338$ and $|\Lambda|/\lambda v^2 = 0.1$ (see Fig. 3).

3. Charged black hole with two horizons

As the scale of cosmological constant $|\Lambda|/\lambda v^2$ becomes smaller than the critical value to support the extremal black hole, we expect to witness a charged black hole with two horizons. One can easily read this phenomenon by examining the integral equation of $B(r)$. The right-hand side of Eq. (3.9) involves two contributions; the positive term is proportional to $|\Lambda|$ while the negative one is proportional to G. Obviously two terms are zero at the origin, and so $B(0)$ is equal to 1 (positive). If the second term dominates due to $|\Lambda|/\lambda v^2 \ll 8\pi G v^2$, then $B(r)$ becomes negative at some intermediate region. However, for sufficiently large r, the first term proportional to the square of the radius r is much larger than the second term of which the leading contribution is logarithmic, and finally $B(r)$ becomes positive again.

From now on, let us discuss details about the existence of two horizons in several steps. The first step is to show that there exists the inner horizon $r_{\rm in}$ where $|\phi|(r_{\rm in}) < v$ and $d|\phi|/dr|_{r_H} > 0$. Let us assume a situation that $2\pi G(\lambda v^4 + 4\phi_0^2 \delta_{1n})$ is much larger than $|\Lambda|$ in Eq. (3.12) and $|\phi|(r) \approx \phi_0 r^n$. Then there exists an appropriate ϕ_0 for a sufficiently small r such that $|\phi|(r) < v$ and $B(r)$ hits the zero point approximately at $r_{\rm in} \approx 1/\sqrt{2\pi G(\lambda v^4 + 4\phi_0^2 \delta_{1n}) - |\Lambda|}$. The second step deals with the outer horizon. Again let us assume that the scaled cosmological constant $|\Lambda|/\lambda v^2$ is much smaller than the ratio of the square of the symmetry-breaking scale and that of the Planck scale $2\pi G v^2$. Then there exists obviously a position r such that Eq. (3.15) without the term of order $1/r^2$ is valid ($r \gg r_c$) and, simultaneously, the $|\Lambda| r^2$ term can also be neglected:

$$(3.15) \sim -8\pi G v^2 n^2 \ln r/r_c - 8G\mathcal{M} + 1$$

$$\sim 4\pi G v^2 n^2 (1 - 2\ln r/r_c) + 1. \qquad (3.41)$$

If the second logarithmic term on the right-hand side of Eq. (3.41) is larger than order 1 for some r, the value of $B(r)$ is negative at this range of r. For an arbitrarily small positive ϕ_∞, $|\phi|(r) < v$. In addition, $d|\phi|/dr|_H$ is positive because of the negativity of both the numerator and denominator in Eq. (3.19). The third step begins with recalling r_m obtained in Eq. (3.18) by minimizing $B(r)$. For the condition $2\pi G v^2 > |\Lambda|/\lambda v^2$, $r_m > r_H^{\rm in}$. Since Eq. (3.15) tells us that $B(r_m) < 0$ when $8\pi G v^2 n^2 > 1$, $r_{\rm out}$ should exist and be larger than r_m. Therefore, the remaining step is to find a smooth scalar field $|\phi|(r)$ to have two horizons $r_H^{\rm in}$ and $r_H^{\rm out}$ through numerical analysis. The above range does not forbid the convexity of the metric function $B(r)$ both for shooting from the outside and for shooting from the inside; it implies the possibility of the existence of a smooth configuration to connect $|\phi|(0) = 0$ and $|\phi|(\infty) = v$. The approximate value
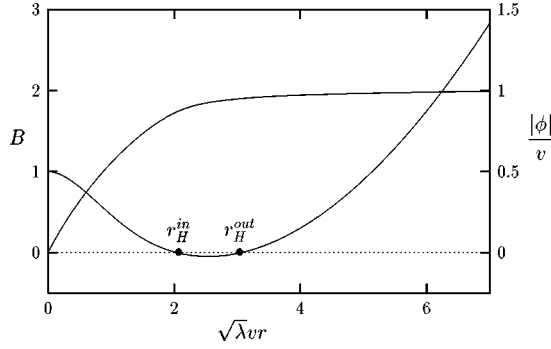
FIG. 4. A charged black hole solution with two horizons r_H^{in} and r_H^{out}: $|\Lambda|/\lambda v^2 = 0.1$ and $8\pi G v^2 = 1.4$.

of the inner horizon is $\sqrt{\lambda} v r_H^{\text{in}} \approx 2.03$ and that of the outer horizon is $\sqrt{\lambda} v r_H^{\text{out}} \approx 3.05$ from Fig. 4.

Now we have three types of global U(1) vortices in anti–de Sitter spacetime. Smooth as the configurations of the scalar field $|\phi|(r)$ are, the forms of the metric function $B(r)$ contain horizons. At this stage, we should make it clear whether the singularity at each horizon is a coordinate artifact or a physical singularity. The answer is given by examining the square of the curvature: Specifically, the Kretschmann scalar is expressed by the Einstein tensors in 2+1 dimensions,

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$= 4 G_{\mu\nu}G^{\mu\nu} = 4 \, \text{Tr}\left[\text{diag}\left(-\frac{1}{2r}\frac{dB}{dr}, -\frac{1}{2r}\frac{dB}{dr} - \frac{B}{r}\frac{dN}{dr}, \right.\right.$$

$$\left.\left. -\frac{1}{2}\frac{d^2B}{dr^2} - \frac{3}{2}\frac{dB}{dr}\frac{dN}{dr} - B\frac{d^2N}{dr^2} - B\left(\frac{dN}{dr}\right)^2 \right)\right]. \quad (3.42)$$

At first glance one can read no curvature singularity at the horizons from the expression (3.42). The forms of Eq. (3.6) and Eq. (3.7) tell us that there is no divergent curvature at the origin as far as the scalar field $|\phi|(r)$ behaves regularly, i.e., $|\phi|(r) \sim r^n$.

