Trace anomaly of dilaton-coupled scalars in two dimensions

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Conformal scalar fields coupled to the dilaton appear naturally in two-dimensional models of black hole evaporation. We show that their trace anomaly is $(1/24\pi)[R-6(\nabla\phi)^2-2\Box\phi]$. It follows that a Russo-Susskind-Thorlacius−type counterterm appears naturally in the one-loop effective action.
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I. INTRODUCTION

In the study of black hole radiation, many useful results have been obtained from two-dimensional (2D) models. It is hoped that the results will extend, at least partly, to the behavior of realistic black holes in four or more dimensions. To make this claim plausible, the 2D actions were usually obtained by a dimensional reduction from a higher-dimensional theory. In the seminal papers of Callan, Giddings, Harvey, and Strominger (CGHS) [1], and of Russo, Susskind, and Thorlacius (RST) [2], the classical action is

$$S=\frac{1}{16\pi G}\int d^2x\sqrt{-g}e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2].$$

CGHS introduced this theory primarily as a toy model, but with the physical motivation that it could be obtained by dimensional reduction of dilaton black holes [3,4]. In order to study black hole radiation, N matter fields are added and a large N limit is taken, in which the quantum fluctuations of ϕ and the 2D metric are neglected. In the spirit of a toy model, CGHS chose the simplest possible matter fields, scalar fields that are minimally coupled in 2D, with the physical justification that this could be obtained by dimensional reduction of Ramond-Ramond fields. Such minimally coupled scalars have the well-known $R/24\pi$ trace anomaly and this determines the effective action up to boundary condition terms.

In other 2D models, however, couplings of the scalars to the dilaton arise naturally. For example, the dimensional reduction of spherically symmetric general relativity by the ansatz

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu+e^{-2\phi}d\Omega^2$$

gives the 2D action

$$S=\frac{1}{16\pi G}\int d^2x\sqrt{-g}e^{-2\phi}[R+2(\nabla\phi)^2]+2-2Q^2e^{2\phi}.$$

Trivedi and Strominger [5,6] used this action with 2D minimally coupled scalars. They had to claim they were bosonizations of 4D fermions. It would be more natural, however, to use a minimally coupled 4D scalar field, f. On dimensional reduction with the above ansatz, its kinetic term acquires an exponential coupling to the dilaton:

$$S_m=-\frac{1}{2}\int d^2x\sqrt{-g}e^{-2\phi}(\nabla f)^2. \tag{1.1}$$

Dilaton coupling can also arise in other ways. Therefore it is of interest to calculate the trace anomaly and effective action for dilaton coupled scalars in two dimensions.

II. METHODS

From the eigenvalues λ_n of the operator A, one defines a generalized zeta function [7,8]

$$\zeta(s)=\mathrm{tr}A^{-s}=\sum_n \lambda_n^{-s}.$$

This sum converges for a sufficiently large real part of s. By analytic extension, it defines a meromorphic function of s, which is regular even in regions where the sum diverges. The one-loop effective action W is given by

$$W=-\frac{1}{2}[\zeta'(0)+\zeta(0)\ln\mu^2], \tag{2.1}$$

where $\zeta'=d\zeta/ds$. Under a rescaling of the operator,

$$A[k]=k^{-1}A, \tag{2.2}$$

the one-loop effective action transforms as

$$W[k]=W-\frac{1}{2}\zeta(0)\ln k. \tag{2.3}$$

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We denote the trace anomaly by T. Let us summarize some of its general properties in D-dimensional spacetimes [7]. If D is odd, the trace anomaly is zero. If D is even, it consists of terms T_i that are generally covariant and homogeneous of order D in derivatives: $T=\Sigma q_i T_i$. The dimensionless numbers q_i are universal, i.e., independent of the background metric. This property will allow us to choose convenient backgrounds to determine their values.

