# Direct $\zeta$-function approach and renormalization of one-loop stress tensors in curved spacetimes

Valter Moretti*
*European Centre for Theoretical Studies in Nuclear Physics and Related Areas Villa Tambosi, Strada delle Tabarelle 286 I-38050 Villazzano, TN, Italy*
*and Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Trento I-38050 Povo, TN, Italy*

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A method which uses a generalized tensorial $\zeta$ function to compute the renormalized stress tensor of a quantum field propagating in a (static) curved background is presented. The method does not use point-splitting procedures or off-diagonal $\zeta$ functions but employs an analytic continuation of a generalized $\zeta$ function. The starting point of the method is the direct computation of the functional derivatives of the Euclidean one-loop effective action with respect to the background metric. It is proven that the method, when available, gives rise to a conserved stress tensor and, in the case of a massless conformally coupled field, produces the conformal anomaly formula directly. Moreover, it is proven that the obtained stress tensor agrees with statistical mechanics in the case of a finite-temperature theory. The renormalization procedure is controlled by the structure of the poles of the stress-tensor $\zeta$ function. The infinite renormalization is automatic due to a ''magic'' cancellation of two poles. The remaining finite renormalization involves locally geometrical terms arising by a certain residue. Such terms are also conserved and thus represent just a finite renormalization of the geometric part of the Einstein equations (customary generalized through high-order curvature terms). The method is checked in several particular cases finding a perfect agreement with other approaches. First the method is checked in the case of a conformally coupled massless field in the static Einstein universe where all hypotheses initially requested by the method hold true. Second, dropping the hypothesis of a closed manifold, the method is checked in the open static Einstein universe. Finally, the method is checked for a massless scalar field in the presence of a conical singularity in the Euclidean manifold (i.e., Rindler spacetimes, large mass black hole manifold, cosmic string manifold). Concerning the last case in particular, the method is proven to give rise to the stress tensor already got by the point-splitting approach for every coupling with the curvature regardless of the presence of the singular curvature. Comments on the measure employed in the path integral, the use of the optical manifold and the different approaches to renormalize the Hamiltonian are made.
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## I. INTRODUCTION

As is well known, the stress tensor of a matter field in a curved spacetime is obtained by computing the functional derivative of the matter field action with respect to the background metric. That is also the stress tensor which appears as a gravitational source into Einstein's equations. Trying to generalize the theory by including quantum aspects of the matter fields at least, one should consider the quantum averaged values of the same stress tensor (considered as an operator) as a gravitational source (see, for example, [1]).

As first proposed by Schwinger [2], dealing with quantum (quasifree) field theory in curved background, the quantum averaged stress tensor is computed by executing metric functional derivatives of the one-loop effective action. Then, the effective action takes account of the quantum state of the matter fields [3]. In fact, considering the averaged stress tensor as gravitational source is the first step in order to perform a semiclassical approach to the quantum gravity [1,3].

One can get the averaged stress tensor also in thermal quantum states dealing with an opportune Euclidean time-periodic continuation of the theory and the corresponding Euclidean effective action, when the Lorentzian manifold is static (i.e., the time of the considered and analytically continued coordinates defines a timelike Killing vector normal to the surfaces at constant time). In this case, the vanishing temperature limit should reproduce the nonthermal stress tensor referred to the vacuum state related to the timelike Killing vector.[1]

The computation of the one-loop *regularized* and *renormalized* quantum Euclidean effective action can be performed employing the $\zeta$-function procedure [4,3,5] that we shall summarize in the following.

One starts with the identity which defines the (Euclidean) effective action

$$S_{\mathrm{eff}}[\phi,g]:=\ln\int \mathcal{D}\phi e^{S[\phi]}=-\frac{1}{2}\ln\det[A/\mu^2],$$

where $S$ is the Euclidean action of the matter field $\phi$ which we can suppose, for sake of simplicity, a real scalar field (the approach also deals with much more complicated cases). The

*Electronic address: moretti@science.unitn.it

[1]Obviously, one has to eventually continue the Euclidean stress tensor into the real time in order to get the physical stress tensor.

 

space-configuration measure which appears in the functional integral is that well-known [4,6,7]

$$\mathcal{D}\phi=\prod_x \{g(x)^{1/4}d\phi(x)\} \tag{1}$$

and the action is built up as

$$S[\phi]=S_A[\phi]:=-\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{g(x)}\phi(x)A\phi(x), \tag{2}$$

where $A$ is an elliptic differential second-order self-adjoint operator positive defined on the Euclidean manifold $\mathcal{M}$. In a thermal theory with a temperature $T$, this manifold is periodic in the Euclidean time being $\beta=1/T$ the period. $\mu$ is a scale parameter necessary from dimensional considerations. This parameter may remain in the final results and thus can be reabsorbed into the renormalized gravitational constant as well as other physically measurable parameters involved in (generalized) Einstein's equations. This is a part of the program of the semiclassical quantum gravity approach [1,3].

We can suppose that the manifold above is closed (namely, compact without boundary) in order to have a discrete spectrum with proper eigenvectors of $A$ and do not consider boundary conditions. Anyhow, the method can be generalized for the nonclosed case (e.g., an infinite volume or presence of boundaries) by considering continuous spectra and boundary conditions [8,5]. We can compute the previous determinant in the framework of the *local* $\zeta$ function regularization [4] by (the reason of that generalized definition will be clear shortly):

$$\ln\det[A/\mu^2]=-\frac{d}{ds}\bigg|_{s=0}\zeta(s|A/\mu^2)=-\frac{d}{ds}\bigg|_{s=0}\zeta(s|A) -2\zeta(0|A)\ln\mu. \tag{3}$$

The $\zeta$ function can be obtained by integrating the *local* $\zeta$ function

$$\zeta(s|A)=\int d^4x\sqrt{g}\,\zeta(s,x|A), \tag{4}$$

where $\phi_n(x)$ is a normalized eigenvector of $A$ with eigenvalue $\lambda_n$:

$$\zeta(s,x|A)=\sum_n \lambda_n^{-s}\phi_n(x)\phi_n^*(x). \tag{5}$$

The expression above is the so-called spectral representation of the local $\zeta$ function. Equivalently,

$$\zeta(s|A)=\sum_n \lambda_n^{-s}. \tag{6}$$

These identities have to be understood in the sense of the analytic continuation of the right-hand sides to values of $s$ by which the series do not converge. The series above converge whenever $\mathrm{Re}\,s>2$ defining analytic functions which can be extended into meromorphic functions defined on the complex $s$ plane except for two simple poles on the real axis, at $s=1$ and $s=2$. We refer to [5] and references therein for a complete report in the general case.

The main reason to define the determinant of $A$ as in Eq. (3) is that, in the finite-dimensional case, this coincide with the usual definition. One can obtain this directly through Eq. (6) which reduces to an ordinary summation in the finite-dimensional case $A$ being an usual matrix.

The $\zeta$ function approach provides us with a good theoretical definition of the determinant of an operator. Moreover, as far as the quantum field theory in a curved background is concerned, the $\zeta$-function approach has been proven to produce the right interpretation of the functional integral and the one-loop renormalized effective action whatever someone was able to perform the previously cited analytical continuation [3,5]. Furthermore, on the theoretical ground, this approach led to very satisfactory results. In particular, the *renormalization* procedure[2] hidden in the $\zeta$-regularization procedure seems to be the correct one in the sense that it agrees with all physical requirements and with different procedures (e.g., dimensional regularization, point-splitting method [3]). The important difference from the other renormalization techniques is that the $\zeta$-function approach leads naturally to finite quantities without any ''by hand'' subtraction of infinite quantities, also maintaining possible terms arising from any *finite* renormalization. Finally, it is worthwhile stressing that $\zeta$-function approaches are currently employed in dealing with black-hole entropy physics, in particular to obtain quantum correction to the Beckenstein-Hawking entropy (e.g., see [9–12]).

In principle, the Euclidean (quantum) stress tensor[3] can be carried out from the one-loop effective action employing the usual definition[4]

$$\langle T_{ab}(x)\rangle=-2g(x)^{-1/2}\frac{\delta S_{\mathrm{eff}}[\phi,g]}{\delta g^{ab}(x)}. \tag{7}$$

The Lorentzian stress tensor is then obtained by the Euclidean one recontinuing analytically the latter into the Lorentzian section of the manifold.

However, it is not so simple to perform the functional derivative in the formula written above, employing the $\zeta$-function regularized effective action, because the local $\zeta$ function is not explicitly expressed in terms of the metric. In general, considering all known methods to regularize the stress tensor, barring (very important) theoretical consider-

[2]We remind the reader that the local averaged quantities as the stress tensor or the effective Lagrangian are affected by divergences also in quantum field theory in a curved background.

[3]When it is not specified otherwise, it is understood that we are employing the *Euclidean* metric, namely, the signature of the metric tensor $g_{ab}$ is (1,1,1,1) throughout this paper.

[4]This is the definition of the *Euclidean* stress tensor when the classical *Euclidean* action is *negative* definite [4]. We adopt such a convention throughout this paper.

ation [3], it is not so simple to use the formula above at all.[5] Other, more indirect, procedures have been found in order compute the stress tensor, e.g., the so-called ''point-splitting'' approach [3] or mixed procedures which involves point-splitting-like methods and off-diagonal local $\zeta$ functions [13,14].

This paper is devoted to propose a generalization of the *local* $\zeta$-function approach in order to use Eq. (7) *directly*.[6] We shall perform all proofs considering stationary Lorentzian manifolds with closed (i.e., compact without boundary) Euclidean sections. Anyhow, we shall see, in concrete examples, that the method works also dropping the requirement of compactness. We shall present a $\zeta$-function direct approach which, when available, produces a conserved stress tensor as well as the well-known and expected conformal anomaly in the case of a conformally coupled massless field. Furthermore, by our approach, one can prove thermodynamical identities usually supposed true without any general proof in a curved spacetime. Obviously, the usual concrete problem remains, one has to perform some analytic continuation explicitly to get the final result and this is not possible, in practice, for all physically interesting cases. At least, the formulas we will find define an alternative procedure among those which already exist. Moreover, it seems that our formulation could be interesting on the theoretical ground in particular. Indeed, as we shall see in this work, one can obtain the general results above-cited by employing a very little amount of calculations and a very clear procedure.

Anyhow, within this paper, we shall consider also several particular applications of the method. First, we shall consider the (thermal) theory of a conformally coupled massless scalar field within the closed Einstein universe. The Euclidean related manifold satisfies completely our initial hypotheses of a closed manifold. Secondly, we shall consider the same field propagating in the open Einstein universe. The related Euclidean manifold is not compact and this is a first nontrivial ground where check our approach assumed by definition. The third case we shall consider is the Euclidean manifold related both to the cosmic string manifold and Rindler space (which can be considered also as the manifold containing a very large mass black hole). That Euclidean manifold is not *ultrastatic* differently from the two manifolds considered above, moreover, it has a conical singularity which, for some aspects, could be considered as a boundary. That singularity involves a lot of difficulties dealing with $\zeta$-function approaches to renormalize the effective action. In particular, stress tensor components built up through the local $\zeta$ function of the effective action in the physical manifold have been obtained making direct use of mechanical-statistical laws or supposing a particular form of the stress tensor *a priori*. These results disagree, at low energies, with those obtained by the point-splitting method (see Sec. II of [12], and the final discussion of [12] for a discussion and references on these topics). In this paper we shall see that, concerning the stress tensor in the conical manifold, it is possible to get the same results arising also from the point-splitting approach, for every value of the coupling parameter $\xi$ by means of our local $\zeta$-function approach. This result will be carried out not depending on the mechanical-statistical laws and without supposing any particular form of the stress tensor *a priori*. Concerning this case in particular but also in the general case, we shall point out also some remarks on the problem of the choice of the configuration-space measure in the path integral to define the partition function of the fields. We shall see that this problem is related to the renormalization procedure involved in defining physical quantities, concerning the Hamiltonian in particular.

The paper is organized as follows. In Sec. II, we shall build up our general approach defining the background where it should work and we shall also stress some features of the method as far as the involved finite renormalization is concerned. In Sec. III, we shall analyze some general features of our theory by employing the heat kernel expansion. In Secs. IV and V, we shall prove that our approach, when available, produces a conserved stress tensor naturally and gives rise to the conformal anomaly directly in the case of a conformally invariant classical action. In Sec. VI, we shall prove that our approach agrees with the statistical mechanics interpretation of the time periodic Euclidean path integral. This result implies some comments on the correct use of the apparently ''wrong'' path-integral phase-space measure (that is an old problem reproposed recently by several authors). In Sec. VII, we shall compute the geometrical tensor related to the finite-renormalization part of the stress tensor in the general case of a conformally invariant scalar field in any static curved spacetime. In Sec. VIII we shall consider the simplest application of our method; namely, we shall compute the (thermal) stress tensor of a massless boson field in a flat-space box. In Sec. IX, we shall consider the (thermal) stress tensor of a conformally coupled massless scalar field propagating in closed Einstein's universe. In Sec. X, we shall compute the (thermal) stress tensor of a conformally coupled massless scalar field propagating in open Einstein's universe. Finally, in Sec. XI, we shall compute the (thermal) stress tensor of a massless field propagating in a manifold containing a conical singularity in the Euclidean section for every coupling with the singular curvature. We shall report also some comments on the thermodynamics and on the renormalization procedure. Section XII contains a summary of the topics dealt with in this paper. The Appendix contains proofs of some useful formulas employed throughout the paper.

## II. THE $\zeta$ FUNCTION OF THE STRESS TENSOR

Let us consider the functional definition of the stress tensor appearing in Eq. (7). In that formula, employing a

[5] Birrell and Davies, on page 190 of their fundamental book [3], wrote (Birrell-Davies' $W_{\rm ren}$ is our $S_{\rm eff}$) ''$[\cdots]$ in a practical calculation it is not possible to follow this route. This is because in order to carry out the functional differentiation of $W_{\rm ren}$ with respect to $g_{\mu\nu}[\cdots]$, it is generally necessary to know $W_{\rm ren}$ for all geometries $g_{\mu\nu}$. This is impossibly difficult.''

[6] A similar attempt appeared in [4], but the way followed there was quite different with respect to our approach because, there, the heat kernel representation of the $\zeta$ function rather than the $\zeta$ function expressed in terms of eigenvalues was considered and no general theory was presented. An important recent work [15] uses the heat-kernel representation and further nonlocal regularization procedures to compute the stress tensor fluctuations in curved spacetimes.

$\zeta$-function approach, the effective action is defined as

$$S_{\rm eff}[\phi,g]=\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\zeta(s|A)+\frac{1}{2}\zeta(0|A)\ln(\mu^2), \qquad (8)$$

where

$$\zeta(s|A)=\sum_n{}' \lambda_n^{-s}. \qquad (9)$$

The prime means that the summation written above does not include any possible null eigenvalues [4].

As we said in Introduction, the identity (9) holds in the sense of the analytic continuation when $\text{Re}\,s<M$, where $M$ is a number obtained by the heat kernel expansion depending on the operator $A$ and the structure of the manifold (usually $M=2$ dealing with Euclidean four-manifolds) [8,5]. We restress that the spectrum of the operator $A$ which appears into the Euclidean action is supposed to be purely discrete as it happens for Hodge–de Rham Laplacian operators in closed manifolds [16]. In other physically interesting cases, one should deal with proper spectral measures, or consider the studied manifolds as opportune limits of closed manifolds, and possibly, one has to take care of possible boundary conditions in defining the self-adjointness domain of the operator $A$. Due to the purely heuristic form of this paper, we shall not consider all mathematical subtleties involved in the $\zeta$-function approach (see [16,5], and Refs. therein).

