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Geodesics, gravitons, and the gauge-fixing problem

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When graviton loops are taken into account, the background metric obtained as a solution to the one-loop corrected Einstein equations turns out to be gauge-fixing dependent. Therefore it is of no physical relevance. Instead we consider a physical observable, namely, the trajectory of a test particle in the presence of gravitons. We derive a quantum corrected geodesic equation that includes backreaction effects and is explicitly independent of any gauge-fixing parameter. [S0556-2821(97)02024-9]

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I. INTRODUCTION

In quantum field theory there are many physical situations where one is interested in the dynamical evolution of fields rather than in S-matrix elements. The effective action (EA) is a useful tool to obtain the equations that govern such dynamics including the backreaction effects due to quantum fluctuations. In the context of gravity, the equations that give the dynamics of the spacetime metric including quantum effects are the so-called semiclassical Einstein equations (SEE) [1]. These have been widely used to analyze different physical situations such as gravitational collapse and black hole evaporation.

Since DeWitt's pioneering work [2], it is known that, at the one-loop level, the quantization of the fluctuations of the gravitational field around a given background is equally as important as the quantization of the matter fields. Therefore, the graviton field contributes to the SEE along with all the other matter fields. In order to avoid the technical complications that take place when gravitons are quantized, their contribution to the SEE is usually neglected. It is a common belief that, once the technical details are solved, one can compute their contribution to the energy momentum tensor and write the full one loop SEE. The solution to these equations would be the quantum corrected metric of spacetime.

In the present paper we will argue that, when gravitons are taken into account, the solution to the SEE is not physical. The reason is simple: any classical device used to measure the spacetime geometry will also feel the graviton fluctuations. As the coupling between the classical device and the metric is nonlinear, the device will not measure the "background geometry" (i.e., the geometry that solves the SEE). As a particular example we will show that a classical particle does not follow a geodesic of the background metric. Instead its motion is determined by a quantum corrected geodesic equation that takes into account its coupling to the gravitons.

This analysis also leads us to find a solution to the so-called gauge-fixing problem. A "technical" obstacle to think of a solution to the SEE as the metric of spacetime is that in general it depends on the gauge fixing of the gravitons. As an example we can mention calculations of compactification radii in Kaluza-Klein theories [3]. The standard approach to tackle this problem is to consider the Vilkovisky-DeWitt effective action [4], which is specifically built to give a reparametrization, gauge-fixing independent action. However, this action suffers from another type of arbitrariness, namely the dependence on the supermetric in the space of fields that is introduced in its definition [5–7]. The aforementioned obstacle is not "technical" but physical: since the classical device couples to gravitons, the solution to the SEE will not, in general, have a clear physical interpretation. We will demonstrate explicitly that while the solution to the SEE is gauge-fixing dependent, the quantum corrected geodesic equation (that takes into account such coupling) does not depend on the gauge fixing. In summary, the solution of the backreaction problem consists of two steps: to solve the semiclassical Einstein equations and to extract the physical quantities from the solution.

In order to illustrate these facts we will consider the calculation of the leading quantum corrections to the Newtonian potential. As has been pointed out in [8–10], when general relativity is looked upon as an effective field theory, low-energy quantum effects can be studied without the knowledge of the (unknown) high-energy physics. The leading long distance quantum corrections to the gravitational interactions are due to massless particles and only involve their coupling at energies low compared to the Planck mass. Using this idea, many authors have calculated the leading quantum corrections to the Newtonian potential computing different sets of Feynman diagrams [8,9,11,12]. Instead of evaluating diagrams and S-matrix elements, we are here concerned with a covariant calculation based on EA and effective field equations. This covariant approach is more adequate to study problems in which one considers fluctuations around nonflat backgrounds. We shall first compute the SEE for the backreaction problem starting from the standard EA and show how they depend on the gauge fixing. Using a

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corrected geodesic equation we will deduce a physical quantum corrected Newtonian potential, which does not depend on the gauge-fixing parameters.

II. THE ONE LOOP EFFECTIVE ACTION FOR GRAVITY+MASS: DIVERGENCES

The Einstein-Hilbert action for pure gravity is[1]

$$S_G=\frac{2}{\kappa^2}\int d^4x\sqrt{-\bar{g}}\bar{R}, \tag{1}$$

where $\bar{R}$ is the curvature scalar, $\bar{g}_{\mu\nu}$ is the metric tensor, $\bar{g}=\det\bar{g}_{\mu\nu}$, and $\kappa^2=32\pi G$, with G being Newton's constant. In the background field method we consider fluctuations of the gravitational field around a background metric, $\bar{g}_{\mu\nu}=g_{\mu\nu}+\kappa s_{\mu\nu}$. Expanding the action up to quadratic order in the graviton fluctuations $s_{\mu\nu}$, the gravitational action reads

$$\begin{aligned}S_G=\int d^4x\sqrt{-g}\Bigg[&\frac{2}{\kappa^2}R+\frac{1}{\kappa}s_{\mu\nu}(g^{\mu\nu}R-2R^{\mu\nu})\\&+\Bigg\{\frac{1}{2}\nabla_\alpha s_{\mu\nu}\nabla^\alpha s^{\mu\nu}-\frac{1}{2}\nabla_\alpha s\nabla^\alpha s+\nabla_\alpha s\nabla_\beta s^{\alpha\beta}\\&-\nabla_\alpha s_{\mu\beta}\nabla^\beta s^{\mu\alpha}+R\left(\frac{1}{4}s^2-\frac{1}{2}s_{\mu\nu}s^{\mu\nu}\right)\\&+R^{\mu\nu}(2s^\lambda_\mu s_{\nu\lambda}-ss_{\mu\nu})\Bigg\}+\cdots\Bigg],\end{aligned} \tag{2}$$

where $s=g^{\mu\nu}s_{\mu\nu}$, and the ellipsis denotes higher order terms in the fluctuations. In order to fix the gauge one chooses a gauge-fixing function $\chi^\mu[g,s]$, and a gauge-fixing action

$$S_{\rm gf}[g,s]=-\frac{1}{2}\int d^4x\sqrt{-g}\chi^\mu g_{\mu\nu}\chi^\nu. \tag{3}$$

The one loop effective action for the background metric is obtained from integrating out quantum fluctuations and implies the evaluation of functional determinants for gravitons and ghosts in the presence of the background fields. It reads

$$\begin{aligned}S_{\rm eff}=S_G&+\frac{i}{2}{\rm Tr}\ \ln\left[\frac{\delta^2 S_G[g]}{\delta g^{\alpha\beta}g^{\gamma\delta}}-\frac{\delta\chi^\mu}{\delta g^{\alpha\beta}}g_{\mu\nu}\frac{\delta\chi^\nu}{\delta g^{\gamma\delta}}\right]\\&-i{\rm Tr}\ \ln\left[-2g_{\sigma\alpha}\nabla_\beta\frac{\delta\chi^\mu}{\delta g^{\alpha\beta}}\right].\end{aligned} \tag{4}$$

The first term is the classical action, the second one stems from graviton fluctuations and the last one is the ghosts contribution. These last two terms are quantum corrections linear in $\hbar$.

