Gravitational waves from inspiraling compact binaries: Angular momentum flux, evolution of the orbital elements, and the waveform to the second post-Newtonian order

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The post-post-Newtonian (2PN) accurate mass quadrupole moment, for compact binaries of arbitrary mass ratio, moving in general orbits is obtained by the multipolar post Minkowskian approach of Blanchet, Damour, and Iyer. Using this, for binaries in general orbits, the 2PN contributions to the gravitational waveform, and the associated far-zone energy and angular momentum fluxes are computed. For quasielliptic orbits, the energy and angular momentum fluxes are averaged over an orbital period, and employed to determine the 2PN corrections to the rate of decay of the orbital elements. [S0556-2821(97)01824-9]

I. INTRODUCTION

Inspiraling compact binaries are one of the most promising sources of gravitational radiation for kilometer size laser interferometric gravitational wave detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) [1] and VIRGO [2]. The method of matched filtering will be employed to detect and extract information of binaries from the inspiral waveforms [3,4]. In this technique one cross correlates the noisy output of a detector with theoretical templates. For this technique to be successful, the templates must remain in phase with the exact—general relativistic— waveform as long as possible. If the signal and template lose phase with each other even by a cycle in the ten thousand as the waves sweep through the bandwidth of the detector their cross-correlation will be significantly reduced and one may lose the event altogether. Detailed works on data analysis aspects [5–8] bears out this inference and one is forced to a description of the evolution of the binary system, using the best available theory of gravity to substantially higher accuracy than that provided by the lowest order Newtonian approximation. The construction of accurate theoretical templates for inspiraling compact binaries involves the solution to two different but related aspects referred to respectively as the "wave generation problem" and the "radiation reaction problem." In the generation problem one computes the gravitational waveforms and the associated energy and angular momentum fluxes emitted by the binary for a fixed, specified orbital motion ignoring the back reaction of the radiation emission on the orbit. In the radiation reaction problem on the other hand, one computes the effect of the emitted radiation on the orbital phase evolution and this is of crucial importance as explained earlier.

Einstein's [9] far field quadrupole equation is the solution to the generation problem to the lowest order but applies only to objects held together by nongravitational forces. Fock [10] and Landau-Lifshiftz [11] provided two very different methods to generalize the above results to weakly self-gravitating systems and the two approaches are the starting points for the two methods available today to calculate gravitational wave generation to higher orders: the Blanchet-Damour-Iyer (BDI) [12] approach and the Epstein-Wagoner-Thorne-Will-Wiseman (EWTWW) [13] approach.

The BDI approach builds on a Fock-type derivation using the double-expansion method of Bonnor [14]. This approach makes a clean separation of the near-zone and the wave-zone effects. It is mathematically well defined, algorithmic, and provides corrections to the quadrupolar formalism in the form of compact support integrals or more generally well-defined analytically continued integrals. The scheme has a modular structure: the final results are obtained by combining an "external zone module" with a "near zone module" and a "radiative zone module." For dealing with strongly self-gravitating material sources such as neutron stars or black holes one needs to use a "compact body module" supplemented by an "equation of motion module" to describe their "conservative" orbital motion. Using this approach the generation of gravitational waves from inspiraling compact binaries of arbitrary mass ratio moving in a *quasicircular* orbit has been computed to 2PN accuracy [15,16] and more recently to 2.5PN accuracy [17]. In this paper, in the first instance, using the BDI approach, we extend the above 2PN treatment to inspiralling compact binaries moving in a *general orbit* and compute the 2PN contributions to the waveform and the energy flux. Unlike for circular orbits, the angular momentum flux from general orbits provides additional, independent information and we also compute the same.

The Epstein-Wagoner-Thorne-Will-Wiseman approach on the other hand, builds on a Landau-Lifshiftz type treatment to derive post-Newtonian corrections to the lowest order quadrupole formula. The combined use of an effective stress energy tensor for the gravitational field (with noncompact support) and of formal post-Newtonian expansions led to the appearance of divergent integrals. The presense of the divergent integrals and the lack of a clear separation between the near zone and the wave zone were unsatisfactory features of this scheme until recently. However, last year, Will and Wiseman [18] have provided a resolution to this problem by taking literally the statement that the solution is a *retarded integral*, i.e., an integral over the entire past null cone of the field point. A careful evaluation of the far-zone contributions, then shows that all integrations are indeed convergent and finite and moreover the tail terms are also correctly recovered. Using this treatment, Will and Wiseman have com-

puted the 2PN accurate waveform and energy flux for *general orbits*. We thus have two approaches to the 2PN generation, which can provide a useful check on the long and tedious algebra.

The most accurate results to date for the generation and the radiation reaction have been obtained in the limit where a test body orbits a very massive central body. In this complementary approach, based on black hole perturbation techniques there exist numerical results that are exact in (v/c) and analytical results accurate to the 5.5PN order, i.e., corrections of $O[(v/c)^{11}]$ for a test particle in a circular orbit around a Schwarzschild black hole [19–26], where v is the orbital velocity of the test particle. For a test particle in a slightly eccentric orbit, around a spinning black hole expressions for the energy and the angular momentum fluxes have also been computed to the 2.5PN order [24].

It is well known that, when gravitational waves from prototype systems such as the binary pulsar 1913+16 enter the bandwidth of the terrestial interferometric detectors, the eccentricity of these binary systems would have been drastically reduced and have become negligible due to the gravitational radiation reaction. A treatment of such systems is simpler since the quasicircular approximation for their orbits is amply adequate and the corresponding waveforms do not depend on the eccentricity parameter of the orbit. However, there exist scenarios in which the eccentricity is no longer negligible and this would require the more general treatment provided in this paper and independently by Will and Wiseman [18]. One such possibility has been discussed by Shapiro and Teukolsky [27] in the context of the formation of supermassive black holes. They consider a cluster of compact objects—neutron stars and black holes—residing at the center of a galactic nucleus. Coulomb scattering and dissipative processes will drive such a cluster to a high density, high redshift state. Once the central redshift becomes sufficiently large, relativistic instability sets in, and the core undergoes catastrophic collapse to form a supermassive black hole. Quinlan and Shapiro [28] have shown that during the final year of the evolution of such a cluster, just prior to the catastrophic collapse, there can be $100-10^4$ evolving black hole binaries in eccentric orbits driven by gravitational radiation reaction, with masses in the range $10-100\,M_\odot$. Ground based interferometric detectors will be sensitive to the gravitational radiation from these binaries in eccentric orbits and such eccentric binaries may prove to be another possible class of gravitational wave sources. More recently, Flanagan and Hughes [29] suggest that intermediate mass black hole binaries of the kind considered above—with total masses in the range $50M_\odot \lesssim M \lesssim$ (a few) $\times 10^3 M_\odot$—may well be the first sources to be detected by LIGO and VIRGO. The other possibility involves compact objects orbiting 10^6 to $10^7 M_\odot$ black holes, that seem fairly common in galactic nuclei. In this case the compact objects could be scattered into very eccentric orbits orbits via gravitational deflections by other stars. However, by the time gravitational radiation reaction becomes the dominant orbital driving force, there is not enough inspiral remaining to fully circularize these orbits. Hills and Bender [30] have argued that the event rates for the above process are very encouraging and the chances of such signals being observed by Laser Interferometric Space Antenna (LISA) [31] appear very good.

The expressions for the far-zone energy and angular momentum fluxes find application in another related but distinct problem—the evolution of the orbital elements of systems such as the binary pulsar 1913+16—most importantly the orbital period, and to a lesser extent the eccentricity and semimajor axis. This application, in addition to the generation results discussed earlier, requires a convenient representation of the post-Newtonian motion of two point masses in elliptical orbits. To 1PN accuracy, such a quasi-Keplerian representation has been provided by Damour and Deruelle [32], while to the 2PN order a generalized quasi-Keplerian representation has been implemented by Damour, Schäfer, and Wex [33–36]. This representation differs from the Keplerian representation of the Newtonian motion through the appearance of three eccentricities instead of one, and a constant measuring the secular advance of the periastron. Starting from the above representation of the orbital elements in terms of the conserved energy and angular momentum, one computes the time variation of the orbital elements. One ends up with a result, in terms of the time variation of the ''conserved'' energy and angular momentum. By a heuristic argument, one replaces these by the corresponding *average* far-zone fluxes which may be computed by averaging the far-zone fluxes over an orbital period, using the quasi-Keplerian orbital representation. The reduction in the orbital period, accurately inferred from the timing data of the binary pulsars is in excellent agreement with the rigorous predictions of general relativity [37–41], which in turn are consistent with the results of the above heuristic approach [42,44,45]. Extending the above approach, Blanchet and Schäfer have obtained the 1PN and the 1.5PN corrections to $\dot{P}$, the rate of decay of the orbital period P [46,47]. They have shown that for PSR 1913+16, the relative 1PN and 1.5PN corrections are numerically equal to $+2.15 \times 10^{-5}$ and $+1.65 \times 10^{-7}$, respectively. These are unfortunately far below the present accuracy in the measurements of $\dot{P}$ for 1913+16. Junker and Schäfer [48] computed the 1PN contributions to the gravitational waveforms, the associated angular and linear momentum fluxes and used it to compute the evolution of the orbital elements in the quasi-Keplerian representation. In the other part of the paper, we extend the above computations to obtain the 2PN corrections to the evolution of orbital elements, taking due care of a new complication at this order that the far-zone fluxes are computed in the harmonic or De Donder coordinates, while the orbital representation is available in the Arnowitt-Deser-Misner (ADM) coordinates.