The integral of the trace anomaly over the manifold is given by

$$\int d^D x\sqrt{g}T=-2\frac{dW}{dk}\bigg|_{k=1}, \tag{2.4}$$

where k is defined as a scale factor of the metric, $\hat{g}^{\mu\nu}=k^{-1}g^{\mu\nu}$. But under this scale transformation, the eigenvalues of A transform as in Eq. (2.2). Therefore, Eq. (2.3) can be used, which yields the elegant result

$$\int d^D x\sqrt{g}T=\zeta(0). \tag{2.5}$$

Given an operator A, one could, in principle, calculate the one-loop effective action directly from Eq. (2.1). In practice, it is often simpler to calculate the trace anomaly from Eq. (2.5), because the zeta function is usually easier to obtain than its derivative. By requiring that Eq. (2.4) hold, the effective action can be inferred up to terms that do not depend on the scale factor. (We shall use W^* to denote a quantity which differs from the effective action only by such terms.) Also, if the effective action is known for the operator kA, one can use Eq. (2.3) to obtain W for the operator A.

III. DILATON COUPLED SCALAR

Variation of the action, Eq. (1.1), with respect to f, yields the equation of motion $Af=0$, with the field operator

$$A=e^{-2\phi}(-\Box+2\nabla^\mu\phi\nabla_\mu). \tag{3.1}$$

The trace anomaly consists of covariant terms with two metric derivatives. For the operator at hand, there are only three such expressions: R, $(\nabla\phi)^2$, and $\Box\phi$. In principle, these terms could still be multiplied by arbitrary functions of ϕ. But consider shifting ϕ by a constant value $\Delta\phi$. This corresponds merely to multiplying the kinetic term in the action by a factor $e^{-2\Delta\phi}$; the trace anomaly will remain the same. Therefore a functional dependence of any of its terms on ϕ can be excluded. Consequently, we can write

$$T=q_1 R+q_2(\nabla\phi)^2+q_3\Box\phi. \tag{3.2}$$

By writing the metric in conformal gauge,

$$ds^2=e^{2\rho(t,x)}(dt^2+dx^2),$$

it is easy to check that this anomaly derives from the effective action

$$W^*=\frac{1}{2}\int d^2x\sqrt{g}\left[\frac{q_1}{2}R\frac{1}{\Box}R+q_2(\nabla\phi)^2\frac{1}{\Box}R+q_3\phi R\right]. \tag{3.3}$$

This follows from Eq. (2.4), since $R=-2\Box\rho$. [A more straightforward result for the last term would be $\Box\phi(1/\Box)R$. It is related to the term we use by two integrations by parts; the difference can at most be a boundary term. It will become clear below why we choose the form ϕR.] We must only determine the universal numbers q_1, q_2, and q_3 to obtain the trace anomaly completely.

First consider the case when ϕ is identically zero. Then Eq. (3.2) simplifies to $T_{\phi\equiv 0}=q_1 R$. But if $\phi\equiv 0$, the operator A in Eq. (3.1) becomes $A_{\phi\equiv 0}=-\Box$. This is the operator for the minimally coupled scalar, for which the trace anomaly is well known [9]: $T_{\min}=R/24\pi$. Therefore, one finds that

$$q_1=\frac{1}{24\pi}.$$

Now consider the case where ϕ is constant, $\phi\equiv\phi_{\rm c}$. Then the one-loop effective action, Eq. (3.3), simplifies to

$$W^*_{\phi\equiv\phi_{\rm c}}=W^*_{\min}+\frac{1}{2}\int d^2x\sqrt{g}q_3\phi_{\rm c}R. \tag{3.4}$$

To make sure that the integral over the Ricci scalar does not vanish, we can specify that a background with the topology of a two-sphere be used. For constant ϕ, the operator A becomes $A_{\phi\equiv\phi_{\rm c}}=-e^{-2\phi_{\rm c}}\Box$. But this is just the minimally coupled operator, rescaled by a constant factor $k^{-1}=e^{-2\phi_{\rm c}}$. Therefore, Eqs. (2.3) and (2.5) yield

$$W^*_{\phi\equiv\phi_{\rm c}}=W^*_{\min}-\phi_{\rm c}\zeta_{\min}(0)=W^*_{\min}-\phi_{\rm c}q_1\int d^2x\sqrt{g}R. \tag{3.5}$$

Comparison with Eq. (3.4) shows that

$$q_3=-2q_1=-\frac{1}{12\pi}.$$

The same consideration also vindicates the choice of ϕR for the last term in the effective action, Eq. (3.3): If $\Box\phi(1/\Box)R$ was used, the last term in Eq. (3.4) would be zero, since ϕ is constant. It would then be impossible to match Eq. (3.4) to Eq. (3.5), in which the last term is nonzero on a two-sphere background.[1]