[1]Our metric has signature $(-+++)$ and the curvature tensor is defined as $\bar{R}^{\mu}_{\cdot\nu\alpha\beta}=\partial_\alpha\Gamma^\mu_{\nu\beta}-\cdots$, $\bar{R}_{\alpha\beta}=\bar{R}^{\mu}_{\cdot\alpha\mu\beta}$ and $\bar{R}=\bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta}$. We use units $\hbar=c=1$.

To proceed further one has to choose a particular gauge-fixing function. The simplest choices of gauge are those called ''minimal'' gauges, which lead to the evaluation of functional traces for gravitons and ghosts of second-order differential operators of the form $F_{AB}(\nabla)=\hat{C}_{AB}\,g^{\mu\nu}\nabla_\mu\nabla_\nu+\hat{Q}_{AB}$, where $\hat{C}_{AB}$ is an invertible matrix and $\hat{Q}_{AB}$ is an arbitrary matrix. For these cases the one loop EA can be expanded in powers of the background dimensionality using the well-known Schwinger-DeWitt expansion, which is local in the background fields (see Appendix A). For the other ''nonminimal'' gauges, in [13] it has been developed a reduction method that generalizes the former technique (see Appendix B).

In the following we shall mainly consider the so-called λ family, which is a one parameter family of gauge-fixing functions

$$\chi^\mu(\lambda)=\frac{1}{\sqrt{1+\lambda}}\left[g^{\mu\gamma}\nabla^\sigma s_{\gamma\sigma}-\frac{1}{2}g^{\gamma\sigma}\nabla^\mu s_{\gamma\sigma}\right]. \tag{5}$$

For gauge-fixing functions linear in the metric fluctuations, ghosts decouple from the fluctuations $s_{\mu\nu}$ and only couple to the background fields. The one loop EA takes the form

$$S_{\rm eff}=S_G+\frac{i}{2}{\rm Tr}\ \ln F^{\alpha\beta,\mu\nu}(\nabla)-i{\rm Tr}\ \ln(\Box\delta^\mu_\nu+R^\mu_\nu), \tag{6}$$

where the second term involves graviton diagrams and the third one involves ghost diagrams. The second-order differential operator is

$$\begin{aligned}F^{\alpha\beta,\mu\nu}(\nabla)=\sqrt{-g}C^{\alpha\beta,\lambda\sigma}\Bigg\{&\Box\delta^{\mu}_{(\lambda}\delta^{\nu}_{\sigma)}-\frac{2\lambda}{1+\lambda}\delta^{(\mu}_{(\lambda}\nabla_{\sigma)}\nabla^{\nu)}\\&+\frac{\lambda}{1+\lambda}g^{\mu\nu}\nabla_{(\lambda}\nabla_{\sigma)}+P^{\mu\nu}_{\lambda\sigma}\Bigg\},\end{aligned} \tag{7}$$

where

$$C^{\alpha\beta,\lambda\sigma}=\frac{1}{4}(g^{\lambda\alpha}g^{\sigma\beta}+g^{\lambda\beta}g^{\sigma\alpha}-g^{\lambda\sigma}g^{\alpha\beta}),$$

$$\begin{aligned}P^{\mu\nu}_{\lambda\sigma}=&2R_{\lambda\ \cdot\ \sigma\ \cdot}^{\ (\mu\ \ \nu)}+2\delta^{(\mu}_{(\lambda}R^{\nu)}_{\sigma)}-g^{\mu\nu}R_{\lambda\sigma}-g_{\lambda\sigma}R^{\mu\nu}-R\delta^{\mu}_{(\lambda}\delta^{\nu}_{\sigma)}\\&+\frac{1}{2}g^{\mu\nu}g_{\lambda\sigma}R.\end{aligned} \tag{8}$$

Here the parentheses denote symmetrization with a 1/2 factor. We see that F does not have the form of a minimal operator due to the presence of the second and third terms. For the special case $\lambda=0$, which is known as DeWitt gauge, we have the simplest case of a minimal operator.

Next we couple gravity to a heavy particle (a classical source) of mass M, which adds a new term to the action

$$S_M=-M\int\sqrt{-\bar{g}_{\mu\nu}dx^\mu dx^\nu}. \tag{9}$$

This coupling introduces an additional contribution to the EA. Expanding the action for the particle up to quadratic order in gravitons, we have

$$S_M = -M\int d\tau\left[1 - \frac{\kappa}{2}s_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{\kappa^2}{8}s_{\mu\nu}s_{\rho\sigma}\dot{x}^\mu\dot{x}^\nu\dot{x}^\rho\dot{x}^\sigma + \cdots\right], \tag{10}$$

where the overdots represent derivatives with respect to the proper time τ, defined as $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$, and the ellipsis are higher order terms in the gravitons fluctuations. Introducing an identity as $1 = \int d^4y\sqrt{-g}\,\delta^4[y - x(\tau)]$, the action can be rewritten in the following way:

$$S_M = -M\int d\tau + \frac{\kappa}{2}\int d^4y\sqrt{-g}s_{\mu\nu}(y)T^{\mu\nu}(y) + \int d^4y\sqrt{-g}s_{\mu\nu}(y)s_{\rho\sigma}(y)\widetilde{M}^{\mu\nu\rho\sigma}(y) + \cdots, \tag{11}$$

where

$$T^{\mu\nu}(y) = M\int d\tau\dot{x}^\mu\dot{x}^\nu\delta^4(y - x(\tau)), \tag{12}$$

and

$$\widetilde{M}^{\mu\nu\rho\sigma}(y) = \frac{M\kappa^2}{8}\int d\tau\delta^4(y - x(\tau))\dot{x}^\mu\dot{x}^\nu\dot{x}^\rho\dot{x}^\sigma. \tag{13}$$

The quadratic terms in Eq. (11) introduce a new contribution to the differential operator $F(\nabla)$, which finally takes the form

$$F^{\alpha\beta,\mu\nu}(\nabla) = \sqrt{-g}C^{\alpha\beta,\lambda\sigma}\left\{\Box\,\delta^\mu_{(\lambda}\delta^\nu_{\sigma)} - \frac{2\lambda}{1+\lambda}\delta^{(\mu}_{(\lambda}\nabla_{\sigma)}\nabla^{\nu)} + \frac{\lambda}{1+\lambda}g^{\mu\nu}\nabla_{(\lambda}\nabla_{\sigma)} + P^{\mu\nu}_{\lambda\sigma} + M^{\mu\nu}_{\lambda\sigma}\right\}, \tag{14}$$

with

$$M^{\mu\nu}_{\lambda\sigma}(y) = (C^{-1})^{\mu\nu\alpha\beta}\widetilde{M}_{\alpha\beta\lambda\sigma}(y) = \frac{M\kappa^2}{8}\int d\tau\delta^4(y - x(\tau)) \times [g^{\mu\nu}\dot{x}_\lambda\dot{x}_\sigma + 2\dot{x}^\mu\dot{x}^\nu\dot{x}_\lambda\dot{x}_\sigma]. \tag{15}$$