Briefly, in this paper we obtain the terms $O(\epsilon^2)$ in the expressions below, where $\epsilon \sim v^2/c^2 \approx Gm/c^2\,r$; m, r, v, being the total mass, the distance between the bodies and the relative velocity of the two bodies, respectively,

$$I_{ij} = (I_{ij})_N \{1 + O(\epsilon) + O(\epsilon^2) + \cdots\}, \qquad (1.1a)$$

$$h_{km}^{TT} = (h_{km}^{TT})_N \{1 + O(\epsilon^{0.5}) + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\}, \qquad (1.1b)$$

$$\frac{d\mathcal{E}}{dt} = \left(\frac{d\mathcal{E}}{dt}\right)_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\}, \qquad (1.1c)$$

$$\frac{d\mathcal{J}}{dt} = \left(\frac{d\mathcal{J}}{dt}\right)_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\},$$

(1.1d)

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle = \left\langle\frac{d\mathcal{E}}{dt}\right\rangle_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\},$$

(1.1e)

$$\left\langle\frac{d\mathcal{J}}{dt}\right\rangle = \left\langle\frac{d\mathcal{J}}{dt}\right\rangle_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\},$$

(1.1f)

$$\left\langle\frac{da_r}{dt}\right\rangle = \left\langle\frac{da_r}{dt}\right\rangle_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\},$$

(1.1g)

$$\left\langle\frac{de_r}{dt}\right\rangle = \left\langle\frac{de_r}{dt}\right\rangle_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\},$$

(1.1h)

$$\frac{dP}{dt} = \left(\frac{dP}{dt}\right)_N \{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + \cdots\},$$

(1.1i)

and where I_{ij} is the mass quadrupole moment for a system of two compact objects moving in general orbits while h_{km}^{TT} is the transverse-traceless (TT) part of the radiation field, representing the deviation of the metric from the flat spacetime. In the above $d\mathcal{E}/dt, d\mathcal{J}/dt$ are the far-zone energy and angular momentum fluxes, $\langle d\mathcal{E}/dt\rangle$ and $\langle d\mathcal{J}/dt\rangle$ represent the averages of the far-zone fluxes over an orbital period, while $\langle da_r/dt\rangle$, $\langle de_r/dt\rangle$ along with dP/dt give the gravitational radiation driven rate of decay of the orbital elements of the

binary in the generalized quasi-Keplerian parametrization. Note that the suffix N denotes Newtonian contribution in all the above equations. For example $(h_{km}^{TT})_N$ denotes the Newtonian contribution to the waveform given by $\{2\,G/(c^4\,R)\}\mathcal{P}_{ijkm}\,I_{ij}^{(2)}$. See Eq. (5.1) for our notation.

The plan of the paper is as follows. In Sec. II, using the BDI approach, we compute the 2PN accurate mass quadrupole moment for two masses moving on general orbits. We also obtain and list the mass and the current moments to the required accuracy, needed to compute the 2PN accurate waveform. In Sec. III we calculate the 2PN contributions to the far-zone energy and angular momentum fluxes and discuss the limiting forms of these expressions. In Sec. IV, for the quasielliptic case, we average the above expressions over an orbital period to obtain the 2PN corrections to $\langle d\mathcal{E}/dt\rangle$, $\langle d\mathcal{J}/dt\rangle$, and the rate of decay of the orbital elements. Section V computes the 2PN contribution to gravitational wave form for general orbits. Section VI contains the summary and a few concluding remarks. In Appendix A we list identities, that are used in the computations, especially, of the waveform. Finally, in Appendix B, we sketch the steps involved in verifying the equivalence of our waveform obtained using the symmetric trace-free (STF) multipoles of the radiative field and the Will-Wiseman one obtained using the Epstein-Wagoner multipoles.

II. MASS AND CURRENT MOMENTS OF COMPACT BINARIES ON GENERAL ORBITS FOR 2PN GENERATION

A. 2PN mass quadrupole moment

The starting point for the computation of the 2PN accurate mass moment is the form of the moment quoted in [16], i.e., Eq. (2.17):

$$I_L(t) = \text{FP}_{B=0} \int d^3\mathbf{x}|\mathbf{x}|^B \left\{ \hat{x}_L \left[\sigma - \frac{4}{c^4}\sigma U_{ss} + \frac{4}{c^4}U\sigma_{ss} \right] + \frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2l+3)}\partial_t^2\sigma - \frac{4(2l+1)\hat{x}_{iL}}{c^2(l+1)(2l+3)}\partial_t\left[\left(1 + \frac{2U}{c^2}\right)\sigma_i \right] \right.$$

$$- \frac{2U_i}{c^2}\sigma + \frac{1}{\pi G c^2}\left(\partial_j U \partial_i U_j - \frac{3}{4}\partial_i U \partial_j U_j \right) \right] + \frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2l+3)(2l+5)}\partial_t^4\sigma - \frac{2(2l+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(l+1)(2l+3)(2l+5)}\partial_t^3\sigma_i$$

$$+ \frac{2(2l+1)\hat{x}_{ijL}}{c^4(l+1)(l+2)(2l+5)}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U \partial_j U \right] + \frac{\hat{x}_L}{\pi G c^4}\left[2U_i\partial_{ij}U_j - U_{ij}\partial_{ij}U - \frac{1}{2}(\partial_i U_i)^2 \right.$$

$$\left. + 2\partial_i U_j \partial_j U_i - \frac{1}{2}\partial_t^2(U^2) + W_{ij}\partial_{ij}U \right] \right\} + O(\varepsilon^5).$$

(2.1)

The symbol $\text{FP}_{B=0}$ in the above stands for "finite part at $B=0$" and denotes a mathematically well-defined operation of analytic continuation. For more details see [16].

As emphasized in [16] though the above expression is mathematically well defined, it is a nontrivial and long calculation to rewrite it explicitly in terms of the source variables only. This is achieved by representing the stress energy tensor of the source as a sum of Dirac δ functions:

$$T^{\mu\nu}(\mathbf{x},t) = \sum_{A=1}^{N} m_A \frac{dy_A^\mu}{dt}\frac{dy_A^\nu}{dt}\frac{1}{\sqrt{-g}}\frac{dt}{d\tau}\delta(\mathbf{x} - \mathbf{y}_A(t)),$$

(2.2)

where m_A denotes the (constant) Schwarzschild mass of the Ath compact body. Evaluating this to 2PN accuracy we obtain for the source variables

$$\sigma(\mathbf{x},t)=\sum_{A=1}^{N}\mu_A(t)\left(1+\frac{\mathbf{v}_A^2}{c^2}\right)\delta+(\mathbf{x}-\mathbf{y}_A(t)),\quad(2.3a)$$

$$\sigma_i(\mathbf{x},t)=\sum_{A=1}^{N}\mu_A(t)v_A^i\delta(\mathbf{x}-\mathbf{y}_A(t)),\qquad(2.3b)$$

$$\sigma_{ij}(\mathbf{x},t)=\sum_{A=1}^{N}\mu_A(t)v_A^iv_A^j\delta(\mathbf{x}-\mathbf{y}_A(t)),\qquad(2.3c)$$

where $v_A^i\equiv dy_A^i/dt$ and

$$\mu_A(t)=m_A\{1+(d_2)_A+(d_4)_A\},\qquad(2.4a)$$

$$d_2\equiv\frac{1}{c^2}\left\{\frac{1}{2}\mathbf{v}^2-V\right\},\qquad(2.4b)$$

$$d_4\equiv\frac{1}{c^4}\left\{\frac{3}{8}\mathbf{v}^4+\frac{3}{2}U\mathbf{v}^2-4U_iv_i-2\Phi+\frac{3}{2}U^2+4U_{ss}\right\}.\qquad(2.4c)$$

In the above V denotes the combination

$$V\equiv U+\frac{1}{2c^2}\partial_t^2X\ ,\qquad(2.5)$$

the potential appearing naturally in the 1PN near-zone metric in harmonic coordinates. The subscript A appearing in Eq. (2.4a) indicates that one must replace the field point $\mathbf{x}$ by the position $\mathbf{y}_A$ of the Ath mass point, while discarding all the ill-defined (formally infinite) terms arising in the limit $\mathbf{x}\rightarrow\mathbf{y}_A$. For instance

$$(U)_A=G\sum_{B\neq A}\frac{\mu_B(t)(1+\mathbf{v}_B^2/c^2)}{|\mathbf{y}_A-\mathbf{y}_B|},\qquad(2.6a)$$

$$(U_{ss})_A=G\sum_{B\neq A}\frac{\mu_B(t)\mathbf{v}_B^2}{|\mathbf{y}_A-\mathbf{y}_B|},\qquad(2.6b)$$

$$(\Phi)_A=G\sum_{B\neq A}\frac{\mu_B(t)(1+\mathbf{v}_B^2/c^2)(U)_B}{|\mathbf{y}_A-\mathbf{y}_B|},\qquad(2.6c)$$

$$(X)_A=G\sum_{B\neq A}\mu_B(t)(1+\mathbf{v}_B^2/c^2)|\mathbf{y}_A-\mathbf{y}_B|.\qquad(2.6d)$$

[Note that the second time derivative appearing in V, Eq. (2.5), must be explicated before making the replacement $\mathbf{x}\rightarrow\mathbf{y}_A(t)$.]

