Colliding axion-dilaton plane waves from black holes

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The colliding plane wave metric discovered by Ferrari and Ibañez to be locally isometric to the interior of a Schwarzschild black hole is extended to the case of general axion-dilaton black holes. Because the transformation maps either black hole horizon to the focal plane of the colliding waves, this entire class of colliding plane wave spacetimes only suffers from the formation of spacetime singularities in the limits where the inner horizon itself is singular, which occur in the Schwarzschild and dilaton black hole limits. The supersymmetric limit corresponding to the extreme axion-dilaton black hole yields the Bertotti-Robinson metric with the axion and dilaton fields flowing to fixed constant values. The maximal analytic extension of this metric across the Cauchy horizon yields a spacetime in which two sandwich waves in a cylindrical universe collide to produce a semi-infinite chain of Reissner-Nordstrom-like wormholes. The focusing of particle and string geodesics in this spacetime is explored. [S0556-2821(97)00224-5]

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I. INTRODUCTION

A static, spherically symmetric black hole spacetime is only static in the regions where $\partial/\partial t$ is timelike. In the trapped region between the two horizons, the metric is quite violently dependent on the timelike radial coordinate, while $\partial/\partial t$ and $\partial/\partial\phi$ act as a pair of purely spacelike commuting Killing vectors. A violently time-dependent spacetime with two commuting spacelike Killing vectors is also a potential description of the spacetime of two colliding plane symmetric gravitational waves. This idea was first recognized and explored in the in the Einstein-Maxwell limit by Chandrasekhar in 1984 [1]. A colliding plane wave metric locally isometric to the interior of a Schwarzschild black hole was obtained by Ferrari, Ibañez, and Bruni in 1987 [2,3]. The direct transformation from a Schwarzschild black hole to a colliding plane wave spacetime was described by Yurtsever in 1988 [4,5].

The purpose of this paper is to extend this analysis to the case of axion-dilaton black holes [6–9] that are $N=4$ supersymmetric solutions of low-energy string theory, and to compare string and particle propagation in the resulting spacetimes.

In Sec. II, we display the transformation between a Schwarzschild trapped region and a colliding plane wave spacetime elucidated by Yurtsever. Then we extend this transformation to the general case of axion-dilaton black holes found in low-energy string theory. We show how these plane wave collisions end in the formation of singularities *only* when they represent transformations of black hole spacetimes where the singularity is touching the trapped region, as in the case of the Schwarzschild and the singular dilaton black holes.

The nonsingular colliding wave spacetimes have Killing-Cauchy horizons instead of singularities. The curvature at the Killing-Cauchy horizon is equal to the curvature of the appropriate black hole horizon locally isometric to that particular plane wave spacetime. The metric can be extended across this horizon in an intuitively appealing manner, but the price of avoiding the singularity is the loss of global hyperbolicity. This is consistent with Hawking-Penrose singularity theorems in that geodesic focusing forces a choice between a local or global pathology.

In Sec. III we review the work done by Yurtsever on the asymptotic structure of colliding plane wave spacetimes, and we show where the transformed black hole solutions fit in this general classification scheme.

In Sec. IV we compare the asymptotic causal structure of axion-dilaton colliding plane waves with that of general colliding plane wave solutions of the vacuum Einstein and Maxwell-Einstein-dilaton equations.

In Sec. V we show that the maximal analytic extension of the general axion-dilaton colliding plane wave spacetime is two collinearly polarized waves propagating in a cylindrical universe to create a black hole with the same causal structure as an infinite chain of wormholes in Reissner-Nordstrom spacetime. In the event that either of the incoming waves has a δ function profile in the incoming regions, the maximal analytic extension degenerates to the extreme dilaton supersymmetric configuration with 1/2 of $N=4$ supersymmetry unbroken.

In Sec. VI we compare particle and string propagation in an exact plane wave background and plot the effects of violation of the principle of equivalence by strings. We also briefly examine the issue of more realistic finite-sized almost-plane waves.

II. COLLIDING WAVES OUT OF BLACK HOLES

Inside the trapped region ($r\leq 2M$) of a Schwarzschild black hole the metric can be written:

$$ds^2=-\frac{r}{2M-r}dr^2+r^2d\theta^2+\frac{2M-r}{r}dt^2+r^2\sin^2\theta d\phi^2. \tag{1}$$

On the other hand, the metric for the interaction region of

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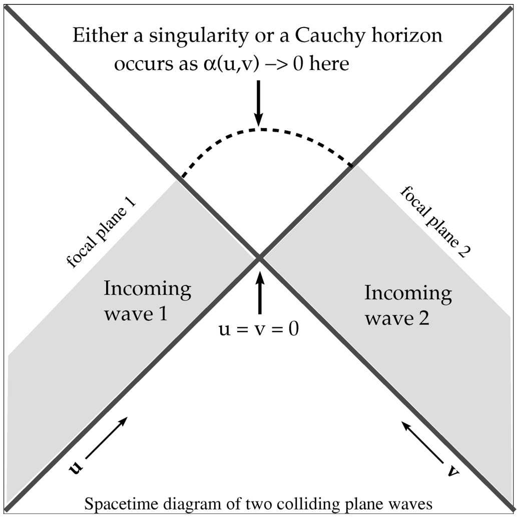

FIG. 1. Spacetime diagram of two colliding plane waves.

two colliding, collinearly polarized, plane symmetric gravity waves (Fig. 1) can be written in the form

$$ds^2=-e^{-M(u,v)}du\,dv+e^{-U(u,v)}(e^{V(u,v)}dx^2+e^{-V(u,v)}dy^2). \tag{2}$$

The former spacetime can be put in the form of the latter using the coordinate transformation [4,5]

$$r\to M[1-\sin(u+v)],\quad \theta\to\frac{\pi}{2}+v-u,\quad t\to x,$$

$$\phi\to 1+\frac{y}{M}, \tag{3}$$

for $u\geqslant 0$, $v\geqslant 0$, and $u+v\leqslant\pi/2$. To make a colliding plane wave spacetime we have to analytically continue y past the cyclic boundary conditions on ϕ. Therefore the resulting metric

$$ds^2=-4M^2[1-\sin(u+v)]^2du\,dv+\frac{\cos^2(u+v)}{[1-\sin(u+v)]^2}dx^2$$
$$+\cos^2(u-v)[1-\sin(u+v)]^2dy^2 \tag{4}$$

is locally, but not globally, isometric to Eq. (1).

There is one slight problem with this metric: it serves as a good description of the interaction region for two colliding plane waves, but it does not describe the spacetime before the two waves have met. Penrose and Khan [11] came up with an effective yet slightly flawed prescription for constructing incoming waves from a metric for a colliding wave interaction region: replace u and v, respectively, by $uH(u)$ and $vH(v)$, where $H(x)$ is the Heaviside step function. Thus an incoming wave in the region $u>0, v<0$ can be written

$$ds^2=-4M^2[1-\sin(u)]^2du\,dv+\frac{\cos^2(u)}{[1-\sin(u)]^2}dx^2$$
$$+\cos^2(u)[1-\sin(u)]^2dy^2. \tag{5}$$

Length scales for the colliding wave system are introduced via $u\to u/a,\ v\to v/b$. The requirement that the metric be continuous with flat spacetime at $u=v=0$ relates the focal lengths and amplitudes of the incoming waves through $ab=4M^2$, which we will see more of later. The spacelike Killing vector $\partial/\partial x$ becomes singular when $u/a+v/b\to\pi/2$, and moreover, $R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}\to\infty$ there, indicating a spacetime singularity. The incoming wave metrics obtained by the above Khan-Penrose prescription have $R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}=0$ but the Weyl tensor component

$$C_{uxux}=\frac{-3\,[\cos(u/2\,a)+\sin(u/2\,a)]^2}{a^2\,[\cos(u/2\,a)-\sin(u/2\,a)]^4} \tag{6}$$

blows up as $u\to\pi a/2$, showing that the incoming waves are singular in some sense before they collide.

A better-behaved metric can be obtained by sending (u,v) to $(-u,-v)$:

$$ds^2=-4M^2[1+\sin(u+v)]^2du\,dv+\frac{\cos^2(u+v)}{[1+\sin(u+v)]^2}dx^2$$
$$+\cos^2(u-v)\,[1+\sin(u+v)]^2dy^2. \tag{7}$$

This metric is also locally isometric to the trapped region of a Schwarzschild black hole, except that $R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}=3/4M^4$ in the limit $u/a+v/b\to\pi/2$. The spacelike Killing vector $\partial/\partial x$ becomes null there, signalling a Cauchy horizon (because initial data that are spatially homogenous in the x direction cease to be so when $\partial/\partial x$ is no longer spacelike). The incoming waves extended from this collision region have

$$C_{uxux}=\frac{-3[\cos(u/2\,a)-\sin(u/2a)]^2}{a^2[\cos(u/2a)+\sin(u/2\,a)]^4}, \tag{8}$$

which vanishes on the incoming focal plane $u=\pi a/2$. These waves are called ''sandwich waves,'' the curvature being neatly sandwiched between the past wave front and the focal plane to the future. The incoming waves in Eq. (5) are not sandwich waves in this sense.