Our proposal is to perform the functional derivative with respect to the metric directly in the right-hand side of Eq. (9) *before* we perform the analytic continuation. This should produce another series and another analytic function. The value at $s=0$ of the $s$ derivative of this new $\zeta$ function should be considered as a possible regularization of the stress tensor.

In practice, we define the $\zeta$ *function of the stress tensor* as

$$Z_{ab}(s,x|A) \text{ ``}:=\text{'' } -2g(x)^{-1/2}\frac{\delta\zeta(s|A)}{\delta g^{ab}(x)}$$

or, more correctly, $Z_{ab}(s,x|A)$ is the analytic continuation in the variable $s$ of the series

$$-2g(x)^{-1/2}\sum_n{}' \frac{\delta\lambda_n^{-s}}{\delta g^{ab}(x)} =2sg(x)^{-1/2}\sum_n{}' \frac{\delta\lambda_n}{\delta g^{ab}(x)}\lambda_n^{-(s+1)}, \qquad (10)$$

supposing that this series converges for $\text{Re}\,s>M'$ similarly to the case of the simple $\zeta$ function. Then, following the spirit of Eqs. (7) and (8), our idea is, when possible, *define* the renormalized stress tensor as

$$\langle T_{ab}(x)\rangle:=\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}Z_{ab}(s,x|A/\mu^2) =\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}Z_{ab}(s,x|A)+\frac{1}{2}Z_{ab}(0,x|A)\ln(\mu^2). \qquad (11)$$

Now, our aim to get a useful expression for the function $Z_{ab}(s,x|A)$. In the Appendix we shall prove the formula

$$\frac{\delta\lambda_n}{\delta g^{ab}(x)}=\frac{\lambda_n}{2}\sqrt{g(x)}g_{ab}(x)\phi_n(x)\phi_n^*(x)-2\frac{\delta S_A[\phi_n^*,\phi_n]}{\delta g^{ab}(x)}, \qquad (12)$$

where, through obvious notations, we defined

$$S_A[\phi_n^*,\phi_n]:=-\frac{1}{2}\int_{\mathcal{M}}d^4x\sqrt{g(x)}\phi_n^*(x)A\phi_n(x). \qquad (13)$$

Let us further define

$$T_{ab}[\phi_n^*,\phi_n](x):=-2g(x)^{-1/2}\frac{\delta_g S_A[\phi_n^*,\phi_n]}{\delta g^{ab}(x)}. \qquad (14)$$

This is nothing but the classical *real scalar field* stress-tensor evaluated on the $n$th mode. A few calculations employing Eqs. (9) and (5) lead us to, for the values of $s$ where the series in the right-hand side converge,

$$-\frac{g(x)^{1/2}}{2}Z_{ab}(s,x|A)=-s\sum_n{}' \lambda_n^{-(s+1)}g(x)^{1/2}\times T_{ab}[\phi_n^*,\phi_n](x)-\frac{s}{2}g^{1/2}(x)g_{ab}(x)\zeta(s,x|A). \qquad (15)$$

For future reference, let us define, in the sense of the analytic continuation in $s$

$$\zeta_{ab}(s,x|A):=\sum_n{}' \lambda_n^{-s}T_{ab}[\phi_n^*,\phi_n](x). \qquad (16)$$

It is finally useful to explicit the form of the function $Z_{ab}(s,x|A)$ in terms of the function $\zeta_{ab}(s+1,x|A)$ and $\zeta(s,x|A)$. We have

$$Z_{ab}(s,x|A)=-2g(x)^{-1/2}\left[-sg(x)^{1/2}\zeta_{ab}(s+1,x|A)-\frac{s}{2}g(x)^{1/2}g_{ab}(x)\zeta(s,x|A)\right] =2s\zeta_{ab}(s+1,x|A)+sg_{ab}(x)\zeta(s,x|A). \qquad (17)$$

We stress that the functions $\zeta$ which appear in the formula above are the analytic continuations of the corresponding series.

An important technical comment is in order. We are considering theories in which the $\zeta$-function approach is available in order to regularize the effective action (Lagrangian).

In such a situation the following two conditions have to hold true: $\zeta(0,x|A)$ and $\zeta'(0,x|A)$ (where the prime indicates the $s$ derivative) must be *finite*. By consequence, the limits of $s\zeta'(s,x|A)$ and $s\zeta(s,x|A)$ as $s\to 0$ have to vanish. The final result which arises performing the derivative in Eq. (11), taking account of the previous remark, reads

$$\langle T_{ab}(x)\rangle = \Big\{ \zeta_{ab}(s+1,x|A) + \frac{1}{2}g_{ab}(x)\zeta(s,x|A) + s[\zeta'_{ab}(s+1,x|A) + \ln(\mu^2) \times \zeta_{ab}(s+1,x|A)] \Big\}_{s=0}. \quad (18)$$

We shall define a ''super-ζ-regular theory'' as a quantum field theory (QFT) on a (Euclidean) manifold which can be regularized by the local ζ-function approach as far as the one-loop action and the stress tensor are concerned and, in particular, producing a $x$-smooth function $\zeta_{ab}(s,x|A)$ which can be analytically continued from values of $s$ where the corresponding series converges to a neighborhood of $s=1$ including this point. Thus, in the case of a super ζ-regular theory, Eq. (18) reads more simply

$$\langle T_{ab}(x)\rangle = \zeta_{ab}(1,x|A) + \frac{1}{2}g_{ab}(x)\zeta(0,x|A). \quad (19)$$

Note that the stress tensor of a super ζ-regular theory is independent of the scale $\mu$. The price one has to pay in order to preserve the $\mu$ dependence is the presence of a divergence in the first term in the right-hand side of Eq. (18). We shall come back to this point shortly.

The second term in the right-hand side of the equation above is quite surprising at first sight. This is because the classical stress tensor (evaluated on the modes) is related only with the first term in the right-hand side. Anyhow, as we shall see later, the unexpected terms in Eqs. (19) and (18) are necessary in order to produce a conserved stress tensor and give raise to the conformal anomaly formula. In particular, notice that the classical stress tensor evaluated on the modes cannot be conserved because the modes do not satisfy the (Euclidean) motion equations (barring null modes).

In general, dealing with physical theories in four-dimensional manifolds, we expect that the function $\zeta_{ab}(s+1,x|A)$ may take a singularity in $s=0$ for two reasons at least. First of all $\zeta_{ab}(s,x|A)$ is related to $\zeta(s,x|A)$ which, dealing with four dimensional manifolds, carries a possible pole as $s\to 1$ and we expect that $x$ derivatives do not change this fact (this will be more clear employing the heat kernel expansion as we shall see in the following). A more physical reason is the following one. As is well known, the matter-field action when renormalized through any procedure, also different from the ζ-function approach (see [3]), results to be affected by an ambiguous part containing an arbitrary scale parameter. That role is played by $\mu$ in the ζ-function approach. This is a *finite* relic of the *infinite* subtraction procedure. These relic terms depend on the geometry locally. For this reason they can be also thought like parts of the gravitational action [3]. In fact, it has been proven that their only role is to renormalize the coupling constants of the Einstein-Hilbert gravitational action opportunely generalized in order to contain also high order terms in the curvatures [3].[7]

We have to expect that similar scale-dependent terms also appears in the renormalized stress tensor. This is because they have to renormalize the same coupling constants which also appear in the geometrical part of (generalized) Einstein's equations of the gravity [3]. In this sense, dealing with the stress tensor renormalization, the arbitrary scale $\mu$ in Eq. (18) should play the same job which it does as far as the ζ regularization of the effective action is concerned. It is worthwhile stressing that such a result is also allowed in Wald's axiomatic approach to characterize physically possible renormalization procedures of the stress tensor in curved spacetime [1]. Indeed, Wald's theorem proves that a geometric ambiguity remains also after one imposed strong requirements on the renormalization procedure. Such an ambiguity can be considered as an ambiguity of the coupling constants appearing in the geometric part of (generalized) Einstein's equations.

Following this insight we are led to assume that, more generally than in the case of a super-ζ-regular theory, when our approach is available

$$\lim_{s\to 0} s\zeta_{ab}(s+1,x|A) = G_{ab}(x|A) \quad \text{(finite quantity)}. \quad (20)$$

This is the only possibility in order to maintain the parameter $\mu$ into the final renormalized stress tensor in Eq. (18). Our assumption implies that the function $\zeta_{ab}(s+1,x|A)$ has a simple pole at $s=0$.

We shall define a ''ζ-regular theory'' as a quantum field theory on a curved spacetime which can be regularized through the local ζ-function approach as far as the effective action is concerned, and produces a $x$-smooth $\zeta_{ab}(s,x|A)$ which can be analytically continued from values of $s$ where the corresponding series converges to a neighborhood of $s=1$, except for the point $s=1$ which is a simple pole.

*A priori*, in the case of a ζ-regular theory, the definition (11)–(18) of the renormalized stress tensor can be employed provided the infinite terms arising from the poles in the first and third term in the right-hand side of Eq. (18) are discarded. Actually a magic fact happens, those two divergences cancel out each other and the function $Z_{ab}(s,x|A)$ results to be analytic also at $s=0$ where $\zeta_{ab}(s+1,x|A)$ has a pole. Indeed, taking into account that the singularity in $\zeta_{ab}(s+1,x|A)$ is a simple pole, a trivial calculation proves that the structure of these singularities as $s\to 0$ are, respectively,

$$\zeta_{ab}(s+1,x|A) \sim \frac{G_{ab}(x|A)}{s} \quad (21)$$

and

[7]Obviously, as for the flat-space renormalization procedures, all measured physical quantities (e.g., dressed coupling constants) are finally independent of the parameter $\mu$. See [3] for a whole discussion.

$$s\zeta'_{ab}(s+1,x|A) \sim -\frac{G_{ab}(x|A)}{s}. \qquad (22)$$

Substituting in Eq. (18), we see that those divergences cancel out each other. We stress that the function $G_{ab}(x|A)$ remains into the finite renormalization term containing the scale $\mu$ in Eq. (18). The difference between super-$\zeta$-regular theories and $\zeta$-regular theories concerns the presence of the scale $\mu$ in the final stress tensor. As a further remark we stress that the function $G_{ab}(x|A)$ which appears in Eq. (20) as well as in Eqs. (21) and (22) contains the whole information about both the *(scale dependent) finite* and *infinite* renormalization of the stress tensor.

In the next sections we shall prove the important identity which holds in case of $\zeta$-regular theories:

$$\nabla^a G_{ab}(x|A) = 0. \qquad (23)$$

We expect that the term $G_{ab}(x|A)$ is built up through the local geometry of the manifolds. This is a consequence of the fact that the function $G_{ab}(x|A)$ can be carried out employing the heat kernel expansion coefficients as we shall see in the next section. All that means that we can consider $G_{ab}(x|A)$ as a correction to the geometrical term in (generalized) Einstein's equations of the gravity. This is in perfect agreement with Wald's theorem [1].

We have dealt with a real scalar field in a closed manifold only. Anyhow, reminding of the general success of the $\zeta$-function approach to regularize the effective action, we expect that our method can be used to regularize the stress tensor in more general situations, simply passing, when necessary, to consider (charged) spinorial modes[8] and continuous spectral measures in Eq. (16). Conversely, the presence of boundaries could involve further problems. The examples we shall consider in Secs. VIII, XI, and XII deal with some possible generalizations.

### III. HEAT-KERNEL EXPANSION ANALYSIS

In this section we shall consider, on a general ground, the behavior of the function $Z_{ab}(s,x|A)$ near the point $s=0$ in the case of a real scalar field whose action is

$$S = -\frac{1}{2}\int d^4\sqrt{g}(\nabla_a\phi\nabla^a\phi + m^2\phi^2 + \xi R\phi^2). \qquad (24)$$

By employing the heat-kernel expansion we shall see that such a theory define a $\zeta$-regular theory (possibly also super-$\zeta$-regular). We shall be also able to relate the residue $G_{ab}(s,x|A)$ to the heat-kernel coefficients.

The operator which correspond to the action above is

$$A = -\Delta + m^2 + \xi R \qquad (25)$$

and the corresponding stress tensor reads

$$T_{ab}(x) = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}(x)(\nabla_c\phi\nabla^c\phi + m^2\phi^2) + \xi\left[\left(R_{ab} - \frac{1}{2}g_{ab}R\right)\phi^2 + g_{ab}\nabla_c\nabla^c\phi^2 - \nabla_a\nabla_b\phi^2\right]. \qquad (26)$$

A few calculations lead to the stress tensor evaluated on the modes

$$T_{ab}[\phi_n^*\phi_n](x) = \frac{1}{2}(\nabla_a\phi_n^*\nabla_b\phi_n + \nabla_b\phi_n^*\nabla_a\phi_n) - \xi\nabla_a\nabla_b|\phi_n|^2 + \left(\xi - \frac{1}{4}\right)g_{ab}\Delta|\phi_n|^2 + \xi R_{ab}|\phi_n|^2 - \frac{1}{2}g_{ab}\lambda_n|\phi_n|^2. \qquad (27)$$

We are able to write down the function $\zeta_{ab}(s,x)$ in the general case considered above. Employing the definition (16) we find

$$\zeta_{ab}(s,x|A) = \frac{1}{2}\sum_n{}' \lambda_n^{-s}(\nabla_a\phi_n^*\nabla_b\phi_n + \nabla_b\phi_n^*\nabla_a\phi_n) + \left[-\xi\nabla_a\nabla_b + \left(\xi - \frac{1}{4}\right)g_{ab}\Delta + \xi R_{ab}\right] \times \zeta(s,x|A) - \frac{1}{2}g_{ab}\zeta(s-1,x|A). \qquad (28)$$

For future reference, it is convenient to define also

$$\overline{\zeta_{ab}}(s,x|A) := \frac{1}{2}\sum_n{}' \lambda_n^{-s}(\nabla_a\phi_n^*\nabla_b\phi_n + \nabla_b\phi_n^*\nabla_a\phi_n), \qquad (29)$$

where we suppose to continue the series above analytically as far as possible in the complex $s$ plane.

We want to study the behavior of the function $\zeta_{ab}(s+1,x|A)$ and hence $Z_{ab}(s,x|A)$ near the possible singularity at $s=0$ and, more generally, we want to study the meromorphic structure of these functions. Let us consider the off-diagonal heat-kernel asymptotic expansion [17] in four dimension, which holds asymptotically for $t\to 0$ and $x$ near $y$ (in a convex normal neighborhood) in closed Euclidean manifolds

$$H(t,x,y|A) \sim (4\pi t)^{-2} e^{-\sigma(x,y)/2t}\sum_{j=0}^{+\infty} a_j(x,y|A)t^j. \qquad (30)$$

$\sigma(x,y)$ is half the square of the geodesical distance from the point $x$ to the point $y$. The heat kernel $H(t,x,y)$ decays very speedly as $t\to+\infty$, the only singularities come out from its behavior near $t=0$ when $x=y$.