The terms in Eq. (2.1) fall into three types: compact terms, Y terms, and W terms. The compact terms, where the three-dimensional integral extends only over the compact support of the material sources—the Y terms involving three-dimensional integral of the product of two Newtonian-like potentials—and the W term involving three-dimensional integrals of terms trilinear in source variables. The evaluation of these different terms proceeds exactly as in the circular case. In fact if the time derivatives are not explicitly implemented the expression in the general case and the circular case would be identical. The difference obtains when the time derivatives are implemented using the equation of

motion. In this section we need to use the general form of the Damour-Deruelle equations of motion rather than the restricted form of the circular orbit equations of motion relevant in [16].

We take up the compact terms first. They are given by

$$I_L^{(C)}=\sum_{A=1}^{N}\left\{\widetilde{\mu}_A\left[1-\frac{4}{c^4}U_{ss}^A+\frac{4}{c^4}U^A(\mathbf{v}_A)^2\right]\hat{y}_A^L\right.$$

$$+\frac{1}{2(2l+3)c^2}\frac{d^2}{dt^2}(\widetilde{\mu}_A\mathbf{y}_A^2\hat{y}_A^L)$$

$$+\frac{1}{8(2l+3)(2l+5)c^4}\frac{d^4}{dt^4}(\widetilde{\mu}_A(\mathbf{y}_A^2)^2\hat{y}_A^L)$$

$$-\frac{4(2l+1)}{(l+1)(2l+3)c^2}\frac{d}{dt}$$

$$\times\left(\left[\mu_A\left(1+\frac{2U^A}{c^2}\right)v_A^i-\frac{2U_i^A}{c^2}\widetilde{\mu}_A\right]\hat{y}_A^{iL}\right)$$

$$-\frac{2(2l+1)}{(l+1)(2l+3)(2l+5)c^4}\frac{d^3}{dt^3}(\mu_Av_A^i\mathbf{y}_A^2\hat{y}_A^{iL})$$

$$\left.+\frac{2(2l+1)}{(l+1)(l+2)(2l+5)c^4}\frac{d^2}{dt^2}(\mu_Av_A^iv_A^j\hat{y}_A^{ijL})\right\}\ ,$$

$$(2.7)$$

in which we have introduced for convenience $\widetilde{\mu}_A\equiv\mu_A(1+\mathbf{v}_A^2/c^2)$. In the above form the moment depends not only on the position and velocity of the bodies but on higher time derivatives. It is in the reduction of these derivatives that we need the 2PN accurate equation of motion for general orbits. We use a harmonic coordinate system in which the 2PN center of mass is at rest at the origin. Using the 2PN accurate center of mass theorem, in the center of mass frame, we can express the individual positions of the two bodies moving in general orbits in terms of their relative position $\mathbf{x}=\mathbf{y}_1-\mathbf{y}_2$ and velocity $\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2$:

$$\mathbf{y}_1=\left\{X_2+\frac{\eta\delta m}{2mc^2}\left[v^2-\frac{Gm}{r}\right]+\frac{\chi_1}{c^4}\right\}\mathbf{x}+\frac{\chi_2}{c^4}\mathbf{v},\quad(2.8a)$$

$$\mathbf{y}_2=\left\{-X_1+\frac{\eta\delta m}{2mc^2}\left[v^2-\frac{Gm}{r}\right]+\frac{\chi_1}{c^4}\right\}\mathbf{x}+\frac{\chi_2}{c^4}\mathbf{v},\quad(2.8b)$$

where $r=|\mathbf{y}_1-\mathbf{y}_2|$ is the harmonic separation between the two bodies. The explicit values of χ_1 and χ_2 are not needed in our calculations and hence not given above. The above equations are obtained by setting equal to zero the conserved mass dipole $\mathbf{G}$ for general orbits. Here we denote

$$m\equiv m_1+m_2,\quad\delta m\equiv m_1-m_2,$$

$$X_1\equiv\frac{m_1}{m},\quad X_2\equiv\frac{m_2}{m}=1-X_1,$$

$$\eta\equiv X_1X_2=\frac{m_1m_2}{m^2}\equiv\frac{\mu}{m}\ .\qquad(2.9)$$

The 2PN accurate equations of motion is written down next for completeness, where finite-size effects, such as spin-orbit, spin-spin, or tidal interactions are ignored [49–51]. For the relative motion we have

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{PN}^{(1)} + \mathbf{a}_{2PN}^{(2)} + O(\epsilon^5), \qquad (2.10)$$

where the subscripts denote the nature of the term, Newtonian (N), post-Newtonian (PN), post-post-Newtonian (2PN), and the superscripts denote the order in ϵ. The explicit expressions for various terms mentioned above are given by

$$\mathbf{a}_N = -\frac{Gm}{r^2}\mathbf{n}, \qquad (2.11a)$$

$$\mathbf{a}_{PN}^{(1)} = -\frac{Gm}{c^2 r^2}\left\{\left[-2(2+\eta)\frac{Gm}{r} + (1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2\right]\mathbf{n} - 2(2-\eta)\dot{r}\mathbf{v}\right\}, \qquad (2.11b)$$

$$\mathbf{a}_{2PN}^{(2)} = -\frac{Gm}{c^4 r^2}\left\{\left[\frac{3}{4}(12+29\eta)\frac{G^2 m^2}{r^2} + \eta(3-4\eta)v^4 + \frac{15}{8}\eta(1-3\eta)\dot{r}^4 - \frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2 \right.\right.$$

$$\left. -\frac{1}{2}\eta(13-4\eta)\frac{Gm}{r}v^2 -(2+25\eta+2\eta^2)\frac{Gm}{r}\dot{r}^2\right]\mathbf{n}$$

$$-\frac{1}{2}\left[\eta(15+4\eta)v^2 - (4+41\eta+8\eta^2)\frac{Gm}{r}\right.$$

$$\left.\left. -3\eta(3+2\eta)\dot{r}^2\right]\dot{r}\mathbf{v}\right\}, \qquad (2.11c)$$

where $\mathbf{n} = \mathbf{x}/r$ and $\dot{r} = dr/dt$.

We have on hand all the ingredients to compute I_c. Though long and tedious the computation is straightforward and yields for the 2PN mass quadrupole:

$$I_{ij}^{[C]} = \eta m\, \text{STF}_{ij}\left\{ x^{ij} + \frac{1}{42c^2}\left\{ x^{ij}\left[29(1-3\eta)v^2 - 6(5-8\eta)\frac{Gm}{r}\right] - 24(1-3\eta)r\dot{r}x^i v^j + 22(1-3\eta)r^2 v^{ij}\right\}\right.$$

$$+ \frac{1}{1512c^4}\left[v^4(759 - 5505\eta + 10635\eta^2) + \frac{G^2 m^2}{r^2}(1758 - 6468\eta + 1878\eta^2) + v^2\frac{Gm}{r}(5818 - 16742\eta - 12166\eta^2)\right.$$

$$\left. -\dot{r}^2\frac{Gm}{r}(2038 - 6662\eta + 146\eta^2)\right]x^{ij} + \frac{1}{378c^4}\left[v^2(123 - 1011\eta + 2199\eta^2) + \frac{Gm}{r}(68 + 434\eta - 2090\eta^2)\right.$$

$$\left.\left. + 30\dot{r}^2(1 - 5\eta + 5\eta^2)\right]r^2 v^{ij} - \frac{1}{378c^4}\left[\frac{Gm}{r}(101 + 287\eta - 1655\eta^2) + v^2(156 - 1212\eta + 2508\eta^2)\right]r\dot{r}x^i v^j\right\}. \qquad (2.12)$$