As is well-known, the EA has divergences. For example, for the pure gravitational part, the one loop divergences in the DeWitt $(\lambda = 0)$ gauge have been calculated long ago using dimensional regularization and turn out to be local terms quadratic in the curvature tensors [14]. They read[2]

$$\Delta S_G^{\rm div}(\lambda = 0) = \frac{2}{(4-d)96\pi^2} \times \int d^4x\sqrt{-g}\left[\frac{53}{15}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2) + \frac{21}{10}R_{\mu\nu}R^{\mu\nu} + \frac{1}{20}R^2\right], \tag{16}$$

where the first term in parentheses is the Gauss-Bonnet term, a topological invariant in $d = 4$ spacetime dimensions. In Appendix A we show how to evaluate the divergence stemming from the massive part for the minimal gauge $\lambda = 0$. It reads

$$\Delta S_M^{\rm div}(\lambda = 0) = \frac{2}{(4-d)64\pi^2}\int d^4x\sqrt{-g}\left[M_{\mu\nu\rho\sigma}M^{\mu\nu\rho\sigma} + 2M_{\mu\nu\rho\sigma}\left(P^{\rho\sigma\mu\nu} + \frac{1}{6}R\,\delta^{\rho(\mu}\delta^{\sigma\nu)}\right)\right]. \tag{17}$$

Now we have to calculate the EA for any member of the λ family gauge-fixing functions other than the $\lambda = 0$ one. The calculation is cumbersome and we leave it for Appendix B. Here we just state the main result that shall concern us (see below), namely, the divergence of the one loop EA that is linear in the extremal $\mathcal{E}^{\mu\nu} = -(2/\kappa^2)(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) + \frac{1}{2}T^{\mu\nu}$,

$$\Delta S^{\rm div}(\lambda) = \Delta S^{\rm div}(\lambda = 0) - \frac{\lambda}{4-d}\frac{\kappa^2}{24\pi^2} \times \int d^4x\sqrt{-g}\left[-5R_{\mu\nu}\mathcal{E}^{\mu\nu} + \frac{5}{2}Rg_{\mu\nu}\mathcal{E}^{\mu\nu}\right], \tag{18}$$

where $\Delta S^{\rm div}(\lambda = 0) = \Delta S_G^{\rm div}(\lambda = 0) + \Delta S_M^{\rm div}(\lambda = 0)$ is the divergence for the DeWitt gauge, that was already calculated. Note that the (ultraviolet) divergences of the EA take the form of local tensors expressed in terms of curvatures and the energy-momentum tensor for the source particle.

III. LONG DISTANCE LEADING QUANTUM CORRECTIONS: THE LOG TERMS

The theory we are considering is not renormalizable, since the divergences cannot be absorbed into the parameters introduced thus far. Additional divergent counterterms (and some accompanying finite parts) quadratic in the curvature tensors must be added to the classical action $S_G + S_M$. However, the nonrenormalizability of the theory is not an impediment for making well defined quantum predictions at low energies/large distances. As we have already remarked, the idea is to treat gravity as an effective field theory, and perform a systematic expansion in the energy. In this approach, the unknown parameters introduced with the various counterterms have to be determined by comparison with experiment, which then allows to make predictions to a given order in an energy expansion. However, the low-energy physics is not contained in these parameters, but rather in a different class of quantum corrections. The leading long distance cor-

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[2] To be precise, the EA contains two additional divergences, one proportional to $\sqrt{-g}$ and another proportional to $\sqrt{-g}R$. As these can be absorbed into a redefinition of the cosmological constant and the Newton constant, we shall not consider them in what follows.

rections stem from the nonlocal, nonanalytic terms in the one loop effective action. These nonlocal terms have been computed in [15,16] expanding the EA in powers of the curvatures, using a resummation procedure of the Schwinger-DeWitt expansion for the action. Keeping up to quadratic order in the curvature tensors, the general form of such terms is $\mathcal{R}G(\Box)\mathcal{R}$, where $\mathcal{R}$ denotes any of the tensors $R, R_{\mu\nu}, M_{\mu\nu\rho\sigma}$, and $G(\Box)$ is a nonlocal form factor. For the theory we are considering, $G(\Box)$ is proportional to $\ln(-\Box)$, and these logarithmic terms are the relevant ones in the low-energy limit. The proportionality constants accompanying the $\ln(-\Box)$ can be read off from the (local) divergences in Eqs. (16), (17), (18) in a manner outlined in [9,15]. One extracts the coefficient of the logarithmic correction from the divergence in the following way:

$$\frac{\alpha}{4-d}\int d^4x\sqrt{-g}(\cdots)\rightarrow-\frac{\alpha}{2}\int d^4x\sqrt{-g}(\cdots)\ln(-\Box). \tag{19}$$

Using this result, the nonlocal part of the EA proportional to the logarithm takes the form $\Delta S=\Delta S_G^{\rm nl}(\lambda=0)+\Delta S_M^{\rm nl}(\lambda=0)+\Delta S^{\rm nl}(\lambda\neq 0)$, with

$$\Delta S_G^{\rm nl}(\lambda=0)=-\frac{1}{96\pi^2}\int d^4x\sqrt{-g}\left[\frac{21}{10}R_{\mu\nu}\ln(-\Box)R^{\mu\nu}+\frac{1}{20}R\ \ln(-\Box)R\right], \tag{20}$$

$$\Delta S_M^{\rm nl}(\lambda=0)=-\frac{1}{64\pi^2}\int d^4x\sqrt{-g}\left[M_{\mu\nu\rho\sigma}\ln(-\Box)M^{\rho\sigma\mu\nu}+2M_{\mu\nu\rho\sigma}\ln(-\Box)\left(P^{\rho\sigma\mu\nu}+\frac{1}{6}R\,\delta^{\rho(\mu}\delta^{\sigma\nu)}\right)\right], \tag{21}$$

$$\Delta S^{\rm nl}(\lambda\neq 0)=\int d^4x\sqrt{-g}[a(\lambda)R_{\mu\nu}\ln(-\Box)\mathcal{E}^{\mu\nu}+b(\lambda)Rg_{\mu\nu}\ln(-\Box)\mathcal{E}^{\mu\nu}], \tag{22}$$

where $a(\lambda)=-5\lambda\kappa^2/48\pi^2$ and $b(\lambda)=5\lambda\kappa^2/96\pi^2$.

