

Domain walls in supersymmetric Yang-Mills theories

A. Kovner*

*Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom
and Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455*

M. Shifman and A. Smilga[†]

Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455

(Received 1 July 1997)

We present a detailed analysis of the domain walls in supersymmetric gluodynamics and SQCD. We use the (corrected) Veneziano-Yankielowicz effective Lagrangians to explicitly obtain the wall profiles and check recent results of Dvali and Shifman: (i) the BPS-saturated nature of the walls and (ii) the *exact* expressions for the wall energy density which depend only on global features of dynamics (the existence of a nontrivial central extension of $N=1$ superalgebra in the theories which admit wall-like solutions). If supersymmetry is softly broken by the gluino mass, the degeneracy of the distinct vacua is gone, and one can consider the decay rate of the “false” vacuum into the genuine one. We do this calculation in the limit of the small gluino mass. Finally, we comment on the controversy regarding the existence of N distinct chirally asymmetric vacua in $SU(N)$ SUSY gluodynamics. [S0556-2821(97)02224-8]

PACS number(s): 11.27.+d, 11.30.Pb, 12.60.Jv

I. INTRODUCTION

Recently it was noted [1] that some supersymmetric gauge theories possess domain walls with rather remarkable properties. The energy density of these domain walls is exactly calculable, in spite of the fact that the theories under consideration are in the strong coupling regime. For supersymmetric gluodynamics, the theory of gluons and gluinos with no matter, the calculation of the energy density was carried out in Ref. [1], in an indirect way. The key ingredient is the central extension of the $N=1$ superalgebra,

$$\{Q_\alpha^\dagger Q_\beta^\dagger\} = \frac{N}{4\pi^2} (\vec{\sigma})_{\alpha\beta} \int d^3x \vec{\nabla}(\text{Tr } \lambda^2), \quad (1)$$

where Q_α^\dagger is the supercharge, λ is the gluino field, and $(\vec{\sigma})_{\alpha\beta} = \{\sigma^3, -i, -\sigma^1\}_{\alpha\beta}$ is a set of matrices converting the vectorial index of the representation (1,0) of the Lorentz group in the spinorial indices.¹ The commutator (1) is given for the $SU(N)$ gauge group; the parameter N reflects this choice of the group. The integral over the full derivative on the right-hand side is zero for all localized field configurations; it does not vanish, however, for the domain walls. Equation (1) implies that the energy density of the domain wall is

$$\varepsilon = \frac{N}{8\pi^2} |\langle \text{Tr } \lambda^2 \rangle_\infty - \langle \text{Tr } \lambda^2 \rangle_{-\infty}|, \quad (2)$$

where the subscript $\pm\infty$ marks the values of the gluino condensate at spatial infinities (say, at $z \rightarrow \pm\infty$ assuming that the domain wall lies in the xy plane). The existence of the exact relation (2) is a consequence of the fact that the domain wall in the case at hand is a Bogomol’nyi-Prasad-Sommerfield-(BPS-) saturated configuration preserving 1/2 of the original supersymmetry.

In this paper we will explore in more detail the issue of the domain walls both in supersymmetric gluodynamics (Sec. II) and in a supersymmetric extension of QCD (SQCD, supersymmetric Yang-Mills theory with matter); see Secs. III–V. We will consider the profiles of the domain wall solutions, and calculate the energy density directly, by analyzing these profiles. The expressions obtained in this way will be confronted with the general results of Ref. [1]. Another issue of interest, to be discussed below, is the dependence of the central charge on the mass parameter of the matter field m_0 . The central charge is a chiral quantity; therefore, the dependence on m_0 should be holomorphic, as in Ref. [2]. The holomorphy implies that as far as the energy density of a BPS-saturated domain wall is concerned, the transition from the weak coupling Higgs regime to the strong coupling supersymmetric gluodynamics is smooth.

If supersymmetry is explicitly (softly) broken, say, by the gluino mass term, the vacuum degeneracy is lifted — we find ourselves in a classical situation with a false vacuum. Now, instead of the domain wall, one can study the decay rate of the false vacuum. If supersymmetry breaking is small, so that it is legitimate to work in the leading order in this parameter, one can obtain an explicit expression for the decay rate of the false vacuum. This problem is discussed in Sec. VI.

Section VII is devoted to an issue which, although related to the domain wall solutions, can be formulated in wider terms. Questioned is the very existence of N distinct chirally asymmetric vacua in the $SU(N)$ -supersymmetric gluodynamics. The chiral Z_{2N} symmetry is a remnant of the anomalous R_0 symmetry of the model. The presence of this symmetry is due to quantization of the topological charge. If the topological charge is quantized in a nonstandard way (i.e.,

*On leave of absence from PPARC.

[†]Permanent address: Institute of Theoretical and Experimental Physics, Moscow 117259, Russia.

¹Equations (9) and (11) in the original version of Ref. [1] contained a misprint; the factor $N/4\pi^2$ was omitted. Further comments regarding Eq. (1) are presented in Sec. IV.

fractional topological charges are allowed²), then the global structure of the theory changes. In particular, only one chirally asymmetric vacuum survives, not N . The extra states disappear as a result of a new superselection rule. This is clearly seen within the effective Lagrangian approach. The issue is being debated in the literature. We discuss the impact of new arguments associated with supersymmetry (SUSY) and smooth (and calculable) behavior of the wall energy density.

II. SUPERSYMMETRIC GLUODYNAMICS

To begin with, we will concentrate on supersymmetric generalization of pure gluodynamics — i.e., the theory of gluons and gluinos. The Lagrangian of the model at the fundamental level has the form [3]

$$\mathcal{L} = -\frac{1}{4g_0^2} G_{\mu\nu}^a G_{\mu\nu}^a + \frac{\vartheta}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a + \frac{1}{g_0^2} [i\lambda^{a\alpha} D_{\alpha\beta} \bar{\lambda}^{a\dot{\beta}}], \quad (3)$$

where the spinorial notation is used. In the superfield language the Lagrangian can be written as

$$\mathcal{L} = \text{Re} \frac{1}{2g^2} \text{Tr} \int d^2\theta W^2, \quad (4)$$

where

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{i\vartheta}{8\pi^2}.$$

In what follows the vacuum angle ϑ will play no role and can be set equal to zero. Our conventions regarding the superfield formalism are summarized, e.g., in a recent review [4]. We will limit ourselves to the $SU(N)$ gauge group [the generators of the group T^a are in the fundamental representation, so that $\text{Tr}(T^a T^b) = (1/2) \delta^{ab}$].

$SU(N)$ -supersymmetric gluodynamics has a discrete symmetry, Z_{2N} , a (nonanomalous) remnant of the anomalous axial symmetry generated by the phase rotations of the gluino field. The formation of the gluino condensate $\langle \lambda\lambda \rangle$ (it follows from certain supersymmetric Ward identities combined with explicit instanton calculations [5]) breaks this discrete chiral symmetry down to Z_2 . Therefore, there exists a set of distinct vacua labeled by the value of the gluino condensate. The field configurations interpolating between different values of $\langle \lambda\lambda \rangle$ at spatial infinities are topologically stable domain walls.

A formal description of these domain walls can be given in the framework of the effective Lagrangian approach. We will exploit the so-called Veneziano-Yankielowicz (VY) effective Lagrangians [6,7]. The original VY Lagrangian does not possess [8] the discrete Z_N invariance of supersymmetric gluodynamics (3). Recently it was shown that the VY expression is incomplete; it was amended to become compatible with all symmetries of supersymmetric gluodynamics in Ref. [9]. The corrected expression exhibits N minima of the

scalar potential corresponding to $Z_{2N} \rightarrow Z_2$ breaking, *plus* an additional minimum at the origin where the gluino condensate vanishes.

For simplicity from now on, if not stated to the contrary, we will consider the case of $SU(2)$, although this restriction is not of any conceptual importance and can be easily lifted.

The Lagrangian realizing the anomalous Ward identities is constructed in terms of the chiral superfield

$$S = \frac{3}{32\pi^2} \text{Tr} W^2, \quad (5)$$

namely,

$$\mathcal{L} = \frac{1}{4} \int d^4\theta C (\bar{S}S)^{1/3} + \frac{1}{3} \int d^2\theta S \left(\ln \frac{S^2}{\sigma^2} + 2\pi i n \right) + \frac{1}{3} \int d^2\bar{\theta} \bar{S} \left(\ln \frac{\bar{S}^2}{\sigma^2} - 2\pi i n \right), \quad (6)$$

where σ is a numerical parameter,

$$\sigma = e\Lambda^3 e^{i\vartheta/2},$$

and Λ is the scale parameter, a positive number of dimension of mass which we will set equal to unity in the following.