The Y terms on the other hand are given by

$$I_{ij}^{[Y]} = -\frac{2Gm_1 m_2}{c^4}\left\{ 2Y_{v_1 v_2}^{ij} - Y_{v_1 v_1}^{ij} - \frac{1}{2}v_1 Y_{v_2}^{ij} + 2v_2 Y_{v_1}^{ij}\right.$$

$$-\frac{1}{2}\partial_t^2(Y^{ij}) - \frac{20}{21}\partial_t\left[v_2 Y_a^{aij} - \frac{3}{4}v_1 Y_a^{aij}\right]$$

$$\left. + \frac{5}{216}\partial_t^2[_a Y_b^{abij}]\right\} + (1 \leftrightarrow 2), \qquad (2.13)$$

where following [16]

$$_{v_1}Y_{v_2}^L = v_1^a v_2^b\, _a Y_b^L, \qquad (2.14a)$$

$$_a Y_b^L = \partial_{y_1^a}\partial_{y_2^b}Y^L, \qquad (2.14b)$$

$$Y^L(\mathbf{y_1},\mathbf{y_2}) = \frac{|\mathbf{y_1}-\mathbf{y_2}|}{l+1}\sum_{p=0}^{l} y_1^{\langle l-p}y_2^{p\rangle}, \qquad (2.14c)$$

so that

$$Y_{v_1 v_2}^{ij} \equiv 2v_1^s v_2^k Y_{sk}^{ij}, \qquad (2.15a)$$

$$Y_{sk}^{ij} = \frac{1}{3}\frac{\partial}{\partial y_2^s}\frac{\partial}{\partial y_2^k}r_{12}(y_1^{ij} + y_1^{(i}y_2^{j)} + y_2^{ij}), \qquad (2.15b)$$

$$r_{12} = |\mathbf{y_1}-\mathbf{y_2}|. \qquad (2.15c)$$

The explication of all the above terms finally leads us to

$$I_{ij}^{[Y]} = -\frac{2m\eta}{63}\frac{Gm}{c^4 r}\, \text{STF}_{ij}\left\{ x^{ij}\left[(v^2 - \dot{r}^2)(37 - 101\eta - 50\eta^2)\right.\right.$$

$$\left. + \frac{Gm}{r}(18 - 54\eta - 3\eta^2)\right] - r^2 v^{ij}(118 - 92\eta + 10\eta^2)$$

$$\left. + r\dot{r}x^i v^j(82 - 362\eta + 16\eta^2)\right\}. \qquad (2.16)$$

The evaluation of the $I^{[W]}$ term, the new feature at 2PN level, was discussed in detail in [16]. The W term has been evaluated there for general orbits and we need to use the same result here. We have

$$I_{ij}^{[W]} = -\frac{\eta m}{c^4}\frac{G^2 m^2}{r^2}\mathrm{STF}_{ij}\{[2+5\eta]x^{ij}\}. \qquad (2.17)$$

Adding up the compact, i.e., C, Y, and W contributions given by Eqs. (2.12), (2.16), and (2.17), we finally obtain the expression for the 2PN accurate mass quadrupole for a system of two bodies moving in general orbits. The final result is written below as a combination of the three possible combinations $x^{ij}, x^i v^j, v^{ij}$ with coefficients which include corrections beyond the Newtonian order at 1PN and 2PN orders:

$$
\begin{aligned}
I_{ij} = \mu\,\mathrm{STF}_{ij}\Bigg\{ & x^{ij}\Bigg[1 + \frac{1}{42c^2}\Big((29-87\eta)v^2 - (30-48\eta)\frac{Gm}{r}\Big) + \frac{1}{c^4}\Bigg(\frac{1}{504}(253-1835\eta+3545\eta^2)v^4 + \frac{1}{756}(2021-5947\eta \\
& -4883\eta^2)\frac{Gm}{r}v^2 - \frac{1}{756}(131-907\eta+1273\eta^2)\frac{Gm}{r}\dot{r}^2 - \frac{1}{252}(355+1906\eta-337\eta^2)\frac{G^2m^2}{r^2}\Bigg)\Bigg] - x^i v^j\Bigg[\frac{r\dot{r}}{42c^2}(24-72\eta) \\
& + \frac{r\dot{r}}{c^4}\Bigg(\frac{1}{63}(26-202\eta+418\eta^2)v^2 + \frac{1}{378}(1085-4057\eta-1463\eta^2)\frac{Gm}{r}\Bigg)\Bigg] + v^{ij}\Bigg[\frac{r^2}{21\,c^2}(11-33\eta) + \frac{r^2}{c^4}\Bigg(\frac{1}{126}(41-337\eta \\
& + 733\eta^2)v^2 + \frac{5}{63}(1-5\eta+5\eta^2)\dot{r}^2 + \frac{1}{189}(742-335\eta-985\eta^2)\frac{Gm}{r}\Bigg)\Bigg]\Bigg\}.
\end{aligned}\qquad (2.18)
$$

The above expression is identical to the one obtained by Will and Wiseman in Appendix E of [18] using the new improved version of the Epstein-Wagoner formalism. In their treatment the Epstein-Wagoner multipoles appear more naturally, using which they compute the STF mass quadrupole moment. Since the approach employed here and in [18] follow algebraically different routes, the above match provides a valuable check on the long and complicated algebra involved in the determination of the crucial mass quadrupole moment for 2PN generation.

B. The other relevant mass and current moments

In this section we list the higher order mass and current multipole moments, required to compute the 2PN contributions to the gravitational waveform and the associated far-zone energy and angular momentum fluxes. They are straightforwardly obtained by explicating the point particle limits of the more general expressions in the earlier BDI papers [12]:

$$
\begin{aligned}
I_{ijk} = -\Big(\mu\,\frac{\delta m}{m}\Big)\mathrm{STF}_{ijk}\Bigg\{ & x^{ijk}\Bigg[1 + \frac{1}{6c^2}\Big((5-19\eta)v^2 \\
& - (5-13\eta)\frac{Gm}{r}\Big)\Bigg] - x^{ij}v^k\Bigg[\frac{r\dot{r}}{c^2}(1-2\eta)\Bigg] \\
& + x^i v^{jk}\Bigg[\frac{r^2}{c^2}(1-2\eta)\Bigg]\Bigg\},
\end{aligned}\qquad (2.19)
$$

$$
\begin{aligned}
I_{ijkl} = \mu\,\mathrm{STF}_{ijkl}\Bigg\{ & x^{ijkl}\Bigg[(1-3\eta) + \frac{1}{110c^2}\Big((103-735\eta \\
& + 1395\eta^2)v^2 - (100-610\eta+1050\eta^2)\frac{Gm}{r}\Big)\Bigg] \\
& - v^i x^{jkl}\Bigg[\frac{72r\dot{r}}{55c^2}(1-5\eta+5\eta^2)\Bigg] + v^{ij}x^{kl}\Bigg[\frac{78r^2}{55c^2} \\
& \times(1-5\eta+5\eta^2)\Bigg]\Bigg\},
\end{aligned}\qquad (2.20)
$$

$$I_{ijklm} = -\Big(\mu\,\frac{\delta m}{m}\Big)(1-2\eta)\mathrm{STF}_{ijklm}\{x^{ijklm}\}, \qquad (2.21)$$

$$I_{ijklmn} = \mu(1-5\eta+5\eta^2)\mathrm{STF}_{ijklmn}\{x^{ijklmn}\}, \qquad (2.22)$$

$$
\begin{aligned}
J_{ij} = -\Big(\mu\,\frac{\delta m}{m}\Big)\mathrm{STF}_{ij}\epsilon_{jab}\Bigg\{ & x^{ia}v^b\Bigg[1 + \frac{1}{28c^2}\Big((13-68\eta)v^2 \\
& + (54+60\eta)\frac{Gm}{r}\Big)\Bigg] + v^{ib}x^a\Bigg[\frac{r\dot{r}}{28c^2}(5-10\eta)\Bigg]\Bigg\},
\end{aligned}\qquad (2.23)
$$

$$J_{ijk} = \mu \mathrm{STF}_{ijk} \epsilon_{kab} \left\{ x^{aij} v^b \left[(1 - 3\eta) + \frac{1}{90c^2} \left((41 - 385\eta \right. \right. \right.$$

$$\left. + 925\eta^2)v^2 + (140 - 160\eta - 860\eta^2)\frac{Gm}{r} \right) \right] + \frac{7r^2}{45c^2}(1$$

$$\left. - 5\eta + 5\eta^2) \, x^a v^{ijb} + \frac{10r\dot{r}}{45c^2}(1 - 5\eta + 5\eta^2)x^{ai}v^{bj} \right\},$$

$$(2.24)$$

$$J_{ijkl} = -\left(\mu \frac{\delta m}{m}(1 - 2\eta) \right) \mathrm{STF}_{ijkl}\{\epsilon_{lab} \, x^{aijk}v^b\},$$

$$(2.25)$$

$$J_{ijklm} = [\mu \, (1 - 5\eta + 5\eta^2)]\mathrm{STF}_{ijklm}\{\epsilon_{mab} \, x^{aijkl}v^b\}.$$

$$(2.26)$$

The mass and the current moments listed above, agree with Eqs. (E3) of [18]. For the case of circular orbits, the above mass and current moments reduce to Eqs. (4.4) of [16].

III. THE FAR-ZONE FLUXES

A. The energy flux

As discussed in [16] the end result of the 2PN accurate generation formalism is an expression relating the radiative mass and current multipole moments U_L and V_L, respectively to the source mass and current multipole moments I_L and J_L, respectively, obtained in the previous section. In particular,

$$U_{ij}(T_R) = I_{ij}^{(2)}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) \right.$$

$$\left. + \frac{11}{12} \right] I_{ij}^{(4)}(T_R - \tau) + O(\varepsilon^5) \quad , \qquad (3.1a)$$

$$U_{ijk}(T_R) = I_{ijk}^{(3)}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{97}{60} \right]$$

$$\times I_{ijk}^{(5)}(T_R - \tau) + O(\varepsilon^5) \quad , \qquad (3.1b)$$

$$V_{ij}(T_R) = J_{ij}^{(2)}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{7}{6} \right] J_{ij}^{(4)}(T_R - \tau)$$

$$+ O(\varepsilon^4) \qquad (3.1c)$$

for the moments that need to be known beyond the 1PN accuracy, and

$$U_L(T_R) = I_L^{(l)}(T_R) + O(\varepsilon^3), \qquad (3.2a)$$

$$V_L(T_R) = J_L^{(l)}(T_R) + O(\varepsilon^3) \qquad (3.2b)$$

for the other ones. The integrals in the above expressions, associated with the gravitational wave tails, contain in addition to the total mass-energy of the source m, a quantity b

which is an arbitrary constant with dimensions of time parametrizing a freedom associated in the construction of the far-zone radiative coordinate system.