The relation between the heat-kernel expansion and the $\zeta$-function theory ([5]) is that the heat kernel $H(t,x,y|A)$ satisfies

[8]The case of noninteger spin could be more complicated. In the case of gauge fields, it is convenient to employ the Hodge-de Rham formalism and one has to take the ghost contribution to the stress tensor into account.

$$\zeta(s,x,y|A)=\frac{1}{\Gamma(s)}\int_0^{+\infty}dt\,t^{s-1}H(t,x,y|A), \qquad (31)$$

where, for Re$s$ sufficiently large,

$$\zeta(s,x,y|A)=\sum_n{}'\ \lambda_n^{-s}\phi_n^*(x)\phi_n(y). \qquad (32)$$

We can decompose the integration above into two parts as

$$\zeta(s,x,y|A)=\frac{1}{\Gamma(s)}\int_0^{1}dt\,t^{s-1}H(t,x,y|A)+\frac{1}{\Gamma(s)}\int_1^{+\infty}dt\,t^{s-1}H(t,x,y|A). \qquad (33)$$

The true difference from the left-hand side and any expression in the right-hand side of Eq. (30) truncated at the order $N>2$ is a regular function as $t\to 0$. Taking into account that fact, one can insert the expansion in Eq. (30) into the first integral in the right-hand side of Eq. (33), obtaining

$$\zeta(s,x,y|A)=\frac{1}{\Gamma(s)}\sum_{j=0}^{N} a_j(x,y|A)\int_0^1 dt\,t^{s-3+j}e^{-\sigma(x,y)/2t}+h(N,s,x,y|A), \qquad (34)$$

where $h(N,s,x,y|A)$ is a unknown $x,y$-smooth and $s$-analytic function. This relation is the starting point of our considerations.

As general remarks we stress the following two facts. First, the coefficients $a_j(x,y|A)$ expressed in *Riemannian coordinates* centered in $x$ [3] are polynomials in $x$-$y$ whose coefficients are algebraic combinations of curvature tensors evaluated at the point $x$. Thus the limit as $x\to y$ of quantities as $a_j(x,y|A)$, $\nabla_a^{(x)}a_j(x,y|A)$, $\nabla_a^{(x)}\nabla_b^{(y)}a_j(x,y|A)$ and so on we shall consider shortly, produces algebraic combinations of (covariant derivatives of) curvature tensors evaluated at the same point $x$. Second, there exists a recursive procedure which permits one to get the coefficients $a_j(x,y|A)$ and their covariant derivatives evaluated in the limit of coincidence of arguments, when one knows the coefficients $a_i(x,y|A)$, their derivative for $0\leqslant i<j$ and the covariant derivatives of the function $\sigma(x,y|A)$, everything evaluated in the argument coincidence limit. Such a procedure can be obtained by a simple generalization of a similar procedure (which does not consider covariant derivatives) presented in [17].

Let us evaluate the pole structure of the function $\overline{\zeta_{ab}}(s,x|A)$ employing Eq. (34) and the following known identities [3,17]:

$$\nabla_a^{(x)}\nabla_b^{(y)}\sigma(x,y)|_{x=y}=-g_{ab(x)},$$

$$\nabla_a^{(x)}\sigma(x,y)|_{x=y}=0,$$

$$\nabla_b^{(y)}\sigma(x,y)|_{x=y}=0.$$

By taking the opportune derivatives in Eq. (34) and posing $y=x$ finally, we find

$$\begin{aligned}\overline{\zeta_{ab}}(s+1,x|A)&=\overline{\zeta_{ab}}(s+1,x|A)_{\text{ analytic}}+\frac{1}{(4\pi)^2\Gamma(s+1)}\\&\times\left[\frac{a_{0(ab)}(x|A)}{s-1}+\frac{a_{1(ab)}(x|A)}{s}+\frac{1}{2}g_{ab}(x)\right.\\&\left.\times\left(\frac{a_0(x|A)}{s-2}+\frac{a_1(x|A)}{s-1}+\frac{a_2(x|A)}{s}\right)\right],\end{aligned} \qquad (35)$$

where we defined

$$a_{j(ab)}(x|A):=\frac{1}{2}[\nabla_a^{(x)}\nabla_b^{(y)}a_j(x,y|A)+\nabla_a^{(y)}\nabla_b^{(x)}a_j(x,y|A)]_{x=y}. \qquad (36)$$

Notice that in the pole expansion written above, an infinite number of apparent poles have been canceled out by corresponding zeros of $[\Gamma(s+1)]^{-1}$.

The function $\zeta(x|A)$ has the well-known similar structure

$$\zeta(s,x|A)=\zeta(s,x|A)_{\text{ analytic}}+\frac{1}{(4\pi)^2\Gamma(s)}\left(\frac{a_0(x|A)}{s-2}+\frac{a_1(x|A)}{s-1}\right). \qquad (37)$$

Employing the results written above to calculating the pole structure of the function $\zeta_{ab}(s+1,x|A)$ through Eq. (28), we find

$$\begin{aligned}(4\pi)^2\zeta_{ab}(s+1,x|A)&=(4\pi)^2\zeta_{ab}(s+1,x|A)_{\text{ analytic}}+\frac{1}{\Gamma(s+1)}\\&\times\left[\frac{a_{0(ab)}(x|A)}{s-1}+\frac{a_{1(ab)}(x|A)}{s}+\frac{g_{ab}(x)}{2}\left(\frac{a_0(x|A)}{s-2}\right.\right.\\&\left.\left.+\frac{a_1(x|A)}{s-1}+\frac{a_2(x|A)}{s}\right)\right]+\frac{1}{\Gamma(s+1)}\left[-\xi\nabla_a\nabla_b\right.\\&\left.+\left(\xi-\frac{1}{4}\right)g_{ab}\Delta+\xi R_{ab}\right]\left(\frac{a_0(x|A)}{s-1}+\frac{a_1(x|A)}{s}\right)\\&-\frac{g_{ab}(x)}{2\Gamma(s)}\left(\frac{a_0(x|A)}{s-2}+\frac{a_1(x|A)}{s-1}\right).\end{aligned} \qquad (38)$$

We stress the presence of a simple pole for $s=0$. The pole expansion above written proves that the considered theory is a $\zeta$-regular theory. The theory is also a *super*-$\zeta$-regular theory when the residue at $s=0$ vanishes.

This residue is just the function $G_{ab}(x|A)$ which reads in terms of heat-kernel coefficients

$$G_{ab}(x|A)=\frac{1}{(4\pi)^2}\left\{a_{1(ab)}(x|A)+\frac{g_{ab}(x)}{2}a_2(x|A)+\left[-\xi\nabla_a\nabla_b+\left(\xi-\frac{1}{4}\right)g_{ab}\Delta+\xi R_{ab}\right]a_1(x|A)\right\}. \qquad (39)$$

Now, it is obvious that $G_{ab}(x|A)$ depends on the geometry locally. In particular it is built up by algebraic combination of curvature tensors and their covariant derivatives. A closer scrutiny, employing the recursive procedure to compute the heat-kernel coefficients cited above, proves that $G_{ab}(x|A)$ contains combinations of products of two curvature tensors at most.[9] Considering that $G_{ab}(x|A)$ is also conserved, this means that it is obtained from an Einstein-Hilbert action improved by including quadratic terms in the curvature tensors. As we said, $G_{ab}(x|A)$ is the part of the renormalized stress tensor which can be changed by finite renormalization. This agrees with all known different stress-tensor renormalization procedures where one finds that, in the case of a scalar field studied here, the finite renormalization of the stress tensor involves only curvature quadratic terms [3]. This agrees with Wald's theorem and the related comments reported in [1], too. We shall return to these facts later.

## IV. CONSERVATION OF THE STRESS TENSOR AND $G_{ab}(X|A)$

Let us prove that, in the case of a super-$\zeta$-regular theory or a $\zeta$-regular theory, the stress tensor obtained from Eq. (18) is conserved. By the same proof, we shall get conservation of $G_{ab}(x|A)$ too.

Our strategy will be the following one. We shall consider the function whose the value at $s=0$ is the renormalized stress tensor

$$\langle T_{ab}(s,x)\rangle := \frac{1}{2}\frac{d}{ds}Z_{ab}(s,x|A)+\frac{1}{2}Z_{ab}(s,x|A)\ln(\mu^2) \tag{40}$$

and we shall evaluate the covariant divergence for the values of $s$ in which the involved $\zeta$ function can be expanded as a series. We shall find that this covariant divergence vanishes. Due to the analyticity of the considered functions, this result can be continued as far as the physical value $s=0$.

Let us consider the $\zeta$ function $Z_{ab}(s,x)$ expressed as the series in Eq. (10):

$$Z_{ab}(s,x)=2sg(x)^{-1/2}\sum_n{}' \lambda_n^{-(s+1)}\frac{\delta\lambda_n}{\delta g^{ab}(x)}.$$

We have

$$\nabla^a Z_{ab}(s,x)=-s\sum_n{}' \lambda_n^{-(s+1)}\nabla^a\left[-2g(x)^{-1/2}\frac{\delta\lambda_n}{\delta g^{ab}(x)}\right]. \tag{41}$$

Let us prove that

$$\nabla^a Z_{ab}(s,x)=0 \tag{42}$$

because

$$\nabla^a\left[-2g(x)^{-1/2}\frac{\delta\lambda_n}{\delta g^{ab}(x)}\right]=0. \tag{43}$$

Due to Eqs. (41) and (40), this proves conservation of the stress tensor by taking the limit at $s=0$.

A nice proof of Eq. (43) is dealt with as follows. Let us consider the new ''action''

$$\Lambda_n[g,\phi^*,\phi] := 2S[g,\phi^*,\phi] - \lambda_n\int_{\mathcal{M}} d^4x\sqrt{g(x)}\,\phi^*(x)\phi(x).$$

This is a diffeomorphism invariant action producing the field equations

$$A\phi(x)=\lambda_n\phi(x)$$

and

$$A\phi^*(x)=\lambda_n\phi^*(x).$$

In particular, these equations are fulfilled by the eigenfunctions $\phi_n(x)$ and $\phi_n^*(x)$. As is well known, due to diffeomorphism invariance of the action, one gets conservation of a stress tensor $T_{nab}(x)$ evaluated on the motion solutions, namely, *on the modes* $\phi_n(x)$ *and* $\phi_n^*(x)$. Again, this stress tensor is obtained as the functional derivative of the action $\Lambda_n$ with respect to the metric [with the overall factor $-2g(x)^{-1/2}$]. A little computation and Eq. (12) get just

$$T_{nab}(x) := -2g(x)^{-1/2}\frac{\delta_g\Lambda_n}{\delta g^{ab}(x)}=2g(x)^{-1/2}\frac{\delta\lambda_n}{\delta g^{ab}(x)}. \tag{44}$$

Conservation of the left-hand side implies Eq. (43) trivially.

An important remark, in the case of a $\zeta$-regular theory, is finally necessary. Conservation of the tensor $Z_{ab}(s,x)$ reads, employing Eq. (17),

$$s\nabla^a\{\zeta_{ab}(s+1,x|A)+g_{ab}(x)\zeta(s,x|A)\}=0.$$

We get, recalling Eq. (21) and performing the limit as $s\to 0$

$$\nabla^a G_{ab}(x|A)=0.$$

This is nothing but Eq. (23).

## V. THE CONFORMAL ANOMALY

Let us prove of the conformal anomaly formula [3] by employing a way similar to that in the previous section, in the case of a super-$\zeta$-regular theory or a $\zeta$-regular theory. As usual, we have to suppose that the classical action $S[\phi]$ is conformally invariant. As is well known, by performing an infinitesimal local conformal transformation on both the metric and the field, the following equations arise:

$$g^{ab}(x)T_{ab}[\phi](x)-g(x)^{-1/2}\phi(x)\frac{\delta S}{\delta\phi(x)}=0.$$

This implies that classically, *working on solutions of the motion equation*, the trace of the stress tensor vanishes. Simi-

[9] In particular, in a flat space and for $m=0$ the residue above vanishes.

larly, dealing with the action evaluated on the modes $\phi_n(x)$, conformal invariance of the action lead us to[10]

$$g^{ab}(x)T_{ab}[\phi_n^*\phi_n](x)-g(x)^{-1/2}\phi_n(x)\frac{\delta S}{\delta\phi_n(x)}$$

$$-g(x)^{-1/2}\phi_n^*(x)\frac{\delta S}{\delta\phi_n^*(x)}=0$$

or, equivalently,

$$g^{ab}(x)T_{ab}[\phi_n^*\phi_n](x)=-\lambda_n\phi_n^*(x)\phi_n(x).$$

From this equation, employing Eqs. (5) and (16) we get

$$g^{ab}(x)\zeta_{ab}(s+1,x|A)=-\zeta(s,x|A), \tag{45}$$

where the involved $\zeta$ functions can be defined as a series.

Holding our hypothesis of a $\zeta$-regular theory, this result can be analytically continued arbitrarily close to the physical value $s=0$. In particular, the left-hand side of Eq. (45) must be *finite* at $s=0$ because so is the right-hand side. This seems quite surprising because $\zeta_{ab}(s+1,x|A)$ may take a pole at $s=0$. We conclude that the pole has to disappear due to trace procedure in case of a conformally invariant action, namely,

$$g^{ab}(x)G_{ab}(x|A)=0. \tag{46}$$

We shall check this fact directly later.

It is worthwhile noticing that the trace procedure, canceling out the pole in $g^{ab}(x)\zeta_{ab}(1,x|A)$, gives rise to vanishing terms $sg^{ab}(x)\zeta'_{ab}(s+1,x|A)$ and $sg^{ab}(x)\zeta_{ab}(s+1,x|A)$ when $s\to 0$. Finally, Eq. (18) through Eq. (45) produces the well-known conformal anomaly formula [4,3]

$$g^{ab}(x)\langle T_{ab}(x)\rangle=\zeta(0,x|A)=\frac{a_2(x|A)}{16\pi^2}. \tag{47}$$

## VI. THERMODYNAMICS AND COMMENTS ON THE PHASE-SPACE MEASURE OF THE PATH INTEGRAL

In this section[11] we prove that for $\zeta$-regular theories or super-$\zeta$-regular theories

$$-\frac{\partial \ln Z_\beta}{\partial\beta}=-\int_\Sigma d\vec{x}\sqrt{-g_L}\langle T^0_{L0}(\vec{x})\rangle_\beta, \tag{48}$$

where we have defined $\ln Z_\beta:=S_{\rm eff}$, provided the (Euclidean and Lorentzian) manifold admits a global (Lorentzian timelike) Killing vector arising from the Euclidean temporal coordinate with a period $\beta=1/T$. $\vec{x}$ represents the spatial coordinates which belong to the spatial section $\Sigma$ and $g_L=-g$ is the determinant of the Lorentzian metric.

As it is clear from the notations, we are trying to interpret $Z_\beta$ as a *partition function*.[12] Notice that all quantities which appear in the formula above do not depend on the Euclidean or Lorentzian time because the manifold is stationary and thus no time dependence arises from the metric. By the same reason, the time dependence in the eigenvectors of the motion operator is exponential and thus it cancels out in all involved local $\zeta$ functions. Finally, $\langle T^0_{L0}(\vec{x})\rangle=\langle T^0_0(\vec{x})\rangle$ by a trivial analytic continuation.