A new element in the Lagrangian (6) is an integer-valued Lagrange multiplier n . In calculating the partition function and all correlation functions the sum over n is implied. The variable n takes only integer values and is not a local field. It does not depend on the space-time coordinates and, therefore, integration over it imposes a global constraint on the topological charge. It is easy to see that (after the Euclidean rotation) the constraint takes the form

$$\nu = \frac{1}{32\pi^2} \int d^4x G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a = Z. \quad (7)$$

While the F term in Eq. (6) is unambiguously fixed, the D term is not specified by the anomalous Ward identities. First, the effective Lagrangian at hand is not Wilsonian; therefore, there are no reasons to discard terms with higher derivatives, generally speaking. A possible example of this type is

$$\left. \frac{(\partial_\mu \bar{S}^{1/3} \partial_\mu S^{1/3})}{(\bar{S}S)^{1/3}} \right|_D. \quad (8)$$

Even leaving aside higher derivatives, one is free to choose any value of the numerical constant C in Eq. (6). As we will see shortly, these ambiguities do have an impact on the profile of the wall. The surface energy density, however, remains intact, in full accordance with the general arguments of Ref. [1]. For the time being we will put $C=1$.

The extra term added to the Veneziano-Yankielowicz Lagrangian is clearly supersymmetric and is also invariant under all global symmetries of the original theory. The single valuedness of the scalar potential and the Z_2 invariance are restored.³ The chiral phase rotation by the angle πk with

²Another way to make the same statement is to say that noncontractable cycles in the space of the gauge fields are shorter than is usually believed.

³The explicit invariance here is Z_2 rather than the complete Z_4 of the original SUSY gluodynamics, since we have chosen to write our effective Lagrangian for the superfield which is invariant under $\lambda \rightarrow -\lambda$.

integer k just leads to the shift of n by k units. Since n is summed over in the functional integral, the resulting Lagrangian for S is indeed Z_2 invariant.

The constraint on $[S - \bar{S}]_F$ following from the Lagrangian (6) results in a peculiar form of the scalar potential. The expression for the scalar potential is given in Eq. (13) of Ref. [9]. It can be considerably simplified in the infinite volume limit. Eliminating, as usual, the F component of S with the help of classical equations of motion at fixed n , the effective potential can be written as

$$U(\phi) = -V^{-1} \ln \left[\sum_n \exp\{-16V(\phi^* \phi)^{2/3} [\ln^2 |\phi| + (\alpha + \pi n)^2]\} \right]. \quad (9)$$

Here V is the total space-time volume of the system, ϕ is the lowest component of the superfield S , and $\alpha = \arg(\phi)$. In the limit $V \rightarrow \infty$ only one term in the sum over n contributes for every value of α . Thus, for $-\pi/2 < \alpha < \pi/2$ the only contribution comes from $n=0$, while for $\pi/2 < \alpha < 3\pi/2$ from $n = -1$. Therefore in the right half-plane,

$$U(\phi) \equiv U_0(\phi) = 16(\phi^* \phi)^{2/3} \ln \phi \ln \phi^* \quad \text{at } \arg \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (10)$$

while in the left half-plane, when $\arg \phi \in (\pi/2, 3\pi/2)$,

$$U(\phi) = U_0(\phi e^{-i\pi}). \quad (11)$$

In other words, the complex ϕ plane is divided into two sectors. The scalar potential in the second sector is just that in the first sector rotated by $-\pi$. The scalar potential itself is continuous, but its first derivative in the angular direction experiences a jump at $\arg \phi = \pm \pi/2$. The scalar potential is ‘‘glued’’ out of two pieces.⁴ The Z_2 symmetry is explicit in this expression. It is quite obvious that the problem at hand has three supersymmetric minima — two at $\phi = \pm 1$, corresponding to a nonvanishing value of the gluino condensate (spontaneously broken discrete chiral symmetry), and a minimum at $\phi = 0$ (unbroken chiral symmetry).⁵

⁴Similar ‘‘glued’’ potentials appear due to the quantization of topological charge in the Schwinger model [10].

⁵The existence of this *additional* vacuum state with vanishing gluino condensate, which does not follow from any symmetry considerations, is probably the most surprising and nontrivial finding of Ref. [9] and is also a very important element of the present study of domain walls. We want to draw an analogy here with the situation in two-dimensional QCD with adjoint fermion studied in Ref. [10]. There too vacuum structure is nontrivial and ‘‘domain walls’’ interpolating between the vacua with different values of the fermion condensate exist. For $SU(N)$ gauge groups with even $N \geq 4$, the existence of different vacua did not follow from symmetry considerations. In Ref. [10], this was formulated as a paradox. It seems now that it is not a logical paradox, but rather a surprising phenomenon which takes place in some sophisticated enough two-dimensional and four-dimensional gauge theories.

III. DOMAIN WALLS IN SUPERSYMMETRIC GLUODYNAMICS

We will now consider the domain wall interpolating between

$$\phi = 0 \quad \text{and} \quad \phi = 1$$

at spatial infinities.

For definiteness, the domain wall is assumed to be in the xy plane. The profile to be determined is a function of one coordinate, z . The corresponding domain wall is purely real ($\text{Im } \phi$ will vanish on the wall profile) and lies completely inside the first sector. In other words, we will only need the potential on the real positive semiaxis of ϕ .

It is convenient to perform a rescaling

$$S = \Phi^3,$$

which casts the kinetic term of the scalar field in the canonic form. Correspondingly,

$$\phi = \varphi^3,$$

where φ is the lowest component of the superfield Φ .

The scalar potential in the right half-plane of ϕ then is

$$U(\varphi) = \left| \frac{\partial \mathcal{W}(\varphi)}{\partial \varphi} \right|^2, \quad (12)$$

where

$$\mathcal{W}(\varphi) = \frac{2}{3} \varphi^3 \ln \frac{\varphi^6}{e^2}. \quad (13)$$

The domain wall is the planar static field configuration $\varphi(z)$ satisfying the boundary conditions $\varphi(-\infty) = 0, \varphi(\infty) = 1$ and minimizing the energy functional. Hence, $\varphi(z)$ satisfies the classical equations of motion with the inversed sign of the potential:

$$\partial_z^2 \varphi = \frac{\partial^2 \bar{\mathcal{W}}}{\partial \varphi^2} \frac{\partial \mathcal{W}}{\partial \varphi}. \quad (14)$$

As we will now show the domain wall we deal with is a BPS-saturated state [1]. This means that a linear combination of supercharges, acting on the wall, annihilates it. One-half of supersymmetry is preserved. The general formula for the supersymmetry transformation is

$$\begin{aligned} \delta \psi^\alpha &= \sqrt{2} i (\partial^{\dot{\beta}\alpha} \varphi) \bar{\epsilon}_{\dot{\beta}} + \sqrt{2} \epsilon^\alpha F, \\ \delta \bar{\psi}^{\dot{\alpha}} &= -\sqrt{2} i (\partial^{\dot{\alpha}\alpha} \bar{\varphi}) \epsilon_\alpha + \sqrt{2} \bar{\epsilon}^{\dot{\alpha}} \bar{F}. \end{aligned} \quad (15)$$

If the parameter of the transformation ϵ^α is chosen in such a way that

$$\sigma_1 \epsilon^* = \pm i \epsilon,$$

then $\delta \psi = 0$ provided that

$$\partial_z \varphi = \pm F, \quad (16)$$

where $F = -\partial \bar{\mathcal{W}} / \partial \bar{\varphi}$.

Equation (16) is the first-order differential equation, a ‘‘square root’’ of the general Eq. (14). The solution with the

boundary conditions we are interested in corresponds to the plus sign in Eq. (16);⁶ under our choice of parameters it is obviously purely real. Therefore, $\overline{\mathcal{W}} = \mathcal{W}$. Even without knowing the explicit form of the solution, the energy density of the wall can be readily calculated. Indeed,

$$\begin{aligned} \varepsilon &= \int_{-\infty}^{\infty} dz \left[\partial_z \overline{\varphi} \partial_z \varphi + \frac{\partial \mathcal{W}}{\partial \varphi} \frac{\partial \overline{\mathcal{W}}}{\partial \overline{\varphi}} \right] \\ &= 2 \int_{-\infty}^{\infty} dz (\partial_z \overline{\varphi}) \left(-\frac{\partial \overline{\mathcal{W}}}{\partial \overline{\varphi}} \right), \end{aligned} \quad (17)$$

where Eq. (16) is used. The right-hand side evidently reduces to

$$\varepsilon = 2(\overline{\mathcal{W}}_{-\infty} - \overline{\mathcal{W}}_{\infty}) = \frac{8}{3}, \quad (18)$$

where the values of the superpotential (13) at $z = \pm\infty$ (i.e., $\varphi = 0$ and 1) are substituted. This result is in full agreement with Eq. (2).⁷

Let us discuss now the wall profile. Combining Eqs. (13) and (16) we arrive at the relation

$$\int_{1/\varphi}^2 \frac{dx}{\ln x} = 12(z - z_0), \quad (19)$$

where z_0 is the wall center. The left-hand side is expressible in terms of the integral logarithm. This does not help us find the explicit form $\varphi(z)$ and we therefore will not use this expression in the following. The asymptotic behavior of the solution is, however, transparent. At large positive z , $\varphi(z)$ approaches unity exponentially,

$$\varphi(z) \rightarrow 1 - \text{const} \times e^{-12(z-z_0)}.$$

At large negative z , $\varphi(z)$ approaches zero as

$$\varphi \sim \frac{1}{12} \frac{1}{z_0 - z}$$

modulo logarithms. This type of behavior was anticipated. Indeed, for positive z we are in the phase with the spontaneous breaking of the chiral symmetry and a mass gap. Hence, the approach is exponential. On the other hand, the phase at $z \rightarrow -\infty$ has no mass gap [9], and the asymptotics is power like.