In terms of the STF radiative moments of the gravitational field the far-zone energy flux to 2PN accuracy is given by [17] (with $U^{(n)} \equiv d^n U/dT_R^n$)

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{far-zone}} = \frac{G}{c^5} \left\{ \frac{1}{5} U_{ij}^{(1)} U_{ij}^{(1)} + \frac{1}{c^2}\left[\frac{1}{189} U_{ijk}^{(1)} U_{ijk}^{(1)} \right. \right.$$

$$+ \frac{16}{45} V_{ij}^{(1)} V_{ij}^{(1)} \right] + \frac{1}{c^4}\left[\frac{1}{9072} U_{ijkm}^{(1)} U_{ijkm}^{(1)} \right.$$

$$\left. \left. + \frac{1}{84} V_{ijk}^{(1)} V_{ijk}^{(1)} \right] + O(\varepsilon^6) \right\}. \qquad (3.3)$$

The 2PN-accurate energy loss given by Eq. (3.3) splits naturally into an ''instantaneous'' contribution and a ''tail'' one. In this paper, we deal only with the instantaneous contribution, which is given by [52,16]

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{far-zone}}^{\text{inst}} = \frac{G}{c^5}\left\{ \frac{1}{5}I_{ij}^{(3)}I_{ij}^{(3)} + \frac{1}{c^2}\left[\frac{1}{189}I_{ijk}^{(4)}I_{ijk}^{(4)} + \frac{16}{45}J_{ij}^{(3)}J_{ij}^{(3)} \right] \right.$$

$$\left. + \frac{1}{c^4}\left[\frac{1}{9072}I_{ijkm}^{(5)}I_{ijkm}^{(5)} + \frac{1}{84}J_{ijk}^{(4)}J_{ijk}^{(4)} \right] \right\}. \qquad (3.4)$$

Here $I_L^{(n)}$ denotes the nth time derivative of STF multipole moment of rank L. All the computations from here onwards are performed, using MAPLE [53]. Evaluating the relevant time derivatives of the multipole moments in Eq. (3.4), using the post-Newtonian equations of motion to the appropriate order we obtain

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{far-zone}}^{\text{inst}} = \dot{\mathcal{E}}_N + \dot{\mathcal{E}}_{1PN} + \dot{\mathcal{E}}_{2PN}, \qquad (3.5a)$$

$$\dot{\mathcal{E}}_N = \frac{8}{15}\frac{G^3 m^2 \mu^2}{c^5 r^4}\{12v^2 - 11\dot{r}^2\}, \qquad (3.5b)$$

$$\dot{\mathcal{E}}_{1PN} = \frac{8}{15}\frac{G^3 m^2 \mu^2}{c^7 r^4}\left\{ \frac{1}{28}\left[(785 - 852\eta)v^4 \right. \right.$$

$$- 2(1487 - 1392\eta)v^2\dot{r}^2 - 160(17 - \eta)\frac{Gm}{r}v^2$$

$$+ 3(687 - 620\eta)\dot{r}^4 + 8(367 - 15\eta)\frac{Gm}{r}\dot{r}^2$$

$$\left. \left. + 16(1 - 4\eta)\frac{G^2 m^2}{r^2} \right] \right\}, \qquad (3.5c)$$

$$\dot{\mathcal{E}}_{2PN} = \frac{8}{15} \frac{G^3 m^2 \mu^2}{c^9 r^4} \left\{ \frac{1}{42}(1692-5497\eta+4430\eta^2)v^6 - \frac{1}{14}(1719-10278\eta+6292\eta^2)v^4\dot{r}^2 - \frac{1}{21}(4446-5237\eta+1393\eta^2) \right.$$

$$\times \frac{Gm}{r} v^4 + \frac{1}{14}(2018-15207\eta+7572\eta^2)v^2\dot{r}^4 + \frac{1}{7}(4987-8513\eta+2165\eta^2)\frac{Gm}{r} v^2\dot{r}^2$$

$$+ \frac{1}{756}(281473+81828\eta+4368\eta^2)\frac{G^2m^2}{r^2} v^2 - \frac{1}{42}(2501-20234\eta+8404\eta^2)\dot{r}^6$$

$$- \frac{1}{63}(33510-60971\eta+14290\eta^2)\frac{Gm}{r}\dot{r}^4 - \frac{1}{252}(106319+9798\eta+5376\eta^2)\frac{G^2m^2}{r^2}\dot{r}^2$$

$$\left. + \frac{2}{63}(-253+1026\eta-56\eta^2)\frac{G^3m^3}{r^3} \right\}. \tag{3.5d}$$

Equations (3.5) are in exact agreement with the results of Will and Wiseman using the new improved Epstein-Wagoner approach [18]. Circular and radial infall limits of Eqs. (3.5) are in agreement with earlier results [54,16,60,18] and discussed further in Sec. III C.

The tail contribution, on the other hand is given by

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{far-zone}}^{\text{tail}} = \frac{2G}{5c^5} \frac{2Gm}{c^3} I_{ij}^{(3)}(T_R) \int_0^{+\infty} d\tau \ln\left(\frac{\tau}{2b_1}\right)$$

$$\times I_{ij}^{(5)}(T_R - \tau), \tag{3.6}$$

where $b_1 \equiv b e^{-11/12}$. A detailed discussion of the tail terms and its implications has been given by Blanchet and Schäfer [47], and we do not discuss it any further in this paper.

B. The angular momentum flux

In terms of the STF radiative multipole moments the far-zone angular momentum flux to 2PN accuracy is given by

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\text{far-zone}} = \frac{G}{c^5}\epsilon_{ipq}\left\{\frac{2}{5}U_{pj}U_{qj}^{(1)} + \frac{1}{c^2}\left[\frac{1}{63}U_{pjk}U_{qjk}^{(1)}\right.\right.$$

$$\left. + \frac{32}{45}V_{pj}V_{qj}^{(1)}\right] + \frac{1}{c^4}\left[\frac{1}{2268}U_{pjkl}U_{qjkl}^{(1)}\right.$$

$$\left.\left. + \frac{1}{28}V_{pjk}V_{qjk}^{(1)}\right]\right\} . \tag{3.7}$$

As before, rewriting the radiative moments in terms of the source moments, allows us to separate the instantaneous and tail contributions and we discuss them independently. We have [52]

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\text{far-zone}}^{\text{inst}} = \frac{G}{c^5}\epsilon_{ipq}\left\{\frac{2}{5}I_{pj}^{(2)}I_{qj}^{(3)} + \frac{1}{c^2}\left[\frac{1}{63}I_{pjk}^{(3)}I_{qjk}^{(4)}\right.\right.$$

$$\left. + \frac{32}{45}J_{pj}^{(2)}J_{qj}^{(3)}\right] + \frac{1}{c^4}\left[\frac{1}{2268}I_{pjkl}^{(4)}I_{qjkl}^{(5)}\right.$$

$$\left.\left. + \frac{1}{28}J_{pjk}^{(3)}J_{qjk}^{(4)}\right]\right\} . \tag{3.8}$$

Computing the required time derivatives of the STF moments, using the post-Newtonian equations of motion to the appropriate order, we obtain

$$\left(\frac{d\mathcal{J}}{dt}\right)_{\text{far-zone}}^{\text{inst}} = \dot{\mathcal{J}}_N + \dot{\mathcal{J}}_{1PN} + \dot{\mathcal{J}}_{2PN}, \tag{3.9a}$$

$$\dot{\mathcal{J}}_N = \frac{8}{5}\frac{G^2m\mu^2}{c^5r^3}\widetilde{\mathbf{L}}_{\mathbf{N}}\left\{2v^2 - 3\dot{r}^2 + 2\frac{Gm}{r}\right\}, \tag{3.9b}$$

$$\dot{\mathcal{J}}_{1PN} = \frac{8}{5}\frac{G^2m\mu^2}{c^7r^3}\widetilde{\mathbf{L}}_{\mathbf{N}}\left\{\frac{1}{84}\left[(307-548\eta)v^4 - 6(74\right.\right.$$

$$-277\eta)v^2\dot{r}^2 - 4(58+95\eta)\frac{Gm}{r}v^2 + 3(95-360\eta)\dot{r}^4$$

$$\left.\left. + 2(372+197\eta)\frac{Gm}{r}\dot{r}^2 - 2(745-2\eta)\frac{G^2m^2}{r^2}\right]\right\}, \tag{3.9c}$$

$$\mathcal{J}_{2PN} = \frac{8}{5} \frac{G^2 m \mu^2}{c^9 r^3} \widetilde{\mathbf{L}}_{\mathbf{N}} \left\{ \frac{1}{504} \left[(2665 - 12355 \eta + 12894 \eta^2) v^6 - 3(2246 - 12653 \eta + 15637 \eta^2) v^4 \dot{r}^2 + (165 - 491 \eta \right. \right.$$