As mentioned previously, a characteristic of BTZ black holes is that they need not contain divergent curvature at real r. We have understood that the Schwarzschild-type solution in Eq. (2.13) is regular everywhere and the charged black hole formed by the neutral vortex discussed in the Sec. III B 3 has also a divergent curvature counterpart of a charged BTZ black hole formed by the electric point source in Eq. (2.19). Here let us clarify the structure of manifolds and the nature of forces due to the global vortices by studying the geodesics of test particles [13].

The geometry depicted by the metric in Eq. (2.8) admits two Killing vectors $\partial/\partial t$ and $\partial/\partial\theta$, and then the constants of motion along the geodesic are

$$\gamma = Be^{2N}\frac{dt}{ds} \quad \text{and} \quad L = r^2\frac{d\theta}{ds}, \quad (3.43)$$

where s is an affine parameter along the geodesic. Note that one cannot interpret γ as the local energy of a particle at spatial infinity, since the spacetime is not asymptotically flat.

Using these constants of motion, we obtain the following geodesic equation for the radial motions:

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 = -\frac{1}{2}\left[B(r)\left(m^2 + \frac{L^2}{r^2}\right) - \frac{\gamma^2}{e^{2N(r)}} \right] = -V(r),$$

$$(3.44)$$

where m can be rescaled to be 1 for any time like geodesic and 0 for a null-like geodesic. The radial equation in Eq. (3.44) is an analogue of Newton's equation for $r \geq 0$ with a conservative effective potential $V(r)$ in which the hypothetical particle has unit mass and zero total mechanical energy. In order to identify the existence of the black hole, a meaningful quantity is the elapsed coordinate time t of the static observer at r_0 for the motion of a test particle from r_0 to r:

$$t = \int_r^{r_0} \frac{dr}{B(r)e^{N(r)}\sqrt{1 - (1/\gamma^2)\left(m^2 + L^2/r^2\right)B(r)e^{2N(r)}}}.$$

$$(3.45)$$

When we analyze the motions of test particles, they are divided into four categories according to whether they have mass $(m=1)$ or not $(m=0)$, or whether their motions are purely radial $(L=0)$ or rotating $(L \neq 0)$.

a. $(m=0, \ L=0)$ (regular solution). For the radial motion of a massless test particle, the effective potential has no explicit dependence on $B(r)$ such that

$$V(r) = -\frac{\gamma^2}{2}e^{-2N(r)} \leq 0. \quad (3.46)$$

The allowed motions are (i) stopped particle motion for $\gamma=0$, which is unstable, and (ii) an unbounded motion for $\gamma \neq 0$ since the potential is negative everywhere even at spatial infinity. Moreover, $N(r)$ is monotonic increasing, and so the radial force is always attractive. Then the test particle starts with initial speed greater than $dr/d\tau = \gamma/\sqrt{2}$ at a point r_0 and approaches the center of the vortex. Obviously there is no horizon [$B(r) > 0$ for all r], and a test particle starting from a finite initial position r_0 arrives at the origin in a finite time measured by the clock of the static observer; i.e., $t = \int_0^{r_0} dr/B(r)e^{N(r)}$ is finite.

b. $(m=0, \ L \neq 0)$ (regular solution). For a rotational motion, the centrifugal force term is introduced in the potential $V(r)$:

$$V(r) = \frac{1}{2}\left(\frac{L^2 B(r)}{r^2} - \frac{\gamma^2}{e^{2N(r)}} \right), \quad (3.47)$$

which forbids the motion near the core of the vortex. From Fig. 5, we read the minimum value of γ, γ_{cr}, which means no motion is allowed for γ smaller than γ_{cr} (see the dotted line in Fig. 5) and the circular orbit is at the radius r_{cr} for $\gamma = \gamma_{\text{cr}}$ (see the solid line in Fig. 5). When $\gamma > \gamma_{cr}$, the motions are classified into two categories according to the values of γ/L: (i) When $\gamma/L \geq \sqrt{|\Lambda|}$, the motion is unbounded in which the speed of the test particle at spatial infinity is $dr/d\tau|_{r=\infty} = \sqrt{L^2|\Lambda| - \gamma^2}$ [see dashed lines (ii) and (iii) in Fig. 5]. (ii) When $\gamma_{\text{cr}}/L < \gamma/L < \sqrt{|\Lambda|}$, it is a bounded orbit
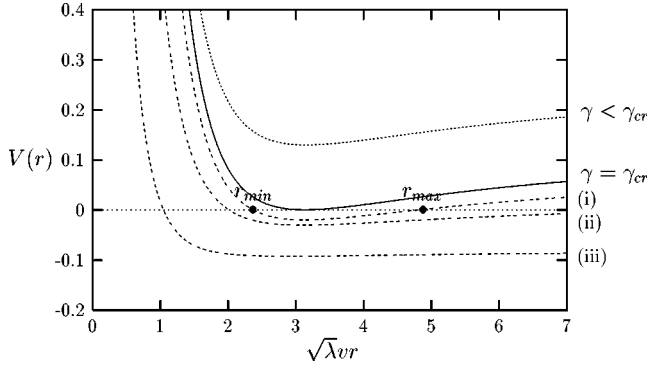
FIG. 5. Schematic shapes of the effective potential $V(r)$ for the rotational motions of the massless test particle when $L=1$. Here $B(r)$ is always positive.

with perihelion r_{min} and aphelion r_{max} [see a dashed line (i) in Fig. 5]. For the bounded motion, the apsidal distance $r_{max}-r_{min}$ is in order $\sqrt{\lambda}v$.