Actually, it is not necessary to interpret $x^0$ as a time coordinate, the same result in Eq. (48) arises also when the Killing vector is associated to the ''spatial'' coordinate $x^i$, provided $\beta$ were changed to $L_i$, the ''spatial'' period of the manifold along the $i$th direction. Assuming both the homogeneity along $x^0$ and $x^i$ we get another expected formula trivially:

$$-\frac{\partial \ln Z_\beta}{\partial L_i}=-\frac{\beta}{L_i}\int_\Sigma d\vec{x}\sqrt{-g_L}\langle T^i_{Li}(\vec{x})\rangle_\beta. \tag{49}$$

Before we start with the proof of Eq. (48), some important remarks are in order. In particular, let us consider a scalar field with an Euclidean action coupled with the scalar curvature, given by

$$S[\phi]=-\frac{1}{2}\int d^4x\sqrt{g(x)}\,\phi(x)A\phi(x)$$
$$=-\frac{1}{2}\int d^4x\sqrt{g(x)}\,\phi(x)[-\nabla_a\nabla^a+m^2+\xi R(x)]\phi(x) \tag{50}$$

and let us assume explicitly that the (both Lorentzian and Euclidean) metric is *static*, namely, $g_{(L)0i}=0$ besides $\partial_0 g_{(L)ab}(x)=0$ (but not necessarily *ultrastatic*). In that case, in principle [4], there is no problem in implementing the canonical-ensemble approach to the thermodynamic and trying the interpretation of the Euclidean time-periodic path integral as a partition function $Z_\beta$, and thus, in principle,

$$-\beta^{-1}S_{\rm eff}=-\beta^{-1}\ln Z_\beta$$

could be interpreted as the free energy of the field in the considered quantum thermal state. The case of a stationary manifold ($g_{L0i}\neq 0$) involves more subtleties also considering the analytic continuation into an Euclidean manifolds which we shall not consider here [4]. Anyhow, it is worthwhile stressing that (48), written in terms of $\langle T^0_0\rangle$ and $g$, holds true in the general case of a stationary Euclidean metric $\ln Z_\beta$ being $S_{\rm eff}$ without assuming that this define any free energy.

Identities such as Eqs. (48) or (49) represent a direct evidence that the definition of the partition function as a path integral on the continued Euclidean manifold, also in the case of a *curved* spacetime, does not lead to thermodynamical inconsistencies in the case of a closed spatial section of the manifold at least. We stress that $-T^0_0$ does not coincide

[10]Notice that we transform the modes employing the same transformation of the field $\phi(x)$. This transformation does not preserve the normalization of the modes but preserves the value of the action.

[11]From now on, we employ the signature $(-1,1,1,1)$ for the Lorentzian metric and Lorentzian quantities shall be labeled by an index $L$.

[12]$T$ is the ''statistical'' temperature, the ''local thermodynamical'' one being given by Tolman's relation $T/\sqrt{g_{00}}$ $(=T/\sqrt{-g_{L00}})$.

with the Hamiltonian density $\mathcal{H}$ which one could expect in the right-hand side of Eq. (48). Anyhow, the difference of these quantities is a spatial divergence which does not produce contributions to the spatial integral, holding our hypothesis of a closed spatial section. Indeed, the case of a *static metric* we have

$$\mathcal{H}=-T_0^0+\xi(-g_L)^{-1/2}\partial_i[(-g_L)^{1/2}(g^{ij}\partial_j\phi^2-\phi^2 w^i)], \tag{51}$$

where $w^a=1/2\nabla^a\ln|g_{L00}|$. Interpreting $\langle\phi^2(x)\rangle$ as the limit of $\zeta(s,x|A)$ as $s\to1$, the previous equation leads to a natural regularization of $\langle\int d\vec{x}\sqrt{g(\vec{x})}\mathcal{H}(x)\rangle$ which coincides with the corresponding integral of $\langle T_0^0(\vec{x})\rangle$ which appears in the right-hand side of Eq. (48).[13]

The validity of Eqs. (48) and (49) is an indirect proof that the canonical measure suggested by Toms [7] in defining the path integral in the phase space

$$\prod_x\{[g^{00}(x)]^{-1/2}d\phi(x)d\Pi(x)\}$$

instead of the apparently more ''natural'' [18,19]

$$\prod_x\{d\phi(x)d\Pi(x)\}$$

can be correctly used in defining the partition function in terms of an Euclidean Hamiltonian path integral. Indeed it is Toms' measure in the phase space which produces, by the usual momentum integration, the configuration space measure (1) which is used as a starting point to the $\zeta$-function interpretation of the configuration space path integral [4,7,6].

As a final comment, it is worthwhile stressing that, already on a classical ground, dropping the requirement of a closed spatial section, the Hamiltonian could not coincide with the integral of $T_0^0$ and the theory would be more problematic. This could be very important in studying the quantum correction of the black-hole entropy, where the spatial section of the spacetime has a boundary represented, in the Lorentzian picture, by the event horizon [20].

To conclude, let us prove the identity (48). We just sketch the way because that is very similar to the proofs in the previous sections. In Appendix we shall prove the identity (where $g_0^0=1$)

$$\frac{\partial\lambda_n}{\partial\beta}=-2\int_\Sigma d\vec{x}\sqrt{g(\vec{x})}\left\{T_0^0[\phi_n^*\phi_n](\vec{x})+\frac{1}{2}g_0^0\lambda_n\phi_n^*(\vec{x})\phi_n(\vec{x})\right\}. \tag{52}$$

From the expression above and employing definitions in Sec. II, we get that, for the values of $s$ where the involved $\zeta$ functions can be expanded as series

$$\frac{\partial\zeta(s|A)}{\partial\beta}=\int_\Sigma d\vec{x}\sqrt{g(\vec{x})}2s\left\{\zeta_0^0(s+1,\vec{x}|A)+\frac{1}{2}g_0^0\zeta(s,\vec{x}|A)\right\}=\int_\Sigma d\vec{x}\sqrt{g(\vec{x})}Z_0^0(s,x|A),$$

and thus we find

$$-\frac{\partial\ln Z_\beta}{\partial\beta}=-\frac{\partial S_{\text{eff}}}{\partial\beta}=\frac{1}{2}\frac{d}{ds}\Big|_{s=0}\int_\Sigma d\vec{x}\sqrt{g(\vec{x})}Z_0^0(s,x)+\frac{1}{2}\ln(\mu^2)\int_\Sigma d\vec{x}\sqrt{g(\vec{x})}Z_0^0(0,x)=-\int_\Sigma d\vec{x}\sqrt{g}\langle T_0^0(\vec{x})\rangle_\beta,$$

that is, Eq. (48). Notice that both $Z_\beta$ and $T_0^0$ may be affected by arbitrary $\mu$-dependent terms. A comparison between both sides of Eq. (48) to make it explicit in terms of $\zeta$ functions leads us to the identity for the factors of $\ln(\mu^2)$

$$\frac{\partial\zeta(0|A)}{\partial\beta}=2\int_\Sigma d\vec{x}\sqrt{g}G_0^0(\vec{x}|A), \tag{53}$$

where $G_{ab}(x|A)$ is the previously introduced residue of $\zeta_{ab}(s+1,x|A)$ at $s=0$ (21).

## VII. EXPLICIT COMPUTATION OF $G_{ab}(X|A)$ IN A $\zeta$-REGULAR THEORY: THE CONFORMALLY COUPLED CASE

Let us consider the case of a massless scalar field conformally coupled in a generic (closed Euclidean) four dimensional spacetime. Because a particular discussion on the form of $\langle T_{ab}\rangle$ depends on the particular manifold we are dealing with, we shall consider, in the general case of a massless conformally coupled field, only the general form of the pole $G_{ab}(s,x|A)$ employing the equations founds in Sec. III. We shall find that $G_{ab}(s,x|A)$ has a vanishing trace (and thus the conformal anomaly formula follows as we saw previously), it is conserved and depends locally on the geometry. In particular it is quadratic in the curvatures and can be thought of as a generalization of the geometrical term in Eintein's equations. Moreover, we shall find that the explicit form of $G_{ab}(x|A)$ is just that required by other renormalization procedures.

We remind the reader the first and the second heat kernel

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[13]One has to be very careful in dealing with the limit as $s\to1$ (I am grateful to Iellici who has focused my attention on this general problem) because as previously discussed, in four dimensions, $\zeta(1,x|A)$ usually diverges as it follows from heat kernel theory [5], except for the case of a massless field conformally coupled to $R$ or a massive field with an opportune coupling with $R$ in a curvature-constant manifold. Actually, one has to calculate *first* the spatial integral for $s\neq1$ and thus all terms containing the integral of the derivative of $\zeta(s,x|A)$ on $\partial\Sigma$ vanish, *then* one can perform the limit as $s\to1$ which is trivial.

off-diagonal coefficient in the case of a massless field. These coefficients appear in [3][14]

$$g(y)^{-1/4}a_1(x,y|A)=\left(\frac{1}{6}-\xi\right)R(x)-\frac{1}{2}\left(\frac{1}{6}-\xi\right)R_{;a}(x)z^a-\frac{1}{3}\mathcal{A}_{ab}(x)z^az^b, \quad (54)$$

$$g(y)^{-1/4}a_2(x,y|A)=\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2R^2(x)-\frac{1}{3}\mathcal{A}^c_c(x), \quad (55)$$

where $z=y-x$ are Riemannian coordinates with the origin on the point $x$, the semicolon indicates the covariant derivative and

$$\mathcal{A}_{ab}(x):=\frac{1}{2}\left(\frac{1}{6}-\xi\right)R_{;ab}(x)+\frac{1}{120}R_{;ab}(x)-\frac{1}{40}R^c_{ab;c}(x)+\frac{1}{30}R^c_a(x)R_{cb}(x)-\frac{1}{60}R^{cd}_{ab}(x)R_{cd}(x)-\frac{1}{60}R^{cde}_a(x)R_{cdeb}(x). \quad (56)$$

Let us consider the conformally coupled case, i.e., $\xi=1/6$. Then

$$a_1(x|A)=0, \quad (57)$$

$$a_2(x|A)=-\frac{1}{3}\mathcal{A}^c_c(x), \quad (58)$$

$$a_{1(a,b)}(x|A)=\frac{2}{3}\mathcal{A}_{ab}(x). \quad (59)$$

Employing Eq. (39) as well as the coefficients above, we find

$$3(4\pi)^2G_{ab}(x|A)=2\mathcal{A}_{ab}(x)-\frac{g_{ab}(x)}{2}\mathcal{A}^c_c(x). \quad (60)$$

It is obvious that, just as we expected,

$$g^{ab}(x)G_{ab}(x|A)=0.$$

As we said previously, this is related to the conformal anomaly.

[14]It is very important to note that the coefficients reported in [3] are referred to the Lorentzian metric. The choice of the signature employed in [3] is $(1,-1,-1,-1)$ and the definition of the Riemann tensor $R^a_{bcd}$ takes the opposite sign with respect to the more usual choice [17] which we are employing. To pass from the Lorentzian convention in [3] to our Euclidean convention is sufficient to use the two formal transformations $R'^a_{bcd}\to -R^a_{bcd}$, $g'_{ab}\to -g_{ab}$, where the primed quantities are those Lorentzians which appear in [3] and the others are our Euclidean quantities. The definitions of $R_{ab}$ and $R$ do not change; we have $R_{ab}:=R^c_{acd}$, $R:=R^c_c$ in both formalisms.

Let us make the form of $G_{ab}(x|A)$ explicit. A few trivial calculations[15] produce the result

$$G_{ab}(x|A)=\frac{1}{60(4\pi)^2}\left[{}^{(2)}H_{ab}(x)-\frac{1}{3}{}^{(1)}H_{ab}(x)\right]. \quad (61)$$

The tensors ${}^{(1)}H_{ab}(x)$ and ${}^{(2)}H_{ab}(x)$ are well-known conserved tensors obtained by varying geometrical actions built up by quadratic curvature tensor terms. The right-hand side of Eq. (61) is, up to constant overall factors, the only linear combination of those tensor which is traceless. Explicitly

$${}^{(1)}H_{ab}(x)=-\frac{1}{g^{1/2}}\frac{\delta}{\delta g^{ab}}\int d^4x\sqrt{g}R^2(x)=2R_{;ab}(x)-2g_{ab}(x)\Delta R(x)+\frac{1}{2}g_{ab}(x)R^2(x)-2R(x)R_{ab}(x)$$

and

$${}^{(2)}H_{ab}(x)=-\frac{1}{g^{1/2}}\frac{\delta}{\delta g^{ab}}\int d^4x\sqrt{g}R^{cd}(x)R_{cd}(x)=R_{;ab}(x)-\frac{1}{2}g_{ab}(x)\Delta R(x)-\Delta R_{ab}(x)+\frac{1}{2}g_{ab}(x)R^{cd}(x)R_{cd}(x)-2R^{cd}(x)R_{cdab}(x).$$

We remind the reader that the term $\ln(\mu^2)G_{ab}(x|A)$ represents the finite renormalization part of our ζ-function renormalization procedure. The expression of the finite renormalization part we have found in Eq. (61) is exactly the same which appears in other regularization and renormalization procedures (e.g., dimensional regularization) [3].

## VIII. THE SIMPLEST CASE: A BOX IN THE FLAT SPACE

Let us consider the simplest example of a super-ζ-regular theory. That is a massless boson gas at the inverse temperature $\beta$ in a flat box with a very large spatial volume $V$. This is the same example considered by Hawking in [4] as far as the ζ-function regularization of the effective action was concerned; rather, we will deal with the stress tensor. For the sake of simplicity, we shall deal with the component $T_{00}$ of the stress tensor only.