$$+ 4022 \eta^2) \frac{Gm}{r} v^4 + 3(3575 - 16805 \eta + 15680 \eta^2) v^2 \dot{r}^4 + (21853 - 21603 \eta + 2551 \eta^2) \frac{Gm}{r} v^2 \dot{r}^2 - 2(10651 - 10179 \eta$$

$$+ 3428 \eta^2) \frac{G^2 m^2}{r^2} v^2 - 28(195 - 815 \eta + 485 \eta^2) \dot{r}^6 - (22312 - 41\,398\,\eta + 96\,95\,\eta^2) \dot{r}^4 \frac{Gm}{r} + 2(8436 - 25102 \eta$$

$$+ 4587 \eta^2) \frac{G^2 m^2}{r^2} \dot{r}^2 \right] + \frac{1}{2268} (170362 + 70461 \eta + 1386 \eta^2) \frac{G^3 m^3}{r^3} \bigg\}, \tag{3.9d}$$

where $\widetilde{\mathbf{L}}_{\mathbf{N}} = \mathbf{r} \times \mathbf{v}$. The 1PN contribution is in agreement with the earlier results of Junker and Schäfer [48]. The 2PN contribution is new and together with the energy flux obtained in the earlier section forms the starting point for the computation the 2PN radiation reaction for compact binary systems—4.5PN terms in the equation of motion [55]—using the refined balance method proposed by Iyer and Will [56,57]. The tail terms, in the angular momentum flux are given by

$$\left(\frac{d\mathcal{J}_i}{dt} \right)^{\text{tail}}_{\text{far-zone}} = \frac{2G}{5c^5} \frac{2Gm}{c^3} \epsilon_{ijk} I^{(3)}_{kl}(T_R) \int_0^{+\infty} d\tau \ln\left(\frac{\tau}{2b_1} \right)$$
$$\times I^{(4)}_{jl}(T_R - \tau) \ . \tag{3.10}$$

We will not be discussing the tails terms here, as they are extensively studied by Rieth and Schäfer [58].

C. Limits

All the complicated formulae, discussed in the earlier sections take more simpler forms for quasicircular orbits. For compact binaries like PSR 1913+16, quasicircular orbits should provide a good description close to the inspiral phase, since gravitational radiation reaction would have reduced the present eccentricity, to vanishingly small values. In this context ''quasi'' implies the slow inspiral caused by the radiation reaction. The quasicircular orbit is characterized by $\ddot{r} = \dot{r} = O(\epsilon^{2.5})$. The 2PN equations of motion become

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} \equiv \frac{d^2\mathbf{x}}{dt^2} = -\omega^2_{2PN} \mathbf{x} + O(\epsilon^{2.5}) \ , \tag{3.11}$$

with ω_{2PN}, the 2PN accurate orbital frequency, is given by

$$\omega^2_{2PN} \equiv \frac{Gm}{r^3} \left\{ 1 - (3 - \eta)\gamma + \left(6 + \frac{41}{4}\eta + \eta^2 \right)\gamma^2 \right\} \ , \tag{3.12}$$

where $\gamma = Gm/c^2 r$. Note that Eqs. (3.11) imply as usual, that $v \equiv |\mathbf{v}| = \omega_{2PN} r + O(\epsilon^{2.5})$, so that from Eq. (3.12) we get

$$v^2 = \frac{Gm}{r} \left\{ 1 - (3 - \eta)\gamma + \left(6 + \frac{41}{4}\eta + \eta^2 \right)\gamma^2 \right\} \ . \tag{3.13}$$

Substituting $\dot{r} = 0$ in Eqs. (3.5), and using Eq. (3.13) we obtain the 2PN corrections to the far-zone energy flux for compact binaries of arbitrary mass ratio, moving in a quasicircular orbit

$$\left(\frac{d\mathcal{E}}{dt} \right)^{\text{inst}}_{\text{far-zone}} = \frac{32}{5} \frac{c^5}{G} \eta^2 \gamma^5 \left\{ 1 - \gamma \left(\frac{2927}{336} + \frac{5}{4}\eta \right) \right.$$
$$\left. + \gamma^2 \left(\frac{293383}{9072} + \frac{380}{9}\eta \right) \right\}. \tag{3.14}$$

Equation (3.14) is consistent with results of [54,16,18].

The energy and angular momentum fluxes are not independent but related in the case of circular orbits. The precise relation may be written following [59] as

$$\left(\frac{d\mathcal{E}}{dt} \right)_{\text{far-zone}} = v^2 \mathcal{J}, \tag{3.15a}$$

where

$$\left(\frac{d\mathcal{J}}{dt} \right)_{\text{far-zone}} \equiv \widetilde{\mathbf{L}}_{\mathbf{N}} \mathcal{J}, \tag{3.15b}$$

where v^2 defined in terms of Gm/r, is given by Eq. (3.13) [16].

The other limiting case we compare to corresponds to the case of radial infall of two compact objects of comparable masses. Equations representing the head-on infall can be obtained from the expressions for the general orbit by imposing the restrictions $\mathbf{x} = z\hat{\mathbf{n}}$, $\mathbf{v} = \dot{z}\hat{\mathbf{n}}$, $r = z$, and $v = \dot{r} = \dot{z}$. We consider two different cases, following Simone, Poisson, and Will [60]. In case (A), the radial infall proceeds from rest at infinite initial separation, which implies that the conserved energy $E(z) = E(\infty) = 0$. In case (B), the radial infall proceeds from rest at finite initial separation z_0, which implies

$$E(z = z_0) = -\mu c^2 \gamma_0 \left\{ 1 - \frac{1}{2}\gamma_0 + \frac{1}{2}\left[1 + \frac{15}{2}\eta \right]\gamma_0^2 \right\}, \tag{3.16}$$

where $\gamma_0 = Gm/z_0 c^2$. Inverting $E(z)$ for $\dot{z}^2$ and using Eq. (3.16) we obtain

$$\dot{z} = -c\left\{ 2(\gamma - \gamma_0)\left[1 - 5\gamma\left(1 - \frac{\eta}{2}\right) + \gamma_0\left(1 - \frac{9\eta}{2}\right)\right.\right.$$

$$+ \gamma^2\left(13 - \frac{81\eta}{4} + 5\eta^2\right) - \gamma\gamma_0\left(5 - \frac{173\eta}{4} + 13\eta^2\right)$$

$$\left.\left. + \gamma_0^2\left(1 - \frac{5\eta}{4} + 8\eta^2\right)\right]\right\}^{1/2}, \qquad (3.17)$$

where $\gamma = Gm/zc^2$. Using the radial infall restrictions and Eq. (3.17) in Eqs. (3.5) we obtain for case (B), the far-zone radiative energy flux

$$\left(\frac{d\mathcal{E}}{dt}\right)^{\text{inst}}_{\text{far-zone}} = \frac{16}{15}c^5\eta^2\gamma^5\left\{1 - x - \frac{1}{7}\left[43 - \frac{111}{2}\eta - x(116\right.\right.$$

$$- 131\eta) + x^2\left(71 - \frac{135\eta}{2}\right)\right]\gamma - \frac{1}{3}\left[\frac{1127}{9}\right.$$

$$+ \frac{803}{12}\eta - 112\eta^2 + \frac{x}{7}\left(\frac{4471}{9} - \frac{15481\eta}{3}\right)$$

$$+ 2864\eta^2\right) - \frac{x^2}{7}\left(1870 - \frac{38521\eta}{6} + \frac{8800\eta^2}{3}\right)$$

$$\left.\left. + x^3\left(83 - \frac{1183\eta}{4} + \frac{872\eta^2}{7}\right)\right]\gamma^2\right\}, \qquad (3.18)$$

where $x = \gamma_0/\gamma$. For case (A), the expressions for $\dot{z}$ and $d\mathcal{E}/dt$ are obtained by setting $\gamma_0 = 0$ in Eqs. (3.17) and (3.18). Equation (3.18), along with corresponding one for case (A) are in agreement with [60].

IV. THE EVOLUTION OF THE ORBITAL ELEMENTS IN THE GENERALIZED QUASI-KEPLERIAN PARAMETRIZATION OF THE BINARY

In this section, we compute the 2PN corrections to the rate of decay of the orbital elements of a compact binary, in quasi-elliptical orbit, i.e., the effect of the 4.5PN radiation reaction on a 2PN accurate conservative elliptical motion, extending the earlier computations [46–48]. The basic ingredients we employ for the calculations are the far-zone energy and angular momentum fluxes in the harmonic coordinates computed in previous sections and a 2PN accurate description of the relative motion of the compact binaries available in a generalized quasi-Keplerian parametrization given in the ADM coordinates [34–36]. Since the De-Donder(harmonic) and the ADM coordinates are different at the 2PN order, we use the coordinate transformations connecting the harmonic and the ADM coordinates [61], to rewrite the far-zone fluxes in the ADM coordinates. The far zone fluxes, in the ADM coordinates are averaged over an orbital period, extending the earlier computations at the 1PN and the 1.5PN order [46–48,58]. The 2PN corrections to the rate of decay of the orbital elements are computed using heuristic arguments based on the conservation of energy and angular momentum to the 2PN order. Before proceeding to the actual computations, in the next two sections, we summarize the generalized quasi-Keplerian description of the binary orbits in the ADM coordinates and the transformations needed, to relate the kinematical variables in the harmonic and the ADM coordinates.