Figure 1 presents the profile of the domain wall for the field $\varphi(z)$.

Note that the wall energy density calculated above is insensitive to the particular choice of the kinetic term. Equation (17) illustrates this insensitivity. Let us return to Eq. (6) and restore the constant C in the kinetic term, $C \neq 1$, as an example of possible ambiguity. It is quite clear that the con-

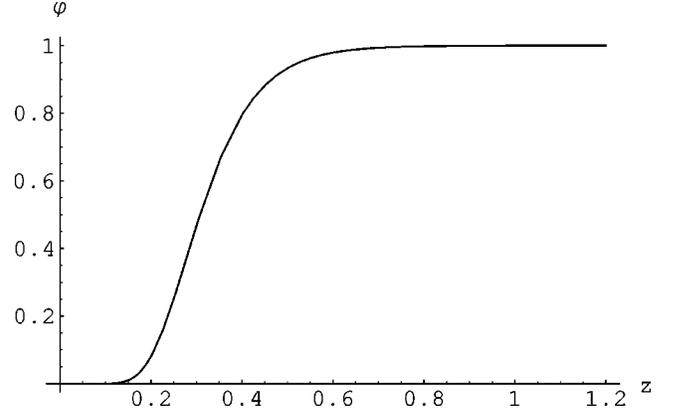


FIG. 1. The domain wall profile in SUSY gluodynamics.

stant C is immediately absorbed in a rescaling of z , and the final expression (18) remains intact.

IV. SQCD WITH ONE FLAVOR

What happens with the domain walls when the matter fields are added? In this section we will consider $SU(2)$ theory with one flavor (two subflavors). The matter fields belong to the fundamental representation of $SU(2)$. This model is described in great detail in a review paper [11], and so we omit all explanations. At the fundamental level the matter sector has the form

$$\mathcal{L}_M = \frac{1}{4} \int d^2\theta d^2\overline{\theta} \overline{Q}^f e^V Q^f + \left(\frac{m_0}{4} \int d^2\theta Q^{\alpha f} Q_{\alpha f} + \text{H.c.} \right), \quad (20)$$

where α is the color and f the subflavor index, $\alpha, f = 1, 2$; Q is the quark superfield. Furthermore, m_0 is the matter mass term. The subscript 0 indicates that it is the bare mass that enters the Lagrangian; this parameter is complex. Certain quantities depend on m_0 in a holomorphic way, which will allow us to exactly trace the evolution of the wall parameters under variations of m_0 [2,12].

By examining the symmetries of the model one derives the corresponding VY effective Lagrangian [7]. The R_0 current, the superpartner of the energy-momentum tensor and supercurrent [13], generates

$$\lambda_\alpha \rightarrow e^{i\beta} \lambda_\alpha, \quad \psi_\alpha^f \rightarrow e^{-(i/3)\beta} \psi_\alpha^f, \quad \phi_\alpha^f \rightarrow e^{(2i/3)\beta} \phi_\alpha^f, \quad (21)$$

where ψ and ϕ are the quark and squark fields, respectively. The R_0 current is anomalous at the quantum level. The anomaly-free R current is a linear combination of the R_0 current and the Konishi current [14]. At the one-loop level it corresponds to [15]⁸

$$\lambda_\alpha \rightarrow e^{i\beta} \lambda_\alpha, \quad \psi_\alpha^f \rightarrow e^{-2i\beta} \psi_\alpha^f, \quad \phi_\alpha^f \rightarrow e^{-i\beta} \phi_\alpha^f. \quad (22)$$

The R current is conserved in the massless limit $m_0 = 0$.

⁶The corresponding supertransformation parameter ϵ^α is $\epsilon^{1,2} = \{1, -i\}$.

⁷We recall that Λ is set equal to unity, so that $3(32\pi^2)^{-1} \text{Tr} \lambda^2 = 1$ in the vacuum with the broken chiral invariance.

⁸In higher orders the form of the conserved R current becomes more complicated [16], but this is unimportant for our purposes.

The VY Lagrangian is constructed in terms of two chiral superfields S [see Eq. (5)] and M ,

$$M = Q^{\alpha f} Q_{\alpha f}. \quad (23)$$

Omitting irrelevant constants one can write

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} \int d^4 \theta (\bar{S} S)^{1/3} + \frac{1}{4} \int d^4 \theta (\bar{M} M)^{1/2} \\ & + \left[\frac{1}{3} \int d^2 \theta S \left(\ln \frac{SM}{e \tilde{\Lambda}^5} + 2\pi i n \right) + \frac{m}{4} \int d^2 \theta M + \text{H.c.} \right]. \end{aligned} \quad (24)$$

Following Ref. [9], we introduced the Lagrange multiplier (integer-valued constant ‘‘field’’ n) in the original expression which can be borrowed, e.g., from [17], to ensure proper quantization of the topological charge. As before, summation over n is implied. *A priori* the mass parameter (the coefficient in front of M_F) is proportional to the quark mass m_0 . The coefficient of proportionality (+1) was established from the Konishi anomaly [14]; see below. Furthermore, $\tilde{\Lambda}$ is the scale parameter of SQCD with one flavor. Its relation to Λ , the scale parameter of supersymmetric gluodynamics, will be established later. Again, as in supersymmetric gluodynamics, the D terms in the Lagrangian (24) are not determined completely by the symmetries of the theory. We have chosen the simplest D term which is as good as any other one for the purpose of illustrating our point.

In the massless limit, $m_0 \rightarrow 0$, the Lagrangian (24) is obviously invariant under the R transformation, $\theta \rightarrow e^{i\beta} \theta$, $S \rightarrow e^{2i\beta} S$, and $M \rightarrow e^{-2i\beta} M$. The variation of the Lagrangian (24) under the anomalous R_0 transformation is

$$\frac{20}{9} i\beta (S - \bar{S})_F,$$

while that of Eq. (6) is

$$\frac{8}{3} i\beta (S - \bar{S})_F,$$

in full accordance with the fact that the first coefficients of the β functions are 5 and 6 in $SU(2)$ SQCD with one flavor and supersymmetric gluodynamics, respectively. If $m_0 \neq 0$, the only chiral invariance left in the Lagrangian (6) is the discrete (anomaly free) Z_4 chiral transformation

$$\lambda_\alpha \rightarrow e^{\pi i k/2} \lambda_\alpha, \quad \psi_\alpha^f \rightarrow \psi_\alpha^f, \quad \phi_\alpha^f \rightarrow e^{\pi i k/2} \phi_\alpha^f,$$

where k is an integer. In the VY Lagrangian (24), amended in accordance with Ref. [9], the discrete chiral symmetry is realized as Z_2 ,

$$\theta \rightarrow e^{\pi i k/2} \theta, \quad S \rightarrow e^{\pi i k} S, \quad M \rightarrow e^{\pi i k} M.$$

The scalar potential, being properly calculated from Eq. (24), is composed of two sectors, much in the same way as in supersymmetric gluodynamics, Sec. II.

The fundamental theory described by the Lagrangian (3) and (20) is in the unified Higgs-confinement phase, since the Higgs field is in the fundamental representation. One can distinguish the strong coupling regime versus the weak cou-

pling regime. If the expectation value of the field M is small (this happens either for large values of m_0 or in the additional vacuum found in Ref. [9]), we are in the strong coupling regime, where the VY effective Lagrangian is expected to properly capture the qualitative picture of the emerging dynamics. If the expectation value of the field M is large, we are in the weak coupling regime [15]. Here the VY Lagrangian provides a good description of the moduli fields M , but does not properly describe the dynamics of the massive W bosons characteristic of the weak coupling regime. Yet it reproduces correctly the vacuum structure in both cases. Correspondingly, the profile of the domain wall following from Eq. (24) will be qualitatively realistic. In this section we will consider the wall solution interpolating between $\phi=1$ and $\phi=0$. The solution that interpolates between two chirally noninvariant vacua at small m_0 will be treated separately in Sec. V. As in supersymmetric gluodynamics, the energy density of the walls will be exact.

Again, we will be interested here in a solution interpolating between the vacuum at the origin and that at finite values of S, M . Therefore, it is sufficient to consider only one sector of the theory. The corresponding vacuum structure is obtained by minimizing the superpotential

$$\mathcal{W} = \frac{2}{3} S \ln \frac{SM}{e \tilde{\Lambda}^5} + \frac{m}{2} M. \quad (25)$$

From $\partial \mathcal{W} / \partial M = 0$ we conclude that, in the standard vacuum,

$$\langle S \rangle = -\frac{3}{4} m \langle M \rangle. \quad (26)$$

Given our definitions of S and M , this is nothing but a consequence of the Konishi relation [14]

$$\begin{aligned} \bar{D}^2 \bar{Q}^{\alpha f} e^V Q^{\alpha f} &= \sum_f \left(4 Q^f \frac{\partial \mathcal{W}}{\partial Q^f} + \frac{1}{2\pi^2} \text{Tr} W^2 \right) \\ &= 4 m_0 Q^{\alpha f} Q_{\alpha f} + \frac{1}{2\pi^2} \text{Tr} W^2. \end{aligned} \quad (27)$$

This explains our choice of the mass parameter in Eq. (24).