We choose a classical static point mass located at the origin. Hence $\dot{x}^\mu=(1,0,0,0)$, $T^{\mu\nu}(x)=M\delta_0^\mu\delta_0^\nu\delta^3(\vec{x})$, and $T_\mu^\mu=-M\delta^3(\vec{x})$. As we will calculate long distance corrections to gravitational interactions (in particular to the Newtonian potential), we can assume the source is a ''point mass,'' although its size should be much larger than its Schwarzschild radius and the Planck length in order to justify the weak field approximation to be done in what follows. With this choice for the source, the different tensors appearing in the massive nonlocal part of the EA take the form

$$\begin{aligned}
M^{\mu\nu\lambda\sigma}(y)&=\frac{M\kappa^2}{8}\delta^3(\vec{y})\ [g^{\mu\nu}+2\delta_0^\mu\delta_0^\nu]\ \delta_0^\lambda\delta_0^\sigma,\\
M_{\mu\nu\rho\sigma}R\,\delta^{\rho(\mu}\delta^{\sigma\nu)}&=\frac{M\kappa^2}{8}R\,\delta^3(\vec{y}),\\
M_{\mu\nu\rho\sigma}P^{\rho\sigma\mu\nu}&=\frac{M\kappa^2}{8}\delta^3(\vec{y})[g^{\mu\nu}P_{00\mu\nu}+2P_{0000}]\\
&=-\frac{M\kappa^2}{8}R\,\delta^3(\vec{y}).
\end{aligned} \tag{23}$$

With the help of these expressions, the contribution of the source to the nonlocal part of the EA is

$$\Delta S_M^{\rm nl}(\lambda=0)=\frac{5M\kappa^2}{1536\pi^2}\int d^4x\sqrt{-g}R\ \ln(-\nabla^2)\delta^3(\vec{x}), \tag{24}$$

where we have used the fact that the mass M is static to replace $\Box\rightarrow\nabla^2$. We have omitted the term that is quadratic in M because it will be irrelevant in the long distance limit.

Adding the classical and quantum contributions of the EA and taking functional derivations with respect to the metric, it is possible to compute the SEE including backreaction of gravitons. As we are neglecting $\mathcal{O}(\mathcal{R}^3)$ terms in the effective action, it makes no sense to retain $\mathcal{O}(\mathcal{R}^2)$ terms in the equations of motion. Therefore, when doing the variation of the action with respect to the metric, it is not necessary to take into account the $g_{\mu\nu}$ dependence of the logarithmic form factors. Moreover it is possible to commute the covariant derivatives acting on a curvature, i.e., $\nabla_\mu\nabla_\nu\mathcal{R}=\nabla_\nu\nabla_\mu\mathcal{R}+\mathcal{O}(\mathcal{R}^2)$. However, if one uses the standard in-out EA calculated thus far, the equations of motion turn out to be neither real nor causal. In order to get the equations for the mean values one can take any of the following routes: to calculate the in-in EA (which involves a doubling of the number of fields) and derive from it the appropiate field equations [17], to take twice the real and causal part of the in-out equations, or to calculate the Euclidean EA and replace in the equations of motion the Euclidean propagators by the retarded ones [18]. Using any of these alternatives, the mean value equations, up to linear order in curvatures, read

$$\frac{1}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=T_{\mu\nu}+\langle T_{\mu\nu}\rangle_{\lambda=0}^G+\langle T_{\mu\nu}\rangle_{\lambda=0}^M+\langle T_{\mu\nu}\rangle_{a(\lambda)}+\langle T_{\mu\nu}\rangle_{b(\lambda)}, \tag{25}$$

where

$$\langle T_{\mu\nu}\rangle^{G}_{\lambda=0}=-\frac{1}{96\pi^2}\left[\frac{21}{10}\ln(-\nabla^2)H^{(2)}_{\mu\nu}+\frac{1}{20}\ln(-\nabla^2)H^{(1)}_{\mu\nu}\right],$$

$$\langle T_{\mu\nu}\rangle^{M}_{\lambda=0}=\frac{5M\kappa^2}{768\pi^2}(\nabla_\mu\nabla_\nu-g_{\mu\nu}\nabla^2)\ln(-\nabla^2)\,\delta^3(\vec{x}),$$

$$\langle T_{\mu\nu}\rangle_{a(\lambda)}=a(\lambda)\left[-\frac{2}{\kappa^2}\ln(-\nabla^2)H^{(2)}_{\mu\nu}+\frac{1}{\kappa^2}\ln(-\nabla^2)H^{(1)}_{\mu\nu}-\frac{1}{2}\nabla^2\ln(-\nabla^2)T_{\mu\nu}\right],$$

$$\langle T_{\mu\nu}\rangle_{b(\lambda)}=b(\lambda)\left[\frac{2}{\kappa^2}\ln(-\nabla^2)H^{(1)}_{\mu\nu}+\nabla_\mu\nabla_\nu\ln(-\nabla^2)T^{\alpha}_{\alpha}-g_{\mu\nu}\nabla^2\ln(-\nabla^2)T^{\alpha}_{\alpha}\right], \tag{26}$$

where we have introduced the tensors $H^{(1)}_{\mu\nu}=4\nabla_\mu\nabla_\nu R-4g_{\mu\nu}\nabla^2 R$ and $H^{(2)}_{\mu\nu}=2\nabla_\mu\nabla_\nu R-g_{\mu\nu}\nabla^2 R-2\nabla^2 R_{\mu\nu}$. The nonlocal operator $\ln(-\nabla^2)$ acts on the δ function as[3] $\ln(-\nabla^2)\delta^3(\vec{x})=-1/2\pi r^3$ [18].

IV. QUANTUM CORRECTIONS TO THE CLASSICAL METRIC

In order to solve the effective Einstein equations for the background metric we shall make perturbations around flat spacetime, $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ with $\eta_{\mu\nu}=\mathrm{diag}(-+++)$. We choose the harmonic gauge $(h_{\mu\nu}-\frac{1}{2}h\,\eta_{\mu\nu})^{;\nu}=0$ for the background perturbation metric. It is worth mentioning that this choice is completely independent of the gauge-fixing problem for the quantum fluctuations. In this gauge, the Ricci tensor is $R_{\mu\nu}=-\frac{1}{2}\nabla^2 h_{\mu\nu}$ and the Ricci scalar $R=-\frac{1}{2}\nabla^2 h$, with $h=\eta^{\mu\nu}h_{\mu\nu}$. Indeces are lowered and raised with the flat metric. The equations of motion take the form

$$\nabla^2\overline{h}_{\mu\nu}=-16\pi G[T_{\mu\nu}+\langle T_{\mu\nu}\rangle^{G}_{\lambda=0}+\langle T_{\mu\nu}\rangle^{M}_{\lambda=0}+\langle T_{\mu\nu}\rangle_{a(\lambda)}+\langle T_{\mu\nu}\rangle_{b(\lambda)}], \tag{27}$$

