Effective Einstein theory from metric-affine gravity models via irreducible decompositions

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The irreducible decomposition technique is applied to the study of classical models of metric-affine gravity (MAG). The dynamics of the gravitational field is described by a 12-parameter Lagrangian encompassing a Hilbert-Einstein term, torsion and nonmetricity square terms, and one quadratic curvature piece that is built up from Weyl's segmental curvature. Matter is represented by a hyperfluid, a continuous medium the elements of which possess classical momentum and hypermomentum. With the help of irreducible decompositions, we are able to express torsion and traceless nonmetricity explicitly in terms of the spin and the shear current of the hyperfluid. Thereby the field equations reduce to an effective Einstein theory describing a metric coupled to the Weyl 1-form (a Proca-type vector field) and to a spin fluid. We demonstrate that a triplet of torsion and nonmetricity 1-forms describes the general and unique vacuum solution of the field equations of MAG. Finally, we study homogeneous cosmologies with a hyperfluid. We find that the hypermomentum affects significantly the cosmological evolution at very early stages. However, unlike spin, shear does not prevent the formation of a cosmological singularity. [S0556-2821(97)07922-8]

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I. INTRODUCTION

Within the framework of the gauge approach to gravity (see, e.g., [1]) the *kinematic* scheme of the metric-affine theory is well understood at present. However, the *dynamic* aspects of metric-affine gravity (MAG) have been rather poorly studied up to now. The choice of the basic Lagrangian of the theory remains an open problem, and this, in turn, prevents a detailed analysis of possible physical effects. (An analysis of physical observations in MAG without specializing to a particular Lagrangian was made in [2].) As a first step, one can use a correspondence principle. It is well known that Einstein's general relativity theory is satisfactorily supported by experimental tests on the macroscopic level. Thus, whereas the gravitational gauge models provide an alternative description of gravitational physics in the microworld, it is natural to require their correspondence with general relativity at large distances. Unfortunately, direct generalization of the standard Hilbert-Einstein Lagrangian yields an unphysical MAG model which is projectively invariant and, accordingly, imposes unphysical constraints on the matter sources.

Another essential difficulty in the development of a dynamical scheme of MAG was, until recently, the lack of self-consistent models which describe physical (quantum, semiclassical, or classical) sources of MAG possessing mass or energy-momentum and hypermomentum. The reader may consult [1,3] which give a modern presentation of the so-called manifield and world spinor approach based on the theory of infinite dimensional representations of the affine and linear groups. However, the main achievements there are again of kinematic nature, and no dynamical model for manifields and world spinors is available.

Recently there has been some progress both in the development of the simplest viable metric-affine Lagrangians that generalize the Hilbert-Einstein model and in the establishment of a variational theory of a *hyperfluid* which seems to represent a reasonable classical model of a continuous medium with energy-momentum and hypermomentum. In the papers [4,5] it was proposed to take as the gravitational Lagrangian the sum of the (generalized) Hilbert-Einstein term and the square of the segmental curvature (thus reviving the old proposals of [6,7]). Further extensions of this model, which include the quadratic invariants of torsion and nonmetricity, were investigated (in vacuum) in [8–12]. A hyperfluid model was developed in [13] along the lines of the Weyssenhoff approach to spin fluids which now reappear as a particular case of the hyperfluid. (Note that a different variational model of a fluid with hypermomentum was suggested in [14].)

Relativistic fluid dynamics covers a vast field of research in gravitation, cosmology, and particle physics. Relativistic fluid models are working tools in high-energy plasma astrophysics and in nuclear physics (where nonideal fluids are extremely successfully applied to the description of heavy ion reactions), see, e.g., [15]. In cosmology, hydrodynamical description of matter is standard both for the early and for the later stages of the evolution of the universe [16]. Spin fluids are used for the consistent statistical treatment of a medium the elements of which are particles with intrinsic angular momentum [17] (cosmological ''soup'' of fundamental particles in the early universe or a fluid of spinning galaxies, clusters of galaxies, turbulent eddies during the later times). In Poincaré gauge gravity, the Weyssenhoff spin fluid [18] provides an adequate description of a continuous medium with spin degrees of freedom. Spin of matter sources proves to be significant in the Einstein-Cartan theory, where the cosmological singularity can be avoided

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due to effective repulsion of spinning particles [19]. It is worthwhile to stress that the spin fluid can be consistently derived from the quantum theory of Dirac particles. [One may ask though: How can this be reconciled with the studies [20] for the Einstein-Cartan-Dirac cosmology where a singularity is not avoided? The answer is that it is misleading to compare a spin fluid cosmology (i.e., a cosmology of *ensemble of large number* of gravitating particles with spin) with a clearly unphysical classical Dirac field ''cosmology''(i.e., a ''cosmology'' of *one* gravitating particle with spin). The correct comparison can only be made with the Einstein-Cartan-Dirac cosmology in which the energy-momentum and spin currents are obtained as macroscopic averages from the quantum density operators. This was done, e.g., in [21] with the help of the relativistic Wigner function formalism, and the effective repulsion was confirmed.]

In the framework of MAG, to the best of our knowledge, the hyperfluid represents the only available self-consistent dynamical model of matter with nontrivial hypermomentum. It generalizes in a natural way the Weyssenhoff spin fluid by including additional degrees of freedom (dilation and shear densities). At the same time, further study is certainly needed for establishing the *fundamental* theory of matter with hypermomentum. The manifields [1,3] seem to be a step in the right direction, but unfortunately no dynamical scheme is known for them (i.e., no Lagrangian, no equations of motion, no precise form of the Noether currents). It is even unclear how the standard Dirac fermion matter can be recovered from the manifields when shear and dilation charges vanish. We are convinced though, that even after the fundamental theory of matter with hypermomentum is completed, the hyperfluid model will remain a tool useful for practical applications (like the relativistic fluid models are amazingly handy in nuclear physics for calculations on heavy ion reactions, despite the fact that a fundamental Dirac theory is always also available [15]).

In this paper we will study the classical dynamics of metric-affine gravitational fields for the general MAG Lagrangian which includes the Hilbert-Einstein term, the segmental curvature square term (of Weyl), and all possible quadratic torsion and nonmetricity contractions. The hyperfluid provides nontrivial energy-momentum and hypermomentum currents which describe classical matter sources in the MAG field equations. We demonstrate the exceptional effectiveness of the technique of irreducible decompositions applied to post-Riemannian geometrical objects. In particular, we show that (i) the separation of Riemannian and post-Riemannian structures and (ii) the subsequent decomposition of the latter into irreducible pieces, leads to the solution of the coupled MAG field equations with respect to torsion and nonmetricity. As a result of this process, we are left with an effective Einsteinian gravitational field equation for the metric which is a direct generalization of the effective equations arising in the Einstein-Cartan theory (cf. [22]). Specializing our results to the vacuum case, we are able to complete the study of the ansatz of the so-called *1-form triplet*, which underlies the results of [10–12], by demonstrating its uniqueness. Namely, for generic MAG models, the solution with a 1-form triplet is not only the most general solution of the *second* field equation (for terminology, see [1]) but it is also unique.

Our basic notations and conventions are those of [1], in particular the signature of the metric is assumed to be $(-,+,+,+)$.