A. Axion-dilaton black holes

In order to better understand this pattern of singular and nonsingular behavior, we will extend the coordinate transformation made for the Schwarzschild black hole to the general case of an axion-dilaton black hole in $d=4$ with N U(1) gauge fields, with the action

$$S_{\rm eff}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\Bigg(-R+\frac{1}{2}\,\frac{\partial_\mu\lambda\,\partial^\mu\overline{\lambda}}{(\mathrm{Im}\lambda)^2}$$
$$-\sum_{n=1}^{N}F^{(n)\star}_{\mu\nu}\widetilde{F}^{(n)\mu\nu}\Bigg), \tag{9}$$

where $\widetilde{F}^{\mu\nu}=e^{-2\phi\star}F^{\mu\nu}-i\psi F^{\mu\nu}$. The axion (ψ) and dilaton (ϕ) fields are combined into $\lambda=\psi+ie^{-2\phi}$, and * is the spacetime dual operation.

In the trapped region the metric can be written

$$ds^2=-\frac{(r^2-|\Upsilon|^2)}{(r_+-r)(r-r_-)}dr^2+(r^2-|\Upsilon|^2)d\theta^2$$
$$+\frac{(r_+-r)(r-r_-)}{(r^2-|\Upsilon|^2)}dt^2+(r^2-|\Upsilon|^2)\sin^2\theta d\phi^2, \tag{10}$$

where

$$r_\pm=M\pm r_0,\quad r_0^2=M^2+|\Upsilon|^2-4\sum|\Gamma^{(n)}|^2,$$

$$\Gamma^{(n)}=\frac{1}{2}(Q^{(n)}+iP^{(n)}), \tag{11}$$

and

$$\Upsilon=\Sigma-i\Delta=-\frac{2}{M}\sum_{n=1}^{N}(\Gamma^{(n)})^2. \tag{12}$$

$\{Q^{(n)},P^{(n)}\}$ are the U(1) electric and magnetic charges, respectively. The entropy of the axion-dilaton black hole is given by 1/4 of the area of the horizon

$$S=\frac{A}{4}=\pi[(r_+)^2-|\Upsilon|^2]. \tag{13}$$

It is important to remember that the coordinate r is now measuring time, so this is a highly time-dependent spacetime, not the placid exterior of a classical black hole. In the extreme limit of $r_+\to r_-\to M$, or $r_0\to 0$, the region over which r is timelike shrinks to zero, and so the amount of violent time dependence inside the black hole shrinks away as well. The area of the extreme black hole is

$$S_{\rm extr}=\frac{A_{\rm extr}}{4}=\pi(M^2-|\Upsilon|^2). \tag{14}$$

In general these axion-dilaton black holes have fascinating properties and relationships to deep symmetries in string theory [6–9]. The parameter r_0 measures how far the black hole is from the extremal limit $r_+=r_-$, where the trapped region threatens to vanish and reveal a naked singularity to the universe. The parameter r_0 also measures the breaking of supersymmetries in the $N=4$ supergravity theory underlying the action (9). The condition $r_0=0$ corresponds to the saturation of the supersymmetry (SUSY) bound via $M=|z_1|>|z_2|$ or $M=|z_2|>|z_1|$ between the black hole mass and the largest of the eigenvalues (z_1,z_2) of the central charge matrix of the $N=4$ theory, restoring 1/4 of the broken $N=4$ supersymmetry. The area of the extreme horizon is proportional to the square of the largest central charge at the ''fixed point'' where the other central charge vanishes.

The full saturation $M=|z_1|=|z_2|$ restores 1/2 of the broken $N=4$ supersymmetry. The $r_0\to 0$ limit of the corresponding black hole is an extreme dilaton black hole with $M=|\Upsilon|$, zero entropy and a singular horizon. Hence supersymmetry serves as a cosmic censor for these black holes as long as not more than 1/4 of the $N=4$ supersymmetry is restored.

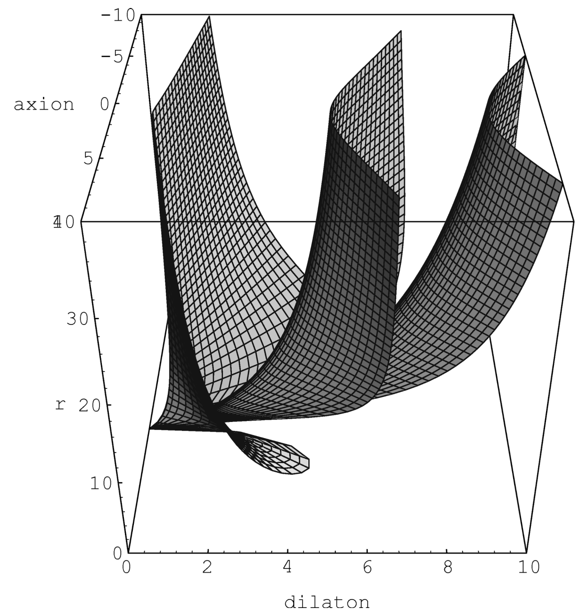

FIG. 2. These contours of constant ϕ_0 show how the axion and dilaton fields lose their dependence on ϕ_0 and ψ_0 and flow to fixed values on the horizon of an extreme axion-dilaton black hole. Here the coordinate r measures distance from the extreme horizon.

The axion and dilaton fields add to this interesting behavior at the horizon in the $r_0=0$ limit (Fig. 2). At the extreme horizon they lose all dependence on their values $\lambda_0=\psi_0+ie^{-2\phi_0}$ at spatial infinity and depend only on the values of quantized conserved charges. For a single extreme black hole of this type with N electric and magnetic charges $Q^{(i)}+iP^{(i)}=e^{\phi_0}(n_i-\overline{\lambda_0 m_i})$, with $(n_i,m_i)\in Z$, the axion and dilaton fields at the horizon reduce to [10]

$$\psi_f=\frac{\sum n_im_i}{\sum m_i^2},\quad e^{-2\phi_f}=\frac{\left(\sum_{i<j}(n_im_j-n_jm_i)^2\right)^{1/2}}{\sum m_i^2}. \tag{15}$$

B. Axion-dilaton colliding waves

The coordinate transformation from (r,θ,t,ϕ) to (u,v,x,y) gives $r(u,v)\to r_\pm$ for the $(\pm)$ branch of the solution as $u/a+v/b\to\pi/2$:

$$r\to M\pm r_0\sin\left(\frac{u}{a}+\frac{v}{b}\right),\quad \theta\to\frac{\pi}{2}\pm\left(\frac{u}{a}-\frac{v}{b}\right),$$
$$t\to xr_0/(M^2-|\Upsilon|^2)^{1/2},\quad \phi\to 1+y/(M^2-|\Upsilon|^2)^{1/2}. \tag{16}$$

The trapped region of the black hole is only locally isometric to the interaction region of the colliding plane wave spacetime, because we are sending the cyclic coordinate ϕ to the noncyclic coordinate y, to represent a plane wave infinite in both the x and y directions.

The axion-dilaton black hole metric in the trapped region now takes the form

$$g_{uv}=\frac{-2\{[M\pm r_0\sin(u/a+v/b)]^2-|\Upsilon|^2\}}{ab}, \tag{17}$$

$$g_{xx}=\frac{(M^2-|\Upsilon|^2)\cos(u/a+v/b)^2}{[M\pm r_0\sin(u/a+v/b)]^2-|\Upsilon|^2}$$

$$g_{yy}=\cos^2\left(\frac{u}{a}-\frac{v}{b}\right)\frac{\{[M\pm r_0\sin(u/a+v/b)]^2-|\Upsilon|^2\}}{M^2-|\Upsilon|^2}.$$

Requiring $g_{\mu\nu}=\eta_{\mu\nu}$ for $u=v=0$ constrains the incoming parameters a and b to satisy

$$ab=4(M^2-|\Upsilon|^2)=\frac{4S_{\rm extr}}{\pi}. \tag{18}$$

Thus only when the mass is larger than the axion-dilaton charge and the entropy of the relevant black hole is not vanishing are both parameters a and b nonvanishing. This constraint is significant because the condition $M>|\Upsilon|$ is a supersymmetry bound that helps enforce cosmic censorship in the black hole system. This bound in the colliding plane wave system tells us that the effective focal length of the colliding axion-dilaton plane wave system is not negative ($f=\pi\sqrt{ab}/2\geqslant 0$) and only approaches zero in the singular extreme dilaton limit $M=|\Upsilon|$. This looks and acts like a supersymmetric enforcement of cosmic censorship, although in the context of the colliding-wave problem, it was derived by requiring that the spacetime be exactly flat before the arrival of each incoming wave.