The condition $\partial \mathcal{W} / \partial S = 0$ implies that the vacuum states are at $\langle S \rangle \langle M \rangle = 1$ (in the units of $\tilde{\Lambda}$ which will be used hereafter, if not stated to the contrary). It is seen that it is convenient to assume the mass parameter m to be real and negative. Then the vacuum value of M will be real. The phase of the parameter m can be adjusted arbitrarily by an appropriate rotation of the fields S and M . From now on we will assume that m is a real negative number; for convenience we will introduce

$$\tilde{m} = -m.$$

Then \tilde{m} is real and positive.

It is convenient to pass to the superfields with the canonic kinetic terms,

$$S \rightarrow \Phi^3, \quad M \rightarrow X^2. \quad (28)$$

The lowest component of Φ is denoted by φ , as in Sec. II; the lowest component of X is χ . The superpotential (25) generates the scalar potential

$$U(\varphi, \chi) = \left| \frac{\partial W}{\partial \varphi} \right|^2 + \left| \frac{\partial W}{\partial \chi} \right|^2 = 4|\varphi^2 \ln(\varphi^3 \chi^2)|^2 + \left| \chi \left(\tilde{m} - \frac{4\varphi^3}{3\chi^2} \right) \right|^2. \quad (29)$$

It is zero at the origin and at the standard minima (26).

Let us analyze the profile of the domain wall connecting the minimum at the origin and one of the standard minima (with the positive sign of $\langle M \rangle = \langle \chi^2 \rangle$). It is again a BPS-saturated state, preserving two out of four supercharges; see Eq. (15). The corresponding equations are

$$\partial_z \varphi = \pm F_\Phi, \quad \partial_z \chi = \pm F_X, \quad (30)$$

where

$$\bar{F}_\Phi = -\frac{\partial \mathcal{W}(\varphi, \chi)}{\partial \varphi}, \quad \bar{F}_X = -\frac{\partial \mathcal{W}(\varphi, \chi)}{\partial \chi}.$$

The following boundary conditions are imposed:

$$\begin{aligned} \varphi = \chi = 0 & \quad \text{at } z \rightarrow -\infty; \\ \varphi \rightarrow \varphi_*, \quad \chi \rightarrow \chi_* & \quad \text{at } z \rightarrow \infty, \end{aligned} \quad (31)$$

where (the real positive) parameters φ_*, χ_* are defined through the expressions

$$\chi_*^2 = \frac{2}{\sqrt{3}} \frac{1}{\sqrt{\tilde{m}}}, \quad \varphi_*^3 = \frac{\sqrt{3}}{2} \sqrt{\tilde{m}},$$

and the scale parameter $\tilde{\Lambda}$ is put to unity, temporarily. These boundary conditions correspond to the plus sign in Eq. (30). After the rescaling $\chi = \tilde{m}^{-1/4} \tilde{\chi}$, $\varphi = \tilde{m}^{1/6} \tilde{\varphi}$, Eqs. (30) take the form

$$\begin{aligned} \partial_z \tilde{\varphi} &= -2\tilde{m}^{1/6} \tilde{\varphi}^2 \ln(\tilde{\varphi}^3 \tilde{\chi}^2), \\ \partial_z \tilde{\chi} &= \tilde{\chi} \left(\tilde{m} - \frac{4\tilde{\varphi}^3}{3\tilde{\chi}^2} \right). \end{aligned} \quad (32)$$

These equations can be solved analytically in two interesting cases, $\tilde{m} \rightarrow \infty$ and $\tilde{m} \rightarrow 0$.

When the mass is large, the behavior of the solution is qualitatively evident. In this case, we can integrate out the heavy matter fields, after which the dynamics of the light (gauge) sector is the same as in supersymmetric gluodynamics. Indeed, at $z = \infty$ the curve starts at φ_*, χ_* ; then it follows the trajectory on which φ^3 is (almost) equal to $3\tilde{m}\chi^2/4$, and the solution for φ is (almost) the same as in Sec. II — exponentially close to φ_* — then quickly φ and χ approach zero as $(z_0 - z)^{-1}$ modulo logarithms. The condition $\varphi^3 = 3\tilde{m}\chi^2/4$ nullifies the (large) second term in the scalar potential (29) and exactly corresponds to freezing the heavy degree of freedom in the spirit of the Born-Oppenheimer approach.

The situation is somewhat nontrivial in the opposite limit when the matter fields are very light. A standard procedure would be to freeze the gauge degrees of freedom, which amounts to imposing the condition $\varphi^3 \chi^2 = 1$ (so that the ‘‘large’’ first term in the scalar potential would disappear).

After this, the effective potential for the light matter fields is obtained [15] which gives a valid description of the ‘‘Higgs’’ phase, $|\chi| \gg 1$. We will analyze in more detail the physics of the Higgs phase in the subsequent section and will show, in particular, that BPS-saturated domain walls which interpolate between two Higgs vacua (26) with different phases of χ do exist. However, the domain walls interpolating between one of the standard vacua and the new vacuum at the origin disappear in this approach.

We want to emphasize that Eqs. (30) and (32) with the boundary conditions (31) *do* have a solution at any value of \tilde{m} . In the limit $\tilde{m} \rightarrow 0$, the solution acquires, however, a peculiar singular form being composed of two pieces. The first piece corresponds to moving from the minimum $\varphi = \chi = 0$ at $z = -\infty$ to the point $\varphi = 0, \chi = \chi_*$ at $z = z_0$ according to the law

$$\chi(z) = \chi_* e^{\tilde{m}(z - z_0)}.$$

Then the trajectory abruptly turns: $\chi(z)$ does not change anymore, but $\varphi(z)$ starts rising from $\varphi(z_0) = 0$ to $\varphi(\infty) = \varphi_*$ as is dictated by the first equation in Eqs. (32) with frozen $\chi(z) = \chi_*$ (it has the same functional form as the wall equation in supersymmetric gluodynamics). This second half of the wall is much broader than the first one. Its width is of order $\tilde{m}^{-1/6}$ compared to the width $\sim 1/\tilde{m}$ of the section of the wall with negative $z - z_0$.

It is clear why this solution is lost in the Born-Oppenheimer analysis. The scalar potential (29) involves a rather high barrier between the Higgs vacua and the origin. The new wall corresponds to climbing this ‘‘mountain ridge.’’ In the vicinity of the ridge the relation $\varphi^3 \chi^2 = 1$ is not valid and the naive Born-Oppenheimer analysis breaks down.

It is not surprising as such that a solution going over this ridge still exists. What is rather remarkable and very specific for the supersymmetric case is that the surface energy density of this nonstandard BPS-saturated wall is not high. It is determined by the generalization of Eq. (17),

$$\begin{aligned} \varepsilon &= \int_{-\infty}^{\infty} dz \left[\partial_z \tilde{\varphi} \partial_z \varphi + \partial_z \tilde{\chi} \partial_z \chi + \frac{\partial \mathcal{W}}{\partial \varphi} \frac{\partial \bar{\mathcal{W}}}{\partial \tilde{\varphi}} + \frac{\partial \mathcal{W}}{\partial \chi} \frac{\partial \bar{\mathcal{W}}}{\partial \tilde{\chi}} \right] \\ &= 2(\bar{\mathcal{W}}_{-\infty} - \bar{\mathcal{W}}_{\infty}). \end{aligned} \quad (33)$$

Using Eq. (25) for the superpotential we conclude that

$$\varepsilon = \frac{4}{\sqrt{3}} (\tilde{m} \tilde{\Lambda}^5)^{1/2}, \quad (34)$$

where the scale parameter $\tilde{\Lambda}$ is restored. The relation (34) holds for any \tilde{m} . It is amusing that, in the limiting case $\tilde{m} \rightarrow 0$, half of the energy density (33) comes from the region $z < z_0$ (a narrow section of the wall) and another half from the region $z > z_0$ (a broad section).

When the mass is neither too large nor too small, Eqs. (30) do not have analytic solutions. They can be solved numerically, however. As an illustration the results for $\tilde{\varphi}(z)$, $\tilde{\chi}(z)$ and the parametric plot in the φ, χ plane with $\tilde{m} = 1.0$ are presented in Figs. 2, 3, and 4.

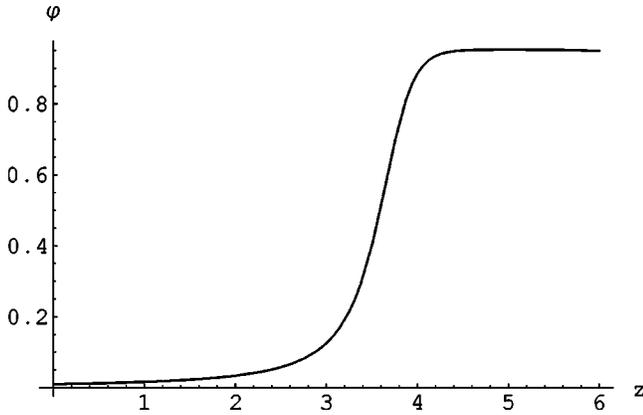


FIG. 2. The profile of the field φ inside the domain wall in SUSY QCD for $\tilde{m}=1$.