II. PRELIMINARIES: METRIC-AFFINE GEOMETRY AND BASIC DECOMPOSITIONS

In this section we recall some basic facts concerning metric-affine geometry in four dimensions. For a more detailed discussion in arbitrary dimensions see [1]. The metric-affine spacetime is described by the metric $g_{\alpha\beta}$, the coframe 1-forms ϑ^α, and the linear connection 1-forms $\Gamma_\beta{}^\alpha$. These are interpreted as the generalized gauge potentials, while the corresponding field strengths are the nonmetricity 1-form $Q_{\alpha\beta}=-Dg_{\alpha\beta}$ and the 2-forms of torsion $T^\alpha=D\vartheta^\alpha$ and curvature $R_\beta{}^\alpha=d\Gamma_\beta{}^\alpha+\Gamma_\gamma{}^\alpha\wedge\Gamma_\beta{}^\gamma$. The general affine connection can always be decomposed into Riemannian and post-Riemannian parts,

$$\Gamma_\beta{}^\alpha=\widetilde{\Gamma}_\beta{}^\alpha+N_\beta{}^\alpha, \tag{2.1}$$

where the *distortion* 1-form $N_{\alpha\beta}$ can be expressed in terms of torsion and nonmetricity:

$$N_{\alpha\beta}=-e_{[\alpha}\rfloor T_{\beta]}+\tfrac{1}{2}(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\vartheta^\gamma+(e_{[\alpha}\rfloor Q_{\beta]\gamma})\vartheta^\gamma+\tfrac{1}{2}Q_{\alpha\beta}. \tag{2.2}$$

We denote the Riemannian connection (the Christoffel symbols) by $\widetilde{\Gamma}_\beta{}^\alpha$, and hereafter the tilde will denote purely Riemannian geometrical objects and operators. Using Eq. (2.1), it is possible to split all quantities in the metric-affine theory into Riemannian and post-Riemannian pieces (for curvature this reads, e.g., $R_\beta{}^\alpha=\widetilde{R}_\beta{}^\alpha+\widetilde{D}N_\beta{}^\alpha+N_\gamma{}^\alpha\wedge N_\beta{}^\gamma$).

The *irreducible decompositions* of torsion and nonmetricity [1] provide a pattern for the decomposition of the gravitational gauge field momenta which enter the field equations of MAG. In order to make the paper self-contained, we reproduce here the basic formulae.

The torsion 2-form T^α can be decomposed into three irreducible pieces:

$$^{(2)}T^\alpha:=\tfrac{1}{3}\vartheta^\alpha\wedge T,\qquad T:=e_\alpha\rfloor T^\alpha, \tag{2.3}$$

$$^{(3)}T^\alpha:=-\tfrac{1}{3}{}^*(\vartheta^\alpha\wedge P),\qquad P:={}^*(T^\alpha\wedge\vartheta_\alpha), \tag{2.4}$$

$$^{(1)}T^\alpha:=T^\alpha-{}^{(2)}T^\alpha-{}^{(3)}T^\alpha. \tag{2.5}$$

The nonmetricity 1-form $Q_{\alpha\beta}$ can be decomposed into four irreducible pieces:

$$^{(2)}Q_{\alpha\beta}:=\tfrac{2}{3}\,{}^*(\vartheta_{(\alpha}\wedge\Omega_{\beta)}), \tag{2.6}$$

$$^{(3)}Q_{\alpha\beta}:=\tfrac{4}{9}(\vartheta_{(\alpha}e_{\beta)}\rfloor\Lambda-\tfrac{1}{4}g_{\alpha\beta}\Lambda), \tag{2.7}$$

$$^{(4)}Q_{\alpha\beta}:=g_{\alpha\beta}Q, \tag{2.8}$$

$$^{(1)}Q_{\alpha\beta}:=Q_{\alpha\beta}-{}^{(2)}Q_{\alpha\beta}-{}^{(3)}Q_{\alpha\beta}-{}^{(4)}Q_{\alpha\beta}. \tag{2.9}$$

Here the shear covector part and the Weyl covector are, respectively, $\Lambda:=\vartheta^\alpha e^\beta\rfloor \not{Q}_{\alpha\beta}$, and $Q:=\tfrac{1}{4}g^{\alpha\beta}Q_{\alpha\beta}$, where $\not{Q}_{\alpha\beta}=Q_{\alpha\beta}-Qg_{\alpha\beta}$ is the traceless piece of the nonmetricity. It seems worthwhile to notice that the 2-form $\Omega_\alpha := \Theta_\alpha - 1/3\, e_\alpha \rfloor (\vartheta^\beta \wedge \Theta_\beta)$, with $\Theta_\alpha := {}^*(\not{Q}_{\alpha\beta} \wedge \vartheta^\beta)$, which describes ${}^{(2)}Q_{\alpha\beta}$, has precisely the same symmetry properties as the 2-form ${}^{(1)}T^\alpha$.

Substituting (2.3)–(2.5) and (2.6)–(2.9) into Eq. (2.2), we find the following general decomposition of the distortion 1-form:

$$N_{\alpha\beta} = \tfrac{1}{2}\{Q_{\alpha\beta} - \tfrac{2}{3}\vartheta_{[\alpha} e_{\beta]} \rfloor (3Q - \Lambda + 2T) - 2e_{[\alpha]} \times ({}^*\Omega_{\beta]} + 2\,{}^{(1)}T_{\beta]}) - (e_\alpha \rfloor e_\beta \rfloor {}^{(3)}T_\gamma)\vartheta^\gamma. \quad (2.10)$$

The first irreducible piece of the torsion and the 2-form Ω_α appear as a linear combination here. This formula is extremely useful in the analysis of the field equations of MAG.

III. A MODEL FOR MAG

In this and subsequent sections we will widely use, along with the coframes ϑ^α, the so called η-basis of the dual coframes. Namely, we define [1] the Hodge dual such that $\eta := {}^*1$ is the volume 4-form. Furthermore $\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = {}^*(\vartheta_\alpha \wedge \vartheta_\beta)$, $\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma\delta} := e_\delta \rfloor \eta_{\alpha\beta\gamma}$. The last expression is thus the totally antisymmetric Levi-Civita tensor.

A. Gravitational Lagrangian

Direct generalization of the Hilbert-Einstein Lagrangian $R_{\alpha\beta} \wedge \eta^{\alpha\beta}$ to metric-affine gravity yields an unphysical model which is invariant under projective transformations of the connection.

Consequently, we turn our attention to a model described by a Lagrangian which generalizes the models studied recently in [4,10–12]:

$$V_{\mathrm{MAG}} = \frac{1}{2\kappa}\Bigg[-a_0\, R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda\,\eta + T^\alpha \wedge {}^*\left(\sum_{I=1}^{3} a_I\, {}^{(I)}T_\alpha\right) + 2\left(\sum_{I=2}^{4} c_I\, {}^{(I)}Q_{\alpha\beta}\right) \wedge \vartheta^\alpha \wedge {}^*T^\beta + Q_{\alpha\beta} \wedge {}^*\left(\sum_{I=1}^{4} b_I\, {}^{(I)}Q^{\alpha\beta}\right) + b_5({}^{(3)}Q_{\alpha\gamma} \wedge \vartheta^\alpha) \wedge {}^*({}^{(4)}Q^{\beta\gamma} \wedge \vartheta_\beta)\Bigg] - \frac{1}{2} z_4\, R^{\alpha\beta} \wedge {}^{*(4)}Z_{\alpha\beta}. \quad (3.1)$$

Here, the coupling constants $a_0, \dots, a_3, c_2, c_3, c_4, b_1, \dots, b_5, z_4$ are dimensionless, κ is the standard Einstein gravitational constant, and λ is the cosmological constant. The segmental curvature is denoted by ${}^{(4)}Z_{\alpha\beta} := \frac{1}{4} g_{\alpha\beta} R_\gamma{}^\gamma$; it is a purely *post*-Riemannian piece.