The Euclidean action of the field is simply

$$S=-\frac{1}{2}\int d^4x\,\delta^{ab}\partial_a\phi\partial_b\phi,$$

[15]Taking also account of the ‘‘topological’’ identity [3,17] $1/2g_{ab}(x)R_{cdef}(x)R^{cdef}(x)-2R_{acde}(x)R^{cde}_b(x)-4\Delta R_{ab}(x)+2R_{;ab}(x)+4R_{ac}(x)R^c_b(x)-4R^{cd}(x)R_{cadb}(x)=-{}^{(1)}H_{ab}(x)+4{}^{(2)}H_{ab}(x)$.

where $\delta^{ab}$ is the usually flat Euclidean metric. Notice that all coordinates define Killing vectors. The stress tensor reads simply

$$T_{ab}(x)=\partial_a\phi(x)\partial_b\phi(x)-\frac{1}{2}\delta_{ab}\partial^c\phi(x)\partial_c\phi(x).$$

We shall consider the box as a torus in order to use our method. The motion operator is the trivial Laplacian with the sign changed $A=-\Delta$, and we have the set of normalized eigenvectors

$$\phi_{\vec{k},n}(x):=\frac{e^{i\vec{x}\cdot\vec{k}-ik_nx^0}}{\sqrt{\beta V}}, \tag{62}$$

where $\vec{x}\equiv(x^1,x^2,x^3)$ and $\vec{k}\equiv(k^1,k^2,k^3)$, each $k^i$ being quantized by the usual torus quantization. Also $k_n$ is quantized trivially by $k_n=2\pi n/\beta$, where $n=0,\pm1,\pm2,\ldots$. Obviously, we have also

$$A\phi_{\vec{k},n}=\lambda_{n,\vec{k}}\phi_{\vec{k},n}(x), \tag{63}$$

where

$$\lambda_{n,\vec{k}}:=\vec{k}^2+k_n^2. \tag{64}$$

The local zeta function reads

$$\zeta(s,x|A)=(\beta V)^{-1}\sum_{n,\vec{k}}\lambda_{n,\vec{k}}^{-s} \tag{65}$$

and finally, the $\overline{\zeta_{00}}(s,x|A)$ function [see Eq. (29)] reads similarly

$$\overline{\zeta_{00}}(s,x|A)=(\beta V)^{-1}\sum_{n,\vec{k}}4\pi^2\beta^{-2}n^2\lambda_{n,\vec{k}}^{-s}. \tag{66}$$

Proceeding as discussed in [4], we can rewrite the formulas above, in the limit of a very large $V$,

$$\zeta(s,x|A)=\frac{4\pi}{(2\pi)^3\beta}\left\{\int_\epsilon^{+\infty}dk k^{2-2s}+2\sum_{n=1}^{+\infty}\int_\epsilon^{+\infty}dk k^2(4\pi^2\beta^{-2}n^2+k^2)^{-s}\right\}$$

and

$$\overline{\zeta_{00}}(s,x|A)=\frac{16\pi^4}{(2\pi)^3\beta^3}\left\{\int_\epsilon^{+\infty}dk k^{2-2s}+2\sum_{n=1}^{+\infty}n^2\int_\epsilon^{+\infty}dk k^2(4\pi^2\beta^{-2}n^2+k^2)^{-s}\right\}.$$

The final results are (see [4])

$$\zeta(s,x|A)=-\frac{8\pi}{(2\pi)^3\beta}(2\pi\beta^{-1})^{3-2s}\zeta_R(2s-3)\times(2-2s)^{-1}\frac{1}{2}\frac{\Gamma(1/2)\Gamma(s-3/2)}{\Gamma(s-1)}, \tag{67}$$

and [through Eq. (28)]

$$\begin{aligned}\zeta_{00}(s+1,x|A)&=\overline{\zeta_{00}}(s+1,x|A)-\frac{1}{2}\zeta(s,x|A)\\&=\frac{32\pi^4}{(2\pi)^3\beta^3}(2\pi\beta^{-1})^{1-2s}\zeta_R(2s-3)\times(-2s)^{-1}\frac{1}{2}\frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)}\\&\quad+\frac{4\pi}{(2\pi)^3\beta}(2\pi\beta^{-1})^{3-2s}\zeta_R(2s-3)\times(2-2s)^{-1}\frac{1}{2}\frac{\Gamma(1/2)\Gamma(s-3/2)}{\Gamma(s-1)}.\end{aligned} \tag{68}$$

We have dropped parts dependent on the infrared cutoff $\epsilon$ by putting $\epsilon\to0^+$ after one has fixed Re$s$ large finite, executed the integration and performed the analytic continuation of this result to $s=0$ (see [4]). $\zeta_R(s)$ is the usual Riemann zeta function which can be analytically continued in the whole complex plane except for the only singular point at $s=1$.

We can analytically continue the functions above in the $s$-complex plane. In particular, notice that both functions can be analytically continued in a neighborhood of $s=0$ *including* this point. The apparent pole of $\zeta_{00}(s+1,x|A)$ at $s=0$ is canceled out by the pole of $\Gamma(s)$ in the denominator; this means that $\zeta_{00}(s+1,x|A)$ takes no poles in $s=0$ and defines a *super*-$\zeta$-regular theory. Conversely, the $\zeta$ function in Eq. (67) vanishes at $s=0$.

As a final comment, we notice that the parameter $\mu$ will disappear from the final renormalized effective action and the final renormalized 00 component of the stress tensor. The 00 component of the renormalized stress tensor can be now computed by Eq. (19), taking the value at $s=0$ of the function in Eq. (68). We have

$$-\langle T_{00}(x)\rangle=-\langle T_0^0(x)\rangle=-\langle T_{L0}^0(x)\rangle=-\zeta_{00}(1,x|A)=\frac{\pi^2}{30\beta^2}. \tag{69}$$

This is the well-known energy density of massless scalar bosons in a large box.

The well-known partition function can be computed by the usual method through $\zeta(s,x|A)$ and reads [4]

$$Z_\beta=e^{\beta^{-3}\pi^2V/90}. \tag{70}$$

It is very simple to verify Eq. (48) by using Eqs. (69) and (70).

## IX. EINSTEIN'S CLOSED STATIC UNIVERSE

The ultrastatic metric of the (Euclidean) Einstein closed static universe is [3]

$$ds^2_{\mathrm{ECS}}=d\theta^2+g_{ij}dx^i dx^j=d\theta^2+a^2(dX^2+\sin^2 X d\Omega_2^2).$$

$X$ ranges from 0 to $\pi$ and $d\Omega_2^2$ is the usual metric on $S_2$. The time coordinate $\theta$ ranges from 0 to $\beta\leqslant+\infty$. $\beta$ is the inverse temperature of the considered thermal state referred to the Killing vector generated by the Lorentzian time $i\theta$. The related vacuum state corresponds to the choice $\beta=+\infty$. The curvature of the space is $R=6/a^2$ and the Ricci tensor reads $R_{ij}=2g_{ij}/a^2$, the remaining components vanish.

This manifold is closed, namely, compact without boundary. Also the spatial section at $\theta=$ constant are closed and their volume is $V=2\pi^2a^3$.

Let us consider a conformally coupled massless scalar field propagating within this manifold. We want to compute its stress tensor referred to the thermal states pointed out above, in particular we want to get the vacuum stress tensor which is known in the literature [3]. Notice that all the required hypotheses to implement the stress-tensor $\zeta$-function approach are fulfilled: the Euclidean manifold is static and closed.

Let us build up the function $\zeta_{ab}(s,x|A)$ necessary to get $\langle T_{ab}(x)\rangle_\beta$ through Eq. (18) or Eq. (19). The general expression of $\zeta_{ab}(s,x|A)$ is given in Eq. (28). We can rewrite it down as

$$\zeta_{ab}(s,x|A)=\overline{\zeta}_{ab}(s,x|A)-\xi\nabla_a\nabla_b\zeta(s,x|A)+(\xi-\tfrac{1}{4})\times g_{ab}(x)\Delta\zeta(s,x|A)+\xi R_{ab}(x)\zeta(s,x|A)-\frac{1}{2}g_{ab}(x)\zeta(s-1,x|A), \quad (71)$$

where, in the sense of the analytic continuation of both sides in the whole $s$ complex plane:

$$\overline{\zeta}_{ab}(s,x|A)=\sum_k{}'\ \lambda_k^{-s}\nabla_a\phi_k^*(x)\nabla_b\phi_k(x). \quad (72)$$

We are interested in the case $\xi=\xi_c:=1/6$ (conformal coupling in four dimensions). The local $\zeta$ function is similarly given by

$$\zeta(s,x|A)=\sum_k{}'\ \lambda_k^{-s}\phi_k^*(x)\phi_k(x). \quad (73)$$

The functions $\phi_k(x)$ define a normalized complete set of eigenvectors of the Euclidean motion operator

$$A\phi_k=\lambda_k\phi_k,$$

where, in our case

$$A=-\partial_\theta^2-a^{-2}\Delta_{S_3}+\xi_c R.$$

The explicit form of the considered eigenvalues and Kroneker's $\delta$-normalized eigenvectors is well known [3]. In particular we have $k\equiv(n,q,l,m)$ where $n=0,\pm1,\pm2,\pm3,\ldots$, $q=1,2,3,\ldots$, $l=0,1,2,\ldots,q-1$, $m=0,\pm1,\pm2,\ldots,\pm l$, and

$$\lambda_k=\left(\frac{2\pi n}{\beta}\right)^2+\left(\frac{q}{a}\right)^2. \quad (74)$$

The following relations, which hold true for normalized eigenvectors, are also useful. We leave the proofs of these to the reader:

$$\sum_{lm}\phi_k^*(x)\phi_k(x)=\frac{q^2}{V\beta}, \quad (75)$$

notice that the right-hand side of the equation above is nothing but the degeneracy of each eigenspace times $1/2\beta V$ (or $1/\beta V$ when $n=0$);

$$\sum_{lm}\partial_i\phi_k^*(x)\partial_j\phi_k(x)=g_{ij}(x)\frac{q^2(q^2-1)}{3V\beta a^2} \quad (76)$$

and ($x^0:=\theta$)

$$\sum_{lm}\partial_0\phi_k^*(x)\partial_0\phi_k(x)=\frac{(2\pi nq)^2}{V\beta^3}. \quad (77)$$

We have also, because of the homogeneity of the space,

$$\zeta(s,x|A)=\frac{\zeta(s|A)}{V\beta}, \quad (78)$$

where $\zeta(s|A)$ is the global $\zeta$ function obtained by summing over $\lambda_k^{-s}$ as usual:

$$\zeta(s|A)=\sum_k{}'\ \lambda_k^{-s}. \quad (79)$$

It is possible to relate the function $\overline{\zeta}_{ab}(s,x|A)$ to the function $\zeta(s,x|A)$. Indeed, we notice that

$$\lambda_k^{-s}\left(\frac{2\pi n}{\beta}\right)^2=\frac{\beta}{2(s-1)}\frac{\partial\lambda_k^{-(s-1)}}{\partial\beta}.$$

The identity above inserted into the definition (72) for $a=b=0$, taking Eq. (77) into account, yields

$$\overline{\zeta}_{00}(s+1,x|A)=\frac{1}{2Vs}\frac{\partial}{\partial\beta}\zeta(s|A), \quad (80)$$

or, equivalently,

$$\zeta_{00}(s+1,x|A)=\overline{\zeta}_{00}(s+1,x|A)=-\frac{a}{2V\beta s}\frac{\partial}{\partial a}\zeta(s|A)+\frac{\zeta(s|A)}{V\beta}, \quad (81)$$

which follows from the identity above taking account of

$$2s\zeta(s|A)=\beta\frac{\partial}{\partial\beta}\zeta(s|A)+a\frac{\partial}{\partial a}\zeta(s|A). \quad (82)$$

The last identity is a simple consequence of the expression of the eigenvalues (74).

Concerning the components $ij$ (the remaining components vanish) we can take advantage from the identity

$$\lambda_k^{-s} q^2 = \frac{3a^3}{2(s-1)} \frac{\partial \lambda_k^{-(s-1)}}{\partial a}. \quad (83)$$

Inserting this into Eq. (72) for $a=i,b=j$, taking Eq. (76) into account, we obtain

$$\overline{\zeta_{ij}}(s+1,x|A) = \frac{g_{ij}(x)}{3V\beta a^2}\left[ -\zeta(s+1|A) + \frac{a^3}{2s}\frac{\partial}{\partial a}\zeta(s|A)\right]. \quad (84)$$

To get the renormalized stress tensor, we have to compute $\zeta(s|A)$ or equivalently $\zeta(s,x|A)$ only. The expansion of the latter over the eigenvalues reads

$$\begin{aligned}\zeta(s,x|A) &= \frac{2}{V\beta}\sum_{q=1}^{+\infty}\sum_{n=1}^{+\infty} q^2\left[\left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{q}{a}\right)^2\right]^{-s} \\ &\quad + \frac{1}{V\beta}\sum_{q=1}^{+\infty} q^2\left[\left(\frac{q}{a}\right)^2\right]^{-s} \\ &= \frac{2}{V\beta}\sum_{q=1}^{+\infty}\sum_{n=1}^{+\infty} q^2\left[\left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{q}{a}\right)^2\right]^{-s} \\ &\quad + \frac{a^{2s}}{V\beta}\zeta_R(2s-2). \end{aligned} \quad (85)$$

The last $\zeta$ function is Riemann's one.

Let us introduce the Epstein function [5] obtained by continuing (into a meromorphic function) the series in the variable $s$:

$$E(s,x,y) := \sum_{n,m=1}^{+\infty} (x^2n^2 + y^2m^2)^{-s}. \quad (86)$$

We obtain trivially

$$\sum_{n,m=1}^{+\infty} m^2(x^2n^2+y^2m^2)^{-s} = -\frac{1}{2y(s-1)}\frac{\partial}{\partial y}E(s-1,x,y).$$

Employing such an identity, we can rewrite the expression (85) of $\zeta(s,x|A)$ as

$$\zeta(s,x|A) = \frac{a^{2s}}{V\beta}\zeta_R(2s-2) + \frac{a^3}{V\beta(s-1)}\frac{\partial}{\partial a}E\left(s-1,\frac{2\pi}{\beta},\frac{1}{a}\right). \quad (87)$$

No expression of the Epstein function in terms of elementary functions exists in literature. There exists a well-known expansion in terms of MacDonald functions [5]

$$\begin{aligned}E(s,x,y) &= -\frac{1}{2}y^{-2s}\zeta_R(2s) + \frac{\sqrt{\pi}\Gamma(s-1/2)}{2x\Gamma(s)}y^{1-2s}\zeta_R \\ &\quad \times(2s-1) + \frac{2\sqrt{\pi}x^{-2s}}{\Gamma(s)}\sum_{m,n=1}^{+\infty}\left(\frac{\pi x m}{yn}\right)^{s-1/2} \\ &\quad \times K_{s-1/2}\left(\frac{2\pi y n m}{x}\right). \end{aligned} \quad (88)$$

Notice that, due to the negative exponential behavior of MacDonalds functions $K_a(x)$ at large arguments, the last series defines a function which is analytic on the whole $s$ complex plane. The structure of the poles of the Epstein function is due to the $\Gamma$ and (Riemann's) $\zeta$ functions in the first line of the formula above. In particular there are only two simple poles at $s=1/2$ and $s=1$.

Taking account of the expression above and Eq. (87), we find

$$\begin{aligned}\zeta(s,x|A) &= \frac{\sqrt{\pi}}{4\pi V}\frac{\Gamma(s-3/2)}{\Gamma(s)}(2s-3)a^{2s-1}\zeta_R(2s-3) \\ &\quad - \frac{a}{V\Gamma(s)}\left(\frac{\beta}{2\pi}\right)^{2s-2}\Xi(s,\beta/a), \end{aligned} \quad (89)$$

where the function $\Xi(s,\beta/a)$ given by

$$\Xi(s,z) = 2\pi\frac{d}{dz}\sum_{m,n=1}^{+\infty}\left(\frac{2\pi^2 m}{zn}\right)^{s-3/2}K_{s-3/2}(nmz), \quad (90)$$

is analytic throughout the s complex plane and, due to the large argument behavior of the MacDonald functions, vanishes as $\beta\to+\infty$ as $(\beta/a)^{5/2-s}\exp-\beta/a$ when $\mathrm{Re}s\geqslant 0$. Reminding the reader of the relation

$$2\frac{d}{du}K_a(u) = K_{a-1}(u) + K_{a+1}(u), \quad (91)$$

the function $\Xi(s,z)$ and its $z$ derivative (see below) can be evaluated numerically at the physical values $s=0$ and $s=1$ (see below).

The expression (89) is very useful as far as the low-temperature thermodynamics in our manifold is concerned. Notice that, changing the role of $x$ and $y$ in the expression (90), one may get an expression for $\zeta(s,x|A)$ useful at large temperatures.

Some remarks on Eq. (89) are in order. First notice that, due to the $\gamma$ functions in the denominators, $\zeta(s,x|A)\to 0$ as $s$ when $s\to 0$, and thus no trace anomaly appears and neither renormalization scale $\mu$ remains in the renormalized effective action. The found $\zeta$ function is analytic throughout the $s$ complex plane except for the point $s=2$ where a simple pole appears. Employing Eqs. (80) and (84) we find that $\zeta_{ab}(s,x|A)$ is analytic at $s=1$ and thus the theory is a *super*-$\zeta$-regular theory.