where $\overline{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}h\,\eta_{\mu\nu}$. The terms in the right-hand side (RHS) are those appearing in Eq. (26) evaluated in the weak field approximation. In this approximation, the Newtonian potential is related to the 00 component of the perturbation metric as $V(r)=-\frac{1}{2}h_{00}$. In order to find h_{00} we solve Eq. (27) for $\overline{h}_{00}$ and the trace of that equation for h. We find a perturbative solution to these equations around the classical solutions. This perturbative approach is the reason for having omitted terms in the EA that are proportional to the square of the extremal $\mathcal{E}^{\mu\nu}$. These contribute to the RHS of Eq. (27) with terms proportional to the classical equations, and therefore vanish identically when the equations are solved perturbatively. We write $\overline{h}_{00}=\overline{h}^{(0)}_{00}+\overline{h}^{(1)}_{00}$, where $\overline{h}^{(0)}_{00}=4GM/r$ is the classical contribution, and $\overline{h}^{(1)}_{00}$ is the quantum correction. We get

$$\overline{h}^{(1)}_{00}=-\frac{2}{15\pi}\frac{G^2M}{r^3}+\frac{5}{3\pi}\frac{G^2M}{r^3}+4a(\lambda)\frac{GM}{r^3}+8b(\lambda)\frac{GM}{r^3}, \tag{28}$$

where the first and second terms come from the pure gravitational and massive part of the EA (for the DeWitt $\lambda=0$ gauge) and the last two terms correspond to other gauges of the λ family. The equation for the trace is

$$\nabla^2 h=16\pi G[T^{\mu}_{\mu}+\langle T^{\mu}_{\mu}\rangle^{G}_{\lambda=0}+\langle T^{\mu}_{\mu}\rangle^{M}_{\lambda=0}+\langle T^{\mu}_{\mu}\rangle_{a(\lambda)}+\langle T^{\mu}_{\mu}\rangle_{b(\lambda)}], \tag{29}$$

whose perturbative solution $h=h^{(0)}+h^{(1)}$ leads to a classical term $h^{(0)}=4GM/r$ and a quantum correction

$$h^{(1)}=-\frac{18}{3\pi}\frac{G^2M}{r^3}+\frac{5}{\pi}\frac{G^2M}{r^3}+4a(\lambda)\frac{GM}{r^3}+24b(\lambda)\frac{GM}{r^3}. \tag{30}$$

The origin of each term is the same as previously discussed. Therefore the 00 component of the perturbation $h_{\mu\nu}$ reads

$$h_{00}=\overline{h}_{00}-\frac{1}{2}h=\frac{2GM}{r}\left[1+\frac{43G}{30\pi r^2}-\frac{5G}{12\pi r^2}+\frac{a(\lambda)-2b(\lambda)}{r^2}\right]. \tag{31}$$

The first term is due to the presence of the classical mass M (for simplicity we consider only the Newtonian limit, that is, we do not include classical corrections from general relativity). The last four terms are quantum corrections. The second one stems from pure gravitational contributions (vacuum polarization) while the remaining ones arise from the coupling of the mass M to gravitons. The Newtonian potential follows through the identity $V(r)=-\frac{1}{2}h_{00}$. We stress again that the nonlocal logarithmic corrections to the effective action give the leading quantum corrections in the long distance limit, that are proportional to r^{-3}. Had we considered additional terms proportional to $\mathcal{R}^2$ in the effective action, we would have obtained additional corrections to the classical metric that vanish exponentially as $r\to\infty$.

From Eq. (31) it is then clear that the metric that solves the backreaction equations for the one loop quantized gravity depends on which particular function one chooses to fix the gauge. It is for this reason that the classical geodesic equation for such metric cannot be physical.

V. QUANTUM CORRECTED GEODESIC EQUATION

Let us consider a classical test particle of mass m in the presence of the quantized gravitational field $\overline{g}_{\mu\nu}$. A physical observable should be the motion of this particle. We consider that the mass of this particle is much smaller than M, which

[3] This expression can also be obtained by means of the Fourier transform $\int[d^3q/(2\pi)^3]e^{-i\vec{q}\cdot\vec{r}}\ln q^2=-1/2\pi r^3$.

allows us to neglect all contributions of the test particle to the solution Eq. (31) of the one loop corrected equation. Now comes the key ingredient: in order to determine how this test particle moves, one also has to take into account the fact that it couples to the quantum metric $\overline{g}_{\mu\nu}$ through the term $-m\int\sqrt{-\overline{g}_{\mu\nu}}(z)dz^{\mu}dz^{\nu}$, where z^{μ} denotes the path of the test particle. Therefore there will be an extra contribution to the one loop EA due to this coupling to gravitons, which in turn will introduce a correction to the geodesic equation. It can be obtained from Eqs. (21), (22) replacing $M_{\mu\nu\rho\sigma}$ by $m_{\mu\nu\rho\sigma}+M_{\mu\nu\rho\sigma}$ and $T^{\mu\nu}$ by $T^{\mu\nu}+T_m^{\mu\nu}$ and keeping terms linear in m. Here the tensor $m_{\mu\nu\rho\sigma}$ is the one given in Eq. (15) with M replaced by m and x_{μ} replaced by z_{μ}, and $T_m^{\mu\nu}$ is the energy-momentum tensor for the test particle, given in Eq. (12), with the same replacement. This contribution is

$$\Delta S_m=\int d^4x\sqrt{-g}\Bigg[-\frac{1}{32\pi^2}m_{\mu\nu\rho\sigma}\ln(-\Box)M^{\rho\sigma\mu\nu} -\frac{1}{32\pi^2}m_{\mu\nu\rho\sigma}\ln(-\Box)\left(P^{\rho\sigma\mu\nu}+\frac{1}{6}R\,\delta^{\rho(\mu}\delta^{\sigma\nu)}\right) +\frac{a(\lambda)}{2}R_{\mu\nu}\ln(-\Box)T_m^{\mu\nu}+\frac{b(\lambda)}{2}Rg_{\mu\nu}\ln(-\Box)T_m^{\mu\nu}\Bigg]. \tag{32}$$

The first two terms correspond to the $\lambda=0$ gauge fixing, and the last two are extra terms appearing for any other gauge.