B. Hyperfluid matter

Let us study the model (3.1) with matter represented by a hyperfluid [13]. The matter Lagrangian reads

$$L_{\mathrm{mat}} = \tfrac{1}{2}\rho\ \mu^A{}_B\, b^B_\alpha\, u \wedge D b^\alpha_A - \varepsilon(\rho, s, \mu^A{}_B)\,\eta + L_{\mathrm{constraints}}, \quad (3.2)$$

where the first two terms on the right-hand side describe the kinetic and the internal energy density ε of the hyperfluid, respectively. The latter depends on the particle density ρ, the specific entropy s, and the specific hypermomentum density $\mu^A{}_B$ (''specific'' means ''per particle''). Here u is the flow 3-form, so that the components of the average fluid velocity are given by $u_\alpha := e_\alpha \rfloor {}^*u$. The first term represents the combined kinetic contribution of the rotational and the strain energy of the fluid elements, the motion of which is described by the angular and strain velocity of a 3-volume spanned by the material triad. It is convenient to describe the latter by two variables: a 1-form b^A with the components $b^A_\alpha := e_\alpha \rfloor b^A$ and a 3-form b_A with the components $b^\alpha_A := {}^*(b_A \wedge \vartheta^\alpha)$. The last term in Eq. (3.2) denotes a set of constraints to be added via Lagrange multipliers. We will not display them here (see [13] for a detailed discussion). Let us only mention that they include the standard normalization constraint for the velocity

$$u \wedge {}^*u = \eta, \quad (3.3)$$

and the law of particle number conservation,

$$d(\rho u) = 0. \quad (3.4)$$

Variation of the Lagrangian (3.2) with respect to the matter variables yields the hypermomentum equation of motion in the form

$$D\Delta^\alpha{}_\beta = -u^\alpha\, u_\lambda D\Delta^\lambda{}_\beta - u_\beta\, u^\lambda D\Delta^\alpha{}_\lambda, \quad (3.5)$$

where the hypermomentum current 3-form

$$\Delta^\alpha{}_\beta = u J^\alpha{}_\beta \quad (3.6)$$

can be expressed in terms of the hypermomentum density

$$J^\alpha{}_\beta = -\frac{1}{2}\rho\ \mu^A{}_B\, b^B_\beta\, b^\alpha_A. \quad (3.7)$$

By construction, this tensor satisfies the generalized Frenkel conditions

$$J^\alpha{}_\beta\, u^\beta = 0, \quad J^\alpha{}_\beta\, u_\alpha = 0. \quad (3.8)$$

Variational derivatives of the material Lagrangian L_{mat} with respect to coframe ϑ^α and connection $\Gamma_\beta{}^\alpha$ 1-forms define the material sources. The canonical hypermomentum current 3-form is given by Eq. (3.6), whereas the canonical energy-momentum 3-form reads

$$\Sigma_\alpha = \varepsilon u u_\alpha + p(\eta_\alpha + u u_\alpha) + 2 u u^\beta g_{\gamma[\alpha}\ {}^*(D\Delta^\gamma{}_{\beta]}), \quad (3.9)$$

with the pressure defined as usual by $p = \rho(\partial\varepsilon/\partial\rho) - \varepsilon$.

C. Field equations

The metric-affine field equations are derived from the total Lagrangian $V_{\mathrm{MAG}} + L_{\mathrm{mat}}$ by independent variations with respect to the coframe ϑ^α and connection $\Gamma_\beta{}^\alpha$ 1-forms. The corresponding so-called *first* and *second* field equations read

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{3.10}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta. \tag{3.11}$$

The left hand sides of Eqs. (3.10) and (3.11) are given by

$$\begin{aligned} M^{\alpha\beta} &:= -2\frac{\partial V_{\mathrm{MAG}}}{\partial Q_{\alpha\beta}} \\ &= -\frac{2}{\kappa}\Bigg[{}^*\Bigg(\sum_{I=1}^{4} b_I{}^{(I)}Q^{\alpha\beta}\Bigg) + \frac{1}{2}b_5\big(\vartheta^{(\alpha}\wedge{}^*(Q\wedge\vartheta^{\beta)}) \\ &\quad -\tfrac{1}{4}g^{\alpha\beta}\,{}^*(3Q+\Lambda)\big) + c_2\ \vartheta^{(\alpha}\wedge{}^*\,{}^{(1)}T^{\beta)} \\ &\quad + c_3\ \vartheta^{(\alpha}\wedge{}^*\,{}^{(2)}T^{\beta)} + \frac{1}{4}(c_3 - c_4)\ g^{\alpha\beta\,*}\,T\Bigg], \end{aligned} \tag{3.12}$$

$$\begin{aligned} H_\alpha &:= -\frac{\partial V_{\mathrm{MAG}}}{\partial T^\alpha} \\ &= -\frac{1}{\kappa}{}^*\Bigg[\Bigg(\sum_{I=1}^{3} a_I{}^{(I)}T_\alpha\Bigg) + \Bigg(\sum_{I=2}^{4} c_I{}^{(I)}Q_{\alpha\beta}\wedge\vartheta^\beta\Bigg)\Bigg], \end{aligned} \tag{3.13}$$

$$H^\alpha{}_\beta := -\frac{\partial V_{\mathrm{MAG}}}{\partial R_\alpha{}^\beta} = \frac{a_0}{2\kappa}\ \eta^\alpha{}_\beta + z_4\ {}^*({}^{(4)}Z^\alpha{}_\beta), \tag{3.14}$$

and

$$\begin{aligned} E_\alpha &:= e_\alpha\rfloor V_{\mathrm{MAG}} + (e_\alpha\rfloor T^\beta)\wedge H_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge H^\beta{}_\gamma \\ &\quad + \tfrac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})M^{\beta\gamma}, \end{aligned} \tag{3.15}$$

$$E^\alpha{}_\beta := -\vartheta^\alpha\wedge H_\beta - M^\alpha{}_\beta. \tag{3.16}$$

We note, see [1], that the equation which arise from the variation of the Lagrangian with respect to the metric turns out to be redundant.

IV. SPECIAL QUADRATIC MAG-LAGRANGIAN

As a preliminary step, let us study the MAG model with the Lagrangian

$$\begin{aligned} V^{(0)} = \frac{a_0}{2\kappa}\Bigg\{ & -R_{\alpha\beta}\wedge\eta^{\alpha\beta} - {}^{(1)}T^\alpha\wedge{}^{*(1)}T_\alpha + 2\,{}^{(2)}T^\alpha\wedge{}^{*(2)}T_\alpha + \frac{1}{2}{}^{(3)}T^\alpha\wedge{}^{*(3)}T_\alpha + {}^{(2)}Q_{\alpha\beta}\wedge\vartheta^\beta\wedge{}^{*(1)}T^\alpha - 2\,{}^{(3)}Q_{\alpha\beta}\wedge\vartheta^\beta \\ & \wedge{}^{*(2)}T^\alpha - 2\,{}^{(4)}Q_{\alpha\beta}\wedge\vartheta^\beta\wedge{}^{*(2)}T^\alpha + \frac{1}{4}{}^{(1)}Q_{\alpha\beta}\wedge{}^{*(1)}Q^{\alpha\beta} - \frac{1}{2}{}^{(2)}Q_{\alpha\beta}\wedge{}^{*(2)}Q^{\alpha\beta} - \frac{1}{8}{}^{(3)}Q_{\alpha\beta}\wedge{}^{*(3)}Q^{\alpha\beta} + \frac{3}{8}{}^{(4)}Q_{\alpha\beta} \\ & \wedge{}^{*(4)}Q^{\alpha\beta} + ({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^\alpha)\wedge{}^*({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_\beta)\Bigg\}. \end{aligned} \tag{4.1}$$

It can be seen that this is a particular case of Eq. (3.1) with the special values of the coupling constants.