c. ($m=1$, $L=0$) (*regular solution*). For the radial motion of a massive test particle, the potential of it is

$$V(r) = \frac{1}{2}\left(B(r) - \frac{\gamma^2}{e^{2N(r)}}\right). \tag{3.48}$$

Since $B(r)\sim|\Lambda|r^2$ for sufficiently large r, all possible motions are bounded; i.e., there exists a r_{max} such that $r\lesssim r_{max}$ for any position r of the test particle. When $B(r)$ is monotonic increasing, $V(r)$ is also monotonic increasing and thereby the force is attractive everywhere. It is physically natural since sufficiently large negative vacuum energy can pervade all space despite the repulsive force at the core of the vortex. Therefore, the motions are allowed only when $\gamma\gtrsim e^{N(0)}$: When $\gamma=e^{N(0)}$, the test particle stops at the origin, and, when $\gamma>e^{N(0)}$, the particle can move inside r_{max} and the stopped test particle starts to go inward. When $B(r)$ has a positive minimum at a nonzero r, the schematic shapes of the effective potential $V(r)$ are given in Fig. 6, and the possible radial motions are classified in Table I since it seems that $\gamma_{stop}<e^{N(0)}$ in our solution.

d. ($m=1$, $L\neq0$) (*regular solution*). For the rotational motions of a massive test particle, the effective potential takes the general form

TABLE I. The radial motions of a test particle of mass $m=1$ for various γ's, when $|\Lambda|/\lambda v^2=1.0$, $Gv^2=1.0$, and L is rescaled to 1.

Fig. 6	γ	Orbit,force
(i)	$0\leqslant\gamma<\gamma_{stop}$	No orbit
(ii)	$\gamma=\gamma_{stop}$	Stopped motion at r_{stop}
(iii)	$\gamma_{stop}<\gamma<e^{N(0)}$	Oscillation between r_{min} and r_{max}
(iv)	$\gamma=e^{N(0)}$	$r_{min}=0$
(v)	$e^{N(0)}<\gamma<\gamma_{cr}$	$r\lesssim r_{max}$, repulsive near the core
(vi)	$\gamma\geqslant\gamma_{cr}$	$r\lesssim r_{max}$, attractive everywhere

$$V(r) = \frac{1}{2}\left[B(r)\left(1+\frac{L^2}{r^2}\right) - \frac{\gamma^2}{e^{2N(r)}}\right]. \tag{3.49}$$

$V(r)\sim\frac{1}{2}|\Lambda|r^2$ for large r, and $V(r)\sim L^2/2r^2$ for small r. Therefore, there exists a critical value of γ, γ_{circ}, for positive $B(r)$ such that there is no orbit for γ smaller than this critical value γ_{circ}. The allowed motions are (i) the circular orbit at r_{circ} when $\gamma=\gamma_{circ}$ and (ii) the bounded orbit between the perihelion r_{min} and aphelion r_{max} when γ is larger than γ_{circ} (See Fig. 7). Similar to the previous bounded orbit motions, the range of allowed region is roughly estimated as a few $1/\sqrt{\lambda}v$.

Now we have the global vortex configurations with horizons, i.e., the points of vanishing $B(r)$. We examine the possible motions of massless and massive test particles under the influence of this geometry and identify these manifolds as those of extremal and Reissner-Nordström-type black holes. Similar to the case of regular solutions, we analyze the orbits for four categories. For the extremal case, there is no distinction from the Reissner-Nordström case when we set $r_H=r_H^{in}=r_H^{out}$.

e. ($m=0$, $L=0$) (*extremal and charged black holes*). For the radial motions of a massless test particle, the effective potential $V(r)$ does not depend on $B(r)$ as in Eq. (3.46), and so the analysis in Sec. III B 1 a is the same as that for this case. However, since $B(r)$ includes a negative region between two horizons r_H^{in} and r_H^{out}, the radial motions are divided into two: one around the vortex core inside the inner horizon ($r<r_H^{in}$) and the other outside the outer horizon
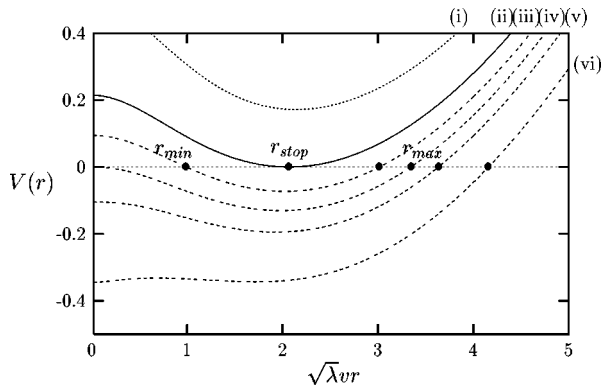

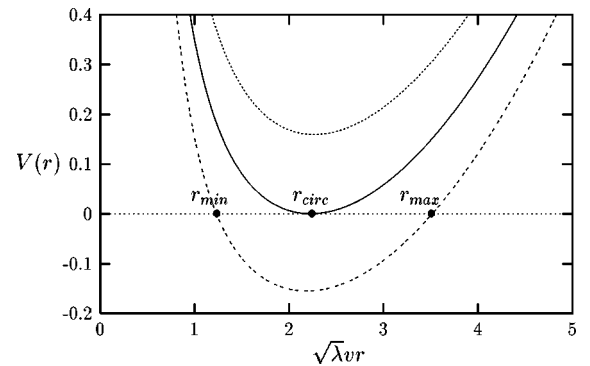FIG. 6. Schematic shapes of the effective potential $V(r)$ for the radial motions of the massive particle. Here $B(r)$ has its positive minimum at a positive r.