The geodesic equation for the test particle can be obtained by taking the functional derivative of the effective action with respect to the coordinates of the particle

$$0=\frac{1}{m}\frac{\delta S_{\mathrm{eff}}}{\delta z_{\rho}}=-\left[\frac{d^2z^{\rho}}{d\tau^2}+\Gamma^{\rho}_{\mu\sigma}\frac{dz^{\mu}}{d\tau}\frac{dz^{\sigma}}{d\tau}\right]+\frac{1}{m}\frac{\delta\Delta S_m}{\delta z_{\rho}}, \tag{33}$$

where $\Gamma^{\rho}_{\mu\sigma}$ is the Christoffel symbol and $d\tau^2=-g_{\mu\nu}dz^{\mu}dz^{\nu}$. In the weak, nonrelativistic Newtonian limit, the quantum corrected geodesic equation reads

$$\frac{d^2\vec{z}}{dt^2}-\frac{1}{2}\vec{\nabla}h_{00}=\frac{1}{m}\frac{\delta\Delta S_m}{\delta\vec{z}}. \tag{34}$$

Note that h_{00}, given in Eq. (31), depends on $a(\lambda)$ and $b(\lambda)$.

Now we proceed to evaluate the RHS of this equation. To that end we first calculate the different terms in ΔS_m. Using the expression for $M^{\mu\nu\rho\sigma}$ corresponding to the static source, the first term of Eq. (32) reads

$$\begin{aligned}\Delta S_{m,M}(\lambda=0)&\equiv-\frac{1}{32\pi^2}\int d^4y\sqrt{-g}m_{\mu\nu\rho\sigma}\ln(-\Box)M^{\rho\sigma\mu\nu}\\ &=-\frac{1}{32\pi^2}\frac{mM\kappa^2}{64}\int d^4y\sqrt{-g}\ln(-\Box)\delta^3(\vec{y})\\ &\quad\times\int d\tau\,\delta^4(y-z(\tau))\\ &\quad\times[2\dot{z}_0\dot{z}_0+2g_{00}\dot{z}_0\dot{z}_0+4\dot{z}_0\dot{z}_0\dot{z}_0\dot{z}_0]\\ &\approx-\frac{mM\kappa^4}{512\pi^2}\int d\tau\ln(-\Box)\delta^3(z(\tau)).\end{aligned} \tag{35}$$

Here we have used the fact that, in the nonrelativistic limit, $\dot{z}_0\approx-1$. As ΔS_m is proportional to $\hbar$, we have also set the metric $g_{\mu\nu}$ in this equation equal to the classical one $\eta_{\mu\nu}$—any other correction would contribute with terms $\mathcal{O}(\hbar^2)$. In a similar fashion

$$\begin{aligned}&\Delta S_{m,R}(\lambda=0)\\ &\quad\equiv-\frac{1}{192\pi^2}\int d^4y\sqrt{-g}m_{\mu\nu\rho\sigma}\ln(-\Box)R\,\delta^{\rho(\mu}\delta^{\sigma\nu)}\\ &\quad=-\frac{m\kappa^2}{1536\pi^2}\int d\tau\ln(-\Box)R(z(\tau)).\end{aligned} \tag{36}$$

The other terms appearing in Eq. (32) are

$$\begin{aligned}\Delta S_{m,P}(\lambda=0)&\equiv-\frac{1}{32\pi^2}\int d^4y\sqrt{-g}m^{\mu\nu\rho\sigma}\ln(-\Box)P_{\rho\sigma\mu\nu}\\ &=-\frac{m\kappa^2}{256\pi^2}\int d\tau[g^{\mu\nu}\dot{z}^{\rho}\dot{z}^{\sigma}+2\dot{z}^{\mu}\dot{z}^{\nu}\dot{z}^{\rho}\dot{z}^{\sigma}]\\ &\quad\times\ln(-\Box)P_{\rho\sigma\mu\nu},\end{aligned} \tag{37}$$

$$\begin{aligned}\Delta S_{m,a(\lambda)}&\equiv\frac{a(\lambda)}{2}\int d^4y\sqrt{-g}R_{\mu\nu}\ln(-\Box)T_m^{\mu\nu}\\ &=a(\lambda)\frac{m}{2}\int d\tau\dot{z}^{\mu}\dot{z}^{\nu}\ln(-\Box)R_{\mu\nu}(z(\tau)),\end{aligned} \tag{38}$$

and

$$\begin{aligned}\Delta S_{m,b(\lambda)}&\equiv\frac{b(\lambda)}{2}\int d^4y\sqrt{-g}Rg_{\mu\nu}\ln(-\Box)T_m^{\mu\nu}\\ &=-b(\lambda)\frac{m}{2}\int d\tau\ln(-\Box)R(z(\tau)).\end{aligned} \tag{39}$$

In these equations the Ricci scalar in the one for the classical metric, i.e., $R(z(\tau))=-\frac{1}{2}\nabla^2h^{(0)}(z(\tau))=8\pi GM\,\delta^3(\vec{z}(\tau))$. The same holds for the Ricci tensor $R_{\mu\nu}(z(\tau))$.

Now we take the variation with respect to $\vec{z}$. We obtain

$$\frac{\delta}{\delta\vec{z}}\Delta S_{m,M}(\lambda=0)=\frac{mMG^2}{\pi}\vec{\nabla}\left(\frac{1}{r^3}\right),$$

$$\frac{\delta}{\delta\vec{z}}\Delta S_{m,R}(\lambda=0)=\frac{mMG^2}{12\pi}\vec{\nabla}\left(\frac{1}{r^3}\right),$$

$$\frac{\delta}{\delta\vec{z}}\Delta S_{m,P}(\lambda=0)=-\frac{mMG^2}{2\pi}\vec{\nabla}\left(\frac{1}{r^3}\right),$$

$$\frac{\delta}{\delta\vec{z}}\Delta S_{m,a(\lambda)}=-a(\lambda)mMG\vec{\nabla}\left(\frac{1}{r^3}\right),$$

$$\frac{\delta}{\delta\vec{z}}\Delta S_{m,b(\lambda)}=2b(\lambda)mMG\vec{\nabla}\left(\frac{1}{r^3}\right), \tag{40}$$

where $r=|\vec{z}|$. Therefore

$$\frac{d^2\vec{z}}{dt^2}-\frac{1}{2}\vec{\nabla}h_{00}=\frac{1}{m}\frac{\delta\Delta S_m}{\delta\vec{z}}=\left[\frac{7G}{12\pi}-a(\lambda)+2b(\lambda)\right]\vec{\nabla}\left(\frac{GM}{r^3}\right). \tag{41}$$

Inserting Eq. (31) into this expression we see that those gauge-fixing-dependent terms arising from the backreaction metric cancel exactly those coming from the coupling of the test particle to gravitons. Note that the terms with $a(\lambda)$ and $b(\lambda)$ cancel separately.