One can verify, by direct calculation, that the gauge field momenta for the Lagrangian (4.1) are given by

$$H^{(0)}_\alpha := -\frac{\partial V^{(0)}}{\partial T^\alpha} \equiv -\frac{a_0}{2\kappa}N^{\mu\nu}\wedge\eta_{\alpha\mu\nu},$$

$$H^{(0)\alpha}{}_\beta := -\frac{\partial V^{(0)}}{\partial R_\alpha{}^\beta} = \frac{a_0}{2\kappa}\ \eta^\alpha{}_\beta, \tag{4.2}$$

and, as a result, it can be straightforwardly proved that

$$DH^{(0)}_\alpha - E^{(0)}_\alpha \equiv \frac{a_0}{2\kappa}\widetilde{R}^{\mu\nu}\wedge\eta_{\alpha\mu\nu}, \quad DH^{(0)\alpha}{}_\beta - E^{(0)\alpha}{}_\beta \equiv 0, \tag{4.3}$$

where, similarly to Eqs. (3.15) and (3.16),

$$\begin{aligned} E^{(0)}_\alpha &:= e_\alpha\rfloor V^{(0)} + (e_\alpha\rfloor T^\beta)\wedge H^{(0)}_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge H^{(0)\beta}{}_\gamma \\ &\quad + \tfrac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})M^{(0)\beta\gamma}, \end{aligned} \tag{4.4}$$

$$E^{(0)\alpha}{}_\beta := -\vartheta^\alpha\wedge H^{(0)}_\beta - M^{(0)\alpha}{}_\beta. \tag{4.5}$$

The *identities* (4.3) can be justified by the fact that

$$V^{(0)} \equiv \frac{a_0}{2\kappa}\{-\widetilde{R}_{\alpha\beta}\wedge\eta^{\alpha\beta} + d[\vartheta^\alpha\wedge{}^*(2T_\alpha - Q_{\alpha\beta}\wedge\vartheta^\beta)]\} \tag{4.6}$$

is, up to an exact form, the *purely Riemannian* Hilbert-Einstein Lagrangian of general relativity theory. However, this observation does not provide a rigorous proof of Eq. (4.3), because in the derivation of the gauge field equations (3.10) and (3.11), see [1], one assumes that the gravitational Lagrangian contains frame derivatives, $d\vartheta^\alpha$, only implicitly in torsion, while Eq. (4.6) contains such terms in the Riemannian connection. Hence, a direct proof is required, and a rather long calculation involving the irreducible decomposition (2.10) of the distortion 1-form $N_{\alpha\beta}$ demonstrates that Eq. (4.3) is true, indeed.

V. DECOMPOSITION OF THE FIELD EQUATIONS OF MAG

Let us write the Lagrangian (3.1) as

$$V_{\rm MAG}=V^{(0)}+\hat{V}, \tag{5.1}$$

where

$$\hat{V}=\frac{1}{2\kappa}\Bigg[-2\lambda\,\eta+T^{\alpha}\wedge{}^*\left(\sum_{I=1}^{3}\alpha_I\,{}^{(I)}T_{\alpha}\right)+2\left(\sum_{I=2}^{4}\gamma_I\,{}^{(I)}Q_{\alpha\beta}\right)\wedge\vartheta^{\alpha}\wedge{}^*\,T^{\beta}+Q_{\alpha\beta}\wedge{}^*\left(\sum_{I=1}^{4}\beta_I\,{}^{(I)}Q^{\alpha\beta}\right)+\beta_5({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^{\alpha})\wedge{}^*({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_{\beta})\Bigg]-\frac{1}{2}z_4\,R^{\alpha\beta}\wedge{}^{*(4)}Z_{\alpha\beta}, \tag{5.2}$$

and

$$\alpha_1=a_1+a_0,\quad \alpha_2=a_2-2a_0,\quad \alpha_3=a_3-\frac{a_0}{2}, \tag{5.3}$$

$$\beta_1=b_1-\frac{a_0}{4},\quad \beta_2=b_2+\frac{a_0}{2},\quad \beta_3=b_3+\frac{a_0}{8},$$
$$\beta_4=b_4-\frac{3a_0}{8},\quad \beta_5=b_5-a_0, \tag{5.4}$$

$$\gamma_2=c_2-\frac{a_0}{2},\quad \gamma_3=c_3+a_0,\quad \gamma_4=c_4+a_0. \tag{5.5}$$

Correspondingly, the field momenta (3.12)–(3.14) and the gauge momentum and hypermomentum (3.15) and (3.16) can be rewritten in the form

$$M^{\alpha\beta}=M^{(0)\alpha\beta}+\hat{M}^{\alpha\beta},\quad H_{\alpha}=H_{\alpha}^{(0)}+\hat{H}_{\alpha},$$
$$H^{\alpha}{}_{\beta}=H^{(0)\alpha}{}_{\beta}+\hat{H}^{\alpha}{}_{\beta}, \tag{5.6}$$

$$E_{\alpha}=E_{\alpha}^{(0)}+\hat{E}_{\alpha},\quad E^{\alpha}{}_{\beta}=E^{(0)\alpha}{}_{\beta}+\hat{E}^{\alpha}{}_{\beta}, \tag{5.7}$$

where

$$\hat{M}^{\alpha\beta}:=-2\frac{\partial\hat{V}}{\partial Q_{\alpha\beta}}=-\frac{2}{\kappa}\Bigg[{}^*\left(\sum_{I=1}^{4}\beta_I{}^{(I)}Q^{\alpha\beta}\right)+\frac{1}{2}\beta_5\left(\vartheta^{(\alpha}\wedge{}^*(Q\wedge\vartheta^{\beta)})-\frac{1}{4}g^{\alpha\beta}\,{}^*(3Q+\Lambda)\right)+\gamma_2\,\vartheta^{(\alpha}\wedge{}^*\,{}^{(1)}T^{\beta)}+\gamma_3\,\vartheta^{(\alpha}\wedge{}^*\,{}^{(2)}T^{\beta)}+\frac{1}{4}(\gamma_3-\gamma_4)\,g^{\alpha\beta}\,{}^*T\Bigg], \tag{5.8}$$

$$\hat{H}_{\alpha}:=-\frac{\partial\hat{V}}{\partial T^{\alpha}}=-\frac{1}{\kappa}\,{}^*\Bigg[\left(\sum_{I=1}^{3}\alpha_I{}^{(I)}T_{\alpha}\right)+\left(\sum_{I=2}^{4}\gamma_I{}^{(I)}Q_{\alpha\beta}\wedge\vartheta^{\beta}\right)\Bigg], \tag{5.9}$$

$$\hat{H}^{\alpha}{}_{\beta}:=-\frac{\partial\hat{V}}{\partial R_{\alpha}{}^{\beta}}=z_4\,{}^*({}^{(4)}Z^{\alpha}{}_{\beta})=\frac{z_4}{2}\,\delta^{\alpha}_{\beta}{}^*dQ, \tag{5.10}$$

and $\hat{E}_{\alpha},\hat{E}^{\alpha}{}_{\beta}$ are defined by putting ''hats'' over corresponding terms in Eqs. (3.15) and (3.16).