FIG. 7. Schematic shapes of the effective potential $V(r)$ for the rotational motions of the massive particle. γ_{circ} is 0.832 and $L=1$.
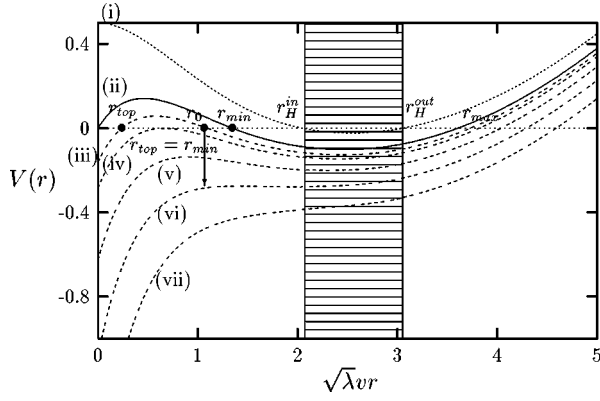
FIG. 8. Schematic shapes of the effective potential $V(r)$ for the radial motions of the massive particle. The shaded region between the inner and outer horizons is forbidden for the test particle.

$(r>r_H^{\text{out}})$. Though the motions at the particle's coordinates resemble those of regular $B(r)$, they are observed with a drastic difference to the static observer. Since $B(r)$ vanishes both at inner and outer horizons, the elapsed time t to reach a horizon is logarithmically divergent in terms of coordinate time for the static observer:

$$t\sim\lim_{\varepsilon\to 0^+}\int_{r_0}^{r_H^{\text{out}}+\varepsilon}\frac{dr}{B(r)} \tag{3.50}$$

$$\sim\left(\lim_{\varepsilon\to 0^+}\int_{r_0}^{r_H^{\text{out}}+\varepsilon}\frac{dr}{r-r_H^{\text{out}}}\right)\times(\text{finite part}), \tag{3.51}$$

where $r_0>r_H^{\text{out}}+\varepsilon$. Since the potential is attractive outside the outer horizon, the ingoing particle takes an infinite time to reach the outer horizon for the static observer. If we replace $r_H^{\text{out}}+\varepsilon$ by $r_H^{\text{in}}-\varepsilon$ and $r_0<r_H^{\text{in}}-\varepsilon$, then one can easily notice that the situation is the same for the case inside the inner horizon. However, one must remember the attractive nature of the force inside the inner horizon, which causes the test particle to move the center of the vortex.

f. ($m=0$, $L\neq 0$) (*extremal and charged black holes*). The rotational motion of a massless test particle is described by the effective potential in Eq. (3.47). Since $B(r)$ is negative between the horizons ($r_H^{\text{in}}<r<r_H^{\text{out}}$) and the $N(r)$ term in Eq. (3.47) is always negative, the shapes of the potential for this case correspond to the dashed lines [(i), (ii), and (iii)] in Fig. 5. Then the allowed regions are as follows: When $\gamma/L<\sqrt{|\Lambda|}$, $r_{\text{min}}\leq r<r_H^{\text{in}}$ and $r_H^{\text{out}}<r\leq r_{\text{max}}$ for two bounded motions. When $\gamma/L\geq\sqrt{|\Lambda|}$, $r_{\text{min}}\leq r<r_H^{\text{in}}$ for a bounded motion inside the black hole and $r_H^{\text{out}}<r$ for an unbounded motion outside the outer horizon. As noted in Eq. (3.51), the time elapsed to reach a horizon for a static observer is infinite, which means no orbital motion.

g. ($m=1$, $L=0$) (*extremal and charged black holes*). As discussed in Sec. III B 3 f, $V(r)$ includes a negative region between $r_{\text{min}}(r_{\text{min}}<r_H^{\text{in}})$ and $r_{\text{max}}(r_{\text{max}}>r_H^{\text{out}})$. Therefore, for various γ values, one can expect two patterns; one is given in the lines (iii)–(vi) in Fig. 6 and the other is summarized below (see Fig. 8). The corresponding solutions are provided in Table I for the former and in Table II for the latter. For the BTZ black hole solutions we obtained, the extremal solution follows Fig. 6 [(iii)–(vi) in Table I] and the BTZ solution with two horizons follows Fig. 8 (Table II).

h. ($m=1$, $L\neq 0$) (*extremal and charged black holes*). For any rotating motion for the massive particle under the influence of $B(r)$ for the vortex black hole, the allowed regions are $r_{\text{min}}\leq r<r_H^{\text{in}}$ in which the force is repulsive, and $r_H^{\text{out}}<r\leq r_{\text{max}}$ in which the force is attractive (see the dashed line in Fig. 7).

Under the metric written as in Eq. (2.1) the conserved quasilocal mass measured by a static observer at r is given by [14]

$$8GM_q=2\sqrt{e^{2N(r)}B(r)}\left[\sqrt{B_0(r)}-\sqrt{B(r)}\right]. \tag{3.52}$$

Here $B_0(r)$ is the background metric g^{rr} which determines the zero point of energy. The background can be obtained simply by setting the integration constant of a particular solution to some specific value that specifies the reference frame. As we discussed previously, the background is the spacetime without the global vortex, specifically, $n=0$ and $|\phi|(r)=v$ and thereby $B_0(r)=|\Lambda|r^2+1$. When the spacetime is asymptotically flat, the usual Arnowitt-Deser-Misner(ADM) mass M_q is determined in the limit $r\to\infty$. For sufficiently large r, Eq. (3.14) and Eq. (3.15) give

TABLE II. The radial motions of a test particle of mass $m=1$ for various γ's, when $|\Lambda|/\lambda v^2=1.4$, $Gv^2=0.1$.