One can perform the same calculation as before for any gauge not belonging to the λ family in a straightforward manner. As it was already mentioned, the difference between the EA for the $\lambda=0$ gauge and that for any other gauge must be proportional to the extremal $\mathcal{E}^{\mu\nu}$, which vanishes on shell. Keeping up to quadratic order in curvature, this requirement fixes the most general form such a difference can have (we concentrate on the nonanalytic log terms)

$$\begin{aligned}\Delta S|_{\text{given gauge}}-\Delta S(\lambda=0)&\\ &=\int d^4x\sqrt{-g}[aR_{\mu\nu}\ln(-\Box)\mathcal{E}^{\mu\nu}\\ &\quad+bRg_{\mu\nu}\ln(-\Box)\mathcal{E}^{\mu\nu}+\mathcal{O}((\mathcal{E}^{\mu\nu})^2)],\end{aligned} \tag{42}$$

where a and b are constants that depend on which particular gauge one uses. For example, for the λ family, $a=a(\lambda)=-5\lambda\kappa^2/48\pi^2$ and $b=b(\lambda)=5\lambda\kappa^2/96\pi^2$, as we have already seen. In view of the above calculations, we conclude that the cancelation of the a- and b-dependent terms takes place for any possible gauge fixing. In this way we obtain a physical, gauge-fixing independent Newtonian potential $V(r)$ which we read from $d^2\vec{z}/dt^2=-\vec{\nabla}V$, namely,

$$V(r)=-\frac{GM}{r}\left[1+\frac{43G\hbar}{30\pi r^2c^3}-\frac{5G\hbar}{12\pi r^2c^3}+\frac{7G\hbar}{12\pi r^2c^3}\right], \tag{43}$$

where we have restored units ($\hbar$ and c). A comparison between Eq. (31) and Eq. (43) shows that the coupling of the test particle with the gravitons produces an additional contribution to the Newtonian potential [the last term in Eq. (43)] and makes it gauge-fixing independent. Note that the long distance quantum correction above is extremely small to be measured. However, the specific number is less important than the conceptual fact that the potential and motion of the test particle are gauge-fixing independent.

VI. CONCLUSIONS

We hope to have convinced the reader that if she/he is interested in solving the backreaction problem including the graviton contribution, it is not enough to solve the semiclassical Einstein equations because they are gauge-fixing dependent and not physical. Rather she/he has to look for physical observables. As an illustration of this point we have chosen the trajectory of a test particle and we have explicitly shown that, in the Newtonian limit, the usual effective action gives a gauge-fixing independent result.

We would like to mention several lines for future research. On the one hand, it is of interest to check whether the Newtonian effective potential derived in this paper does not depend on reparametrizations of the variables chosen to perform the perturbative expansion. When working within the Vilkovisky-DeWitt approach, the potential should not depend on the supermetric defined on the space of fields. On the other hand, it would be interesting to find the quantum corrected geodesic equation in a cosmological setting (desirably, beyond the Newtonian approximation). Finally, we would like to point out that similar ideas to the one proposed here can be applied to the analysis of the mean value equations of any gauge theory, for example, when computing gluon backreaction effects on classical solutions to Yang Mills theories.

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APPENDIX A: DIVERGENCES FOR MINIMAL GAUGES

In this appendix we calculate the divergence of the one-loop EA for the DeWitt gauge $\lambda=0$. We follow closely the methods thoroughly explained in [13]. For DeWitt's gauge, the second-order differential operator $F(\nabla)$ is

$$F^{\alpha\beta,\mu\nu}(\nabla|\lambda=0)=\sqrt{-g}C^{\alpha\beta,\lambda\sigma}\{\Box\delta^{\mu}_{(\lambda}\delta^{\nu}_{\sigma)}+P^{\mu\nu}_{\lambda\sigma}+M^{\mu\nu}_{\lambda\sigma}\}, \tag{A1}$$

and the one loop EA has the following expression:

$$S_{\text{eff}}=S_{\text{class}}+\frac{i}{2}\text{Trln}F^{\alpha\beta,\mu\nu}(\nabla)-i\text{Trln}(\Box\delta^{\mu}_{\nu}+R^{\mu}_{\nu}), \tag{A2}$$

the first term being the classical action. In the gauge under consideration, both the differential operator for the gravitons and the one for the ghosts have a minimal form, which in matrix notation reads

$$\hat{\mathcal{F}}(\nabla)=\Box+\hat{\mathcal{Q}}-\frac{1}{6}R\hat{1}. \tag{A3}$$

Indeed, for the gravitons the matrix $\hat{\mathcal{Q}}$ is given by $\hat{\mathcal{Q}}=\hat{P}+\hat{M}+\frac{1}{6}R\hat{1}$, while for the ghosts $\hat{\mathcal{Q}}=\hat{R}+\frac{1}{6}R\hat{1}$. In order to calculate the functional traces, we make use of the Schwinger-DeWitt (SDW) technique, to get

$$\mathrm{Tr}\ln\hat{\mathcal{F}}=\frac{i}{(4\pi)^{\frac{d}{2}}}\int_0^\infty\frac{ds}{s^{\frac{d}{2}+1}}\int d^dx\,\mathrm{Tr}\sum_{n=0}^{\infty}(is)^n\hat{a}_n(x), \tag{A4}$$

where the $\hat{a}_n(x)$'s are the coincidence limit of the SDW coefficients. The divergent part of the EA for any minimal operator in $d=4$ dimensions is determined by the first three SDW coefficients. The divergences coming from $\hat{a}_0$ and $\hat{a}_1$ can be absorbed into a redefinition of the cosmological constant and the Newton constant. In what follows it will be relevant the divergence coming from the second SDW coefficient. It reads

$$\hat{a}_2(x)=\frac{1}{180}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-R_{\mu\nu}R^{\mu\nu}+\Box R)\hat{1}+\frac{1}{2}\hat{\mathcal{Q}}^2+\frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}+\frac{1}{6}\Box\hat{\mathcal{Q}}, \tag{A5}$$

where $\hat{\mathcal{R}}_{\mu\nu}$ is the commutator of covariant derivatives. Inserting the definition of the operators $\hat{Q}$ for the graviton and the ghost parts into the formula for the second SDW coefficient, one can extract the divergence coming from pure gravity and the corresponding one due to the massive terms. The former one gives the well-known result [14]

$$\Delta S_G^{\mathrm{div}}(\lambda=0)=\frac{2}{(4-d)96\pi^2}\int d^4x\sqrt{-g}\left[\frac{53}{15}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2)+\frac{21}{10}R_{\mu\nu}R^{\mu\nu}+\frac{1}{20}R^2\right], \tag{A6}$$

where the first term in brackets is the Gauss-Bonnet term, a topological invariant in $d=4$ spacetime dimensions. The divergence due to the presence of M is read from its contribution to the second SDW coefficient, namely, $\frac{1}{6}\Box\hat{M}+\frac{1}{2}\hat{M}^2+\hat{M}(\hat{P}+\frac{1}{6}R\hat{1})$. It has the following form:

$$S_M^{\mathrm{div}}(\lambda=0)=\frac{2}{(4-d)64\pi^2}\int d^4x\sqrt{-g}\left[M_{\mu\nu\rho\sigma}M^{\mu\nu\rho\sigma}+2M_{\mu\nu\rho\sigma}\left(P^{\rho\sigma\mu\nu}+\frac{1}{6}R\,\delta^{\rho(\mu}\delta^{\sigma\nu)}\right)\right], \tag{A7}$$

where we have omitted the boundary term.