Taking into account Eq. (5.10) and the identities (4.3), one can transform the field equations of MAG (3.10) and (3.11) into

$$\frac{a_0}{2}\widetilde{R}^{\mu\nu}\wedge\eta_{\alpha\mu\nu}=\kappa(\Sigma_{\alpha}-D\hat{H}_{\alpha}+\hat{E}_{\alpha}), \tag{5.11}$$

$$\frac{z_4}{2}g_{\alpha\beta}\,d{}^*dQ+\vartheta_{(\alpha}\wedge\hat{H}_{\beta)}+\hat{M}_{\alpha\beta}=\Delta_{(\alpha\beta)}, \tag{5.12}$$

$$\vartheta_{[\alpha}\wedge\hat{H}_{\beta]}=\Delta_{[\alpha\beta]}. \tag{5.13}$$

The last two equations are clearly the symmetric and the antisymmetric parts of Eq. (3.11).

Observe that the splitting in Eqs. (5.1)–(5.7) has two important consequences: With the help of the identities (4.3), the first MAG equation reduces to the Einstein equation with some effective source on the right-hand side (5.11), whereas the gauge field momenta (5.8) and (5.9) are linear combinations of irreducible parts of torsion and nonmetricity. Thus, in order to solve the second MAG field equation (5.12) and (5.13), we require an irreducible decomposition (similar to those established for torsion and nonmetricity in Sec. II) of the gauge field momenta.

A. Irreducible decomposition of ${}^*\hat{H}_{\alpha}$.

It turns out that technically it is more convenient not to decompose the gauge field momentum, but rather its Hodge dual, ${}^*\hat{H}_{\alpha}$. This quantity is a vector-valued 2-form, exactly like the torsion form, and hence its irreducible decomposition has the same structure: ${}^*\hat{H}_{\alpha}={}^{(1)}({}^*\hat{H}_{\alpha})+{}^{(2)}({}^*\hat{H}_{\alpha})+{}^{(3)}({}^*\hat{H}_{\alpha})$, where the three irreducible pieces are defined along the same lines as Eqs. (2.3)–(2.5). After some algebra, we derive from Eq. (5.9) the following expressions:

$${}^{(1)}({}^*\hat{H}^{\alpha})=\frac{1}{\kappa}(\alpha_1{}^{(1)}T^{\alpha}-\gamma_2{}^*\Omega^{\alpha}), \tag{5.14}$$

$${}^{(2)}({}^*\hat{H}^{\alpha})=\frac{1}{3\kappa}\vartheta^{\alpha}\wedge(\alpha_2\,T+\gamma_3\,\Lambda-3\gamma_4\,Q), \tag{5.15}$$

$${}^{(3)}({}^*\hat{H}^{\alpha})=\frac{1}{\kappa}\alpha_3\,{}^{(3)}T^{\alpha}. \tag{5.16}$$

As is well known (see, e.g., [1] Appendix A.1.7), an algebraic equation of the type (5.13) can be solved explicitly with respect to the gauge field momentum. The solution reads

$$\hat{H}^{\alpha}=-2e_{\beta}\rfloor\Delta^{[\alpha\beta]}+\tfrac{1}{2}\vartheta^{\alpha}\wedge(e_{\mu}\rfloor e_{\nu}\rfloor\Delta^{[\mu\nu]}). \tag{5.17}$$

In our case, the spin current 3-form $\Delta^{[\alpha\beta]}$ is given in terms of the hyperfluid expression (3.6). Substituting this into the dual of the right-hand side of Eq. (5.17), decomposing it into irreducible pieces, and using Eqs. (5.14)–(5.16) for the dual of the left-hand side of Eq. (5.17), we find that

$$\alpha_1\ {}^{(1)}T^{\alpha}-\gamma_2\ {}^*\Omega^{\alpha}=\frac{4\kappa}{3}\ u^{(\alpha}\tau^{\mu)\nu}\ \vartheta_{\mu}\wedge\vartheta_{\nu}, \tag{5.18}$$

$$\alpha_2 T+\gamma_3\Lambda-3\gamma_4\ Q=0, \tag{5.19}$$

$$\alpha_3\ {}^{(3)}T^{\alpha}=-\frac{\kappa}{2}u^{[\alpha}\tau^{\mu\nu]}\ \vartheta_{\mu}\wedge\vartheta_{\nu}, \tag{5.20}$$

where we have defined $\tau_{\mu\nu}:=J_{[\mu\nu]}$ and used the Frenkel condition (3.8).

B. Irreducible decomposition of ${}^*\hat{M}_{\alpha\beta}$.

We observe that the Hodge dual of the gauge momentum, ${}^*\hat{M}_{\alpha\beta}$, is a symmetric tensor-valued 1-form, exactly like the nonmetricity $Q_{\alpha\beta}$. Hence, we can decompose this quantity into four irreducible parts ${}^*\hat{M}_{\alpha\beta}={}^{(1)}({}^*\hat{M}_{\alpha\beta})+{}^{(2)}({}^*\hat{M}_{\alpha\beta})+{}^{(3)}({}^*\hat{M}_{\alpha\beta})+{}^{(4)}({}^*\hat{M}_{\alpha\beta})$, the structure of which is determined by the pattern (2.6)–(2.9). From Eq. (5.8) we find

$${}^{(1)}({}^*\hat{M}_{\alpha\beta})=-\frac{2}{\kappa}\beta_1\ {}^{(1)}Q_{\alpha\beta}, \tag{5.21}$$

$${}^{(2)}({}^*\hat{M}_{\alpha\beta})=-\frac{2}{\kappa}\,{}^*\left(\vartheta_{(\alpha}\wedge\left[\frac{2}{3}\beta_2\ \Omega_{\beta)}+\gamma_2\ {}^{*(1)}T_{\beta)}\right]\right), \tag{5.22}$$

$${}^{(3)}({}^*\hat{M}_{\alpha\beta})=-\frac{2}{\kappa}\left(\vartheta_{(\alpha}e_{\beta)}\rfloor\mathcal{M}-\frac{1}{4}g_{\alpha\beta}\mathcal{M}\right), \tag{5.23}$$

$${}^{(4)}({}^*\hat{M}_{\alpha\beta})=-\frac{2}{\kappa}g_{\alpha\beta}\left(\beta_4\ Q-\frac{1}{8}\beta_5\ \Lambda-\frac{1}{4}\gamma_4\ T\right), \tag{5.24}$$

where the 1-form $\mathcal{M}$ is defined by

$$\mathcal{M}:=\frac{4}{9}\beta_3\ \Lambda+\frac{1}{3}\gamma_3\ T-\frac{1}{2}\beta_5\ Q. \tag{5.25}$$

Let us analyze the symmetric equation (5.12). Separating out the trace yields

$$z_4\ d\ {}^*dQ+\frac{1}{\kappa}\,{}^*\left(-4\beta_4\ Q+\frac{1}{2}\beta_5\Lambda+\gamma_4 T\right)=\frac{1}{2}\Delta, \tag{5.26}$$

where $\Delta:=\Delta^{\alpha}{}_{\alpha}$ denotes the dilation current 3-form. For the hyperfluid we find $\Delta=J^{\alpha}{}_{\alpha}\ u$. Notice that, in view of the Frenkel condition (3.8), $\vartheta^{\alpha}\wedge\hat{H}_{\alpha}=0$ is an immediate consequence of Eq. (5.17). Subtracting, Eq. (5.26) from Eq. (5.12) yields a traceless algebraic equation which relates torsion and nonmetricity to the pure shear current $\not{\Delta}_{(\alpha\beta)}:=\Delta_{(\alpha\beta)}-\frac{1}{4}g_{\alpha\beta}\Delta$. Substituting Eq. (5.17) into Eq. (5.12) and decomposing the Hodge dual of the resulting traceless equation, we find, after some algebra and on comparison with Eqs. (5.21)–(5.23),

$$\beta_1{}^{(1)}Q_{\alpha\beta}=-\frac{\kappa}{2}\left(u_{(\alpha}\zeta_{\beta\gamma)}-\frac{\zeta}{6}\ u_{(\alpha}g_{\beta\gamma)}\right)\vartheta^{\gamma}, \tag{5.27}$$

$$\frac{2}{3}\beta_2\Omega_{\alpha}+\gamma_2{}^{*(1)}T_{\alpha}=-\frac{\kappa}{3}\left(u_{\mu}\left[\zeta_{\alpha\nu}-\frac{1}{3}\ \zeta\ g_{\alpha\nu}\right]+2u_{(\alpha}\tau_{\mu)\nu}\right)\eta^{\mu\nu}, \tag{5.28}$$

$$\mathcal{M}=\frac{\kappa}{18}\ {}^*\Delta, \tag{5.29}$$

where $\zeta_{\alpha\beta}:=J_{(\alpha\beta)}$ is the strain (shear plus dilation) density and $\zeta:=\zeta^{\alpha}{}_{\alpha}=J^{\alpha}{}_{\alpha}$ the pure dilation density.