Fig. 8	γ	Orbit force
(i)	$\gamma<e^{N(0)}$	$r_{\text{min}}\leq r<r_H^{\text{in}}$ (repulsive), $r_H^{\text{out}}<r\leq r_{\text{max}}$ (attractive)
(ii)	$\gamma=e^{N(0)}$	$r_{\text{min}}\leq r<r_H^{\text{in}}$ (repulsive), $r_H^{\text{out}}<r\leq r_{\text{max}}$ (attractive)
		Stopped motion at $r=0$
(iii)	$e^{N(0)}<\gamma<\gamma_{\text{top}}$	$r_{\text{min}}\leq r<r_H^{\text{in}}$ (repulsive), $r_H^{\text{out}}<r\leq r_{\text{max}}$ (attractive)
		$r<r_{\text{top}}<r_{\text{min}}$ (attractive)
(iv)	$\gamma=\gamma_{\text{top}}$	$r<r_{\text{top}}$ (attractive), stopped motion at r_{top}
		$r<r_H^{\text{in}}$ (repulsive), $r_H^{\text{out}}<r\leq r_{\text{max}}$ (attractive)
(v)	$\gamma_{\text{top}}<\gamma<\gamma_{\text{flat}}$	$r<r_H^{\text{in}}$ (attractive-repulsive), $r_H^{\text{out}}<r<r_{\text{max}}$ (attractive)
(vi)	$\gamma=\gamma_{\text{flat}}$	$r<r_H^{\text{in}}$ (no force at r_0, attractive elsewhere)
		$r_H^{\text{out}}<r<r_{\text{max}}$ (attractive)
(vii)	$\gamma_{\text{flat}}<\gamma$	$r<r_H^{\text{in}}$, $r_H^{\text{out}}<r<r_{\text{max}}$ (attractive everywhere)

$$M_q \overset{r\to\infty}{\to} \pi n^2 v^2 \ln r/r_c + \mathcal{M}. \qquad (3.53)$$

Thus the quasilocal mass determined at $r\to\infty$ contains two terms: finite negative mass from the core of the vortex and the logarithmically divergent one from the topological sector of Goldstone degree. This coincides approximately with the mass formula of the global vortex in flat spacetime, which is obtained by the spatial integration of the time-time component of the energy-momentum tensor. Though M_q cannot be identified as ADM mass due to the hyperbolic structure of global spacetime of our interest, its form looks natural once we recall the nonpropagation of a $(2+1)$-dimensional graviton in anti–de Sitter gravity.

IV. TOPOLOGICAL CHARGE AS A BLACK HOLE CHARGE

In the previous section we have shown that the long tail of a neutral static vortex can provide the black hole charge in a $(2+1)$-dimensional BTZ black hole. By use of the duality transformation [15], we construct the direct relationship between the topological charge n of the neutral vortex and the electric charge of the dual transformed theory. Through this analysis the reason why the vorticity n of neutral objects can play the same role as (electric) charge of the BTZ black hole will be manifest.

The path integral of our theory is written as

$$Z = \int [dg_{\mu\nu}][d\bar{\phi}][d\phi]\exp i\left\{\int d^3x\sqrt{g}\left[-\frac{1}{16\pi G}(R+2\Lambda)\right.\right.$$
$$\left.\left.+\frac{1}{2}g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi - V(|\phi|)\right]\right\}. \qquad (4.1)$$

Rewrite the scalar field in the path integral in terms of the radial variables $\phi = |\phi|e^{i\Omega}$ and linearize the term of the scalar phase such that

$$\int [d\Omega]\exp\left\{i\int d^3x\sqrt{g}\frac{g^{\mu\nu}}{2}|\phi|^2\partial_\mu\Omega\partial_\nu\Omega\right\}$$

$$= \prod_x |\phi|^{-3}g^{1/4}\int [d\Omega][dC_\mu]$$

$$\times\exp\left\{i\int d^3x\sqrt{g}\left[-\frac{g^{\mu\nu}}{2}\left(\frac{C_\mu C_\nu}{|\phi|^2}-2C_\mu\partial_\nu\Omega\right)\right]\right\}. \qquad (4.2)$$

Let us divide the configurations of the scalar phase by the topological sector Θ which is $(\epsilon^{\mu\nu\rho}/\sqrt{g})\,\partial_\nu\partial_\rho\Theta\neq 0$, and the single-valued part η which satisfies $(\epsilon^{\mu\nu\rho}/\sqrt{g})\,\partial_\nu\partial_\rho\eta = 0$: $\Omega = \Theta + \eta$ and $[d\Omega] = [d\Theta][d\eta]$. Integrating out η and using $[d\partial_\mu\eta] = [d\eta]$ up to a field-independent Jacobian factor, we have

$$(4.2) = \prod_x |\phi|^{-3}g^{1/4}\int [d\Theta][d\eta_\mu]$$

$$\times[dC_\mu]\delta\left(\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}\partial_\nu\eta_\rho\right)\exp\left\{i\int d^3x\sqrt{g}\right.$$

$$\left.\times\left[-\frac{g^{\mu\nu}}{2|\phi|^2}C_\mu C_\nu + g^{\mu\nu}C_\mu(\partial_\nu\Theta + \eta_\nu)\right]\right\}. \qquad (4.3)$$