APPENDIX B: DIVERGENCES FOR NONMINIMAL GAUGES

In this appendix we sketch the calculation of the one loop EA and its divergences for the λ family of gauge-fixing functions. As we have already remarked in the text, when $\lambda\neq 0$ we have a nonminimal gauge. For these nonminimal gauges, a reduction method has been developed in [13] which generalizes the Schwinger-DeWitt expansion. It also consists in a local expansion in the background fields, and it has been calculated up to second order in the curvature tensors. The starting point is to note that, since the theory as a whole is gauge independent on the mass shell, the difference of the effective action in any gauge from that in a given minimal gauge is always proportional to the extremal, i.e., the left-hand side (LHS) of the classical field equation. With this idea in mind, that difference can be expressed in terms of nonminimal Green's functions for gravitons and ghosts, which are expanded in terms of the background dimensionality.

One special easy case of nonminimal gauge families is that when the gauge-breaking action differs from the minimal one only by an overall factor. This is indeed the case for the λ family, since $\chi^\mu(\lambda)=(1/\sqrt{1+\lambda})\chi^\mu(\lambda=0)$. Following the methods of [13], the EA for any member of this family of gauge-fixing functions is

$$S_{\mathrm{eff}}(\lambda)=S_{\mathrm{eff}}(\lambda=0)+\frac{i}{2}\lambda[\mathrm{Tr}V^{\mu}_{1\nu}(\nabla)-\mathrm{Tr}V^{\mu}_{2\nu}(\nabla)]-\frac{i}{4}\lambda^2\mathrm{Tr}[V^{\mu}_{1\nu}(\nabla)]^2+\mathcal{O}((\mathcal{E}^{\mu\nu})^2), \tag{B1}$$

where the external $\mathcal{E}^{\mu\nu}$ is given by

$$\mathcal{E}^{\mu\nu}=\frac{\delta(S_G+S_M)}{\delta g^{\mu\nu}}=-\frac{2}{\kappa^2}\left(R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}\right)+\frac{1}{2}T^{\mu\nu}, \tag{B2}$$

and $V^{\mu}_{1\nu}(\nabla)$ and $V^{\mu}_{2\nu}(\nabla)$ are tensors that are linear and quadratic in the extremal, respectively. Their action on a test function ζ^ν is given by

$$V^{\mu}_{1\nu}(\nabla)\zeta^\nu=2\kappa^2\,Q^\mu_\alpha\,\nabla_\beta\,\Gamma^{(\alpha}_{\rho\sigma}(\nabla)\,\mathcal{E}^{\rho\beta)}\,Q^\sigma_\nu\,\zeta^\nu,$$

$$V^{\mu}_{2\nu}(\nabla)\zeta^\nu=-\kappa^2g^{\mu\omega}\,Q^\gamma_\omega\,\mathcal{E}^{(\alpha\rho}\,\Gamma^{\beta)}_{\rho\gamma}(\nabla)\times G_{\alpha\beta,\varphi\theta}(\nabla)\,\Gamma^{(\varphi}_{\delta\sigma}(\nabla)\,\mathcal{E}^{\theta)\delta}\,Q^\sigma_\nu\,\zeta^\nu. \tag{B3}$$

In these expressions, $\Gamma^\nu_{\rho\sigma}(\nabla)=\delta^\nu_\rho\nabla_\rho-2\delta^\nu_\sigma\nabla_\rho$, and $G_{\alpha\beta,\varphi\theta}(\nabla)$ and Q^σ_μ are the Green's functions for the gauge field and ghost field, respectively, evaluated for the DeWitt gauge

$$F^{\gamma\sigma,\alpha\beta}(\nabla|\lambda=0)\,G_{\alpha\beta,\varphi\theta}(\nabla)=-\delta^{\gamma\sigma}_{\varphi\theta},$$

$$(\Box\delta^\mu_\alpha+R^\mu_\alpha)Q^\sigma_\mu=\delta^\sigma_\alpha. \tag{B4}$$

We are interested just in the contribution to the EA that is linear in the extremal (see main text). Therefore we concentrate ourselves on $\mathrm{Tr}V^{\mu}_{1\nu}(\nabla)$, which is given by

$$\mathrm{Tr}V^{\mu}_{1\nu}(\nabla)=2\kappa^2\int d^4x[R^{\alpha}_{\cdot\,\gamma\beta\sigma}\mathcal{E}^{\gamma\beta}-\mathcal{E}^{\beta\gamma}\delta^{\alpha}_{\sigma}\nabla_{\beta}\nabla_{\gamma}] \times(\Box\delta^{\sigma}_{\alpha}+R^{\sigma}_{\alpha})^{-2}\delta(x,y)|_{y=x}. \tag{B5}$$

In order to calculate the divergent part of this expression we use the methods explained in [13]. It is worth recalling that we are working up to quadratic order in curvatures, so that for the contribution of the first term in brackets we can approximate $(\Box\delta^{\sigma}_{\alpha}+R^{\sigma}_{\alpha})^{-2}$ by $\Box^{-2}\delta^{\sigma}_{\alpha}$. The two divergences that appear are

$$\Box^{-2}\delta^{\sigma}_{\alpha}\delta(x,y)|^{\mathrm{div}}_{y=x}=\frac{i}{8\pi^2}\frac{1}{4-d}\sqrt{-g}, \tag{B6}$$

$$\nabla_{\beta}\nabla_{\gamma}(\Box\delta^{\sigma}_{\alpha}+R^{\sigma}_{\alpha})^{-2}\delta(x,y)|^{\mathrm{div}}_{y=x} =\frac{i}{8\pi^2}\frac{1}{4-d}\sqrt{-g}\left[\frac{1}{6}\left(R_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}R\right)\delta^{\sigma}_{\alpha} +\frac{1}{2}R^{\sigma}_{\cdot\ \alpha\beta\gamma}-\frac{1}{2}g_{\beta\gamma}R^{\sigma}_{\alpha}\right]. \tag{B7}$$

Finally the total divergence reads

$$\mathrm{Tr}V^{\mu}_{1\nu}(\nabla)|^{\mathrm{div}}=\frac{2i}{4-d}\frac{\kappa^2}{24\pi^2}\int d^4x\sqrt{-g} \times\left[-5R_{\mu\nu}\mathcal{E}^{\mu\nu}+\frac{5}{2}Rg_{\mu\nu}\mathcal{E}^{\mu\nu}\right]. \tag{B8}$$

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