Abbreviating $r_\pm(u,v)=M\pm r_0\sin(u/a+v/b)$, the axion and dilaton fields become

$$\psi(u,v)=\frac{\psi_0\{\Delta^2+[\Sigma+r_\pm(u,v)]^2\}-2e^{-2\phi_0}\Delta\ r_\pm(u,v)}{\Delta^2+[\Sigma+r_\pm(u,v)]^2} \tag{19}$$

and

$$e^{-2\phi(u,v)}=e^{-2\phi_0}\frac{r_\pm(u,v)^2-(\Sigma^2+\Delta^2)}{\Delta^2+[\Sigma+r_\pm(u,v)]^2}, \tag{20}$$

where Σ and Δ are given by Eq. (12).

Transforming from (t,ϕ) to (x,y) by Eq. (16), the N U(1) potentials with electric and magnetic charges $(Q^{(n)},P^{(n)})$ are transformed from $(A_t^{(n)},A_\phi^{(n)})$ to

$$A_x^{(n)}=\frac{e^{\phi_0}r_0\{P^{(n)}\ \Delta+Q^{(n)}[\Sigma+r_\pm(u,v)]\}}{\sqrt{M^2-|\Upsilon|^2}[r_\pm(u,v)^2-\Delta^2-\Sigma^2]}, \tag{21}$$

$$A_y^{(n)}=-\frac{e^{\phi_0}P^{(n)}\cos(u/a-v/b)}{\sqrt{M^2-|\Upsilon|^2}}. \tag{22}$$

The value of $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ in the limit $u/a+v/b\to\pi/2$ is equal to value of $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ for the equivalent axion-dilaton black hole, evaluated at $r=r_\pm=M\pm r_0$. One can see from the following equation that this quantity will only blow up in two limits: the Schwarzschild limit $|\Upsilon|=0, r=r_-=0$ and the extreme dilaton limit $r_-=|\Upsilon|$:

$$R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}=\frac{8(M^4+4\,M^2\ r_0{}^2+12Mr_0{}^3+7\,r_0{}^4-2M^2|\Upsilon|^2+2r_0{}^2|\Upsilon|^2+|\Upsilon|^4)}{(r_\pm^2-|\Upsilon|^2)^4}. \tag{23}$$

C. Extreme limit of axion-dilaton colliding waves

For axion-dilaton black holes the limit $r_0\to 0$ corresponds to the apparent vanishing of the trapped region between $r_+=M+r_0$ and $r_-=M-r_0$. This also corresponds to the restoration of 1/4 of the broken $N=4$ supersymmetry in the background supergravity theory and fixed values for the axion and dilaton fields at the extreme horizon. For axion-dilaton colliding waves the $r_0\to 0$ limit gives the Bertotti-Robinson colliding plane wave spacetime

$$ds^2=-du\ dv+\cos^2\left(\frac{u}{a}+\frac{v}{b}\right)dx^2+\cos^2\left(\frac{u}{a}-\frac{v}{b}\right)dy^2. \tag{24}$$

In this limit the axion and dilaton fields reduce to

$$\psi_f=\frac{\psi_0[\Delta^2+(\Sigma+M)^2]-2e^{-2\phi_0}\Delta M}{\Delta^2+(\Sigma+M)^2},$$

$$e^{-2\phi_f}=e^{-2\phi_0}\frac{M^2-(\Sigma^2+\Delta^2)}{\Delta^2+(\Sigma+M)^2}, \tag{25}$$

and reduce to Eq. (15) when written in terms of the Dirac-quantized conserved charges. The axion and dilaton are constant and take their critical values over the entire Bertotti-Robinson spacetime, even in the flat region before either wave has passed. Note that the axion and dilaton fields for $r_0\neq 0$ also take their fixed constant values in the flat region before the waves have arrived, but evolve to their values at $r_\pm$ on the focal planes of the incoming and colliding waves.

The incoming wave obtained from the above Bertotti-Robinson metric via the Khan-Penrose prescription for $u>0,\ v<0$ is

$$ds^2=-du\ dv+\cos^2\left(\frac{u}{a}\right)dx^2+\cos^2\left(\frac{u}{a}\right)dy^2. \tag{26}$$

(The other incoming wave is the same as above with $u\to v,\ a\to b$.) Using the coordinate transformation

$$u=U,$$

$$v=V-\frac{1}{a}\tan\left(\frac{u}{a}\right)(X^2+Y^2), \tag{27}$$

$$x = X/\cos\left(\frac{u}{a}\right),$$
$$y = Y/\cos\left(\frac{u}{a}\right), \tag{28}$$

and setting $\Delta U \equiv \pi a/2$, the wave metric (27) becomes

$$ds^2 = -dUdV - \left(\frac{\pi}{2\Delta U}\right)^2 (X^2+Y^2)dU^2 + dX^2 + dY^2,$$
$$0 \leqslant U \leqslant \Delta U$$
$$ds^2 = -dUdV + dX^2 + dY^2, \quad U<0,\, U>\Delta U. \tag{29}$$

This incoming wave extension of a Bertotti-Robinson spacetime is a pulse of constant curvature of duration $\Delta U = \pi a/2$ and magnitude $1/a^2 = (\pi/2\Delta U)^2$. The focal length of the wave is $f = \Delta U = \pi a/2$, meaning that null geodesics from an event at $U = -\infty$ focus at the edge of the wave itself. The relation for colliding waves $ab = 4(M^2 - |\Upsilon|^2)$ is a relation between the curvatures of the incoming waves in the $r_0 \to 0$ limit and the mean focal length of the colliding system. It is curious that this relationship is also enforced away from $r_0 = 0$.

We will compare plots of test particle and test string null geodesics for this metric truncated to $d=3$ in Sec. IV.

III. PROPERTIES OF GENERAL COLLIDING PLANE WAVE SPACETIMES

A. Properties of vacuum solutions

In the interaction region ($u>0$, $v>0$) any collinearly polarized colliding plane wave spacetime can be written in the form [5]

$$ds^2 = \frac{l_1 l_2}{\sqrt{\alpha}} e^{Q(\alpha,\beta)/2}(-d\alpha^2 + d\beta^2) + \alpha(e^{V(\alpha,\beta)}dx^2 + e^{-V(\alpha,\beta)}dy^2), \tag{30}$$

where the coordinate transformation from (u,v) in Eq. (2) to (α,β) is defined by

$$\alpha = e^{-U(u,v)}, \quad \beta_u = -\alpha_u, \quad \beta_v = \alpha_v. \tag{31}$$

The vacuum Einstein equations reduce to:

$$V_{\alpha\alpha} + \frac{V_\alpha}{\alpha} - V_{\beta\beta} = 0, \tag{32}$$

$$Q_\alpha = -\alpha({V_\alpha}^2 + {V_\beta}^2), \tag{33}$$

$$Q_\beta = -2\alpha V_\alpha V_\beta, \tag{34}$$

plus constraints for the initial data along $(u=0,v)$ and $(u,v=0)$. Equations (32)–(34) are solved by functions whose limits as $\alpha \to 0$ are singular like

$$V(\alpha,\beta) \sim \epsilon(\beta)\ln\alpha + \mu(\beta),$$
$$Q(\alpha,\beta) \sim -\epsilon^2(\beta)\ \ln\alpha + \delta(\beta), \tag{35}$$

where $\epsilon(\beta)$ is determined by an integral over the boundary between the interaction region and the incoming waves.

The β dependence in the metric is ignorable if we are only looking at the structure of the singular terms in the metric for $\alpha \to 0$. Counting powers of α and then changing coordinates from (α,β) to (t,z), the metric behaves like the Kasner homogeneous, anisotropic cosmology:

$$ds^2 \sim -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{36}$$

where the Kasner exponents $\{p_i\}$ satisfy

$$p_1 = \frac{2(1+\epsilon)}{\epsilon^2+3},\ \ p_2 = \frac{2(1-\epsilon)}{\epsilon^2+3},\ \ p_3 = \frac{\epsilon^2-1}{\epsilon^2+3},$$

and

$$\sum p_i = \sum {p_i}^2 = 1. \tag{37}$$

This metric has curvature squared

$$R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} = \frac{4p_1p_2p_3}{t^4}, \tag{38}$$

and so is singular as $t \to 0$ unless $\epsilon(\beta) = \pm 1$, in which limit the Kasner metric reduces to a slice of Rindler spacetime.

B. How this applies to axion-dilaton colliding waves

Axion-dilaton colliding plane waves do not obey the vacuum Einstein equations. However, the metric obtained through the transformation (16) fits the form of the metric (2) and the coordinate transformation (31) is still valid. (This transformation determines the existence of a foliation of the interaction region into spacelike hypersurfaces $\alpha = \text{const}$ and works for $R_{\mu\nu} \neq 0$ as long as the plane waves are collinearly polarized.)

Remarkably enough (but not so remarkable once one recalls that this is still essentially a two-dimensional problem), the functions $V(\alpha,\beta)$ and $Q(\alpha,\beta)$ still behave like Eq. (35) in the limit $\alpha \to 0$. Therefore, the Kasner asymptotic limit also applies to axion-dilaton colliding plane waves.