Let us rewrite the δ functional in Eq. (4.3) by introducing the dual vector field A_μ, i.e.,

$$\delta\left(\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}\partial_\nu\eta_\rho\right) = \int[dA_\mu]\exp\left\{-iv\int d^3x\sqrt{g}\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}A_\mu\partial_\nu\eta_\rho\right\},$$

and integrate out η_μ. Then we obtain a relation from the δ functional, $C_\mu = (v/2)\sqrt{g}\epsilon_{\mu\nu\rho}F^{\nu\rho}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Finally if we do the integration over the vector auxiliary field C_μ, then Eq. (4.1) becomes

$$Z = \int [g^{3/4}dg_{\mu\nu}][|\phi|^{-2}d|\phi|][dA_\mu][d\Theta]$$

$$\times\exp\left\{i\int d^3x\sqrt{g}\left[-\frac{1}{16\pi G}(R+2\Lambda)\right.\right.$$

$$+\frac{1}{2}g^{\mu\nu}\partial_\mu|\phi|\partial_\nu|\phi| - V(|\phi|) - \frac{v^2}{4|\phi|^2}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}$$

$$\left.\left.+\frac{v\,\epsilon^{\mu\nu\rho}}{2\sqrt{g}}F_{\mu\nu}\partial_\rho\Theta\right]\right\}. \qquad (4.4)$$

This duality transformation can be achieved in arbitrary $(D+1)$ dimensions by use of an antisymmetric tensor field of rank $(D-1)$, and so the Maxwell-like term in Eq. (4.4) becomes nothing but the Kalb-Ramond action [16] in $(2+1)$D. The Euler-Lagrange equations read

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu|\phi|) = \frac{v^2}{2|\phi|^3}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{dV}{d|\phi|}, \qquad (4.5)$$

$$\frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}\frac{v^2}{|\phi|^2}F^{\mu\nu}\right) = \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}\partial_\nu\partial_\rho\Theta, \qquad (4.6)$$

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}(R+2\Lambda) = \frac{v^2}{4|\phi|^2}g^{\rho\sigma}(g_{\mu\nu}g^{\tau\kappa} - 4g_\mu^\tau g_\nu^\kappa)F_{\rho\tau}F_{\sigma\kappa}$$

$$+\left(g_\mu^\rho g_\nu^\sigma - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\right)\partial_\rho|\phi|\partial_\sigma|\phi|$$

$$+g_{\mu\nu}V. \qquad (4.7)$$

Since we are interested in neutral objects which do not carry a global U(1) charge $(C_0 = g_{0\mu}|\phi|^2\partial^\mu\Omega = 0)$, they do not carry a dual magnetic field $[F^{ij} = (\epsilon^{0ij}/\sqrt{g})\,C_0 = 0]$. Thus

the spatial components of the equation for the dual gauge field are automatically satisfied. The time component of Eq. (4.6) is nothing but Gauss' law in the asymptotic region for large r ($|\phi| \to v$): For the rotationally symmetric vortex solutions $\Theta = n\theta$, it is

$$\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} F^{0i}) \approx n \frac{1}{\sqrt{g}} \delta^{(2)}(\vec{x}). \qquad (4.8)$$

The next order term of the scalar amplitude due to the small perturbation from the vacuum value v does not contribute to the charge, and so one can easily identify the vorticity n as the electric charge of the dual gauge field. Similarly, since the scalar amplitude terms, which are the second and third terms on the right-hand side of Eq (4.7), fall rapidly as the radial coordinate r increases, the time-time component of Einstein equations in Eq. (4.7) has the leading contribution from the negative cosmological constant term and the next leading term from the electric energy for large r:

$$G_{00} \approx e^{2N} B \left(\Lambda + \frac{v^2}{2|\phi|^2} (F_{0i})^2 \right) \sim e^{2N} B \left(\Lambda + v^2 \frac{n^2}{r^2} \right). \qquad (4.9)$$

Obviously, the electric field can be identified as that of the point charge at the origin. The self-energy in flat spacetime contains a logarithmic divergence. Therefore this topological charge can constitute the charge of the BTZ black holes: $B(r) \approx |\Lambda| r^2 - 8\pi G v^2 n^2 \ln r - \mathcal{M}$ for large r.

At the core of the vortex, the nonvanishing component of the dual electric field F_{0r} is regular: For small r,

$$F_{0r} \sim -n \frac{\phi_0}{v^2} r^{2n-1}, \qquad (4.10)$$

since $|\phi|(r) \sim \phi_0 r^n$. The dual electric field term of the energy-momentum tensor in Eq. (4.9) is also regular:

$$T_{00} \propto e^{2N} B \frac{v^2}{2|\phi|^2} (F_{0r})^2 \sim \frac{n^2}{2} \frac{\phi_0^2}{v^2} e^{2N(0)} r^{2(n-1)}. \qquad (4.11)$$

Therefore, the role of $1/|\phi|^2$ in Eq. (4.6) and Eq. (4.7) is a regulator of the soliton at its core.

Now we have an understanding that the addition of the global vortex of vorticity n to the center of the Schwarzschild-type BTZ black hole produces a Reissner-Nordström-type BTZ black hole of electric charge n. This implies that spinless static vortices with finite energy in flat spacetime, e.g., the topological charge of Abrikosov-Nielsen-Olesen vortices in the Abelian Higgs model or that of topological lumps in the O(3) nonlinear σ model, cannot give rise to an additional BTZ black hole (electric) charge since they do not carry long tail of energy density. However, it is an open question as to whether the spinning charged solitons, e.g., Q lumps (or nontopological global vortices) [17] or topological or nontopological vortices in Chern-Simons theories [18,19], can constitute a BTZ black hole with both charge and spin.

V. PHYSICAL RELEVANCE AS A BLACK COSMIC STRING IN (3+1)D

We have considered the global vortices in (2+1)D curved spacetime: however, these point-particle-like extended objects on the spatial plane may describe the straight global U(1) strings along the z direction [4]. If we consider a static metric of cylindrically symmetric string along the z axis

$$ds^2 = B(r) e^{2N(r)} (dt^2 - dz^2) - \frac{dr^2}{B(r)} - r^2 d\theta^2, \qquad (5.1)$$

which also has boost invariance in the z direction, the previous analysis moves to (3+1)-dimensional anti–de Sitter spacetime with no change since we already adjusted the dimension of fields and constants to those in 3+1 dimensions.