We solved also the equations for other values of \tilde{m} and observed how the profiles approach the asymptotic curves in the limit when the mass parameter tends to zero or infinity.

If \tilde{m} is large, the quarks can be integrated out, and SQCD reduces to supersymmetric gluodynamics. It is not difficult to obtain a relation between the scale parameters $\tilde{\Lambda}$ and Λ . In the fundamental theory this can be done by exploiting the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β function [18]; in the effective theories one compares to this end the expressions for $\langle S \rangle$ in the standard vacuum in SQCD and supersymmetric gluodynamics, respectively. In this way we arrive at

$$\Lambda^3 = \frac{\sqrt{3}}{2} (\tilde{m} \tilde{\Lambda}^5)^{1/2}. \tag{35}$$

The scale parameters are introduced via Eqs. (6) and (24), respectively. Thanks to holomorphy, the square root dependence in Eq. (35) is exact, not approximate. This relation is exact. Equation (35) implies, in turn, that in SQCD $\varepsilon = (4\pi^2)^{-1} \text{Tr} \lambda^2$ — this is exactly the same relation we observed in supersymmetric gluodynamics. The fact that the explicit m dependence disappears — it is completely hidden in the gluino condensate — is no coincidence. The reason is rather transparent: The central extension of superalgebra (1) in $SU(N)$ SQCD with matter takes the form

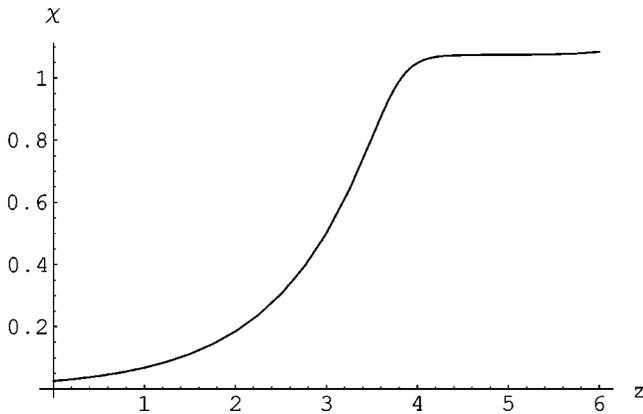


FIG. 3. The profile of the field χ inside the domain wall in SUSY QCD for $\tilde{m}=1$.

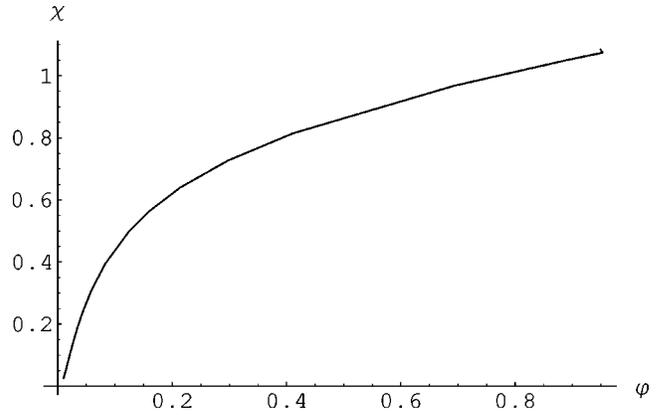


FIG. 4. The parametric plot of χ vs φ inside the domain wall in SUSY QCD for $\tilde{m}=1$.

$$\{Q_\alpha^\dagger Q_\beta^\dagger\} = 4(\vec{\sigma})_{\dot{\alpha}\dot{\beta}} \int d^3x \vec{\nabla} \left\{ \sum_{\mathbb{H}} \left[\frac{m_0}{2} Q^{\alpha f} Q_{\alpha f} + \frac{1}{16\pi^2} \text{Tr} W^2 \right] - \frac{N}{16\pi^2} \text{Tr} W^2 \right\}_{\theta=0}, \tag{36}$$

plus full (super)derivatives. The sum over flavors on the right-hand side assumes that one may have an arbitrary number of flavors (the expression above refers to the fundamental representation; we recall that each flavor requires two sub-flavors). Note that the expression in the square brackets is the Konishi anomaly (27) itself, and as such, is a full super-derivative that gives no contribution in the central charge. Thus, in the theory with any number of flavors the central charge is given by the last term in the braces; i.e., we return to Eq. (1), which holds universally, irrespective of the values of N_f and m .

A brief remark is in order here concerning derivation of Eq. (36). The commutator consists of two parts. The first, tree-level part is proportional to the matter superpotential and is entirely due to the matter fields. It is trivially obtained from the canonic commutation relations. The part containing $\text{Tr} W^2$ is an anomaly. In principle, it could be obtained by a direct calculation of the relevant one-loop graphs, with both the matter and gauge fields in the loop, provided these graphs are regularized in the ultraviolet and infrared in such a way that all symmetries of the model (including supersymmetry) are preserved. We did not attempt to carry out this program in full (although some steps in this direction are reported in Ref. [19]). An indirect way is based on the observation that there are no essentially new “geometric” anomalies other than that in the divergence of the R_0 current and the trace of the energy-momentum tensor. A superfield expression for this standard “geometric” anomaly is

$$\bar{D}^\alpha J_{\alpha\dot{\alpha}} = \frac{1}{3} D_\alpha \left\{ \left[3\mathcal{W} - \sum_i Q_i \frac{\partial \mathcal{W}}{\partial Q_i} \right] - \left[\frac{3N - N_f}{16\pi^2} \text{Tr} W^2 + \frac{1}{8} \sum_i \gamma_i \bar{D}^2 (\bar{Q}_i^+ e^V Q_i) \right] \right\}, \tag{37}$$

where γ_i are the anomalous dimensions of the quark superfields Q_i ,

$$\gamma_i = -\partial \ln Z_i / \partial \ln \mu = -\frac{N^2 - 1}{2N} \frac{\alpha}{\pi} + \dots$$

[see, e.g., Eq. (26) in Ref. [16]]. First, one may discard the last term in the second square brackets, since this term is a full superderivative. To make the tree-level part of this standard anomaly compatible with Eq. (36) one must subtract from the former full superderivatives which obviously does not affect the value of the central charge anyway. The appropriate superderivative is the Konishi relation (27) itself. In this way we arrive at Eq. (36); see Ref. [19] for further details. An independent check is that at $N_f=0$ Eq. (36) coincides with Eq. (1), which, in turn, can be readily derived, e.g., from the VY Lagrangian. Equation (36) guarantees the smooth transition to the large m limit, with no explicit m dependence of the wall energy density. This feature was anticipated in Ref. [1]. Another independent check is the fact that the anomaly part of the central charge (i.e., the coefficient in front of W^2) is proportional to $N_f - N_c$. If we start from $N_f=0$ and gradually increase this parameter, we, eventually, come to a point where $N_f=N_c$; here the coefficient in front of the anomaly part vanishes. The vanishing of the anomaly part at $N_f=N_c$ could have been expected. Indeed, from Ref. [15] we know that at $N_f < N_c$ a nonperturbative superpotential is generated in the massless SQCD, while at $N_f=N_c$, although the superpotential could have appeared, it is not generated. In our approach this is due to the vanishing of the coefficient in front of W^2 in the central charge.

V. SQCD IN THE WEAK COUPLING REGIME (HIGGS PHASE)

If the mass parameter m_0 in Eq. (20) is small *and* the characteristic values of the matter field M are assumed to be large, the Higgs description of the model is more appropriate [15]. In this case the S superfield can be integrated out, the superfield M is light (at the classical level M describes a flat direction if $m_0=0$), and one can obtain a genuinely Wilsonian effective Lagrangian for the field M (certain restrictions apply in the domain of small M). We recall that the VY Lagrangians are not Wilsonian constructions.

The superpotential of the low energy theory for the would-be moduli field M is

$$\mathcal{W} = -\frac{2}{3} \frac{\tilde{\Lambda}^5}{M} - \frac{\tilde{m}}{2} M. \quad (38)$$

In the fundamental theory it is generated by instantons [15] (for a review see Ref. [11]). Note that the superpotential (38) has a discrete Z_2 invariance $M \rightarrow -M$, as it ought to. The region of small M is not legitimate for consideration in this language; therefore, we will consider the wall solution interpolating between the vacua with $\langle M \rangle = \pm (4\tilde{\Lambda}^5/3\tilde{m})^{1/2} \equiv \pm \chi_*^2$, anticipating that this solution will never go through the region of small M .

The superpotential (38) can be obtained from Eq. (25) by eliminating S through the equation of motion, $\ln SM=0$. Note that no trace is left of the two-sector structure charac-

teristic of the VY Lagrangian in the SU(2) theory. The superpotential for M is perfectly holomorphic.