Combining (31) and (35), we see how to calculate $\epsilon(\beta)$ without integrating over the initial data:

$$\epsilon(\beta) = -\lim_{\alpha\to 0} V(\alpha,\beta)/U = \lim_{\alpha\to 0} \frac{\ln g_{xx} - \ln g_{yy}}{\ln g_{xx} + \ln g_{yy}}. \tag{39}$$

The coordinate transformation (31) for the metric under consideration can be solved exactly, giving

$$\alpha(u,v) = \frac{1}{2}\left(\cos\frac{2u}{a} + \cos\frac{2v}{b}\right),$$

$$\beta(u,v) = \frac{1}{2}\left(-\cos\frac{2u}{a} + \cos\frac{2v}{b}\right), \tag{40}$$

and this is easily invertible to give $[u(\alpha,\beta), v(\alpha,\beta)]$. Taking the limit (39) yields $\epsilon(\beta) = 1$, which means that these metrics are in general nonsingular. However, the nonvanishing part of $V(\alpha,\beta)$ as $\alpha\to 0$ consists of the singular function $\sim\epsilon(\beta)\ \ln\alpha$ plus a function $\mu(\beta)$, which for this metric is

$$\mu(\beta)=\ln\frac{(1-\beta^2)(M^2-|\Upsilon|^2)}{(M\pm r_0)^2-|\Upsilon|^2}. \tag{41}$$

This term results in a curvature singularity in two limits: the Schwarzshild limit where $r_0=M$ and $\Upsilon=0$, and the extreme dilaton limit where $r_0=0$ and $M=|\Upsilon|$. This happens because the coordinate transformation (16) maps either r_+ or r_- to $\alpha=0$. The value of $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ as $\alpha\to 0$ is the same as $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ at either $r_\pm$. The only axion-dilaton black holes where the curvature at $r_\pm$ is not finite are the Schwarzshild and singular dilaton black holes, with spacetime singularities as $r\to r_-$, as described above.

IV. RELATION TO EINSTEIN-MAXWELL-DILATON COLLIDING PLANE WAVE SPACETIMES

Bretón, Matos, and García [12] discovered a large class of colliding plane wave metrics that also obey the equations of motion for the action

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}\{-R+2(\nabla\Phi)^2+e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu}\}, \tag{42}$$

which for $\alpha=1$ is the same as the $N=1$, $\psi=0$ limit of the action (9). The metric takes the form

$$ds^2=\frac{e^{k(\alpha,\beta)/2}}{f}(-d\alpha^2+d\beta^2)+(\alpha^2/f)dx^2+fdy^2. \tag{43}$$

For solutions that overlap with those discussed in this paper, the function $f(\alpha,\beta)$ has the form

$$f=\frac{f_0e^{\lambda(\alpha,\beta)}}{(a_1\Sigma_1+a_2\Sigma_2)}, \tag{44}$$

the dilaton field is

$$\kappa^2=e^{-2\Phi}=\kappa_0^2(a_1\Sigma_1+a_2\Sigma_2)e^{\lambda}, \tag{45}$$

and the Maxwell potential has the form

$$A=A_y=\frac{(a_3\Sigma_1+a_4\Sigma_2)}{(a_1\Sigma_1+a_2\Sigma_2)}. \tag{46}$$

The functions $\Sigma_{1,2}$ that reproduce the $N=1$, $\psi=0$ limit of axion-dilaton colliding waves are

$$\Sigma_1=e^{q_1\tau(\alpha,\beta)},\quad \Sigma_2=e^{-q_2\tau(\alpha,\beta)}. \tag{47}$$

For this class of solutions, the functions $\tau(\alpha,\beta)$, $\lambda(\alpha,\beta)$, and $k(\alpha,\beta)$ satisfy

$$\tau_{\alpha\alpha}+\frac{\tau_\alpha}{\alpha}-\tau_{\beta\beta}=0,\quad \lambda_{\alpha\alpha}+\frac{\lambda_\alpha}{\alpha}-\lambda_{\beta\beta}=0, \tag{48}$$

$$k_\alpha=\frac{\alpha}{2}(\lambda_\alpha{}^2+\lambda_\beta{}^2)+q_1^2(\tau_\alpha{}^2+\tau_\beta{}^2),$$

$$k_\beta=-\alpha(\lambda_\alpha\lambda_\beta+q_1^2\tau_\alpha\tau_\beta), \tag{49}$$

and the constants f_0,a_i,κ_0 obey

$$f_0a_1a_2=\kappa_0^2(a_3a_2-a_1a_4)^2. \tag{50}$$

Since $\tau(\alpha,\beta)$ and $\lambda(\alpha,\beta)$ obey Eq. (32), as $\alpha\to 0$

$$\tau(\alpha,\beta)\sim\epsilon_1(\beta)\ln\alpha,\quad \lambda(\alpha,\beta)\sim\epsilon_2(\beta)\ln\alpha, \tag{51}$$

$$k(\alpha,\beta)\sim-[q_1{}^2\epsilon_1{}^2(\beta)+\epsilon_2{}^2(\beta)]\ln\alpha. \tag{52}$$

There are two conditions under which the metric and fields given above exhibit the same nonsingular Kasner asymptotic limit as exhibited by the $N=1$, $\psi=0$ limit of axion-dilaton colliding waves. The Schwarzschild limit with constant dilaton and Maxwell potential requires $q_1=q_2$ and $\lambda(\alpha,\beta)=-q_1\tau(\alpha,\beta)$, with $\epsilon_2(\alpha,\beta)=1$ or 0. The Einstein-Maxwell dilaton limit of axion-dilaton colliding waves is reachable only if $q_2=-q_1$ and $\epsilon_2(\alpha,\beta)=|q_1\epsilon_1(\alpha,\beta)|=1$.

V. MAXIMAL ANALYTIC EXTENSIONS OF AXION-DILATON COLLIDING PLANE WAVES

In Sec. III we showed that the asymptotic causal structure of the axion-dilaton colliding plane wave spacetime near the Killing-Cauchy horizon at $u/a+v/b=\pi/2$ is that of the Kasner metric

$$ds^2=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^z+t^{2p_3}dz^2 \tag{53}$$

in the limit $p_1=1$, $p_2=p_3=0$, corresponding to the wedges of Minkowski spacetime in Rindler coordinates that are ''behind the horizon'' for the usual constantly accelerating observer. This insight was derived using the general asymptotic structure of colliding plane graviational waves in [5], but it is more easily derived using black hole coordinates. The proper time from $r=r_\pm$ as measured by a nearby freely falling observer is approximately

$$\tau_+^2(r)\sim 2(r_+-r)\left(\frac{r_+^2-|\Upsilon|^2}{r_0}\right),$$

$$\tau_-^2(r)\sim 2(r-r_-)\left(\frac{r_-^2-|\Upsilon|^2}{r_0}\right). \tag{54}$$

Changing coordinates by assigning $\chi_\pm=tr_0/(r_\pm^2-|\Upsilon|^2)$, the metric becomes

$$ds^2\sim-d\tau_\pm^2+\tau_\pm^2d\chi_\pm^2+R(r_\pm)^2d\Omega. \tag{55}$$

In the (τ,χ) plane the metric is the wedge of Rindler spacetime defined in Minkowski coordinates by

$$T^2-X^2=\tau_\pm^2,\quad \frac{X}{T}=\tanh\chi_\pm. \tag{56}$$

The axion-dilaton colliding plane wave maps to the wedges of Rindler spacetime in the ''trapped regions'' II and IV and the maximal analytic extension across $\tau_\pm=0$ gives back the parts of Rindler space that correspond to the non-trapped regions I and III. It is important to remember that χ is proportional to x, and that the spacelike Killing vector

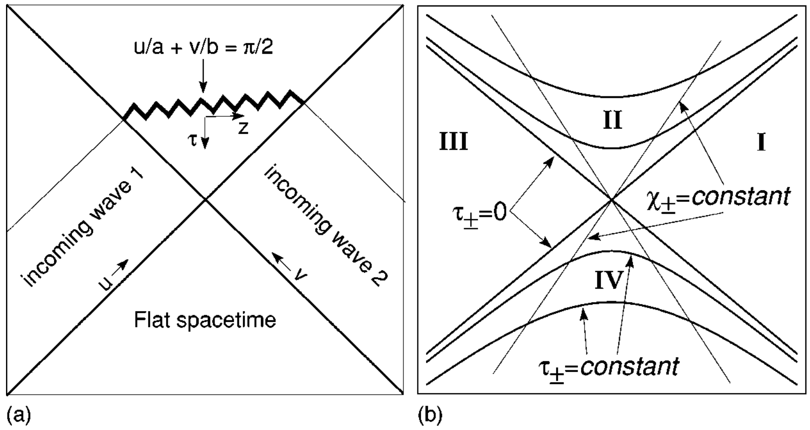

FIG. 3. (a) shows the wave collision in the (u,v) or (τ,z) plane. (b) shows how the metric near $u/a+v/b\rightarrow\pi/2$ looks in the (τ,χ) plane. The lines $\chi_\pm=$const are lines of constant x that cross on the Killing-Cauchy horizons $\tau_\pm=0$, where $\partial/\partial x$ becomes null.