Employing the definition (19), Eq. (71), and the obtained expression for $\zeta_{ab}(s,x|A)$, a few calculations lead us to

$$\langle T^b_{La}(x)\rangle_\beta = \langle T^b_a(x)\rangle_\beta \equiv T(\beta)\left(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \quad (92)$$

where

$$\begin{aligned}T(\beta) &= -\frac{1}{2V}\frac{\partial}{\partial\beta}\frac{\zeta(s|A)}{s}\bigg|_{s=0} = \frac{1}{480a^4\pi^2} \\ &\quad + \frac{1}{a^4}\frac{d}{dz}\bigg|_{z=\beta/a}\frac{\Xi(0,z)}{z}. \end{aligned} \quad (93)$$

Notice that the last derivative term vanishes very fast at low temperatures.

Now, one can prove very simply that the obtained stress tensor is conserved, has a vanishing trace and reduces to the well-known vacuum stress tensor in the closed Einstein universe [3] as $\beta\to+\infty$

$$\langle T^b_a(x)\rangle_{\text{vacuum}}\equiv\frac{1}{480a^4\pi^2}\left(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right). \tag{94}$$

Taking account of $\zeta(0|A)=0$, we can rewrite Eq. (93) as

$$T(\beta)=-\frac{1}{2V}\frac{\partial}{\partial\beta}\zeta'(0|A)=-\frac{1}{V}\ln Z_\beta,$$

where the prime means the $s$ derivative. Hence, the relation (48) holds true trivially. The general relation between the Hamiltonian density and the stress-tensor energy density in case of static coordinates reads[16]

$$\mathcal{H}=-T^0_0+\xi g^{-1/2}\partial_i[g^{1/2}(g^{ij}\partial_j\phi^2-\phi^2w^i)], \tag{95}$$

where $w^a=1/2\nabla^a\ln g_{00}$. $w^a$ vanishes in the present case. Let us employ such a relationship to evaluate the averaged value of the quantum Hamiltonian. We have to interpret Eq. (95) as

$$\langle\mathcal{H}\rangle_\beta=-\langle T^0_0\rangle_\beta+\xi g^{-1/2}\partial_i[g^{1/2}(g^{ij}\partial_j\langle\phi^2\rangle_\beta-\langle\phi^2\rangle_\beta w^i)]. \tag{96}$$

As is well known, provided the local $\zeta$ function is regular at $s=1$, we can define $\langle\phi^2(x)\rangle=\zeta(1,x|A)$. This is the case and we find

$$\langle\phi^2(x)\rangle_\beta=-\frac{1}{48\pi^2a^2}-\frac{1}{2\pi^2a^2}\Xi(1,\beta/a).$$

This reduces to the known value as $\beta\to+\infty$ [3]. Notice that, due to the homogeneity of the space, there is not dependence on $x$ and thus all derivatives in Eq. (96) vanish yielding $\langle\mathcal{H}\rangle_\beta=-\langle T^0_0\rangle_\beta$. Then Eq. (48) can be rewritten in terms of the averaged Hamiltonian in the right-hand side

$$-\frac{\partial\ln Z_\beta}{\partial\beta}=\langle H\rangle_\beta. \tag{97}$$

## X. EINSTEIN'S OPEN STATIC UNIVERSE

The ultrastatic metric of the (Euclidean) Einstein closed static universe is [3]

$$ds^2_{\text{EOS}}=d\theta^2+g_{ij}dx^idx^j=d\theta^2+a^2(dX^2+\sinh^2X d\Omega^2_2).$$

$X$ ranges from 0 to $+\infty$ and $d\Omega^2_2$ is the usual metric on $S_2$. The time coordinate $\theta$ ranges from 0 to $\beta\leq+\infty$. Again, $\beta$ is the inverse temperature of the considered thermal state referred to the Killing vector generated by the Lorentzian time $i\theta$ and the related vacuum state corresponds to the choice $\beta=+\infty$. The curvature of the space is $R=-6/a^2$ and the Ricci tensor reads $R_{ij}=-2g_{ij}/a^2$, the remaining components vanish. This manifold is not closed and the spatial sections do not have a finite volume.

Let us consider a conformally coupled massless scalar field propagating within this manifold. As in the previously considered case, we want to compute its stress tensor referred to the thermal states, in particular we want to get the vacuum stress tensor. Notice that not all the required hypotheses to implement the stress-tensor $\zeta$-function approach are fulfilled. The manifold has no boundary but it is not compact. We expect to find a continuous spectrum as far as the Euclidean motion operator is concerned.

However, we shall find that our method does work also in this case. Notice that, now, we have to assume Eq. (18) or Eq. (19) by definition and check on the obtained results finally.

The form of the eigenvalues $\lambda_k$ of the conformally coupled massless Euclidean motion operator

$$A=-\partial^2_\theta-a^{-2}\Delta_{H_3}+\xi_cR,$$

is well known [23,3]. We have, exactly as in the previous case,

$$\lambda_k=\left(\frac{2\pi n}{\beta}\right)^2+\left(\frac{q}{a}\right)^2, \tag{98}$$

where $k\equiv(n,q,l,m)$ and $n=0,\pm1,\pm2,\pm3,\ldots,$ $q\in[0,+\infty)$, $l=0,1,2,3,\ldots,$ $m=0,\pm1,\pm2,\ldots,\pm l$. The degeneracy depends only on the indexes $l$ and $m$.

The following relations which hold true for eigenvectors $\phi_k(x)$ (which are Dirac's $\delta$ normalized in $q$ and Kroneker's $\delta$ normalized in the remaining variables) are also useful. We leave the proofs of these to the reader (see also [23]):

$$\sum_{l,m}\phi^*_k(x)\phi_k(x)=\frac{q^2}{2\pi^2a^3\beta}, \tag{99}$$

$$\sum_{l,m}\partial_i\phi^*_k(x)\partial_j\phi_k(x)=g_{ij}(x)\frac{q^2(q^2+1)}{6\pi^2a^5\beta}, \tag{100}$$

and $(x^0:=\theta)$

$$\sum_{l,m}\partial_0\phi^*_k(x)\partial_0\phi_k(x)=\frac{(2\pi nq)^2}{2\pi^2a^3\beta^3}. \tag{101}$$

Notice that the global $\zeta$ function simply does not exist because the infinite spatial volume of the manifold. Anyhow, we can compute the local $\zeta$ function as

$$\zeta(s,x|A):=\int_0^{+\infty}dq\sum_{l,m,n}\phi^*_k(x)\phi_k(x)\lambda_k^{-s}. \tag{102}$$

It is convenient to separate the contribution due to the terms with $n=0$ and introduce, as far as these terms are concerned, a cutoff $\epsilon$ at low $q$. A few trivial manipulations of the expression above yields

[16]Notice that we are writing Lorentzian relations employing the Euclidean metric. We could pass to use the more usual Lorentzian metric simply through the identities $g=-g_L$, $g_{00}=-g_{L00}$ and $g^{ij}=g^{ij}_L$.

$$\zeta(s,x|A)=\frac{a^{2s-3}}{2\pi^2\beta}\int_\epsilon^{+\infty}dq\,q^{2-2s}+\frac{1}{4\pi^2\beta}\left(\frac{\beta}{2\pi}\right)^{2s-3}\times\zeta_R(2s-3)\frac{\Gamma(1/2)\Gamma(s-3/2)}{\Gamma(s)}. \tag{103}$$

The apparent divergent integral as $\epsilon\to0^+$ can be made harmless as in [4] putting $\epsilon\to0^+$ after one has fixed Re$s$ large finite, executed the integration and performed the analytic continuation of this result to $s=0$. This procedure generalize the finite volume prescription to drop the null eigenvalues in defining the $\zeta$ function for the case of an infinite spatial volume. We have finally

$$\zeta(s,x|A)=\frac{1}{8\pi^2\sqrt{\pi}}\left(\frac{\beta}{2\pi}\right)^{2s-4}\zeta_R(2s-3)\frac{\Gamma(s-3/2)}{\Gamma(s)}. \tag{104}$$

Notice that $\zeta(0,x|A)=0$ and thus no renormalization scale appears in the (infinite) partition function.

Let us evaluate $\overline{\zeta_{ab}}(s,x|A)$. The only nonvanishing components are 00 and $ij$. In the first case we have directly from the definitions (omitting the terms with $n=0$ as above)

$$\zeta_{00}(s+1,x|A)=\overline{\zeta_{00}}(s+1,x|A)=\int dq\sum_{l,m,n}\left(\frac{2\pi n}{\beta}\right)^2\lambda_k^{-s}\phi_k^*(x)\phi_k(x)=\frac{1}{8\pi^2\sqrt{\pi}}\left(\frac{\beta}{2\pi}\right)^{2s-4}\zeta_R(2s-4)\frac{\Gamma(s-1/2)}{\Gamma(s+1)}. \tag{105}$$

In order to compute the remaining components of $\overline{\zeta_{ab}}$ we can use Eq. (100) and the relation in Eq. (83) once again. We find

$$\overline{\zeta_{ij}}(s+1,x|A)=\frac{1}{3a^5}g_{ij}(x)\zeta(s+1,x|A)+\frac{1}{2s}g_{ij}(x)\zeta(s,x|A). \tag{106}$$

We have found that $\zeta_{ab}(s,x|A)$ is analytic in $s=1$, hence the theory is a *super*-$\zeta$-regular theory once again. We can use Eq. (19) to compute the stress tensor.

Through Eqs. (71) and (19) we find finally

$$\langle T^b_{La}\rangle_\beta=\langle T^b_a\rangle_\beta\equiv T(\beta)\left(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \tag{107}$$

where

$$T(\beta)=\frac{\pi^2}{30\beta^4}. \tag{108}$$

The stress tensor in Eq. (107) is conserved and traceless as we expected from the general theory. $\langle T^b_a\rangle_\beta$ vanishes as $\beta\to+\infty$, this agrees with the known result [3] that the stress tensor in the vacuum state of the open Einstein universe vanishes. Notice that the found stress tensor, in the considered components, is exactly the same than in Minkowski spacetime.

Let us finally consider Eq. (48). In this case the left-hand side of Eq. (48) does not exist because that simply diverges. Nevertheless, we can notice that the divergence of the partition function is due to the volume divergence only and the remaining factor does not depend on the position on the spatial section; namely,

$$\ln Z_\beta=V\ln\mathcal{Z}_\beta=V\times\left(\beta\frac{1}{2}\zeta'(0,x|A)\right)=V\times\frac{\pi^3}{90\beta^3}, \tag{109}$$

where $V$ diverges and, actually, $\zeta'(0,x|A)$ does not depend on $x$ due to the homogeneity of the spatial manifold. This is the same situation than arises in the Minkowski spacetime. We expect that, although Eq. (48) does not make sense, a local version could yet make sense. Indeed, one can get very simply from Eqs. (107) and (109)

$$-\frac{\partial V\ln\mathcal{Z}_\beta}{\partial\beta}=-V\langle T^0_0\rangle_\beta=-\int_V d\vec{x}\sqrt{g}\langle T^0_{L0}(\vec{x})\rangle_\beta \tag{110}$$

on any finite volume $V$. As in the previously discussed case, $\langle\phi^2(x)\rangle_\beta$ can be obtained by evaluating the local $\zeta$ function at $s=1$, we get

$$\langle\phi^2(x)\rangle_\beta=\frac{1}{12\beta^2}. \tag{111}$$

Notice that this vanishes as $\beta\to+\infty$; namely, in the vacuum state as is known [23]. Furthermore, it does not depend on $x$ and thus, through Eq. (95) and noticing that $w^a=0$ (see the Einstein closed universe case), $\langle T^0_0\rangle_\beta=-\langle\mathcal{H}\rangle_\beta$. We can write finally, with an obvious meaning

$$-\frac{\partial V\ln\mathcal{Z}_\beta}{\partial\beta}=\langle H_V\rangle_\beta. \tag{112}$$

## XI. THE CONICAL MANIFOLD

Let us consider the Euclidean manifold $\mathcal{C}_\beta\times\mathcal{R}^2$ endowed with the metric

$$ds^2=r^2d\theta^2+dr^2+dz_1^2+dz_2^2, \tag{113}$$

where $(z_1,z_2)\in\mathcal{R}^2$, $r\in[0,+\infty)$, $\theta\in[0,\beta)$ when 0 is identified with $\beta$. $\mathcal{C}_\beta\times\mathcal{R}^2$ is a cone with deficit angle given by $2\pi-\beta$. That is the Euclidean manifold corresponding to the finite temperature $(T=1/\beta)$ quantum field theory in the Rindler space. In such a case $\theta$ is the Euclidean time of the theory. This is also a good approximation of a large mass black hole near the event horizon. Equivalently, considering $z_1$ as the Euclidean time, the metric above defines the Euclidean section (at zero temperature) of a cosmic string background. In this case $(2\pi-\beta)/8\pi G$ is the mass of the string.

The metric in Eq. (113), considered as the Rindler Euclidean metric, is static but not *ultrastatic*. Another important point is that such a metric is not homogeneous in the spatial section. The considered manifold is flat everywhere except for conical singularities which appear at $r=0$ whenever $\beta \neq 2\pi$. These singularities produce well-known Dirac's $\delta$ singularities in the curvatures of the manifolds at $r=0$ [24]. The physics involved in such anomalous curvature is not completely known. Actually, we shall see shortly that one can ignore completely the anomalous curvature dealing with the stress tensor renormalization also considering nonminimal coupling with the scalar curvature.

As is well known, the particular value $\beta_H=2\pi$ defines the Hawking-Unruh temperature in the Rindler/large-mass-black-hole interpretation, the corresponding thermal state being nothing but the Minkowski vacuum or Hartle-Hawking (large mass) vacuum. The thermal Rindler stress tensor (renormalized with respect to the Minkowski vacuum) which coincides, in the Euclidean approach, to the zero-temperature cosmic-string stress tensor (renormalized with respect to the Minkowski vacuum) has been computed by the point splitting approach [25].

Such results have only been partially reproduced by some $\zeta$-function or (local) heat kernel approach [26,11]. This is because these approaches were employed to renormalize the effective action only, and thus the stress tensor was computed assuming further hypotheses on its form or assuming some statistical-mechanical law as holding true [26,11].

Recently, in [14], also the massive case has been considered by employing an off-diagonal $\zeta$-function approach and a subtraction procedure similar to that is employed within the point-splitting framework. Here, we shall consider the massless case only. We shall check our approach for every value of the curvature coupling proving that the same results got by the point-splitting approach naturally arise. The important point is that, due to the complete independence of the method from statistical mechanics, we shall be able to discuss the statistical mechanics meaning (if it exists) of our results *a posteriori*.

Let us consider first the case of the *minimal* coupling $\xi=0$. This avoids all problems involved dealing with the singular curvature on the tip of the cone generated by the conical singularity. The $\zeta$ function of the effective action in conic backgrounds has been computed by several authors [27] also in the massive scalar case [14] and for photons and gravitons [12].

Discarding the singular curvature by posing $\xi=0$, a complete normalized set of eigenvectors of the massless Euclidean motion operator[17] $A=-\Delta_{\mathcal{C}_\beta\times\mathcal{R}^2}$ is [27]

$$\phi_q(x)=\frac{1}{2\pi}\sqrt{\frac{\lambda}{\beta}}e^{ikz}e^{i(2\pi n/\beta)\theta}J_{2\pi|n|/\beta}(\lambda r), \tag{114}$$

where $z=(z_1,z_2)$, $q=(n,k,\lambda)$, $n=0,\pm1,\pm2,\dots$, $k=(k_1,k_2)\in\mathcal{R}^2$ $\lambda\in[0,+\infty)$. The considered eigenfunctions are Kroneker's $\delta$ normalized in the index $n$ and Dirac's $\delta$ normalized in the remaining indices. The corresponding eigenvalues are

$$\lambda_q=\lambda^2+k^2. \tag{115}$$

The $\zeta$ function of $A$ has been computed explicitly and reads

$$\zeta(s,x|A)=\frac{r^{2s-4}}{4\pi\beta\Gamma(s)}I_\beta(s-1). \tag{116}$$

$I_\beta(s)$ is a well-known meromorphic function [27] carrying a simple pole at $s=1$. Known values are also

$$I_\beta(0)=\frac{1}{6\nu}(\nu^2-1), \tag{117}$$

$$I_\beta(-1)=\frac{1}{90\nu}(\nu^2-1)(\nu^2+11), \tag{118}$$

where we defined $\nu:=2\pi/\beta$.