VI. GENERIC SOLUTIONS FOR TORSION AND NONMETRICITY

We are now in a position to determine the irreducible parts of torsion and nonmetricity as solutions of the second field equation of MAG which has been separated into its symmetric and antisymmetric parts, Eqs. (5.12) and (5.13), respectively. In order to achieve this, we have to take the final step and resolve the combined system of algebraic equations (5.18)–(5.20), (5.27)-(5.29). Firstly, let us assume that the coupling constants $\alpha_1,\beta_2,\gamma_2$ are such that

$$k_3:=2\alpha_1\beta_2-3\gamma_2^2\neq 0. \tag{6.1}$$

Then, Eqs. (5.18) and (5.28) yield

$${}^{(1)}T_{\alpha}=\kappa\left(\frac{2\left(\frac{4}{3}\beta_1+\gamma_2\right)}{k_3}u_{(\alpha}\tau_{\mu)\nu}+\frac{\gamma_2}{k_3}u_{\mu}\left[\zeta_{\alpha\nu}-\frac{1}{3}\ \zeta\ g_{\alpha\nu}\right]\right)\vartheta^{\mu}\wedge\vartheta^{\nu}, \tag{6.2}$$

$${}^*\Omega_{\alpha}=\kappa\left(\frac{2(\alpha_1+2\gamma_2)}{k_3}u_{(\alpha}\tau_{\mu)\nu}+\frac{\alpha_1}{k_3}u_{\mu}\left[\zeta_{\alpha\nu}-\frac{1}{3}\ \zeta\ g_{\alpha\nu}\right]\right)\vartheta^{\mu}\wedge\vartheta^{\nu}. \tag{6.3}$$

Next, let us introduce three more constants

$$k_0:=4\alpha_2\beta_3-3\gamma_3^2,\quad k_1:=9\left(\frac{1}{2}\alpha_2\beta_5-\gamma_3\gamma_4\right),$$
$$k_2:=3\left(4\beta_3\gamma_4-\frac{3}{2}\beta_5\gamma_3\right), \tag{6.4}$$

and assume that $k_0\neq 0$.

Then Eqs. (5.19) and (5.29), considered as an algebraic system for Λ and T [recall (5.25)], yield

$$\Lambda = \frac{k_1}{k_0}Q + \kappa\frac{\alpha_2}{2k_0}\,{}^*\Delta, \tag{6.5}$$

$$T = \frac{k_2}{k_0}Q - \kappa\frac{\gamma_3}{2k_0}\,{}^*\Delta. \tag{6.6}$$

Substituting this into Eq. (5.26), we find

$$z_4(d{}^*dQ + m^2\,{}^*Q) = \frac{1}{2}\left(1 - \frac{k_1}{9k_0}\right)\Delta, \tag{6.7}$$

where we have denoted

$$m^2 := \frac{1}{z_4\kappa}\left(-4\beta_4 + \frac{k_1}{2k_0}\beta_5 + \frac{k_2}{k_0}\gamma_4\right). \tag{6.8}$$

Thus, all the post-Riemannian geometrical quantities are now determined. The Weyl 1-form Q satisfies the *Proca-type* differential equation (6.7), which describes a covector particle of mass m interacting with the dilation current $\Delta = \zeta\, u$. The remaining irreducible torsion and nonmetricity pieces are constructed algebraically from the Weyl covector Q, the spin current $\Delta_{[\alpha\beta]} = \tau_{\alpha\beta}\, u$, and the strain current $\Delta_{(\alpha\beta)} = \zeta_{\alpha\beta}\, u$.

To summarize, the first, second, and third pieces of the torsion are described by Eqs. (6.2), (6.6), and (5.20), respectively, whereas the 1st, 2nd, and 3rd pieces of the nonmetricity are given by Eqs. (5.27), (6.3), and (6.5), respectively.

This completes the solution of the second field equation of MAG (5.12)–(5.13). We now turn to the analysis of the first field equation of MAG which has the form of an effective Einstein equation (5.11).

VII. EFFECTIVE EINSTEIN THEORY

It is a straightforward task to substitute the results of the previous section into the right-hand side of Eq. (5.11), but an extremely lengthy calculation is required to simplify the result. We have to expand the covariant exterior derivatives D [which appear in Eqs. (3.9) and in (5.11)] in terms of the Riemannian operator $\widetilde{D}$ and possible contributions from the distortion 1-form (2.2). At this stage the decomposition (2.10) is most useful. Another (related) point is the substitution of torsion and nonmetricity into the covariant exterior derivatives in the equations of motion of the hypermomentum (3.5), which then reduce to

$$\widetilde{D}(\tau_{\alpha\beta}u) = -u_\alpha u^\lambda \widetilde{D}(\tau_{\lambda\beta}u) + u_\beta u^\lambda \widetilde{D}(\tau_{\lambda\alpha}u), \tag{7.1}$$

$$\widetilde{D}(\sigma_{\alpha\beta}u) = -u_\alpha u^\lambda \widetilde{D}(\sigma_{\lambda\beta}u) - u_\beta u^\lambda \widetilde{D}(\sigma_{\lambda\alpha}u) - 2\kappa(A+B)\tau^\lambda{}_{(\alpha}\sigma_{\beta)\lambda}\,\eta, \tag{7.2}$$

where the constants A, B are given below in Eqs. (7.8) and (7.9).

As a result of this calculation we find that the effective Einstein equation (5.11) reads

$$\frac{a_0}{2}\,\eta_{\alpha\beta\gamma}\wedge \widetilde{R}^{\beta\gamma} + \lambda\,\eta_\alpha = \kappa(\Sigma_\alpha^{\text{fluid}} + \Sigma_\alpha^{\text{weyl}}), \tag{7.3}$$

where the effective energy-momentum currents are given by

$$\Sigma_\alpha^{\text{fluid}} := \varepsilon_{\text{eff}}\, u u_\alpha + p_{\text{eff}}(\eta_\alpha + u u_\alpha) + \eta^\beta\{2(g^{\mu\nu} - u^\mu u^\nu)e_\mu\rfloor \widetilde{D}(u_{(\alpha}\tau_{\beta)\nu}\}. \tag{7.4}$$

$$\Sigma_\alpha^{\text{weyl}} := \frac{z_4}{2}\{(e_\alpha\rfloor dQ)\wedge{}^*dQ - (e_\alpha\rfloor{}^*dQ)\wedge dQ + m^2[(e_\alpha\rfloor Q){}^*Q + (e_\alpha\rfloor{}^*Q)\wedge Q]\}, \tag{7.5}$$