$\partial/\partial x$ becomes null on the Killing-Cauchy horizon at $\tau_\pm=0$. This signals the breakdown of spatial translation invariance in the x direction just as $\partial/\partial t$ becoming null in regions I and III of Fig. 3(b) below signals the breakdown of time-translation invariance there.

From this point the maximal analytic extension of the axion-dilaton colliding plane wave metric follows the same steps as for the generic axion-dilaton black hole, which has the same causal structure and maximal analytic extension of a Reissner-Nordstrom black hole, except in the extreme dilaton limit to which we will return later. The Schwarzschild limit was described by Yurtsever in [5].

There is, however, one problem — we have broken the cyclic boundary conditions on ϕ in the coordinate transformation $\phi\rightarrow 1+y/(M^2-|\Upsilon|^2)^{1/2}$. The cyclic boundary conditions on ϕ, as extended across the surfaces $\tau_\pm=0$, can be restored by compactifying spacetime in the y direction for the incoming waves on a circle of radius $\sqrt{M^2-|\Upsilon|^2}$. If we insist that the maximal extension of the axion-dilaton colliding plane wave spacetime be analytic, compactification of the y-direction is forced on the incoming waves [5].

The maximal analytic extension of the axion-dilaton colliding plane wave metric has two sandwich waves with translation symmtery in the x and y directions propagating in a universe where the y coordinate lives on a circle of radius $\sqrt{M^2-|\Upsilon|^2}$. The waves collide to form either an event horizon at r_+ or a Cauchy horizon at r_- of an axion-dilaton black hole spacetime, from which the spacetime extends into the relevant nontrapped region of the relevant black hole spacetime. In the diagram above, the $(+)$ region is where the $(+)$ branch of the axion-dilaton colliding plane wave metric extends to the black hole spacetime to give an asymptotically flat universe plus an axion-dilaton black hole to the future (Fig. 4). The $(-)$ region is where the $(-)$ branch of the colliding wave metric extends from the trapped region II into the axion-dilaton black hole spacetime to the future (Fig. 4).

VI. PARTICLE VS STRING PROPAGATION IN THIS SPACETIME

A. How do test particles propagate through the focal plane?

The source of nearly all singularities and causal pathologies that occur in classical general relativity is the inevitability of the gravitational field to cause light cones to focus in on themselves [13]. Hawking-Penrose-type singularity theorems chiefly express the conflict in general relativity when the local existence and uniqueness of extremal length curves breaks down due to the above focusing and threaten the existence of some desired global causal structure in that spacetime.

A simple illustration of this breakdown is gravitational lensing with multiple images. Suppose we are looking at a spacetime where this occurs. A light flashes at spacetime event $E_i=(t_i,\vec{x}_i)$ and the light leaving E_i is lensed by the

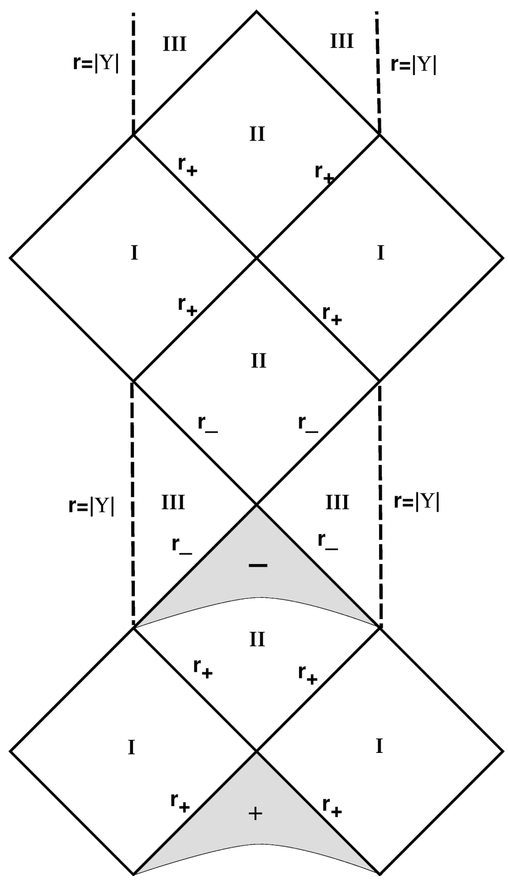

FIG. 4. The axion-dilaton colliding plane wave metrics analytically extend from the shaded regions of the above Penrose diagram into the black hole metric above it.

spacetime geometry so that an observer O at spatial location $\vec{x}_O$ sees two images of the flash from E_i. The two images seen by O represent two different null geodesics γ_1 and γ_2, both of which leave $\vec{x}_i$ at $t=t_i$. The geodesic γ_1 crosses $\vec{x}_O$ at $t=t_1$ and the geodesic γ_2 crosses $\vec{x}_O$ at $t=t_2$.

If $t_2>t_1$ there is a problem. The events $(t_1,\vec{x}_O)$ and $(t_2,\vec{x}_O)$ cannot *both* lie on the future light cone of the event E_i, because the timelike observer O experiences both events. Therefore the geodesic γ_1 must lie on the future light cone of E_i, while γ_2 started out on the future light cone of E_i and somehow left it. Since the problem goes away only when $t_1=t_2$, it must be true that this is where the problem starts and where null geodesics begin to fail to determine causal boundaries in spacetime.

In general, if two null geodesics γ_1 and γ_2 intersect once at some spacetime event E_i and then reintersect at later spacetime event E_c, then both γ_1 and γ_2 leave the ''boundary of the causal future'' of E_i when they cross again at E_c, and any event E_f at $t_f>t_c$ along γ_1 or γ_2 can be reached by a timelike curve from E_i.

This geodesic focusing is not a problem as long as there exists a discrete number of multiple images. Geodesic focusing at the continuum level is more dangerous and hence more interesting. In general relativity the expansion scalar θ determines when geodesic focusing is going to interfere with the unique delimitation of causal boundaries. If n^a is a tangent vector to a null geodesic γ, then θ is defined by $\theta=\nabla_a n^a$. The rotation ω_{ab} and shear σ_{ab} tensors are the antisymmetric and symmetric parts of $\nabla_a n_b$, respectively.[1]

The evolution equation for θ with respect to the affine parameter τ along γ is

$$n^c\nabla_c\theta=\frac{d\theta}{d\tau}=-\frac{1}{2}\theta^2-\sigma_{ab}\sigma^{ab}+\omega_{ab}\omega^{ab}-R_{cd}\xi^c\xi^d. \tag{57}$$

Spacetimes with $\omega_{ab}\neq 0$ are not foliatable into spacelike hypersurfaces and hence are not stably causal, so that term is zero if we exclude such spacetimes from consideration. Since $\sigma_{ab}\sigma^{ab}\geqslant 0$, if $R_{cd}\;n^c n^d\geqslant 0$, it follows that

$$\frac{d\theta}{d\tau}+\frac{1}{2}\theta^2\leqslant 0 \quad\rightarrow\quad \theta^{-1}\geqslant\theta_0{}^{-1}+\frac{1}{2}\tau. \tag{58}$$

Since $\theta\sim(1/V_\perp)(dV_\perp/d\tau)$, where $V_\perp$ is the transverse volume of a bundle of ''nearby geodesics,'' we do not want the right-hand side (RHS) of the above inequality to cross through zero. If the expansion $\theta_0<0$ at some proper time τ_0 along some geodesic, then $\theta\rightarrow-\infty$ along that geodesic within a proper time $\tau\leqslant 2/|\theta_0|$. So $V_\perp\rightarrow 0$ in a finite amount of proper time after the bundle begins to focus, or converge, at τ_0. When this happens to geodesics that are initially intersecting at some previous proper time $\tau<\tau_0$, either the initial value problem breaks down or the geodesics fail to be extendible past the focal plane. The latter alternative defines a spacetime singularity and is accompanied by the blowing up of curvature invariants in that region.