Let us again introduce the superfield $X=M^{1/2}$ with the standard kinetic term. The BPS wall equations corresponding to the superpotential (38) for the wall between the vacua $\langle \chi^2 \rangle = \pm \chi_*^2$ have the form

$$\partial_z \chi = \frac{\partial \tilde{\mathcal{W}}}{\partial \bar{\chi}} = -\tilde{m} \bar{\chi} + \frac{4\tilde{\Lambda}^5}{3\bar{\chi}^3}, \quad (39)$$

and its complex conjugate [there is, of course, also a pair of equations with negative sign with a solution which is the mirror image of the solution of Eq. (39)]. Substituting here $\chi(z) = \rho(z) e^{i\alpha(z)}$, one observes after some simple transformations that $\rho(z) = \chi_*$ is just constant and the phase $\alpha(z)$ satisfies the equation

$$\partial_z \alpha = 2\tilde{m} \sin 2\alpha(z). \quad (40)$$

It is not difficult to solve this equation with the boundary conditions $\alpha(-\infty)=0$, $\alpha(\infty)=\pi/2$. The wall profile thus obtained is

$$\chi(z) = \chi_* \frac{1 + i e^{4\tilde{m}(z-z_0)}}{\sqrt{1 + e^{8\tilde{m}(z-z_0)}}}. \quad (41)$$

The energy density of this wall is

$$\varepsilon = \frac{8}{\sqrt{3}} (m\tilde{\Lambda}^5)^{1/2}, \quad (42)$$

twice the value in Eq. (34). The reason is quite obvious. The wall (42) is not similar to the walls discussed in the previous sections. The latter interpolate between the vacuum with $\langle \lambda^2 \rangle = 0$ and the standard chirally asymmetric vacuum, while the former interpolates between two different chirally asymmetric vacua [in the SU(2) theory].

An interesting question is what happens with the wall (42) in the case when the ratio $\tilde{m}/\tilde{\Lambda}$ is not sent to zero, but has a finite value. We do not have an analytic solution in this case. But, as was the case for the walls interpolating between a chirally asymmetric vacuum and the chirally symmetric one which we discussed in the preceding section, the solution can be found numerically [20]. Not dwelling on details, we only mention here that the BPS equations (30) admit the wall solutions⁹ only in some range of masses, $|m| \leq m_* = 4.67059\dots\tilde{\Lambda}$. At $\tilde{m} > m_*$, the BPS solution disappears. Moreover, if $\tilde{m} > m_{**} \approx 4.83$, no nontrivial complex wall connecting different chirally asymmetric vacua exists within the framework of the VY effective Lagrangian.

⁹Though the existence of the wall solution of the equations of motion is obvious, the statement that the wall satisfies also the first order BPS equations is nontrivial. There is no theorem that the domain walls are always BPS saturated. A simple model example when it is not the case was given in [21].

VI. DECAY OF THE FALSE VACUUM

Let us now discuss what happens if we softly break supersymmetry by adding to the Lagrangian a gluino mass term

$$\Delta \mathcal{L}_m = \frac{m_\lambda}{g_0^2} [\text{Tr } \lambda^2 + \text{H.c.}]. \quad (43)$$

Note that in our notation [see Eq. (3)] $\text{Tr } \lambda^2$ is a renormalization-group-invariant operator, and so is the ratio m_λ/g_0^2 (to the leading order). We will briefly comment on the impact of subleading terms later. It is assumed that m_λ is real and positive (the phase of m_λ is equivalent to a ϑ angle and is irrelevant).

The degeneracy of the three vacuum states of the $SU(2)$ -supersymmetric gluodynamics is lifted. We will limit ourselves to the effects linear in the soft supersymmetry breaking. In the linear in m_λ approximation the chirally symmetric vacuum stays at zero, the energy density of one of the chirally asymmetric vacua becomes positive and that of another negative,

$$\mathcal{E}_\pm = \pm \frac{2m_\lambda \Sigma}{g_0^2}, \quad (44)$$

where

$$\Sigma = \langle \text{Tr } \lambda^2 \rangle_+,$$

and the subscript $+$ in this definition of Σ indicates that here we mean the gluino condensate in the vacuum with the positive value of $\langle \text{Tr } \lambda^2 \rangle$. In other words, Σ is a real positive parameter of dimension (mass)³.

Thus, we deal with two false vacuum states that can decay into the true vacuum through formation of ‘‘bubbles’’ [22].

The decay rate of the false vacuum into the genuine vacuum can be easily evaluated using the general results of Ref. [22]. According to this work, the decay rate of the false vacuum is proportional to

$$\Gamma \propto \exp \left\{ - \frac{27}{2} \pi^2 \frac{\varepsilon^4}{(\Delta \mathcal{E})^3} \right\}, \quad (45)$$

where $\Delta \mathcal{E}$ is the difference of the vacuum energy densities in the false and true vacua, and ε is the surface energy density of the domain wall. The estimate (45) is valid with the exponential accuracy in the ‘‘thin wall limit,’’ i.e., when the radius of the critical bubble is much larger than the characteristic thickness of the wall or, in other words, when the absolute value of the exponent in Eq. (45) is large.

To find the decay rate of the chirally symmetric vacuum in the true vacuum with negative energy density (the one with positive energy density would decay in two stages), we have to substitute in Eq. (45) the expression (44) for $\Delta \mathcal{E}$ and the expression (2) for ε (assuming $N=2$, $\langle \text{Tr } \lambda^2 \rangle_\infty = \Sigma$ and $\langle \text{Tr } \lambda^2 \rangle_{-\infty} = 0$). We then obtain

$$\Gamma \propto \exp \left\{ - \frac{27}{4096 \pi^6} \frac{\Sigma}{(m_\lambda/g_0^2)^3} \right\}. \quad (46)$$

Note a rather small numerical factor in the exponent. The quasiclassical formula (46) is valid when $m_\lambda/g_0^2 \ll 0.02 \Sigma^{1/3}$. It applies both to supersymmetric gluodynamics and to SQCD.

If the gluino mass term has a phase (or if $\vartheta \neq 0$, which is the same),

$$m_\lambda \rightarrow |m_\lambda| e^{i\alpha},$$

the parameter m_λ in the exponent of Eq. (46) is substituted by

$$|m_\lambda \cos \alpha|.$$

At $\alpha = \pi/2$ the exponent becomes infinite. The reason is obvious: For a purely imaginary gluino mass term the vacuum degeneracy is not lifted, and there is no false vacuum decay.

The false vacuum decay rate is a physical quantity, and as such it must be independent of the normalization point μ . It was already mentioned that Σ is renormalization-group (RG) invariant. The ratio m_λ/g_0^2 is RG invariant only in the leading logarithmic approximation. Beyond the leading approximation the exact RG-invariant combination is [23]

$$m_\lambda \left(\frac{1}{g_0^2} - \frac{1}{4\pi^2} \right). \quad (47)$$

Therefore, m_λ/g_0^2 in the exponent in Eq. (46) is actually substituted by the combination (47); see [23] for further details.

VII. WALLS VS TORONS

All the previous discussion was based on the assumption that the topological charge can only be an integer. There is a lasting controversy in the literature as to the question of existence of configurations with fractional topological charge in pure glue $SU(N)$ gauge theories.

In supersymmetric $SU(N)$ Yang-Mills theories the question of the existence (or nonexistence) of domain walls interpolating between different chirally asymmetric vacua is in one-to-one correspondence with the question of the proper quantization of the topological charge (7). If the minimal nontrivial topological charge is unity, the presence of N different vacuum states implies the spontaneous breaking of the physical discrete symmetry $Z_{2N} \rightarrow Z_2$ and, correspondingly, the appearance of domain walls. An alternative, with a new superselection rule replacing the physical Z_{2N} invariance, arises if fractional values of the topological charge are possible. Let us elucidate this assertion in more detail.

If the $SU(N)$ Yang-Mills theory is compactified on a four-dimensional torus, with a finite size L , the topological charge is quantized fractionally. The so-called toron field configurations, with ν being multiple integer of $1/N_c$, do exist [24]. An *assumption* that such configurations survive and contribute in the path integral in the large-volume limit $L \rightarrow \infty$ leads to the conclusion that the Z_N transformation connects vacua with different ϑ rather than physically distinct degenerate vacua. If large gauge transformations can change the winding number of the gauge field configuration by $1/N_c$, while the vacuum angle ϑ is defined in the ‘‘old’’ way [see Eq. (3)], then the vacuum angle ϑ varies within the

range $0 \leq \vartheta \leq 2\pi N_c$, and the sectors with different ϑ do not communicate with each other. A new superselection rule should be imposed. Alternatively one may say that a new vacuum angle should be defined, $\tilde{\vartheta} = N_c^{-1} \vartheta$. Then $\tilde{\vartheta}$ would vary between zero and 2π , as is appropriate for the vacuum angle. N_c chirally asymmetric vacua of supersymmetric gluodynamics with $\langle \text{Tr } \lambda^2 \rangle \propto \exp(2\pi i k / N_c)$ would correspond to different values of $\tilde{\vartheta} = 0, 2\pi/N_c, 4\pi/N_c, \dots$ and could not coexist in one and the same universe (see Refs. [25] and [26] for a detailed discussion). If it were true, one could not speak of domain walls between the different supersymmetric vacua. In particular, the expression (46) for the decay rate of the metastable vacuum in a theory where supersymmetry is slightly broken would not make sense.