Here let us take into account a perfect situation: A global U(1) static string straight along the z axis was generated in some symmetry-breaking scale v and has evolved safely to a static object in the present universe. Inserting the Newton constant and the present lower bound of the cosmological constant into Eq. (3.39), we have the critical value for an extremal black hole in our room temperature scale, $|\Lambda|/2\pi\lambda v^4 \approx 0.3$ eV. This implies that, when the cosmological constant is negative and bounded by the experimental lower limit in the present universe [$-(0.34-0.99) \times 10^{-83}$ (GeV)$^2 < \Lambda < (0.68-1.98) \times 10^{-83}$ (GeV)^2 [20]], the global strings produced at almost all the scales remain as charged black strings. However, the global vortices made in ''relativistic'' ^{4}He superfluid are regular [21]. The characteristic scale r_H in Eq. (3.38) is 10^6 pc for the grand unified scale $v \sim 10^{15}$ GeV, and is 10^{-2} a.u. for electroweak scale. The underlying physics for the reason why we reached this enormous size of horizon is easy: The mass density of a black cosmic string per unit length is given by the ratio of the scalar mass and the Planck scale, $2\sqrt{\pi G} v$, but the negative vacuum energy density inside the horizon is given by the ratio of the square root of the absolute value of the cosmological constant and the scalar mass, $\sqrt{|\Lambda|}/\sqrt{\lambda} v$. The scale of cosmic string generation is large, but the lower bound of the present cosmic vacuum energy is extremely small. Then the scale for this black cosmic string characterized by the horizon scale should be very large. Though these values are obtained under a perfect presumed toy situation without taking into account fluctuations around the black cosmic string, the huge radius of it, namely, the radius of the black cosmic string produced in the grand unified theory (GUT) scale ($\sim 10^6$ pc) is larger than the diameter of our galaxy ($\sim 5 \times 10^4$ pc), and may imply a difficulty for the survival of the charged black cosmic strings produced in such an early universe in the present universe with an extremely small bound for the cosmological constant. Once a global cosmic string is produced, it starts to radiate gapless Goldstone bosons. This dominant mechanism for energy loss makes the lifetime of a typical string loop very short [22]: A global string loop oscillates about 20 times before radiating most of its energy which is contrasted with gravitational radiation where the oscillation lasts about 10^4 times. The space outside the horizon of the black cosmic string is almost flat except for a tiny attractive force due to the negative cosmological constant as shown in Eq. (3.44) and Eq. (3.15), and then the massless

Goldstone bosons can be radiated outside the horizon. However, almost all the energy accumulated inside the horizon remains eternally. This ''black'' nature of the global U(1) string in the anti–de Sitter spacetime is remarkable at least for the case of straight cosmic strings.

Finally, let us emphasize again that we have two mass scales, the core mass and the inverse of the horizon, which are determined by three energy scales with large differences, namely, the Plank scale ($1/\sqrt{G} \sim 10^{19}$ GeV), the present bound of the cosmological constant ($\sqrt{|\Lambda|} \sim 10^{-42}$ GeV), and the symmetry-breaking scale (from $v \sim 10^{19}$ GeV to $v \sim 0.3$ eV). Therefore, the very existence of this horizon is expected to change drastically the physics related to the dynamics of global U(1) strings, e.g., the intercommuting of two strings or the production of wakes by moving long strings [4].

VI. CONCLUSION

In this paper, we have considered a scalar field model with a spontaneously broken U(1) global symmetry in (2+1)-dimensional anti–de Sitter spacetime, and investigated the cylindrically symmetric vortex solutions. We have found regular topological soliton configurations of which base manifolds constitute smooth hyperbolic space, extremal BTZ black holes, and charged BTZ black holes according to the decreasing magnitude of the negative cosmological constant. Different from the zero cosmological constant space supported by the global U(1) vortex, which cannot avoid a physical singularity, the obtained anti–de Sitter spaces are (physical) singularity free. Because of the logarithmic long tail of the Goldstone mode, the BTZ black hole also carries a charge which is identical to the case of an electric Maxwell charge. This identification was constructed by the duality transformation. All possible geodesic motions of massive and massless test particles were analyzed. Since the asymptotic space is hyperbolic, all the motions of massive particles are bounded. However, some massless test particles can escape to the spatial infinity of a hyperbola.

In 3+1 dimensions, the obtained global vortex-BTZ black hole depicts a straight charged black cosmic string. We brought up a toy model situation that these objects formed through a cosmological phase transition in the early universe (from the grand unification scale to the standard model scale) and survive in the present universe assumed with an allowably small magnitude of the negative cosmological constant ($|\Lambda| \sim 10^{-83}$ GeV^2). The corresponding scale of the horizon r_H is in order from 10^6 pc to 10^{-2} a.u. Then it implies that the observation of a black cosmic string in the present universe may relate the bound of the negative cosmological constant to the production of global U(1) vortices in the early universe.

Three brief comments are now in order. (i) For the vortices in the Abelian Higgs model or O(3) nonlinear σ model, they have finite energy in flat spacetime. A question of interest is whether they can form black holes in anti–de Sitter space. Until now we do not have an answer to this question [23]. If we find them, such BTZ black holes must be Schwarzschild type without an electric charge. (ii) Static charged BTZ black holes can also be obtained in dilaton gravity. Therefore, the global U(1) vortices coupled to dilaton and anti–de Sitter gravity may have some relevance in stringy cosmology [24]. (iii) For more realistic models of straight static black cosmic strings, a general metric of the form $ds^2 = B(r)e^{2N(r)}(dt - C(r)dz)^2 - B(r)^{-1}dr^2 - r^2 d\theta^2 - D(r)dz^2$ has to be taken into account.