In the axion-dilaton colliding wave spacetime, a null vector $\mathbf{n}$ tangent to a null geodesic γ can be written

$$\mathbf{n}=\dot{u}\frac{\partial}{\partial u}+\dot{v}\frac{\partial}{\partial v}+\frac{p_x}{g_{xx}}\frac{\partial}{\partial x}+\frac{p_x}{g_{xx}}\frac{\partial}{\partial y}, \tag{59}$$

where p_x and p_y are constants of motion along γ. The condition $\mathbf{n}\cdot\mathbf{n}=0$ yields

$$2g_{uv}\dot{u}\dot{v}=-\left(\frac{p_x^2}{g_{xx}}+\frac{p_y^2}{g_{yy}}\right). \tag{60}$$

If we look at null geodesics along which $p_x=p_y=0$, then we can choose $\dot{v}=0$, and $\mathbf{n}=\dot{u}(\partial/\partial u)$. The geodesic equation $\ddot{u}+\Gamma^u_{uu}\;\dot{u}^2=0$ is solved by $\dot{u}=-g^{uv}$, and the expansion scalar

$$\begin{aligned}\theta=\nabla_a n^a&=\frac{-1}{\sqrt{g}}\frac{\partial}{\partial u}(\sqrt{g}g^{uv})\\ &=\frac{1}{|g_{uv}|\sqrt{g_{xx}g_{yy}}}\frac{\partial}{\partial u}(\sqrt{g_{xx}g_{yy}})\rightarrow-\infty\end{aligned} \tag{61}$$

as $u/a+v/b\rightarrow\pi/2$, where a and b are the focal lengths defined in Sec. II.

This focusing of initially parallel light rays defines the Killing-Cauchy horizon on the focal plane of the collision region. Parallel light rays delimit causal boundaries of events to the infinite past, so information from the infinite past of the colliding wave spacetime is focused together on the focal plane. This spacetime is on the edge of being singular. Instead of having infinite curvature at the focal plane, the curvature is finite and coordinates can be extended across it, but there is instead the global pathology of a Killing-Cauchy horizon. Small plane-symmetric perturbations of the incoming waves lead to the generic singular solutions.

Note that $V_\perp=\sqrt{g_{xx}g_{yy}}=|\cos(u/a+v/b)\cos(u/a-v/b)|$ is independent of r_0, so the focusing is controlled by the supersymmetric limit $r_0\rightarrow 0$ of the Bertotti-Robinson colliding plane wave system. Therefore comparisons of test particle and test string propagation can be made using the incoming wave extended from the Bertotti-Robinson collision region via the Khan-Penrose prescription, and the results should apply to axion-dilaton colliding plane waves with $r_0\neq 0$.

In order to plot these geodesics, it is convenient to truncate the metric (25) to $d=3$. Changing to harmonic coordinates gives the metric

$$\begin{gathered}ds^2=dUdV+h(U)X^2dU^2-dX^2,\\ h(U)=\left(\frac{\pi}{2\Delta U}\right)^2,\quad 0<U<\Delta U=0,\quad U<0,\quad U>\Delta U,\end{gathered} \tag{62}$$

where $\Delta U=\pi a/2$. The geodesic equations are

$$\begin{gathered}\ddot{V}+\frac{\partial h}{\partial U}X^2\;\dot{U}^2+4h(U)\dot{U}X\dot{X}=0,\quad \ddot{U}=0,\\ \ddot{X}+h(U)\dot{U}^2X=0,\end{gathered} \tag{63}$$

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[1] These quantities are defined on the two-dimensional quotient space of vectors orthogonal to $\mathbf{n}$ modulo multiples of $\mathbf{n}$.

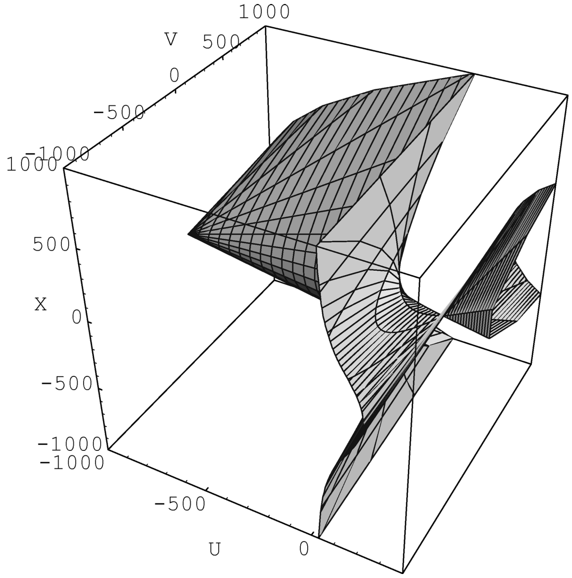

FIG. 5. Null geodesics from $(-1000,0,0)$ pass through the wave between $U=0$ and $U=200$ and are focused to a point. A similar picture was shown by Penrose in [14].

and the null condition gives

$$\dot{U}\,\dot{V}+h(U)X^2\,\dot{U}^2-\dot{X}^2=0. \tag{64}$$

The above equations are invariant under rescaling the affine parameter by $\tau\to\alpha\tau$, so the paths of massless test particles are the same for particles of all energies, a general feature of Einstein relativity. Therefore it is convenient and proper to choose for the above spacetime $U=\tau$, after which the equations are easily solved. Null geodesics passing through this wave take the form

$$U<0, \quad X(\tau)=p_0\tau, \quad V(\tau)=p_0^2\tau,$$

$$0<U<\Delta U, \quad X(\tau)=c_0\sin(\omega_0\tau)+d_0\cos(\omega_0\tau),$$

$$V(\tau)=\int \dot{X}(\tau)^2\,d\tau+v_{02},$$

$$U>\Delta U, \quad X(\tau)=p_f\,\tau+x_f, \quad V(\tau)=p_f^2\,\tau+v_f, \tag{65}$$

where $\omega_0=\pi/2\Delta U$ and the parameter p_0 represents the test particle momentum in the x direction. The six constants above are determined by the continuity of $X(\tau)$, $\dot{X}(\tau)$, and $V(\tau)$ [but not $\dot{V}(\tau)$] across the surfaces $U=0$ and $U=\Delta U$. The geodesics were plotted below using MATHEMATICA.

In Fig. 5 the plane wave passes between $U=0$ and $U=200$. After the null geodesics focus at $U\sim216$, they fail to determine the boundary of the causal future of the initial event, and the light cone is expanded out along the direction *parallel* to the wave. Null geodesics from an event at $U=-\infty$ would focus exactly at $f=\Delta U=200$.

Because of the extreme distortion of the light cone by the plane wave, every spacelike hypersurface in this spacetime intersects at least one null geodesic more than once. A global Cauchy surface cannot be defined, but for local calculations one can define a partial Cauchy surface and compute field theory Bogolyubov coefficients. Gibbons [15] showed that although the quantum theory of a scalar field in a single plane wave background is easily calculable and yields no particle creation, the theory itself becomes singular at the focal plane where the Cauchy horizon can no longer be neglected.

B. How do test strings propagate through the focal plane?

Plane gravitational waves are interesting string backgrounds to explore because the metric fields provide exact conformally invariant couplings on the string world sheet. This is because all the higher-derivative terms that could add (worldsheet) quantum corrections vanish identically [16]. String propagation through gravity waves has been fruitfully explored in the past in the context of scattering amplitudes. A notion of ''stringy singularity'' based on infinite string excitation was examined by Horowitz and Steif [17,18], Sanchez and de Vega [19] and others. While this looks like a good operational definition of singular string propagation, it does not shed light on the nature of causal volume delimitation in string theory and the potential physically relevant pathologies that could occur when causal volumes are delimited by solutions to worldsheet rather than world line mathematics. For this reason, we step back to that earlier work and reexamine it from a geodesic rather than an S-matrix point of view.

In extending the geodesic picture to string theory, the test particle geodesics that define the boundary of the test particle light cone are represented by the zero mode of the string. This is the center-of-mass coordinate that obeys that standard geodesic equation. If we only look at the geodesics of test string zero modes, then the singularities and causal pathologies of general relativity remain with minor modifications (in the cases where we trust the background spacetime approximation, at least.)

This is basically telling us that test particles propagate in ''stringy general relativity'' rather similarly to how they propagate in ordinary general relativity. The biggest difference comes from the rescaling of the stringy affine parameter relative to the test particle affine parameter by $e^{-2\phi}$. This has a noticeable effect mainly in the case of a dilaton black hole with purely electric charge [17].

If we take all string modes into account, the counterpart to a geodesic equation in string theory becomes

$$\Box X^{\mu}+\Gamma^{\mu}_{\nu\lambda}[X(\tau,\sigma)]\partial_a X^{\nu}\partial^a X^{\lambda}=0. \tag{66}$$

In the single plane wave metric (25) the equations reduce to

$$\ddot{V}-V''+\frac{\partial h}{\partial U}X^2(\dot{U}^2-U'^2)+4h(U)X(\dot{U}\dot{X}-U'X')=0,$$

$$\ddot{U}-U''=0, \quad \ddot{X}-X''+h(U)(\dot{U}^2-U'^2)X=0. \tag{67}$$

The mass shell constraints come from the vanishing of the world sheet stress tensor and automatically satisfy the first equation above. If we choose the gauge $U=U(\tau)$ we get

$$\dot{U}\dot{V}=-h(U)X^2\dot{U}^2+(\dot{X}^2+X'^2), \quad \dot{U}V'=2\dot{X}X' \tag{68}$$

and the remaining second order equation reduces to

$$\ddot{X}-X''+h(U)\dot{U}^2X=0. \tag{69}$$

These equations do not allow the rescaling of string proper affine parameter, so if we further fix the gauge by $U=p\tau$ and try to rescale p out of the equations through $\tau'=p\tau$, factors of p end up in the X' terms. Setting $L_{\rm string}=1$ and expanding in open string modes using $X(\tau,\sigma)=\Sigma X_n(\tau)\cos(n\sigma)$, we get

$$\ddot{X}_n(\tau)=-\left(\frac{n^2}{p^2}+h(U)\right)X_n(\tau), \tag{70}$$

with $V(\tau,\sigma)=\Sigma V_n(\tau)\cos(n\sigma)$ obtainable by straightforward integration of Eq. (68). As pointed out in [20], string null trajectories are momentum dependent and hence fail to satisfy the principle of equivalence observed in particle geodesics. So causal boundaries as determined by propagating strings become momentum-dependent.