A. The second post-Newtonian motion of compact binaries

The generalized quasi-Keplerian description for the general binary orbits to the 2PN order, developed by Damour, Schäfer, and Wex [34–36] is best suited for the calculation we propose to do in the following sections and we summarize it in what follows. Let $r_A(t_A), \phi_A(t_A)$ be the planar relative motion of the two compact objects in usual polar coordinates associated with the ADM coordinates. The radial motion $r_A(t_A)$ is conveniently parametrized by

$$r_A = a_r(1 - e_r\cos u), \qquad (4.1a)$$

$$n(t_A - t_0) = u - e_t\sin u + \frac{f_t}{c^4}\sin v + \frac{g_t}{c^4}(v - u), \quad (4.1b)$$

where u is the ''eccentric anomaly'' parametrizing the motion and the constants $a_r, e_r, e_t, n,$ and t_0 are some 2PN semimajor axis, radial eccentricity, time eccentricity, mean motion, and initial instant, respectively. The angular motion $\phi_A(t_A)$ is given by

$$\phi_A - \phi_0 = \left(1 + \frac{k}{c^2}\right)v + \frac{f_\phi}{c^4}\sin 2v + \frac{g_\phi}{c^4}\sin 3v, \quad (4.2a)$$

where

$$v = 2\tan^{-1}\left\{\left(\frac{1 + e_\phi}{1 - e_\phi}\right)^{1/2}\tan\left(\frac{u}{2}\right)\right\}. \qquad (4.2b)$$

In the above ϕ_0, k, e_ϕ are some constant, periastron precession constant, and angular eccentricity, respectively. All the parameters n, k, a_r, e_t, e_r, e_ϕ, f_t, g_t, f_ϕ, and g_ϕ are functions of the 2PN conserved energy and angular momentum per unit reduced mass μ: To avoid additional notations following [34–36], these are also denoted as E and $|\mathbf{J}|$. Their explicit functional forms, given in [35] are displayed below:

$$a_r = -\frac{Gm}{2E}\left\{1 + \frac{1}{2c^2}(7 - \eta)E + \frac{1}{c^4}\left[\frac{1}{4}(1 + 10\eta + \eta^2)E^2\right.\right.$$

$$\left.\left. + \frac{1}{2}(17 - 11\eta)\frac{E}{h^2}\right]\right\}, \qquad (4.3a)$$

$$e_r^2 = 1 + 2Eh^2 - \frac{1}{c^2}\{2(6 - \eta)E + 5(3 - \eta)E^2h^2\}$$

$$+ \frac{1}{c^4}\left\{(26 + \eta + \eta^2)E^2 - 2(17 - 11\eta)\frac{E}{h^2}\right.$$

$$\left. + (80 - 55\eta + 4\eta^2)E^3h^2\right\}, \qquad (4.3b)$$

$$n = \frac{(-2E)^{3/2}}{Gm}\left\{1 + \frac{1}{4c^2}(15-\eta)E + \frac{1}{c^4}\left[\frac{1}{32}(555+30\eta\right.\right.$$

$$\left.\left. + 11\eta)E^2 - \frac{3}{2}(5-2\eta)\frac{(-2E)^{3/2}}{h}\right]\right\}, \qquad (4.3c)$$

$$e_t^2 = 1 + 2Eh^2 + \frac{1}{c^2}\{4(1-\eta)E + (17-7\eta)E^2h^2\} + \frac{1}{c^4}\left\{2(2\right.$$

$$+ \eta + 5\eta^2)E^2 - (17-11\eta)\frac{E}{h^2} + (112-47\eta$$

$$\left. + 16\eta^2)E^3h^2 - 3(5-2\eta)(1+2Eh^2)\frac{(-2E)^{3/2}}{h}\right\},$$

$$(4.3d)$$

$$f_t = -\frac{1}{8h}\eta(4+\eta)(1+2Eh^2)^{1/2}(-2E)^{3/2}, \quad (4.3e)$$

$$g_t = \frac{3}{2}(5-2\eta)\frac{(-2E)^{3/2}}{h}, \qquad (4.3f)$$

$$k = \frac{3}{h^2}\left\{1 + \frac{1}{2c^2}\left[(5-2\eta)E + \frac{5}{2h^2}(7-2\eta)\right]\right\}, \quad (4.3g)$$

$$f_\phi = \frac{1}{8}\frac{\eta}{h^4}(1-3\eta)(1+2Eh^2), \qquad (4.3h)$$

$$g_\phi = -\frac{3}{32}\frac{\eta^2}{h^4}(1+2Eh^2)^{3/2}, \qquad (4.3i)$$

$$e_\phi^2 = 1 + 2Eh^2 - \frac{1}{c^2}\{12E + (15-\eta)E^2h^2\} - \frac{1}{8c^4}$$

$$\times\left\{4(16-88\eta-9\eta^2)E^2 - 4(160-30\eta+3\eta^2)E^3h^2\right.$$

$$\left. + (408-232\eta-15\eta^2)\frac{E}{h^2}\right\}, \qquad (4.3j)$$

where $h = |\mathbf{J}|/(Gm)$. Note that $n = 2\pi/P$, where P is the period of the binary. Using these parametric equations of the motion, we compute $\dot{r}_A^2, v_A^2$ to the 2PN order in terms of $E, h^2, (1-e_r\cos u)$ using

$$\frac{dt_A}{du} = \frac{\partial t_A}{\partial u} + \frac{\partial t_A}{\partial v}\frac{dv}{du}, \qquad (4.4a)$$

$$\dot{r}_A^2 = \left(\frac{dr_A}{du} \Big/ \frac{dt_A}{du}\right)^2, \qquad (4.4b)$$

$$\dot{\phi}_A^2 = \left(\frac{d\phi_A}{dv}\frac{dv}{du} \Big/ \frac{dt_A}{du}\right)^2, \qquad (4.4c)$$

$$v_A^2 = \dot{r}_A^2 + r_A^2\dot{\phi}_A^2. \qquad (4.4d)$$

The subscript A present in Eqs. (4.4) is a reminder that the expressions refer to the ADM gauge. We have

$$\dot{r}_A^2 = \left\{-1 + \frac{2}{(1-e_r\cos u)} + \frac{2}{(1-e_r\cos u)^2}Eh^2\right\}(-2E) + \frac{1}{c^2}\left\{-3+9\eta + \frac{1}{(1-e_r\cos u)}[38-30\eta] - \frac{1}{(1-e_r\cos u)^2}[40-20\eta\right.$$

$$- (36-28\eta)Eh^2] - \frac{1}{(1-e_r\cos u)^3}[(64-24\eta)Eh^2]\bigg\}E^2 - \frac{1}{c^4}\bigg\{4-19\eta+16\eta^2$$

$$- \frac{1}{(1-e_r\cos u)}\left[168-326\eta+98\eta^2 - \frac{1}{Eh^2}(34-22\eta)\right] + \frac{1}{(1-e_r\cos u)^2}[496-712\eta+164\eta^2 - (213-298\eta$$

$$+ 85\eta^2)Eh^2] - \frac{1}{(1-e_r\cos u)^3}[212-332\eta+80\eta^2 - (800-932\eta+188\eta^2)Eh^2] - \frac{1}{(1-e_r\cos u)^4}[528-528\eta$$

$$+ 96\eta^2]Eh^2 + \frac{1}{(1-e_r\cos u)^5}[32+8\eta^2]\eta E^2h^4\bigg\}(-E)^3, \qquad (4.5a)$$

$$v_A^2 = \left\{-1 + \frac{2}{(1-e_r\cos u)}\right\}(-2E) - \frac{1}{c^2}\left\{3-9\eta - \frac{1}{(1-e_r\cos u)}[38-30\eta] + \frac{1}{(1-e_r\cos u)^2}[40-20\eta]\right.$$

$$+ 8\frac{1}{(1-e_r\cos u)^3}\eta Eh^2\bigg\}E^2 - \frac{1}{c^4}\bigg\{4-19\eta+16\eta^2 - \frac{1}{(1-e_r\cos u)}\left[168-326\eta+98\eta^2 - \frac{1}{Eh^2}(34-22\eta)\right]$$

$$+ \frac{1}{(1-e_r\cos u)^2}[428-668\eta+164\eta^2] - \frac{1}{(1-e_r\cos u)^3}[212-332\eta+80\eta^2 - (76-84\eta)\eta Eh^2] - \frac{1}{(1-e_r\cos u)^4}[80$$

$$- 128\eta]\eta Eh^2 + 72\frac{1}{(1-e_r\cos u)^5}\eta^2E^2h^4\bigg\}(-E)^3. \qquad (4.5b)$$

These expressions for $\dot{r}_A^2$ and v_A^2 are consistent with Eqs. (6) and (7) of [62].