Notice that $\zeta(0,x|A)=0$ and thus no scale remains into the renormalized local effective action. $\langle\phi^2(x)\rangle$ can be computed by evaluating the local $\zeta$ function at $s=1$.

The function $\overline{\zeta}_{ab}(s,x|A)$ can be computed making use of intermediate results contained in [14]. A few calculations lead us to

$$\overline{\zeta}_{\theta\theta}(s,x|A)=\frac{r^{2s-4}\Gamma(s-3/2)}{4\pi\sqrt{\pi}\beta\Gamma(s)}H_\beta(s-1), \tag{119}$$

$$\overline{\zeta}_{rr}(s,x|A)=\frac{1}{2r}\partial_r r\partial_r\zeta(s,x|A)-\frac{1}{r^2}\overline{\zeta}_{\theta\theta}(s,x|A)+4\pi(s-2)\zeta_{D=6}(s,x|A), \tag{120}$$

$$\overline{\zeta}_{z_1z_1}(s,x|A)=\overline{\zeta}_{z_2z_2}(s,x|A)=2\pi\zeta_{D=6}(s,x|A). \tag{121}$$

All remaining components vanish. The meromorphic function $H_\beta(s)$ has been defined in [14], it has a simple pole at $s=2$ and known values are

$$H_\beta(0)=\frac{1}{120\nu}(\nu^4-1), \tag{122}$$

$$H_\beta(1)=-\frac{1}{12\nu}(\nu^2-1). \tag{123}$$

The function $\zeta_{D=6}(s,x|A)$ is the $\zeta$ function of the effective action in $\mathcal{C}_\beta\times\mathcal{R}^4$ [14]. It reads

$$\zeta_{D=6}(s,x|A)=\frac{r^{2s-6}}{(4\pi)^2\beta\Gamma(s)}I_\beta(s-2). \tag{124}$$

From the above equations and Eq. (71) it follows that $\overline{\zeta}_{ab}(s,x|A)$ is analytic at $s=1$ and thus the theory is super-$\zeta$ regular once again. Hence, we can use (19) to compute the stress tensor. Trivial calculations employing Eq. (71) with $\xi=0$ and Eq. (19) produce

[17] We are considering a particular self-adjoint extension of the formally self-adjoint Laplace-Beltrami operator in the conical manifold. The general theory of these extensions has been studied in [28].

$$\langle T_a^b(x)_{\xi=0}\rangle_\beta \equiv \frac{1}{1440\pi^2 r^4}\left\{\left[\left(\frac{2\pi}{\beta}\right)^4 - 1\right]\mathrm{diag}(-3,1,1,1) - 20\left[\left(\frac{2\pi}{\beta}\right)^2 - 1\right]\mathrm{diag}\left(\frac{3}{2},-\frac{1}{2},1,1\right)\right\}. \tag{125}$$

This is the correct result arising by the point-splitting approach [25] in the case of the minimal coupling. Let us prove that our method reproduces also the remaining cases.

In general, the relationship between the minimally coupled stress tensor and the generally coupled stress tensor can be trivially obtained by varying the action containing the usual coupling with the curvature, it reads

$$T_{ab}(x)_\xi = T_{ab}(x)_{\xi=0} + \xi\left[\left(R_{ab} - \frac{1}{2}g_{ab}R\right)\phi^2(x) + g_{ab}\Delta\phi^2(x) - \nabla_a\nabla_b\phi^2(x)\right]. \tag{126}$$

It is worthwhile stressing that the last $\xi$-parametrized term appears also when the manifold is flat. Within the quantum field theory, we can interpret quantitatively this relationship as

$$\langle T_{ab}(x)_\xi\rangle = \langle T_{ab}(x)_{\xi=0}\rangle + \xi\langle Q(x)_{ab}\rangle,$$

where

$$\langle Q(x)_{ab}\rangle := \left[\left(R_{ab}(x) - \frac{1}{2}g_{ab}(x)R(x)\right)\langle\phi^2(x)\rangle + g_{ab}\Delta\langle\phi^2(x)\rangle - \nabla_a\nabla_b\langle\phi^2(x)\rangle\right]. \tag{127}$$

Now $\langle T_{ab}(x)_{\xi=0}\rangle_\beta$ is known by Eq. (125), $R_{ab}(x)=0$, $R(x)=0$ and thus we can compute $\langle T_{ab}(x)_\xi\rangle_\beta$ employing the known value of $\langle\phi^2(x)\rangle_\beta$. We have, through Eq. (116)

$$\langle\phi^2(x)\rangle_\beta = \zeta(1,x|A) = \frac{1}{48\pi^2 r^2}\left[\left(\frac{2\pi}{\beta}\right)^2 - 1\right]. \tag{128}$$

The final result is exactly that of the point-splitting approach:

$$\langle T_{La}^b(x)_\xi\rangle_\beta = \langle T_a^b(x)_\xi\rangle_\beta \equiv \frac{1}{1440\pi^2 r^4}\left\{\left[\left(\frac{2\pi}{\beta}\right)^4 - 1\right]\mathrm{diag}(-3,1,1,1) + 20(6\xi-1)\left[\left(\frac{2\pi}{\beta}\right)^2 - 1\right]\mathrm{diag}\left(\frac{3}{2},-\frac{1}{2},1,1\right)\right\}. \tag{129}$$

The same result arises by employing the definition of $\zeta_{ab}(s,x|A)$ given in Eq. (71) with the chosen value of $\xi$, provided $\overline{\zeta_{ab}}(s,x|A)$ and $\zeta(s,x|A)$ are those computed in the *minimal coupling case*. This means that, concerning the renormalization of the stress tensor, the presence of the conical singularity which determines a singular curvature on the tip of the cone is completely irrelevant. Concerning the quantum state, there is no difference between different couplings with the curvature. The $\xi$-parametrized term remains as a relic in the stress tensor because of the *classical* formula (126). This term does not come out from the quantum state once one fixed the renormalization procedure. We see that the renormalization of the stress tensor can be managed completely by our Euclidean $\zeta$-function approach on the physical manifold instead of the *optical* manifold not depending on the presence of the conical singularity in the Euclidean manifold.

The knowledge of the averaged and renormalized stress tensor makes us able to compute the averaged and renormalized Hamiltonian of the system. The Hamiltonian of the theory should not depend on the parameter $\xi$ because that cannot appear into the Lorentzian action, the manifold being flat. Notice that there is no conical singularity in the Lorentzian theory. Not depending on $\xi$, the classical Hamiltonian density coincides with the changed sign energy component of the stress tensor in the minimal coupling. Indeed, employing Eq. (95), we can write down

$$\langle\mathcal{H}(x)\rangle_\beta = -\langle T_0^0(x)_{\xi=0}\rangle_\beta = \frac{3}{1440\pi^2 r^4}\left[\left(\frac{2\pi}{\beta}\right)^4 - 10\left(\frac{2\pi}{\beta}\right)^2 - 11\right]. \tag{130}$$

Let us finally consider the problem of the validity of the relation Eq. (48) in some sense. The spatial section is neither finite nor homogeneous, we could have problems with the use of cutoffs. It is not obvious that such a relation as Eq. (48) can hold true in our case considering cutoff smeared quantities as[18]

$$\ln Z_{\beta\epsilon} := \int_{r>\epsilon} d^4x\sqrt{g}\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\zeta(s,x|A), \tag{131}$$

$$\mathcal{Q}_\epsilon(\beta) := \int_{r>\epsilon} d^3x\sqrt{g}\langle Q_0^0(x)\rangle_\beta, \tag{132}$$

$$\langle H_\epsilon\rangle_\beta := \int_{r>\epsilon} d^3x\sqrt{g}\langle\mathcal{H}\rangle_\beta, \tag{133}$$

and, finally,

$$\mathcal{E}_{\epsilon\xi}(\beta) := -\int_{r>\epsilon} d^3x\sqrt{g}\langle T_0^0(x)_\xi\rangle_\beta = \int_{r>\epsilon} d^3x\sqrt{g}\langle\mathcal{H}\rangle_\beta - \xi\mathcal{Q}_\epsilon(\beta). \tag{134}$$

In particular we have from Eq. (116)

$$\ln Z_{\beta\epsilon} = \frac{A\beta}{2880\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4 + 10\left(\frac{2\pi}{\beta}\right)^2 - 11\right], \tag{135}$$

where $A$ is the area of the event horizon, the regularized volume of the spatial section is $V_\epsilon = A/(2\epsilon^2)$. Notice that,

[18] Notice that also the area $A$ of the horizon is a cutoff because the actual area is infinite. This cutoff is a trivial overall factor. We shall omit this cutoff as an index in the following formulas for sake of simplicity.

actually, the conserved charge $\mathcal{Q}_\epsilon(\beta)$ is a boundary integral which diverges on the conical singularity. Indeed, it can be expressed by the integration of Eq. (95) and it should be discarded if the manifold were regular. Notice that the choice of values of $\xi$ determines different values of $\mathcal{E}_{\xi\epsilon}$ due to the $\xi$-parametrized boundary term $\xi\mathcal{Q}_\epsilon$ in the stress tensor. Conversely, $\ln Z_{\beta\epsilon}$ does not depend on $\xi$.

If something similar to Eq. (48) holds true for a fixed value of $\epsilon$, it does just for a particular and unique value of $\xi$. Actually, a few calculations through Eq. (129) prove that, *not depending on the value of* $\epsilon$,

$$-\frac{\partial \ln Z_{\beta\epsilon}}{\partial\beta}=\mathcal{E}_{\epsilon\xi=1/9}(\beta)+\mathcal{E}_\epsilon=\langle H_\epsilon\rangle_\beta-\frac{1}{9}\mathcal{Q}_\epsilon(\beta)+\mathcal{E}_\epsilon . \tag{136}$$

The last term is an opportune constant energy

$$\mathcal{E}_\epsilon=\frac{A}{120\pi^2\epsilon^2}.$$

The presence of such an added constant could be expected from the fact that the energy $\mathcal{E}_{\epsilon\xi}$ is renormalized to vanish at $\beta=2\pi$ instead of $\beta=+\infty$. Conversely, there is no trivial explanation of the presence of the $\beta$-*dependent* term $-1/9\mathcal{Q}_\epsilon(\beta)$. Then, in the considered case, in the right-hand side of Eq. (48) does not appear the Hamiltonian which, at least classically, corresponds to the value $\xi=0$ as discussed above.

One could wonder whether or not $Z_{\beta\epsilon}$ defined in Eq. (131) can be considered a (regularized) partition function of the system. The simplest answer is obviously not because a fundamental relationship of statistical mechanics does not hold true.

In general, one could think that this negative result arises because we have dropped a contribution due to the conical singularity. This singularity produces a Dirac $\delta$ in the curvature on the tip of the cone in the Euclidean manifold. The integral of the Lagrangian get a contribution from this term in the case of a nonminimal coupling with the curvature. The problem of the contributions of these possible terms, in particular in relation to the black-hole entropy has been studied by several authors (see [20,21,30–33], and references therein), anyhow, in this paper we shall not explore such a possibility.

In any cases, it is worthwhile stressing that the found Euclidean effective action (135) is the correct one in order to get the *thermal* renormalized stress tensor by (formal) variation with respect to the background metric. We restress that the obtained stress tensor is exactly that obtained by the point-splitting approach.

The question of whether or not the effective action computed by the $\zeta$ function defines also the logarithm of the partition function (renormalized with respect to the Minkowski vacuum) is not a simple question. The problem is interesting on a physical ground also because the partition function of the field around a black hole (we remind the reader that the Rindler metric represents a large mass black hole) is used to compute the quantum corrections to the Bekenstein-Hawking entropy as early suggested by 't Hooft [29] or to give a reason for the complete BH entropy in the framework of the induced gravity considering massive fields nonconformally coupled [20–22].

As noticed in Sec. VI, on a more general ground, the considered problem is also interesting because there exist two not completely equivalent approaches to implement the statistical mechanics of a quantum field in a curved spacetime through the use of a path integral techniques and, up to the knowledge of the author, there is not a definitive choice of the method. In this work, we have employed the path integral in the physical manifold instead of in the *optical* related manifold. We remind the reader that in the case of a *static* but not *ultrastatic* spacetime, the naive approach based on the phase-space path integral leads one to a definition of the partition function as an Euclidean path integral performed in the configuration space within the *optical* manifold[19] instead of the physical one [19]. Other approaches [7] lead one to the definition of the partition function as a path integral in the physical manifold.

When the spatial section of the space is regular (e.g., closed) and thus the path integral regularized through the $\zeta$-function approach yields a finite result, formal manipulations of the path integral prove that these two different definitions lead to the same result up to the renormalization of the zero point energy [8]. In such a case these definitions are substantially equivalent. When the manifold is not regular, e.g., it has spatial sections with an infinite volume or has boundaries, in principle one may loose such an equivalence. Indeed, as far as the effective actions are concerned in our case we have

$$\ln Z_{\beta\epsilon}=\frac{A\beta}{2880\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4+10\left(\frac{2\pi}{\beta}\right)^2-11\right],$$

and

$$\ln Z^{\text{opt}}_{\beta\epsilon}=\frac{A\beta}{2880\pi^2\epsilon^2}\left(\frac{2\pi}{\beta}\right)^4. \tag{137}$$

The latter result can be directly obtained noticing that the optical manifold of the Rindler space is the open Einstein static universe [3]. Hence the latter effective action above is nothing but that computed previously in the open Einstein universe (in the conformal coupling). Considering the effective action computed as a path integral in the optical manifold we have

$$-\frac{\partial \ln Z^{\text{opt}}_{\beta\epsilon}}{\partial\beta}=\mathcal{E}_{\epsilon\xi=1/6}(\beta)+\mathcal{E}'_\epsilon . \tag{138}$$

One could conclude that, once again, there is not the Hamiltonian in the right-hand side, also discarding the constant energy. Actually, this result involves more subtle considerations. Indeed, we shall prove that this naive conclusion is not correct.

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[19]This is the ultrastatic manifold conformally related to the physical manifold by defining the optical metric through $\tilde{g}_{ab}:=g_{ab}/g_{00}$. The Euclidean action employed on the optical manifold is the physical action conformally transformed (including the matter fields) following the conformal transformation written above.

Let us suppose to implement canonical QFT [3] for a massless field conformally coupled directly on the optical manifold, namely, in the open Einstein static universe as it were the physical manifold. Obviously, we should get exactly the effective action which appears in Eq. (138). Furthermore, Eq. (138) is nothing but Eq. (97) and the right-hand side of Eq. (138) is nothing but the averaged $\epsilon$-regularized Hamiltonian of the QFT in the open Einstein universe. Such a Hamiltonian can also be obtained as a thermal average of the Hamiltonian operator obtained from the canonical QFT employing the *normal order prescription* and employing the usual definition of the partition function (summing the Boltzmanian exponential in the energy levels of the states in the canonical ensemble) [34].