In conformal gauge the field operator will take the form

$$A=e^{-2\phi-2\rho}\left[-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+2\left(\frac{\partial\phi}{\partial t}\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\right)\right].$$

Consider a Euclidean background manifold of toroidal topology, in which t and x are periodically identified, with period 2π. The integral over the Ricci scalar is a topological invariant and vanishes on a torus. Since $\Box\phi$ is a total divergence, its integral vanishes as well. Thus,

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[1]Nojiri and Odintsov [10] suggest a more general form for the effective action, in which the last term is given by $q_3[a\phi R+(1-a)\Box\phi(1/\Box)R]$. This would give a different value of q_3.

$$\zeta(0)=\int d^2x\sqrt{g}T=q_2\int d^2x\sqrt{g}(\nabla\phi)^2. \tag{3.6}$$

Therefore we can determine q_2 by calculating $\zeta(0)$ from the operator eigenvalues in a conveniently chosen toroidal background, and dividing the result by $\int d^2x\sqrt{g}(\nabla\phi)^2$.

A useful choice of background is the field configuration $\phi=-\rho=\epsilon\sin t$, where $\epsilon\ll 1$. The operator takes the form

$$A=-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+2\epsilon\cos t\frac{\partial}{\partial t}.$$

For $\epsilon=0$, this operator is just the flat space Laplacian, for which $\zeta(0)$ is known to vanish. The integral on the right-hand side of Eq. (3.6) yields $2\pi^2\epsilon^2$. Thus we can proceed as follows. The eigenvalues of A will be found perturbatively in ϵ. This will allow us to expand $\zeta(s)$ to second order in ϵ:

$$\zeta(s)=\zeta^{(0)}(s)+\epsilon\zeta^{(1)}(s)+\epsilon^2\zeta^{(2)}(s).$$

Since $\zeta^{(0)}(0)=0$, we have

$$\epsilon\zeta^{(1)}(0)+\epsilon^2\zeta^{(2)}(0)=2\pi^2q_2\epsilon^2.$$

Consistency requires that $\zeta^{(1)}(0)=0$; we will indeed find that to be the case. Therefore,

$$q_2=\frac{1}{2\pi^2}\zeta^{(2)}(0). \tag{3.7}$$

For $\epsilon=0$, the eigenvalues of the operator A are $\Lambda_{kl}^{(0)}=k^2+l^2$, with degeneracies

$$d(k,l)=\begin{cases}4 & \text{if } k\geqslant 1,\ l\geqslant 1,\\ 2 & \text{if } k\geqslant 1,\ l=0 \text{ or } k=0,\ l\geqslant 1,\\ 1 & \text{if } k=l=0.\end{cases}$$

Clearly, the zeta function

$$\zeta(s)=\sum_{k,l=0}^{\infty}d(k,l)(\Lambda_{kl}^{(0)})^{-s}$$

contains an ill-defined term: $k=l=0$. This problem can be dealt with by introducing a mass term into the operator A: $A\to A+M^2$. Then $\zeta(0)$ can be defined in the limit as $M\to 0$.

Now take $\epsilon\neq 0$, and consider the eigenvalue equation $Af=\Lambda f$. With $f(t,x)=T(t)X(x)$ the equation separates into

$$-X''=\sigma X,\qquad -\ddot{T}+2\epsilon\cos t\dot{T}=\lambda T.$$

Standard perturbation theory yields that, to second order in ϵ, the eigenvalues of A are

$$\Lambda_{kl}=k^2+l^2+M^2+\epsilon^2\frac{2l^2}{4l^2-1},$$

with the same degeneracies $d(k,l)$ as in the unperturbed case. The zeta function is given by

$$\begin{aligned}\zeta(s)&=\sum_{k,l=0}^{\infty}d(k,l)(\Lambda_{kl}^{(0)})^{-s}\left(1+\epsilon^2\frac{\lambda_l^{(2)}}{\Lambda_{kl}^{(0)}}\right)^{-s}\\&=\sum_{k,l=0}^{\infty}d(k,l)(\Lambda_{kl}^{(0)})^{-s}\left(1-\epsilon^2 s\frac{\lambda_l^{(2)}}{\Lambda_{kl}^{(0)}}\right)\\&=\zeta^{(0)}(s)-\epsilon^2 s\sum_{k=0,l=1}^{\infty}d(k,l)\frac{\lambda_l^{(2)}}{(\Lambda_{kl}^{(0)})^{1+s}},\end{aligned}$$