Assigning $\omega_n=\sqrt{n^2/p2+h_0}$ and $\omega_0=\sqrt{h_0}=\pi/2\Delta U$, it is convenient to expand in the basis:

$$U<0 \quad X_0(\tau)=p_0\tau,$$

$$X_n(\tau)=a_n\cos(n\tau/p)+b_n\sin(n\tau/p), \tag{71}$$

$$0<U<\Delta U \quad X_0(\tau)=c_0\sin(\omega_0\tau)+d_0\cos(\omega_0\tau), \tag{72}$$

$$X_n(\tau)=c_n\sin(\omega_n\tau)+d_n\cos(\omega_n\tau), \tag{73}$$

$$U>\Delta U \quad X_0(\tau)=p_f\tau+x_f,$$

$$X_n(\tau)=e_n\cos(n\tau/p)+f_n\sin(n\tau/p), \tag{74}$$

with $U=\tau$ and the $V_n(\tau)$ obtained by integrating Eq. (68). This is related to the more common expansion for strings in flat spacetime

$$U=p\tau, \quad X(\tau,\sigma)=X_0(\tau)+i\sum_n \frac{\alpha_n}{n}e^{-in\tau}\cos(n\sigma) \tag{75}$$

through

$$a_n=-\frac{2p}{n}\mathrm{Im}\alpha_n, \quad b_n=\frac{2p}{n}\mathrm{Re}\alpha_n. \tag{76}$$

Applying continuity equations across the wave boundaries at $U=0$ and $U=\Delta U$ gives the linear transformation between incoming and outgoing mode constants (a_n,b_n) and (e_n,f_n):

$$\begin{aligned} e_n=a_n\bigg[&\cos(n\Delta U/p)\cos(\omega_n\Delta U)\\ &+\left(\frac{\omega_n\, p}{n}\right)\sin(n\Delta U/p)\sin(\omega_n\Delta U)\bigg]\\ +b_n\bigg[&-\sin(n\Delta U/p)\cos(\omega_n\Delta U)\\ &+\left(\frac{n}{\omega_n\, p}\right)\cos(n\Delta U/p)\sin(\omega_n\Delta U)\bigg], \end{aligned} \tag{77}$$

$$\begin{aligned} f_n=a_n\bigg[&\sin(n\Delta U/p)\cos(\omega_n\Delta U)\\ &-\left(\frac{\omega_n p}{n}\right)\cos(n\Delta U/p)\sin(\omega_n\Delta U)\bigg]\\ +b_n\bigg[&\cos(n\Delta U/p)\cos(\omega_n\Delta U)\\ &+\left(\frac{n}{\omega_n p}\right)\sin(n\Delta U/p)\sin(\omega_n\Delta U)\bigg]. \end{aligned} \tag{78}$$

Transforming back to the basis (α_n,α_{-n}) by undoing Eq. (76), the Bogolyubov coefficients B_n obtained match those obtained for $d=4$ in [21], which according to the conventions used here is

$$|B_n|^2=\frac{1}{4}\left(\frac{p}{n\omega_n}\right)^2\omega_0^4\sin^2(\omega_n\Delta U). \tag{79}$$

It is significant that this coefficient is zero in scalar quantum field theory [15]. As Gibbons explained, there is no mixing between in and out bases in that case because there is a global null Killing vector guaranteeing that frequencies can be measured in the same way before and after the wave's passage. Strings are excited because they have extended structure. String in and out bases are getting mixed in outright defiance of this target space Killing vector that has such a powerful restrictive effect on quantum fields.

The limit $\Delta U\rightarrow 0$ leads to a wave profile $h(U)\rightarrow(\pi/2)\delta(U)$, which in [17,21] was shown for bosonic strings to satisfy the definition of a singularity in terms of string propagation because the mass operator for the ''out'' state in the ''in'' vacuum diverges like $\Sigma(1/n)$.

For the single wave under consideration $\Delta U=\pi a/2$. The axion-dilaton colliding wave metric requires $ab=4(M^2-|\Upsilon|^2)$, so the limit in which one or both incoming waves has the profile $h(U)\rightarrow(\pi/2)\delta(U)$ is also the limit in which the maximal analytic extension of the collision region gives an extreme dilaton black hole with zero entropy but infinite curvature at the horizon and 1/2 of the $N=4$ supersymmetry unbroken [6].

String motion through the wave represented by Eq. (25) looks the same globally as the particle motion when plotted at the same scale as in Fig. 5. The main difference becomes visible in the focusing region when the momentum is varied, as shown in Fig. 6.

The plot above shows that the focal region as determined by strings becomes smeared by strings as the momentum decreases. This does not mean that string trajectories are no longer leaving the boundary of the causal future after they cross. This still has to be true at large distances. String effects obscure the location of the focal plane but not the effects of geodesic focusing itself.

The geodesic focusing that determines the location of the focal plane of the extreme single plane wave in Eq. (24) was shown in Eq. (61) to control the focusing of the Killing-Cauchy horizon in the collision region as $u/a+v/b\rightarrow\pi/2$. The Killing-Cauchy horizon for axion-dilaton colliding plane wave system is mapped to $r=r_\pm$ in the axion-dilaton black hole via the coordinate transformation (16). The quantity

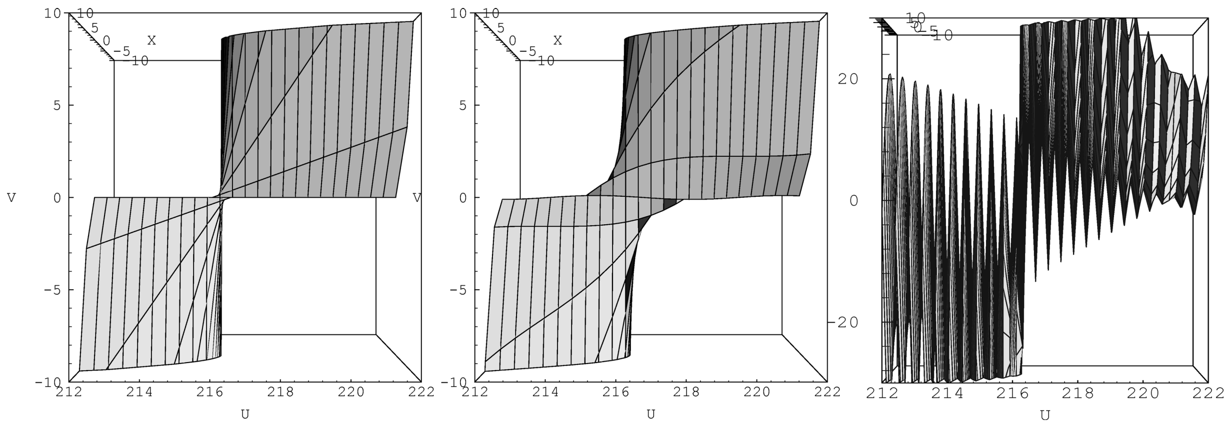

FIG. 6. The surface swept out by $X(\tau,0)$ plotted for $p=1000$, $p=2$, and $p=0.01$.

$V_\perp=\sqrt{g_{xx}\, g_{yy}}$ is locally identified with $\sqrt{-g_{tt}\, g_{\phi\phi}}$. This demonstrates a relationship between infinite geodesic focusing in the colliding wave system and the infinite red shift in the black hole system.

Consider the ''stringy stretched horizon'' as elucidated in [22]. An observer accelerating at constant r [a fiducial observer (FIDO)] near a black hole event horizon sees a passing freely falling string with a time resolution that decreases like $\varepsilon\sim e^{-ct}$, with $c=(r_+-r_-)/2(r_+^2-|\Upsilon|^2)$. But we know that a measurement of the size of a string cut off at mode N grows like $\ln N$, i.e., strings fill more space as we try to measure them with greater time resolution. Since the resolution $\varepsilon\sim 1/N$, the FIDO would see the passing string begin to grow like ct until it filled the horizon area. In [22] the authors fixed $p=1$ and looked at $\varepsilon(N)$. In the plots in Fig. 6 we fixed $N=1$ and varied p instead, finding that as we try to look at the string with decreasing resolution $\varepsilon=p/N$, the string gets longer and fills more space.