In the derivation of Eqs. (7.3) and (7.4), the equation of motion (7.1) of spin was used. The effective energy and pressure are defined by

$$\varepsilon_{\text{eff}} = \varepsilon - \frac{\kappa}{2}(A\tau_{\mu\nu}\tau^{\mu\nu} + B\zeta_{\mu\nu}\zeta^{\mu\nu} - C\,\zeta^2), \tag{7.6}$$

$$p_{\text{eff}} = p - \frac{\kappa}{2}(A\tau_{\mu\nu}\tau^{\mu\nu} + B\zeta_{\mu\nu}\zeta^{\mu\nu} - C\,\zeta^2), \tag{7.7}$$

where we denoted

$$A = \frac{3\alpha_1 + 12\gamma_2 + 8\beta_2}{3k_3} + \frac{1}{6\alpha_3}, \tag{7.8}$$

$$B = \frac{\alpha_1}{3k_3} + \frac{1}{12\beta_1}, \tag{7.9}$$

$$C = \frac{\alpha_1}{9k_3} + \frac{1}{72\beta_1} - \frac{\alpha_2}{36k_0}. \tag{7.10}$$

It is straightforward to see from Eq. (7.1) and (7.2) that the quadratic invariants constructed from spin and strain satisfy

$$u\wedge d(\tau_{\mu\nu}\tau^{\mu\nu}) = 2\tau_{\mu\nu}\tau^{\mu\nu}\, du, \tag{7.11}$$

$$u\wedge d(\zeta_{\mu\nu}\zeta^{\mu\nu}) = 2\zeta_{\mu\nu}\zeta^{\mu\nu}\, du. \tag{7.12}$$

As usual, the translational equations of motion for matter can be obtained from the (effective) Einstein equation. Since the covariant differential $\widetilde{D}$ of the left-hand side of Eq. (7.3) vanishes, one finds

$$\begin{aligned}\widetilde{D}(\Sigma_\alpha^{\text{fluid}} + \Sigma_\alpha^{\text{weyl}}) = {}& u_\alpha[-u\wedge d\varepsilon_{\text{eff}} + (\varepsilon_{\text{eff}} + p_{\text{eff}})du] \\ & - (\varepsilon_{\text{eff}} + p_{\text{eff}})u\wedge \widetilde{D}u_\alpha - (\eta_\alpha + u u_\alpha)dp_{\text{eff}} \\ & - (e_\alpha\rfloor \widetilde{R}_{\mu\nu})\wedge\Delta^{[\mu\nu]} - \frac{1}{2}\left(1 - \frac{k_1}{9k_0}\right) \\ & \times (e_\alpha\rfloor dQ)\wedge\Delta + 2\widetilde{D}(\Delta_{[\alpha\beta]}u^\lambda e_\lambda\rfloor \widetilde{D}u^\beta) \\ = {}& 0,\end{aligned} \tag{7.13}$$

where Eq. (6.7) and (7.1), and the Ricci identity $\widetilde{R}_{\mu\nu}\wedge\vartheta^{\nu}=0$ were used. Contracting Eq. (7.13) with u^{α} and using Eqs. (7.11) and (7.12), one recovers the standard continuity equation

$$u\wedge d\varepsilon-(\varepsilon+p)du=0. \tag{7.14}$$

VIII. GENERAL VACUUM SOLUTION

The main aim of this paper is to consider the hyperfluid as a specific example for a material source of MAG. However, it is also interesting to study the *vacuum case* of the MAG model (3.1), which is recovered by putting all the material variables equal to zero, $\varepsilon=p=\tau_{\alpha\beta}=\zeta_{\alpha\beta}=0$. In this case, our decomposition analysis provides us with the exact general vacuum solution for the post-Riemannian pieces. Namely, it follows from the Eqs. (5.18)–(5.20) and Eqs. (5.27)–(5.29) that, in the *generic case*,

$$\alpha_3\neq 0,\quad \beta_1\neq 0,\quad k_0\neq 0,\quad k_3\neq 0, \tag{8.1}$$

[see Eq. (6.1)] the general solution for torsion and nonmetricity reads

$$^{(1)}T^{\alpha}={}^{(3)}T^{\alpha}=0,\quad {}^{(1)}Q_{\alpha\beta}={}^{(2)}Q_{\alpha\beta}=0, \tag{8.2}$$

$$Q=k_0\phi,\quad \Lambda=k_1\phi,\quad T=k_2\phi, \tag{8.3}$$

where ϕ is a 1-form. We have used Eq. (6.5) and (6.6) to derive the last line. Substituting this into Eqs. (6.7) and (7.5), we are left with the Einstein-Proca system of equations for the metric and the ϕ field,

$$\frac{a_0}{2}\,\eta_{\alpha\beta\gamma}\wedge\widetilde{R}^{\beta\gamma}+\lambda\,\eta_{\alpha}=\kappa\Sigma_{\alpha}^{\{\phi\}}, \tag{8.4}$$

$$d\,{}^{*}d\phi+m^2\,{}^{*}\phi=0, \tag{8.5}$$

where $\Sigma_{\alpha}^{\{\phi\}}=\frac{1}{2}z_4k_0^2\{(e_{\alpha}\rfloor d\phi)\wedge{}^{*}d\phi-(e_{\alpha}\rfloor{}^{*}d\phi)\wedge d\phi+m^2[(e_{\alpha}\rfloor\phi){}^{*}\phi+(e_{\alpha}\rfloor{}^{*}\phi)\wedge\phi]\}$. Using the codifferential δ and the Laplace-Beltrami operator $\square:=d\delta+\delta d$, one can rewrite Eq. (8.5) in the equivalent form

$$(\square+m^2)\phi=0,\quad \delta\phi=0. \tag{8.6}$$

The *1-form triplet* (8.3), first discovered in [10,11], was shown to yield the effective Einstein-Proca system in [12]. We have now obtained a much stronger result: Equations (8.2) and (8.3) are not merely a convenient ansatz which describes a particular vacuum solution of the MAG model (3.1), but is, in fact, its unique and the most general vacuum solution.

For some special choices of the coupling constants, the condition (8.1) may be violated; in [12], e.g., the special case $\alpha_3=0$ was considered. Then *in vacuum*, as was noticed in [12], Eq. (5.20) allows for an arbitrary 3rd irreducible torsion piece, $^{(3)}T^{\alpha}$ (or, equivalently, the pseudotrace P 1-form). However, such degenerate special MAG models are clearly unphysical, because, *in the presence of matter*, an unacceptable constraint will be imposed on the source. The above mentioned $\alpha_3=0$ yields, via Eq. (5.20), the vanishing of the spin current, $\Delta_{[\alpha\beta]}=\tau_{\alpha\beta}u=0$. Hence, we should confine our attention to the generic models satisfying Eq. (8.1), and we discard the nongeneric cases as unphysical. (In this way, one avoids unphysical solutions with free functions, which is a well-known problem in the double duality approach to Poincaré gravity [25].)

IX. COSMOLOGY WITH HYPERFLUID

As an example of nonvacuum dynamics of MAG, let us consider a cosmological model with a hyperfluid as material source. As is well known, the hydrodynamical description of cosmological matter is considered as a reasonable approximation to a realistic physical source both in the early and in the final stages of the universe's evolution. While the cosmology in Einstein's general relativity is confined to an ideal fluid with structureless elements, in MAG the hyperfluid represents a less trivial medium with microstructure, see [23].