Implementing the canonical quantization in the Rindler space for a massless scalar field, one trivially finds that an isomorphism exists between the Fock space built up on the Fulling-Rindler vacuum and the Fock space built up on the natural vacuum of the QFT in the open Einstein static universe (in the conformal coupling). Indeed, this isomorphism arises from the conformal relationship between the wave functions of the particles related to the quantized fields. This relation defines a one-to-one map from the one-particle Hilbert space of the Einstein open universe to the one-particle Hilbert space of the Rindler space which maintains the value of the corresponding indefinite scalar products [3]. This map defines a unitary isomorphism between the two Fock spaces provided one require that this isomorphism transform the vacuum state of the Einstein open universe into the Fulling-Rindler vacuum. In particular, also the Hamiltonian operators are unitarily identified provided one use the normal order prescription in both cases.

As a result we find that the right-hand side of Eq. (138) coincides also with the averaged Hamiltonian operator built up in the framework of the canonical quantization in the Rindler space with respect to the Fulling-Rindler vacuum. In this sense Eq. (138) is the usual statistical-mechanical relationship between the canonical energy and the partition function in the Rindler space.

The central point is that the renormalization scheme employed is the normal order prescription with respect to the Fulling-Rindler vacuum and not the point-splitting procedure. We can finally compare the averaged Rindler Hamiltonian of the canonical quantization $\langle H_\epsilon^{\rm can}\rangle_\beta$ which is renormalized by the *normal order prescription* in the Fulling-Rindler vacuum with the averaged Rindler Hamiltonian $\langle H_\epsilon\rangle_\beta$ obtained by integrating Eq. (130). The latter is renormalized with respect the Minkowski vacuum by the *point-splitting procedure*. We find

$$\langle H_\epsilon\rangle_\beta - \langle H_\epsilon^{\rm can}\rangle_\beta = -\frac{3}{2880\pi^2\epsilon^2} - \frac{30}{2880\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^2 - 1\right]$$
$$= -\frac{1}{960\pi^2\epsilon^2} - \frac{1}{6}\mathcal{Q}_\epsilon(\beta). \tag{139}$$

The first term in the right-hand side is trivial: it takes account of the difference of the zero-point energy. The second term is quite unexpected. It proves that the point-splitting procedure (or equivalently our $\zeta$-function procedure) to renormalize the stress tensor and hence the Hamiltonian is not so trivial as one could expect, this is because it involves terms which do not represent a trivial zero-point energy renormalization.

Concerning the conical manifold, the conclusion is that the theory in the optical manifold leads us naturally to an effective action which can be considered the logarithm of the partition function provided we renormalize the theory with respect to the Fulling-Rindler vacuum. Conversely, the effective action evaluated in the physical manifold is the correct effective action which produces the thermal stress tensor by formal variations with respect to the metric. This stress tensor is that obtained also by the point-splitting procedure and thus renormalizing with respect to the Minkowski vacuum.

## XII. SUMMARY

In this paper we have presented a new approach to renormalize the one-loop stress tensor in a curved background based on an opportune $\zeta$-function regularization. The procedure has been developed within the Euclidean formalism and in the hypothesis of a closed manifold and a real scalar field.

We do not think that our approach should change dramatically relaxing such hypotheses. This is because the same $\zeta$-function approach to renormalize the effective action was born in a similar context and has been successively developed into a very general context. In fact, we have used the method also in cases where the initially requested hypotheses do not hold true obtaining correct results.

Our approach, differently from all other approaches, is directly founded to the definition of the stress tensor as functional derivative of the effective action with respect to the background metric. All proofs contained in this paper are substantially based on that direct definition.

We have seen that, although it is not possible performing the analytic continuations involved in the method in all concrete cases (this is the same drawback of the $\zeta$ function regularization of the effective action), the method is well managed on a theoretical ground. Indeed, within our approach, the proof of the conservation of the stress tensor, the conformal anomaly formula, several thermodynamical identities are actually very easy to carry out. The infinite renormalization is made harmless by an automatic cancellation and the finite part is clearly highlighted as a residue of a pole of the stress tensor $\zeta$ function. It is furthermore clear that the renormalizing terms are conserved and depend on the geometry locally and thus can be thought as parts of geometrical side of the Einstein equations. Their explicit form can be obtained by the heat kernel expansion as outlined previously.

We have checked the method considering several concrete cases obtaining a perfect agreement with other renormalization procedures.

Particular attention has been paid considering the conical manifold, where some unresolved problems concerning the physical interpretation of the obtained results remain when one considers the conical manifold as the Euclidean-thermal Rindler space.

Concerning the general features of the method presented within this paper, many ways remain to explore for the future. An important point to study in depth should be the relation between Wald's axioms concerning any renormalized stress tensor [3,17,1] and the stress tensor arising from our approach. Moreover, the relation between our approach and the usual point-splitting approach based on short-distance Hadamard's behavior of the two-point functions should be investigated.

Other possible generalizations may concern integer or half-integer spinorial fields and gauge theories.

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## APPENDIX: MAIN FORMULAS

Let us consider an Euclidean $N$ manifold $\mathcal{M}$. Suppose that $\mathcal{M}$ is closed (namely, compact without boundary). Let $A$ be a second-order elliptic (positive-definite) self-adjoint differential operator working on smooth real scalar fields of $L^2(\mathcal{M},d\mu_g)$, $\mu_g$ being the usual Riemannian measure induced by the Euclidean metric. Let us finally suppose that the spectrum of the operator is discrete. This holds, for example, in the case of the Laplace-Beltrami operator with the sign changed, namely, the 0-forms Hodge–de Rham Laplacian; in such a case the multiplicity is also finite.

All that we attempt to describe should be more or less generalizable by relaxing some of the conditions above, employing opportune spectral measures, and so on. In particular one could consider the operator $A$ working on $n$ forms and deal with the Hodge–de Rham formalism also in manifolds noncompact or with boundary. Anyhow, this latter case could be more complicated to deal with. We leave to the mathematicians all these considerations.

Our goal is to determine how the generic eigenvalue $\lambda_n$ changes due to local changes of the metric $g_{ab}$ of the manifolds keeping fixed the topology. Let us introduce the Euclidean action

$$S_A[\phi,\phi]:=S_A[\phi]:=-\frac{1}{2}\int_{\mathcal{M}} d^N x\sqrt{g(x)}\,\phi(x)A\phi(x). \tag{A1}$$

Thus we have

$$\frac{\delta S_A}{\delta\phi(x)}=-\sqrt{g}A\phi(x). \tag{A2}$$

Letting $\lambda_n$ be the eigenvalue of the normalized eigenvector $\phi_n$, it holds that

$$A\phi_n=\lambda_n\phi_n\ \ ,\qquad \int_{\mathcal{M}} d^N x\sqrt{g(x)}\,\phi_n^*(x)\phi_n(x)=1, \tag{A3}$$

$$\lambda_n=-2S[\phi_n^*,\phi_n]. \tag{A4}$$

One may change the metric as $g_{ab}(x)\to g'_{ab}(x)=g_{ab}(x)+\delta g_{ab}(x)$. Obviously, provided that opportune mathematical conditions are satisfied, we expect to find a corresponding variation $\lambda_n\to\lambda'_n=\lambda_n+\delta\lambda_n$. We are interested in evaluating the rate of the variation of the eigenvalues with respect to the metric. In fact, we want to compute the functional derivative

$$\frac{\delta\lambda_n}{\delta g^{ab}(x)}=-2\frac{\delta S_A[\phi_n^*,\phi_n]}{\delta g^{ab}(x)}, \tag{A5}$$

where we employed Eq. (A4).

Starting from the identity just written above, we have

$$-\frac{\delta\lambda_n}{\delta g^{ab}(x)}=2\int d^N y\frac{\delta S_A}{\delta\phi_n^*(y)}\frac{\delta\phi_n^*(y)}{\delta g^{ab}(x)} +2\int d^N y\frac{\delta S_A}{\delta\phi_n(y)}\frac{\delta\phi_n(y)}{\delta g^{ab}(x)}+2\frac{\delta_g S_A}{\delta g^{ab}(x)}. \tag{A6}$$

Using the formula corresponding to Eq. (A2) for $\phi_n$ and $\phi_n^*$ (notice that a further factor 1/2 appears in this case), we obtain

$$-\frac{\delta\lambda_n}{\delta g^{ab}(x)}=-\lambda_n\int d^N y\sqrt{g(y)}\left(\phi_n\frac{\delta\phi_n^*(y)}{\delta g^{ab}(x)} +\phi_n^*\frac{\delta\phi_n(y)}{\delta g^{ab}(x)}\right)+2\frac{\delta_g S_A}{\delta g^{ab}(x)}. \tag{A7}$$

Let us look at the first term in the right-hand side of the equation above. We can rewrite it as

$$\begin{aligned}-\lambda_n\int d^N y\sqrt{g(y)}\frac{\delta}{\delta g^{ab}(x)}[\phi_n(y)\phi_n^*(y)]\\ =-\lambda_n\frac{\delta}{\delta g^{ab}(x)}\int d^N y\sqrt{g(y)}\,\phi_n(y)\phi_n^*(y)\\ +\lambda_n\int d^N y\frac{\delta\sqrt{g(y)}}{\delta g^{ab}(x)}\phi_n(y)\phi_n^*(y).\end{aligned} \tag{A8}$$

The first term in the last line vanishes due to the normalization condition in Eq. (A3) which is supposed to hold during the variational process. Eventually, a few of elementary calculations produces the well-known result

$$\frac{\delta\sqrt{g(y)}}{\delta g^{ab}(x)}=\frac{\partial\sqrt{g(x)}}{\partial g^{ab}(x)}\delta(x-y)=-\frac{1}{2}\sqrt{g(x)}g_{ab}(x)\delta(x-y).$$

Coming back to the variational derivative of $\lambda_n$ with respect to the metric and making use of the obtained results in Eq. (A6) we get our main equation (12):

$$\frac{\delta\lambda_n}{\delta g^{ab}(x)}=\frac{\lambda_n}{2}\sqrt{g(x)}g_{ab}(x)\phi_n(x)\phi_n^*(x)-2\frac{\delta_g S_A[\phi_n^*,\phi_n]}{\delta g^{ab}(x)}.$$

We finally remark that in [35] a similar relation has been found in a different context as far as eigenvalues of Dirac's operator is concerned.

Let us finally prove Eq. (52). We suppose that our closed manifold is *stationary*, namely, a global coordinate system exists in where the Euclidean metric looks similar to

$$ds^2 = g_{00}(\vec{x})dx^0dx^0 + 2g_{0i}(\vec{x})dx^0dx^i + g_{ij}(\vec{x})dx^idx^j, \tag{A9}$$

where $\vec{x} \equiv x^i \in \Sigma$. Notice that $\partial_0$ is a Killing vector. We suppose also that the manifold (the metric) is periodic in the coordinate $x^0$ with a period $\beta$.

Our action reads, in the considered coordinates,

$$S[\phi] := \int_0^\beta dx^0 \int_\Sigma d\vec{x}\sqrt{g(\vec{x})}\,\phi(x)A\phi(x).$$

Because it will be very useful shortly, we can consider the new coordinate set given by $y^0 := x^0/\beta$, $\vec{y} := \vec{x}$. In those coordinates, posing $\psi(y) := \phi(x)$ the action reads

$$S[\psi] := \int_0^1 dy^0 \int_\Sigma \sqrt{f(\vec{y})}\,\psi(y)B\psi(y), \tag{A10}$$

where $B$ is obviously defined with respect to the metric $f_{ab}(y)$ which reads $f_{00}(y) := g_{00}(x)/\beta^2$ and $f_{0i}(y) := g_{0i}(x)/\beta$, $f_{ij}(y) := g_{ij}(x)$. Now, we observe that, in Eq. (A10), variations of the parameter $\beta$ can be thought of as variations of the metric of the manifold, keeping fixed the topology.

As for the previous proof it is convenient starting with the usual identity

$$\lambda_n = -2S[\psi_n^*, \psi_n, f].$$

From that it follows

$$\frac{\partial\lambda_n}{\partial\beta} = -2\int d^4y\left\{\frac{\delta S}{\delta f^{ab}(y)}\frac{\partial f^{ab}}{\partial\beta} + \frac{\delta S}{\delta\psi_n^*(y)}\frac{\partial\psi_n^*}{\partial\beta} + \frac{\delta S}{\delta\psi_n(y)}\frac{\partial\psi_n}{\partial\beta}\right\}$$

$$= -2\int d^4y\sqrt{f(y)}\left\{\frac{-2}{\beta^3}g(y)^{-1/2}\frac{\delta S}{\delta f^{00}(y)}f^{00}(y)\beta^2 + \frac{-2}{\beta^2}g(y)^{-1/2}\frac{\delta S}{\delta f^{0i}(y)}f^{0i}(y)\beta\right\} - 2\int d^4y\sqrt{f(y)}$$

$$\times\left\{\frac{\partial\psi_n^*}{\partial\beta} - \frac{\lambda_n}{2}\psi_n(y)\frac{\partial\psi_n^*}{\partial\beta} - \frac{\lambda_n}{2}\psi_n^*(y)\frac{\partial\psi_n}{\partial\beta}\right\}$$

$$= -\frac{2}{\beta}\int d^4y\sqrt{f(y)}\,\overline{T_0^0}[\psi_n^*\psi_n](y) + \lambda_n\int d^4y\sqrt{f(y)}\frac{\partial\psi_n^*(y)\psi_n(y)}{\partial\beta}. \tag{A11}$$

Above, $\overline{T_{ab}}(y)$ is the stress tensor evaluated in the coordinate $y^a$.

Let us consider the second term in Eq. (A11). We can also write that as

$$\lambda_n\int d^4y\frac{\partial\sqrt{f(y)}\psi_n^*(y)\psi_n(y)}{\partial\beta} - \lambda_n\int d^4y\frac{\partial\sqrt{f(y)}}{\partial\beta}\psi_n^*(y)\psi_n(y)$$

$$= \lambda_n\frac{\partial}{\partial\beta}\int d^4y\sqrt{f(y)}\psi_n^*(y)\psi_n(y) - \frac{\lambda_n}{\beta}\int d^4y\sqrt{f(y)}\psi_n^*(y)\psi_n(y).$$

The first term in the right-hand side of the equation above vanishes due to the invariant normalization condition of the modes. The second term, as well as the remaining term in Eq. (A11), can be translated into the initial coordinates obtaining

$$\frac{\partial\lambda_n}{\partial\beta} = -\frac{2}{\beta}\int d^4x\sqrt{g(\vec{x})}T_0^0[\phi_n^*\phi_n](\vec{x}) + \frac{\lambda_n}{\beta}\int d^4x\sqrt{g(\vec{x})}\phi_n^*(x)\phi_n(x).$$

Notice that, as we said above, both the integrands do not depend on $x^0$ because the metric is stationary, and thus the integration on the temporal variable produces only a factor $\beta$. The final formula is then Eq. (52):

$$\frac{\partial\lambda_n}{\partial\beta} = -2\int_\Sigma d\vec{x}\sqrt{g(\vec{x})}\left\{T_0^0[\phi_n^*\phi_n](\vec{x}) + \frac{1}{2}g_0^0\lambda_n\phi_n^*(\vec{x})\phi_n(\vec{x})\right\}.$$


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