The Killing-Cauchy horizon formed by axion-dilaton colliding plane waves maps to a *past* horizon of an axion-dilaton black hole. (See Fig. 4.) So the ''stringy stretched focal plane'' can be viewed as the time-reversed version of the ''stringy stretched horizon'' described in [22]. In other words, suppose we are in the maximally extended colliding plane wave spacetime described in Sec. V, where two waves in a cylindrical universe collide to produce the axion-dilaton black hole spacetime in Fig. 4 at $r=r_+$. A FIDO close to r_+ in region I would see a test string emerging from the collision region at $t=-\infty$ filling the past horizon of the white hole created by the collision and then shrinking rapidly. This is the time-reversed version of what the FIDO at the future horizon sees.

It is important to remember, however, that these nonsingular colliding plane wave metrics are unstable. The singular term in the curvature (38) only vanishes when the product $p_1p_2p_3$ is precisely zero everywhere, which only happens if the initial data is specified with an arbitrarily high precision.

C. Scattering of almost-plane waves

It was shown by Yurtsever [23,24] that two finite-sized gravity waves that are ''nearly plane symmetric'' over some transverse size $L_T\sim L_{1T}\sim L_{2T}$ will collapse through plane-symmetric processes if the average focal length $f=\sqrt{f_1f_2}$ of the incoming waves satisfies $L_T\gg f$. There is a causality-based argument for this: The proper time in the collision region for the singularity or Cauchy horizon to form is $\Delta\tau\sim f$. So if $L_T\gg f$, the asymptotic evolution becomes dominated by *infinite* plane wave dynamics before gravitational shock waves containing the information that the incoming waves are *finite* in extent could have time to reach the collision region.

The mass-energy density contained in each incoming wave of thickness a_i and average curvature $\sim R_i$ would be on the order of $E_i/(a_iL_T^2)\sim R_i$. The focal length $f_i\sim a_i/(a_i^2R_i)$, which gives $E_i\sim L_T^2/f_i$. (So the mass energy per unit area in a finite, nearly plane symmetric gravity wave is $E_i/A_i\sim 1/f_i$.) The total mass energy in the collision region then would be $E_{\rm CW}\sim\sqrt{E_1E_2}\sim L_T^2/\sqrt{f_1f_2}=L_T^2/f$, so the condition $L_T\gg f$ implies $E_{\rm CW}\gg L_T$. In other words, the mass energy in the colliding wave system is contained well within its Schwarzschild radius when the two waves meet and the final product of this collision ought to be a black hole of size $\sim L_T^2/f$.

In the case of axion-dilaton colliding waves, these spacetimes are in general nonsingular and hence are believed to be unstable, in that small plane-symmetric perturbations on the initial data propagate to cause the Killing-Cauchy horizon to become singular. However, $f=\pi\sqrt{ab}/2=\pi\sqrt{M^2-|\Upsilon|^2}$. For the incoming waves, $R_i\sim 1/a_i^2$. Therefore $f_i\sim a_i$, and the limit $f_i\to 0$ is also the limit $R_i\to\infty$. The condition $L_T\gg f$ implies $E_{\rm CW}\gg L_T\gg f=\pi\sqrt{M^2-|\Upsilon|^2}$. This suggests that the collision of these finite waves could nucleate not one, but several axion-dilaton black holes, and in the maximally supersymmetric limit of $M\to|\Upsilon|$, the result could be an explosion of extreme dilaton black holes, which are not really black holes because the event horizon is singular. For that to happen at least one of the incoming waves would have zero thickness and infinite curvature. Such an incoming wave is already singular if we use the operative definition of a singular wave in string theory as a background in which the Bogolyubov coefficient for string excitation becomes infinite.

VII. CONCLUSIONS

The local coordinate transformation between the trapped region of a Schwarzschild black hole and a colliding plane gravitational wave discovered by Ferrari and Ibañez [2,3] extends naturally to the class of axion-dilaton black holes that are classical solutions to the electric-magnetic duality-invariant action (9):

$$S_{\mathrm{eff}}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\Bigg(-R+\frac{1}{2}\frac{\partial_\mu\lambda\,\partial^\mu\overline{\lambda}}{(\mathrm{Im}\lambda)^2}-\sum_{n=1}^{N}F^{(n)\star}_{\mu\nu}\widetilde{F}^{(n)\mu\nu}\Bigg).$$

The local coordinate transformation (16)

$$r\to M\pm r_0\sin\left(\frac{u}{a}+\frac{v}{b}\right),\quad \theta\to\frac{\pi}{2}\pm\left(\frac{u}{a}-\frac{v}{b}\right),$$

$$t\to xr_0/(M^2-|\Upsilon|^2)^{1/2},\quad \phi\to 1+y/(M^2-|\Upsilon|^2)^{1/2}$$

transforms an axion-dilaton black hole metric characterized by mass M and complex axion-dilaton charge Υ to the collision region of a colliding axion-dilation plane wave metric (17)

$$g_{uv}=\frac{-2\{[M\pm r_0\sin(u/a+v/rb)]^2-|\Upsilon|^2\}}{ab},$$

$$g_{xx}=\frac{(M^2-|\Upsilon|^2)\cos(u/a+v/b)^2}{[M\pm r_0\sin(u/a+v/b)]^2-|\Upsilon|^2}$$

$$g_{yy}=\cos^2\left(\frac{u}{a}-\frac{v}{b}\right)\frac{\{[M\pm r_0\sin(u/a+v/b)]^2-|\Upsilon|^2\}}{M^2-|\Upsilon|^2}.$$

The constants a and b represent the focal lengths of the incoming waves obtained from above through the Khan-Penrose prescription [11] and satisfy the relation $ab=4(M^2-|\Upsilon|^2)$. This metric has a Killing-Cauchy horizon at $u/a+v/v=\pi/2$, where the spatial translation Killing vector $\partial/\partial x$ becomes null. The curvature at the Killing-Cauchy horizon is equal to the curvature at $r=r_\pm$ of the correspoonding axion-dilaton black hole and so is finite except in the Schwarzschild and extreme electrically or magnetically charged dilaton limits where the curvature at r_- diverges.

The limit $r_0\to 0$, which for the black hole metrics corresponds to an extreme black hole, takes the axion-dilaton colliding plane wave metric to the Bertotti-Robinson metric (25)

$$ds^2=-du\,dv+\cos^2\left(\frac{u}{a}+\frac{v}{b}\right)dx^2+\cos^2\left(\frac{u}{a}-\frac{v}{b}\right)dy^2,$$

which has a finite average focal length $ab=4(M^2-|\Upsilon|^2)$ despite the fact that the trapped region of the corresponding black hole has become infinitesimal. The product ab of the nonvanishing parameters describing the colliding waves is related to the entropy of a nonsingular extreme black hole with 1/4 unbroken $N=4$ supersymmetry through

$$ab=\frac{4S_{\mathrm{extr}}}{\pi}. \tag{80}$$

An incoming wave obtained from the Bertotti-Robinson collision region can be described in harmonic coordinates as a shock wave of thickness $\Delta U=\pi a/2$, where a is the focal length of that wave, and constant curvature of magnitude $1/a^2=(\pi/2\Delta U)^2$. If we send $a\to 0$ while keeping the other incoming focal length b finite, then the constraint $ab=4(M^2-|\Upsilon|^2)$ says that $M=|\Upsilon|$. The limit $a\to 0$ corresponds to a δ function incoming wave. The black hole corresponding to the $M=|\Upsilon|$ limit has a singular horizon, zero entropy and 1/2 of $N=4$ supersymmetry unbroken. This correspondence between a δ-function gravity wave and this extreme dilaton configuration with zero entropy is like a type of wave-particle duality in string theory, albeit not the usual one.

The maximal analytic extension of the metric (17) across the Killing-Cauchy horizon gives back the nontrapped regions of the corresponding axion-dilaton hole, but requires that the y coordinate live on a circle of radius $\sqrt{M^2-|\Upsilon|^2}$. The resulting spacetime has two plane-symmetric single waves propagating in a cylindrical universe that collide and form a past horizon of an axion-dilaton black hole, shown in Fig. 4.

The propagation of test particle and test strings in a plane gravitational wave were compared. Geodesic focusing for the axion-dilaton colliding wave system is controlled by the supersymmetric Bertotti-Robinson limit. The single plane waves obtained from this collision region metric therefore make good toy backgrounds to study stringy geodesic focusing. The string equivalent of a massless geodesic equation does not allow for rescaling the affine parameter; consequently light cones as delimited by strings depend on momentum. This introduces a time resolution dependence into string geodesic focusing that is the same time resolution dependence that was analyzed in the stretched black hole horizon by Susskind in [22], suggesting that a ‘‘stretched focal plane’’ is the colliding plane wave analog of a stretched horizon for the black hole.

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