Before starting the discussion, let us specialize our general model (3.1) a bit. Although the Lagrangian (3.1) involves 11 coupling constants (a_I,b_J,c_K), they can be combined, as we have seen, into only four essential parameters, m^2,A,B,C, which completely determine the dynamics of the effective Einstein-Proca-hyperfluid system. Hence there is some freedom in the choice of the coupling constants without basically changing the physical content of the model. In this section we will make use of this freedom in order to study more closely the model which has attracted most attention in the literature, see [6,7,4,5,10–12]. Consequently, let us specialize to the case

$$a_I=0,\ I=1,2,3,\quad b_J=0,\ J=1,2,3,5,\quad c_K=0,\ K=2,3,4, \tag{9.1}$$

so that only $b_4\neq 0$. Then the Lagrangian (3.1) reduces to a more manageable form

$$V_{\rm dil}=\frac{1}{2\kappa}\,(-a_0\,R^{\alpha\beta}\wedge\eta_{\alpha\beta}+4b_4Q\wedge{}^{*}Q)-\frac{1}{2}z_4\,R^{\alpha\beta}\wedge{}^{*(4)}Z_{\alpha\beta}. \tag{9.2}$$

Substituting Eq. (9.1) into Eqs. (5.3)–(5.5), (6.1), (6.4), (6.8), (7.8)–(7.10), we find

$$k_0=-4a_0^2,\quad k_1=0,\quad k_2=6a_0^2,\quad k_3=\frac{1}{4}a_0^2, \tag{9.3}$$

$$m^2=-\frac{4b_4}{z_4\kappa},\quad A=B=\frac{1}{a_0},\quad C=\frac{3}{8a_0}. \tag{9.4}$$

As we can see from Sec. VII, the gravitationally interacting hyperfluid in the MAG model (3.1) produces an effect similar to that of matter with spin [1] in the usual Einstein-Cartan theory: The total hypermomentum density contributes quadratic terms which modify the energy and pressure according to Eqs. (7.6) and (7.7). Assuming the absence of the strain current, we recover the Einstein-Cartan theory interacting with a Proca-like Weyl covector Q. The dilation density ζ ''counteracts'' the spin and shear, both of which produce an effective repulsion. The resulting dynamics of the gravitational field depends crucially on the relative values of the quadratic terms in Eqs. (7.6) and (7.7).

Since the effect of pure spin (effective repulsion) is well known in cosmology, let us concentrate here on the particular case of hyperfluid with diagonal specific hypermomentum density, namely $\mu^A{}_B = \mu\ \delta^A_B$. Then Eq. (3.2) reduces to the dilation hyperfluid, the elements of which have only one ''internal'' degree of freedom: they can (in an element's rest frame) change uniformly their scale. Examples of such media are well known in nonrelativistic continuum mechanics. These are, e.g., continua with finely dispersed spherical voids and liquids with nondiffusing gas bubbles [23]. The hypermomentum current (3.6) is then determined by the hypermomentum density

$$J^\alpha{}_\beta = \zeta^\alpha{}_\beta = \frac{1}{3}(\delta^\alpha_\beta + u^\alpha u_\beta)\zeta, \qquad \tau^\alpha{}_\beta = 0, \tag{9.5}$$

so that the effective term in the energy and pressure [recall Eq. (9.4)] reads

$$\frac{1}{2a_0}\left(\zeta_{\mu\nu}\zeta^{\mu\nu} - \frac{3}{8}\zeta^2\right) = -\frac{1}{48a_0}\zeta^2. \tag{9.6}$$

Therefore we conclude that purely dilational matter amplifies gravitational attraction. In particular, it accelerates rather than retards the possible collapse of a system. This happens, though, at extremely small distances due to the smallness of the correction (9.6) which enters Eqs. (7.6) and (7.7) with the gravitational constant κ.

In the general case, a massive dilation (or Weyl) field affects gravitation in a nontrivial way. However, in homogeneous cosmology, there are solutions with $R_\alpha{}^\alpha = 2dQ = 0$. In that case the kinetic terms of the type $(dQ)^2$ in the effective Einstein equation (7.5) disappear, whereas pure mass terms $(Q)^2$ simply supply new corrections to energy and pressure. Let us be more specific and look for the standard cosmological solutions with the space-time interval in the Friedman form,

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{9.7}$$

Substituting Eq. (9.7) into the effective Einstein equations (7.3) and taking Eq. (7.12) into account, we find

$$3\left(\frac{\dot{R}^2}{R^2} + \frac{K}{R^2}\right) = \kappa\left[\varepsilon + \frac{\kappa}{48a_0}\left(1 - \frac{3a_0}{b_4}\right)\frac{\zeta_0^2}{R^6}\right], \tag{9.8}$$

$$-2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{K}{R^2} = \kappa\left[p + \frac{\kappa}{48a_0}\left(1 - \frac{3a_0}{b_4}\right)\frac{\zeta_0^2}{R^6}\right], \tag{9.9}$$

where ζ_0 is an integration constant. In accordance with Eqs. (6.7), (3.5), and (7.2), we obtain

$$Q = -\frac{\kappa\zeta_0}{8b_4}\frac{dt}{R^3(t)}. \tag{9.10}$$

Evidently $dQ = 0$.

Supplementing Eqs. (9.8) and (9.9) by the equation of state $p = p(\varepsilon)$, we can solve Eq. (7.14) explicitly. Let us consider the case $p = \gamma\ \varepsilon$ with constant γ. Then Eq. (7.14) yields

$$\varepsilon = \frac{\varepsilon_0}{R^{3(1+\gamma)}}, \tag{9.11}$$

where ε_0 is a positive integration constant. Equation (9.9) is redundant, as follows from Eqs. (9.8) and (9.11). Thus the dynamics of the scale factor $R(t)$ is determined by the first order equation (9.8), with Eq. (9.11) inserted. Interestingly, for the coupling constant $b_4 = 3a_0$, this dynamics turns out to be completely standard, yielding the well-known cosmological solutions of general relativity theory. However, if one wants to interpret the $(Q)^2$ term in (3.1) as the mass term for the dilation field, then one must take a negative b_4, see, Eq. (9.4). Consequently, the dilation correction $\sim 1/R^6$ enters into the right-hand side of (9.8) with a positive coefficient, which corresponds to an additional effective *attractive* force dominating during the very early stages of evolution. Near the singularity

$$R^3(t) \approx \left(\frac{\kappa\zeta_0}{4}\sqrt{\frac{1}{a_0} - \frac{3}{b_4}}\right)t. \tag{9.12}$$

This is true for any value of the spatial curvature K and for an arbitrary equation of state with $0 \leqslant \gamma < 1$.

X. CONCLUSION

In this paper we have applied the irreducible decomposition technique to the study of the classical MAG model (3.1) which has recently attracted quite some attention in the literature. Our main observations are as follows:

Torsion and traceless nonmetricity are explicitly expressible in terms of the spin and shear currents of the hyperfluid. This enables us to reduce the general MAG field equations to the effective Einstein theory (7.3) with a source represented by the energy-momentum tensors of the Weyl (Proca-type) covector field (7.5) and of the effective (Weyssenhoff-type) spin fluid (7.4).

In vacuum, the 1-form triplet (8.3) describes the general and unique solution of the field equations of MAG. This result completes previous studies of the 1-form triplet [10–12].

As an example of a nontrivial case with matter, we have studied homogeneous cosmologies with hyperfluid. Like in the Einstein-Cartan theory, we conclude that the hypermomentum affects significantly the cosmological evolution only in the very early stages. However, contrary to the effect of spin, shear does not prevent the formation of a cosmological singularity but rather promotes it. Homogeneous cosmologies in MAG models with *ideal* fluid were recently studied in [24].

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