The most serious argument in favor of this viewpoint comes from the calculation of the gluino condensate on the small torus [27]. In the $SU(N)$ -supersymmetric gluodynamics the gluino condensate turns out to be saturated by the toron field configurations. The expression has the form

$$\langle \text{Tr } \lambda^2 \rangle \sim \frac{1}{L^3 g^2(L)} \exp\left\{-\frac{8\pi^2}{Ng^2(L)}\right\}, \quad (48)$$

where $L \ll \Lambda^{-1}$. Using the exact NSVZ β function [18] we find that the condensate actually does not depend on L ; the toron result is equal to a numerical constant times Λ^3 . From this one could tentatively conclude that the toron contribution to the condensate survives for large L [28,25].

Clearly, if we allow the toron configurations, the Z_N as a physical symmetry disappears from the VY Lagrangian. Say, in the $SU(2)$ theory, Eq. (6) must be modified: $2\pi i n$ on the right-hand side must be replaced by $4\pi i n$. Correspondingly, instead of two sectors of the scalar potential [see Eq. (9) and the following discussion] we will have just one sector extending in the interval $\arg \phi \in (-\pi, \pi)$. This scalar potential would have only two minima: one chirally asymmetric at $\phi = 1$ and the second chirally symmetric minimum at $\phi = 0$.

Now, the argument in favor of torons is not free of inconsistencies (see Ref. [10] for a recent detailed discussion). First, the existence of the field configurations with fractional topological charge relies on the fact that the theory is compactified on the torus. If the theory is compactified on S^4 , only field configurations with integer topological charge are admissible and there are no torons. But the physics should not depend on whether the theory is compactified on a sphere, on a torus, or on some other manifold, if the size of the manifold is much larger than the characteristic scale Λ^{-1} .

Second, in the theories with higher orthogonal and exceptional groups, configurations with fractional topological charge do not exist even on a torus, and it is *not* possible to interpret the vacuum degeneracy in these theories in the language of new superselection rules.

Third, if the matter fields in the fundamental representation are introduced, torons disappear since the twisted boundary conditions on a torus necessary for their existence cannot be imposed. When the matter mass term tends to infinity we return back to supersymmetric gluodynamics. If the transition is smooth — and we have seen that, for the walls connecting a chirally asymmetric vacuum and the chirally symmetric one, it is smooth — the energy density of

the domain wall remains finite. Then the torons must be irrelevant in supersymmetric gluodynamics too.

Finally, the exact solution of $N=2$ supersymmetric gluodynamics found recently [29] shows no traces of the presence of torons and fractional topological charges. Although this solution is not rigorously proved, it is perfectly self-consistent and goes through numerous indirect checks.

Our present viewpoint is that the domain walls are real — they *do* exist in supersymmetric gluodynamics and SQCD. The cleanest argument comes from the consideration of the theory with matter in the weak coupling regime $\tilde{m} \ll \tilde{\Lambda}$. In this case the vacua $\chi^2 = 2/\sqrt{3\tilde{m}}$ and $\chi^2 = -2/\sqrt{3\tilde{m}}$ acquire a large classical Higgs vacuum average, and the walls separating them exist by the same token as in the most trivial model of one real scalar field with the double-well potential $V(\phi) = \lambda(\phi^2 - v^2)^2$.

In the theory with fundamental matter there is no place for an argument: Field configurations with fractional topological charges do not exist no matter how the theory is compactified since the twisted boundary conditions on a torus necessary for their existence cannot be consistently imposed. A remarkable corollary of supersymmetry is the holomorphic dependence (42) of the wall energy density on the mass. As was explained before, the energy density [expressed via the physical scale (35)] remains finite in the limit $m \rightarrow \infty$ when the matter fields decouple and we are left with the pure supersymmetric gluodynamics. This means that in this case, the proper topological classification should involve only integer topological charges.

It is very instructive to confront this situation with a simple (nonsupersymmetric) two-dimensional model where a similar question can be posed and exactly answered, but the answer is precisely opposite. Consider two-dimensional QED (the Schwinger model) with two fermion flavors, a massless fermion ψ with the charge e , and a massive fermion Ψ with the charge $e/2$:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} (i\gamma_\mu \partial_\mu - e\gamma_\mu A_\mu) \psi \\ & + \bar{\Psi} \left(i\gamma_\mu \partial_\mu - \frac{e}{2} \gamma_\mu A_\mu - M \right) \Psi, \end{aligned} \quad (49)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \epsilon_{\mu\nu} F.$$

In the Euclidean version of the theory, the topological charge

$$\nu_2 = \frac{e/2}{2\pi} \int F(x) d^2x = Z \quad (50)$$

is quantized. The minimal flux is determined by the boundary condition on the field Ψ , i.e., by the charge $e/2$ of the heavy fermion field. The minimal topological charge is unity.

In the limit $M \rightarrow \infty$, heavy fermions decouple. We must consider now only the boundary conditions on the field ψ . The gauge field configurations with half-integer flux ν_2 become allowed. This simulates the situation in four-dimensional supersymmetric gauge theories with and without fundamental matter. When SQCD involves dynamical

fields in the fundamental representation, only integer topological charges (7) are admissible. In the limit $m \rightarrow \infty$, we are left with the adjoint gauge fields and adjoint fermions (gluinos), and, on a torus, gauge field configurations with fractional topological charge show up.

Let us now address the question of “domain walls” in our two-dimensional model. Quotation marks are used above to recall the reader that due to the lack of extra two dimensions these objects are not really “walls,” but rather localized soliton configurations. The existence of such solitons was recently discovered in [30]. It is best seen by bosonizing the theory according to the rules [31,32]

$$\begin{aligned} \bar{\psi}\gamma_\mu\psi &\rightarrow \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\phi, & \bar{\Psi}\gamma_\mu\Psi &\rightarrow \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\chi, \\ \bar{\psi}\psi &\rightarrow -\mu_\psi\cos(2\sqrt{\pi}\phi), & \bar{\Psi}\Psi &\rightarrow -\mu_\Psi\cos(2\sqrt{\pi}\chi), \end{aligned} \quad (51)$$

where the constants μ carrying the dimension of mass depend on a particular normalization procedure for the scalar fermion bilinears. Assuming $M \gg e$ and integrating out the gauge fields, one arrives at the bosonized Lagrangian involving only the physical degrees of freedom:

$$\begin{aligned} \mathcal{L}_{\text{bos}} &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{e^2}{2\pi}\left(\phi + \frac{\chi}{2}\right)^2 \\ &+ cM^2[\cos(2\sqrt{\pi}\chi) - 1]. \end{aligned} \quad (52)$$

In contrast to the four-dimensional case where the effective boson Lagrangians are approximate, here the Lagrangian (52) is *exactly* equivalent to Eq. (49) in the sense that the spectrum and all other physical properties of the theories (49) and (52) coincide.¹⁰

We see that the potential in Eq. (52) involves an infinite set of minima at $\chi = -2\phi = n\sqrt{\pi}$ with integer n . The theory admits finite energy static solutions which interpolate between $\phi = \chi = 0$ at $x = -\infty$ and, say, $\chi = -2\phi = \sqrt{\pi}$ at $x = \infty$. The physical meaning of such a soliton is clear. It is a kind of a “heavy meson” (cf. Ref. [33] where such meson solutions were obtained in a theory where the charges of the light and heavy fields were equal) composed of the original heavy quark Ψ and a cloud of massless fermion fields ψ which neutralize its charge (a “constituent quark” in terminology of Ref. [33]). The fact that integer-charged light fermions manage to screen a heavy fractional charge [34,30] crucially depends on the masslessness of the light fermion. If $m_\psi \neq 0$, the energy of such a “meson” becomes infinite and heavy fractional charges are confined.

¹⁰Generally speaking, the Lagrangian (52) is not quite correct. As in the case of the standard Schwinger model [10] we have to separate the zero spatial Fourier harmonics of the “massive photon” field $\phi + \chi/2$ and write down a “glued” Lagrangian similar to that in SUSY gluodynamics, invariant under the transformation $\phi + \chi/2 \rightarrow \phi + \chi/2 + \sqrt{\pi}/2$. This is irrelevant for the discussion which follows: The combination $\phi + \chi/2$ inside the wall does not cross the boundary, $|\phi + \chi/2|$ is always less than $\sqrt{\pi}/2$.