[1] A. Staruszkiewicz, Acta. Phys. Pol. **24**, 735 (1963); S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984); J. Gott and M. Alpert, Gen. Relativ. Gravit. **16**, 243 (1984); S. Giddings, J. Abbot, and K. Kuchar, *ibid.* **16**, 751 (1984).

[2] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) **153**, 405 (1984).

[3] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).

[4] For a review, see A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994); M. B. Hindmarsh and T. W. B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).

[5] K. Lee, V. P. Nair, and E. Weinberg, Phys. Rev. Lett. **68**, 1100 (1992); Phys. Rev. D **45**, 2751 (1992); M. E. Ortiz, *ibid.* **45**, R2586 (1992); P. Breitenlohner, P. Forgács, and D. Maison, Nucl. Phys. **B383**, 357 (1992); for a review, see E. Weinberg, in *Proceedings of the 13th Symposium on Theoretical Physics*, edited by J. E. Kim (Min Eum Sa, Seoul, 1995); K. Maeda, in *Proceedings of the First Seoul Workshop on Gravity and Cosmology*, edited by S. W. Kim, P. Oh, and J. Lee (Korean Physical Society, Seoul, 1995).

[6] A. Vilenkin and A. E. Everett, Phys. Rev. Lett. **48**, 1867 (1982); E. P. S. Shellard, Nucl. Phys. **B283**, 624 (1987); D. Harari and P. Sikivie, Phys. Rev. D **37**, 3438 (1988).

[7] R. Gregory, Phys. Lett. B **215**, 663 (1988); A. G. Cohen and D. B. Kaplan, *ibid.* **215**, 67 (1988); G. W. Gibbons, M. E. Ortiz, and F. Ruiz Ruiz, Phys. Rev. D **39**, 1546 (1989).

[8] N. Kim, Y. Kim, and K. Kimm, SNUTP/97-022, gr-qc/9707011.

[9] If we solve the static objects with vanishing $i0$ and ij components of the energy-momentum tensor, e.g., massive but spinless point particles or self-dual vortices without angular momentum, in anti–de Sitter spacetime, the stationary metric solutions contain another possibility in addition to the two known solutions (Deser-Jackiw type or BTZ type). This will be discussed in a paper by Y. Kim and K. Kimm, SNUTP/97-111.

[10] E. L. Ince, in *Ordinary Differential Equations* (Dover, New York, 1956), Chap. XIV, p. 334.

[11] G. Clement, Phys. Rev. D **50**, 7119 (1994); Phys. Lett. B **367**,

70 (1996); J. S. F. Chan, K. C. K. Chan, and R. B. Mann, Phys. Rev. D **54**, 1535 (1996).

[12] D. Harari and C. Loustó, Phys. Rev. D **42**, 2626 (1990); Y. Kim, K. Maeda, and N. Sakai, Nucl. Phys. **B481**, 453 (1996).

[13] C. Farina, J. Gamboa, and A. J. Segui-Santonja, Class. Quantum Grav. **10**, L193 (1993); N. Cruz, C. Martínez, and L. Peña, *ibid.* **11**, 2731 (1994).

[14] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993); J. D. Brown, J. Creighton, and R. B. Mann, *ibid.* **50**, 6394 (1994).

[15] Y. Kim and K. Lee, Phys. Rev. D **49**, 2041 (1994); K. Lee, *ibid.* **49**, 4265 (1994); C. Kim and Y. Kim, *ibid.* **50**, 1040 (1994).

[16] E. Witten, Phys. Lett. **153B**, 243 (1985); R. L. Davis and E. R. S. Shellard, *ibid.* **214**, 219 (1988).

[17] R. A. Leese, Nucl. Phys. **B366**, 283 (1991); C. Kim, S. Kim, and Y. Kim, Phys. Rev. D **47**, 5434 (1993).

[18] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64**, 2230 (1990); R. Jackiw and E. J. Weinberg, *ibid.*, **64**, 2234 (1990); R. Jackiw, K. Lee, and E. J. Weinberg, Phys. Rev. D **42**, 3488 (1990).

[19] P. Valtancoli, Int. J. Mod. Phys. A **18**, 4335 (1992); D. Cangemi and C. Lee, Phys. Rev. D **46**, 4768 (1992); G. Clément, *ibid.* **54**, 1844 (1996).

[20] Particle Data Group, R. M. Burnett *et al.*, Phys. Rev. D **54**, 1 (1996), p. 66.

[21] Suppose that the model of our consideration is the relativistic counterpart of superfluid models and then we can use the phase transition scales for ^{4}He ($\sim 10^{-4}$ eV) and ^{3}He ($\sim 10^{-7}$ eV) as values for v. From the critical condition for black vortex formation [$|\Lambda| \approx 2\pi G\lambda v^4$ from Eq. (3.38)], we can compute the bounds of the negative cosmological constant: For $\lambda = 1$, $|\Lambda| > 10^{-110}$ (GeV)^2 for ^{4}He and $|\Lambda| > 10^{-122}$ (GeV)^2 for ^{3}He in order not to form a black vortex. Of course, this naive result should be refined by taking into account realistic condensed matter models, the specific form of the gravitational interaction, and the systematic nonrelativistic expansion. However, our study may have some relevance to the works by G. E. Volovik (see cond-mat/9706149, cond-mat/9706172).

[22] R. L. Davis, Phys. Rev. D **32**, 3172 (1985); A. Vilenkin and T. Vachaspati, *ibid.* **35**, 1138 (1987); for a review, see Ref. [4] and references therein.

[23] Y. Kim and S. Moon, SNUTP/97-112.

[24] N. Kaloper, Phys. Rev. D **48**, 2598 (1993); K. C. K. Chan and R. B. Mann, *ibid.* **50**, 6385 (1994); **52**, 2600(E) (1995).