where a Taylor expansion to second order in ϵ was used. The sum in the last line does not include $l=0$ because $\lambda_0^{(2)}=0$. Since this excludes $k=l=0$, it is safe to drop M at this point. Thus we have

$$\zeta^{(2)}(0)=-\lim_{s\to 0}sU(s), \tag{3.8}$$

where we view the double sum as a meromorphic function of s:

$$U(s)=\sum_{k=0,l=1}^{\infty}d(k,l)\frac{2l^2}{(k^2+l^2)^{1+s}(4l^2-1)}.$$

We need to find only the principal part of the Laurent series of U around $s=0$, $\mathrm{Pr}[U(s);0]$, because the regular part will be annulled by the factor of s in Eq. (3.8). But $\mathrm{Pr}[U(s);0]=\mathrm{Pr}[U(s)+V(s);0]$ for any function $V(s)$ which is regular at $s=0$. Thus, by adding suitable finite terms to the double sum, we can bring it into a form which can be evaluated.

First, we note that the contribution from $k=0$ is finite at $s=0$:

$$2\sum_{l=1}^{\infty}2(4l^2-1)^{-1}=2.$$

After its subtraction, all summations start at 1:

$$\mathrm{Pr}[U(s);0]=2\,\mathrm{Pr}\left[\sum_{k,l=1}^{\infty}\frac{4l^2}{(k^2+l^2)^{1+s}(4l^2-1)};0\right],$$

where we have used $d(k,l)=4$. Next, we subtract 1 in the numerator; this is possible since

$$\sum(k^2+l^2)^{-1-s}(4l^2-1)^{-1}$$

is finite at $s=0$ (the upper bound

$$-1+\frac{\pi^2}{12}+\left(\ln 2-\frac{1}{2}\right)\pi\coth\pi\approx 0.43$$

is easily found). This cancels the $(4l^2-1)$ factor. The remaining double sum evaluates to

$$\sum_{k,l=1}^{\infty}\frac{1}{(k^2+l^2)^{1+s}}=\frac{1}{4}Z_2(2+2s)-\zeta_{\mathrm{R}}(2+2s),$$

where

$$Z_2(p)=\sum_{k,l=-\infty}^{\infty}{}' \,(k^2+l^2)^{-p/2}$$

is a generalized zeta function of Epstein type; the prime denotes the omission of the $k=l=0$ term in the sum. Epstein showed in Ref. [11] that $Z_2(p)$ is analytic except for a simple pole at $p=2$, with residue 2π. Since the Riemann zeta function $\zeta_R(2+2s)$ is finite for $s=0$, we find

$$2\,\mathrm{Pr}[U(s);0]=\frac{1}{2}\,\mathrm{Pr}[Z_2(2+2s);0]=\frac{1}{2}\left(\frac{2\pi}{2s}\right)=\frac{\pi}{2s}.$$

Therefore, by Eq. (3.8), we find that $\zeta^{(2)}(0)=-\pi/2$, and, by Eq. (3.7), we obtain the result

$$q_2=-\frac{1}{4\pi}.$$

We have thus shown that a 2D conformal scalar field with exponential dilaton coupling has the trace anomaly

$$T=\frac{1}{24\pi}[R-6(\nabla\phi)^2-2\Box\phi].$$

The scale factor-dependent part of the one-loop effective action is

$$W^*=+\frac{1}{48\pi}\int d^2x\sqrt{g}\left[\frac{1}{2}R\frac{1}{\Box}R-6(\nabla\phi)^2\frac{1}{\Box}R-2\phi R\right].$$

It is interesting to note that the last term was inserted by hand in the RST model, albeit with a different coefficient. By Eq. (2.1), the effective action will also contain a term $-(1/2)\zeta(0)\ln\mu^2$. The R and $\Box\phi$ terms in the trace anomaly give only a topological contribution to $\zeta(0)$, which does not affect the equations of motion. The term $(1/8\pi)\ln\mu^2\int d^2x\sqrt{g}(\nabla\phi)^2$, however, must be taken into account.

With our result, it will be possible to study black hole radiation in a large class of 2D models, including those that derive from 4D general relativity.

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