In the context of our discussion here, two facts are important. The light quark condensate $\langle \bar{\psi}\psi \rangle \propto \langle \cos(2\sqrt{\pi}\phi) \rangle$ changes sign in passing from $x = -\infty$ to $x = \infty$ along the soliton solution. Thereby, these solitons are very much analogous to the four-dimensional supersymmetric domain walls separating different vacua. Indeed, the Lagrangian (52) involves a discrete Z_2 symmetry corresponding to the positive or negative sign of the light quark condensate. This symmetry is broken spontaneously.¹¹

Now, we come to a crucial distinction of this two-dimensional model as compared to the four-dimensional supersymmetric theory. In the two-dimensional model at hand, when M is large, the energy of the soliton is of order of the mass of the heavy fermion M (the “constituent quark” has an energy of order of $e \ll M$). This means that, in the limit $M \rightarrow \infty$, the mass of solitons becomes infinite. Correspondingly, the sectors with different signs of $\langle \bar{\psi}\psi \rangle$ cease to talk to each other, which nicely conforms with the standard topological classification of the Schwinger model with one dynamical massless fermion of charge e , where the flux (50) (a half of the standard flux) is quantized to half-integer values.

As was repeatedly mentioned, in the four-dimensional SU(2)-supersymmetric Yang-Mills theory, *both* the walls interpolating between chirally symmetric and asymmetric vacua *and* the walls interpolating between different chirally asymmetric vacua are BPS saturated. By virtue of the exact theorem (1) and its corollary (2) that guarantees that their energy density remains finite also in the limit $m \rightarrow \infty$. By combining two walls of the type we found — one interpolating between 1 and 0, and another between 0 and -1 — we may build a wall interpolating between two chirally asymmetric vacua in the SU(2)-supersymmetric gluodynamics. This wall just consists of two independent components. This “superposition” of two BPS-saturated solutions in the SU(2)-supersymmetric gluodynamics, going through $\phi = 0$ along the real axis, is almost a BPS-saturated solution in itself. That is, the energy of this configuration approaches the BPS bound when the two components of the wall are far apart. For SU(3) and higher groups superimposing two solutions (one goes from 1 to 0, and another goes from 0 to $e^{2\pi i/N}$ along the straight lines in the complex ϕ plane) also gives a two-component domain wall with the interacting components. Strictly speaking, this configuration is not a solution of equations of motion, although it approaches a solution in the limit of infinitely large separations between the components of such a wall. The truly BPS-saturated walls, if they exist in this case, should go through the complex ϕ plane in a nontrivial manner.

Returning to the torons, our consideration implies that, for large volumes, the relevant topological classification in SUSY gluodynamics is exactly the same as in SQCD — *the*

¹¹For those interested in the in-depth coverage we note that the spontaneous breaking takes place only at zero temperature. At any nonzero temperature, the “domain walls” which mix the distinct vacua appear in the heat bath and the symmetry is restored. The density of the soliton states is $\propto \exp\{-E_{\text{sol}}/T\}$. The “domain walls” have finite energy here due to the absence of extra transverse dimensions. The situation is exactly the same as in the one-dimensional Ising model (see, e.g., Ref. [35]) and in QCD₂ with adjoint fermions for higher unitary groups [10]: In all these cases we have a first-order phase transition at $T = 0$.

topological charge (7) is strictly integer and there are no torons.

The arguments presented here, although rather convincing to our mind, are unfortunately indirect. To resolve the paradox completely one should explain why the toron configurations which are essential for small tori [see Eq. (48)] disappear in the limit $L \rightarrow \infty$. We think that is very much probable, but at the moment do not see a technical reason for it.

VIII. CONCLUSIONS

In this work we have studied domain walls in supersymmetric gluodynamics and in SQCD. There are two basic types of walls in these theories: a wall that interpolates between two chirally asymmetric vacua and a wall that interpolates between a chirally asymmetric vacuum and a symmetric vacuum at the origin of the field space. We have shown explicitly that both those types of walls are BPS-saturated. The energy density of BPS-saturated walls in supersymmetric theories is unambiguously determined by the difference between the asymptotic values of the chiral condensate on the two sides of the wall. We have calculated this energy density explicitly from the (corrected) Veneziano-Yankielowicz effective Lagrangian and found that it indeed conforms with the exact relation, Eq. (2).

When a small supersymmetry-breaking mass is added to the SUSY gluodynamics Lagrangian, the degeneracy between the vacua is lifted. All vacuum states except for one become metastable. We have calculated the decay rate of these false vacua to leading order in the SUSY-breaking mass. It is an interesting question whether some of these metastable states survive at large mass and therefore exist also in pure nonsupersymmetric Yang-Mills theory.

Although the explicit calculations in this paper were performed for the gauge group $SU(2)$, the exact relation, Eq. (2), is valid for any $SU(N)$ group and we do not expect any qualitative changes in the character of the wall solutions for higher N .

ACKNOWLEDGMENTS

M.S. is grateful to M. Green and P. Van Baal for useful discussions. A.S. acknowledges warm hospitality extended to him at Theoretical Physics Institute, University of Minnesota. This work was supported in part by DOE under Grant No. DE-FG02-94ER40823, by INTAS Grants Nos. 93-0283 and 94-2851, by the U.S. Civilian Research and Development Foundation under Grant No. RP2-132, and by Schweizerischer National Fonds Grant No. 7SUPJ048716 and by the RFFI grant 97-02-16131.

-
- [1] G. Dvali and M. Shifman, Phys. Lett. B **396**, 64 (1997); **407**, 452(E) (1997).
 - [2] M. Shifman and A. Vainshtein, Nucl. Phys. **B296**, 445 (1988).
 - [3] S. Ferrara and B. Zumino, Nucl. Phys. **B79**, 413 (1974).
 - [4] M. Shifman, Prog. Part. Nucl. Phys. **39**, 1 (1997).
 - [5] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. **B229**, 407 (1983); G. Rossi and G. Veneziano, Phys. Lett. **138B**, 195 (1984).
 - [6] G. Veneziano and S. Yankielowicz, Phys. Lett. **113B**, 231 (1982).
 - [7] T. Taylor, G. Veneziano, and S. Yankielowicz, Nucl. Phys. **B218**, 493 (1983).
 - [8] G. Veneziano, *Classical vs. Quantum Vacua: from Mechanics to Superstrings*, in *Vacuum and Vacua: The Physics of Nothing*, Proceedings of the International School of Subnuclear Physics, Erice, Sicily, 1995, edited by A. Zichichi (World Scientific, Singapore, 1996), p. 16.
 - [9] A. Kovner and M. Shifman, Phys. Rev. D **56**, 2396 (1997).
 - [10] A. V. Smilga, Phys. Rev. D **54**, 7757 (1996).
 - [11] A. Vainshtein, V. Zakharov, and M. Shifman, Usp. Fiz. Nauk **146**, 683 (1985) [Sov. Phys. Usp. **28**, 709 (1985)].
 - [12] M. Shifman and A. Vainshtein, Nucl. Phys. **B359**, 571 (1991); N. Seiberg, Phys. Lett. B **318**, 469 (1993).
 - [13] S. Ferrara and B. Zumino, Nucl. Phys. **B87**, 207 (1975).
 - [14] K. Konishi, Phys. Lett. **135B**, 439 (1984); K. Konishi and K. Shizuya, Nuovo Cimento A **90**, 111 (1985).
 - [15] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B241**, 493 (1984); **B256**, 557 (1985).
 - [16] I. Kogan, M. Shifman, and A. Vainshtein, Phys. Rev. D **53**, 4526 (1996).
 - [17] D. Amati, K. Konishi, Y. Meurice, G. Rossi, and G. Veneziano, Phys. Rep. **162**, 557 (1988).
 - [18] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. **B229**, 381 (1983); Phys. Lett. **166B**, 334 (1986); for a recent review see M. Shifman, Int. J. Mod. Phys. A **11**, 5761 (1996).
 - [19] B. Chibisov and M. Shifman, Phys. Rev. D (to be published).
 - [20] A. Smilga and A. Veselov, hep-th/9706217 [Phys. Rev. Lett. (to be published)]; hep-th/9710123.
 - [21] G. Dvali and M. Shifman, hep-th/9611213.
 - [22] M. B. Voloshin, I. Yu. Kobzarev, and L. B. Okun, Sov. J. Nucl. Phys. **20**, 644 (1975).
 - [23] J. Hisano and M. Shifman, Phys. Rev. D **56**, 5475 (1997).
 - [24] G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).
 - [25] H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607 (1992).
 - [26] M. A. Shifman and A. V. Smilga, Phys. Rev. D **50**, 7659 (1994).
 - [27] E. Cohen and C. Gomez, Phys. Rev. Lett. **52**, 237 (1984).
 - [28] A. R. Zhitnitsky, Nucl. Phys. **B340**, 56 (1990).
 - [29] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B430**, 485(E) (1994).
 - [30] D. Gross, I. Klebanov, A. Matsitsyn, and A. Smilga, Nucl. Phys. **B461**, 109 (1996).
 - [31] A. Luther and V. J. Emery, Phys. Rev. Lett. **33**, 589 (1974).
 - [32] S. Coleman, Phys. Rev. D **11**, 2088 (1975); S. Mandelstam, *ibid.* **11**, 3026 (1975).
 - [33] J. Ellis, Y. Frishman, A. Hanany, and M. Karliner, Nucl. Phys. **B382**, 189 (1992).
 - [34] S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) **93**, 267 (1971).
 - [35] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, New York, 1982).