B. The transformation between De-Donder (harmonic) and ADM gauges

As pointed out earlier, the far-zone fluxes obtained in previous sections are in the harmonic coordinates, whereas, the 2PN accurate orbital description given by Eqs. (4.1), (4.2), and (4.3) are in the ADM coordinates. For the purpose of averaging the far zone fluxes using the the 2PN accurate orbital representation, we need to go from the De-Donder (harmonic) to the ADM gauge, and rewrite the expressions for the far-zone fluxes in the ADM coordinates. These follow straightforwardly from the transformation equations in [61] and we list below the transformation equations, relating the harmonic(De-Donder) variables to the corresponding ADM variables:

$$\mathbf{r}_D = \mathbf{r}_A + \frac{Gm}{8c^4 r}\left\{\left[(5v^2 - \dot{r}^2)\eta + 2(1+12\eta)\frac{Gm}{r}\right]\mathbf{r}\right.$$
$$\left. - 18\eta r\dot{r}\mathbf{v}\right\}, \tag{4.6a}$$

$$t_D = t_A - \frac{Gm}{c^4}\eta\dot{r}, \tag{4.6b}$$

$$\mathbf{v}_D = \mathbf{v}_A - \frac{Gm\dot{r}}{8c^4 r^2}\left\{\left[7v^2 + 38\frac{Gm}{r} - 3\dot{r}^2\right]\eta + 4\frac{Gm}{r}\dot{r}\right\}\mathbf{r}$$
$$- \frac{Gm}{8c^4 r}\left\{\left[5v^2 - 9\dot{r}^2 - 34\frac{Gm}{r}\right]\eta - 2\frac{Gm}{r}\right\}\mathbf{v}, \tag{4.6c}$$

$$(\mathbf{L}_N)_D = (\mathbf{L}_N)_A\left\{1 + \frac{Gm}{4c^4 r}\left[(2+29\eta)\frac{Gm}{r} + 4\eta\dot{r}^2\right]\right\}, \tag{4.6d}$$

$$r_D = r_A + \frac{Gm}{8c^4}\left\{5\eta v^2 + 2(1+12\eta)\frac{Gm}{r} - 19\eta\dot{r}^2\right\}, \tag{4.6e}$$

$$v_D^2 = v_A^2 - \frac{Gm}{4c^4 r}\left\{[5v^4 - 2v^2\dot{r}^2 - 3\dot{r}^4]\eta - [2(1+17\eta)v^2\right.$$
$$\left. - (4+38\eta)\dot{r}^2]\frac{Gm}{r}\right\}, \tag{4.6f}$$

$$\dot{r}^2_D = \dot{r}^2_A - \frac{Gm}{2c^4 r}\dot{r}^2\left\{15(v^2 - \dot{r}^2)\eta + (1+2\eta)\frac{Gm}{r}\right\}. \tag{4.6g}$$

The subscript D denotes quantities in the De-Donder (harmonic) coordinates. Note that in all the above equations the differences between the two gauges are of the 2PN order. As there is no difference between the harmonic and the ADM coordinates to 1PN accuracy, in Eqs. (4.6), for the 2PN terms no suffix is used. The 2PN extension of the evolution of the orbital elements thus requires more technical care than the 1PN case due to the differences in the ADM and harmonic

coordinates given by Eqs. (4.6). Finally using the above equations we have verified that the expressions given by Eqs. (2.8), relating the individual locations of the two bodies to the center of mass coordinate are consistent with the corresponding choice in ADM coordinates, given by Eqs. (A5)–(A8) of [36].

C. 2PN corrections to $\langle d\mathcal{E}/dt\rangle$ and $\langle d\mathcal{J}/dt\rangle$

Starting from Eqs. (3.5) and (3.9) for the far-zone fluxes in the harmonic coordinates we use Eqs. (4.6), to obtain $d\mathcal{E}/dt$ and $d\mathcal{J}/dt$ in the ADM coordinates. For economy of presentation, we write the results as (flux)$_A$ = (flux)$_O$ + "corrections," where (flux)$_A$ represent the far-zone flux in the ADM coordinates. (Flux)$_O$ is a short hand notation for expressions on the right-hand side (RHS) of Eqs. (3.5) and (3.9), where $v^2, \dot{r}, r$ are the ADM variables $v_A^2, \dot{r}_A, r_A$, respectively. For example, the Newtonian part of $(d\mathcal{E}/dt)_O$ will be $\frac{8}{15}G^3m^2\mu^2/c^5 r_A^4\{12v_A^2 - 11\dot{r}_A^2\}$. The "corrections" represent the differences at the 2PN order, that arise due to the change of the coordinate system, given by Eqs. (4.6). As the two coordinates are different at the 2PN order, the "corrections" come only from the leading Newtonian terms in Eqs. (3.5) and (3.9):

$$\left(\frac{d\mathcal{E}}{dt}\right)_A = \left(\frac{d\mathcal{E}}{dt}\right)_O - \frac{G^4 m^3 \mu^2}{15c^9 r_A^5}\left\{[(48+336\eta)v_A^2\right.$$
$$- (36+232\eta)\dot{r}_A^2]\frac{Gm}{r_A} + [360v_A^4 - 1840v_A^2\dot{r}_A^2$$
$$\left. + 1424\dot{r}_A^4]\eta\right\} \tag{4.7a}$$

$$\left(\frac{d\mathcal{J}}{dt}\right)_A = \left(\frac{d\mathcal{J}}{dt}\right)_O + \frac{G^3 m^2 \mu^2 (\tilde{\mathbf{L}}_N)_A}{5c^9 r_A^4}\left\{\left[(4+68\eta)v_A^2\right.\right.$$
$$\left. - (8+76\eta)\frac{Gm}{r_A} + (2+82\eta)\dot{r}_A^2\right]\frac{Gm}{r_A}$$
$$\left. + (363v_A^2\dot{r}_A^2 - 50v_A^4 - 363\dot{r}_A^4)\eta\right\}. \tag{4.7b}$$

Note that all the variables on the RHS of Eqs. (4.7) are in the ADM coordinates. In the circular limit energy and angular momentum fluxes are again related as in Eqs. (3.15), via the corresponding v^2 in the ADM coordinates given by

$$v_A^2 = \frac{Gm}{r_A}\left\{1 - (3-\eta)\frac{Gm}{c^2 r_A} + \frac{1}{8}(42 - 5\eta + 8\eta^2)\frac{G^2 m^2}{c^4 r_A^2}\right\}. \tag{4.8}$$

From this point onwards, in this section, we work exclusively in the ADM gauge and hence we drop the subscript A for the ease of presentation. We now have all the ingredients needed to calculate the 2PN corrections in $\langle d\mathcal{E}/dt\rangle$ and $\langle d\mathcal{J}/dt\rangle$. We explain in detail, the procedure to compute $\langle d\mathcal{E}/dt\rangle$ and only display the final expression for $\langle d\mathcal{J}/dt\rangle$, as the procedure is the same in both the cases. Starting from Eqs. (4.7), (3.5), and (3.9) which give the far zone fluxes as

functions of v^2, $\dot{r}^2$, and Gm/r, we use the 2PN accurate orbital representation, to rewrite $d\mathcal{E}/dt$ as a polynomial in $(1-e_r\cos u)^{-1}$. This polynomial is of the form

$$\frac{d\mathcal{E}}{dt} = \frac{du}{ndt}\sum_{N=2}^{8}\frac{\alpha_N(E,h)}{(1-e_r\cos u)^{(N+1)}}, \tag{4.9}$$

where for the convenience we have factored out du/ndt given by

$$\frac{du}{ndt} = \frac{1}{(1-e_r\cos u)}\left\{1-\frac{E}{c^2}(8-3\eta)\left(1-\frac{1}{(1-e_r\cos u)}\right)\right.$$

$$+\frac{1}{2c^4}\left[E^2\left((56-63\eta+6\eta^2)-\frac{1}{Eh^2}(17-11\eta)\right.\right.$$

$$\times\left(1-\frac{1}{(1-e_r\cos u)}\right)-\frac{1}{(1-e_r\cos u)}$$

$$\times(184-159\eta+24\eta^2)+\frac{1}{(1-e_r\cos u)^2}$$

$$\times(68-76\eta+17\eta^2)-\frac{2Eh^2}{(1-e_r\cos u)^3}\eta(4+\eta)\right)$$

$$\left.\left.+\frac{3}{h}(-2E)^{3/2}(5-2\eta)\right]\right\}. \tag{4.10}$$

It is a straightforward algebra to show that the coefficients $\alpha_N(E,h)$ in Eq. (4.9) take the form

$$\alpha_N(E,h) = \frac{\eta^2}{Gc^5}(-E)^5\beta_N(E,h), \tag{4.11}$$

where $\beta_N(E,h)$ for $N=1,2,\ldots,8$ are given by

$$\beta_2 = -\frac{256}{15}+\frac{1}{105c^2}(29\,824-15\,488\eta)E$$

$$+\frac{1}{c^4}\left\{-\frac{1}{315}(791\,168-874\,624+179456\eta^2)E^2\right.$$

$$\left.+\frac{128}{5}(17-11\eta)\frac{E}{h^2}+\frac{1}{5}(640-256\eta)\frac{(-2E)^{3/2}}{h}\right\}, \tag{4.12a}$$

$$\beta_3 = \frac{512}{15}-\frac{1}{35c^2}(263\,68-19968\eta)E+\frac{1}{c^4}\left\{\left[\frac{2716928}{315}\right.\right.$$

$$\left.-\frac{13040896}{945}\eta+\frac{538496}{135}\eta^2\right]E^2-\frac{896}{15}(17-11\eta)\frac{E}{h^2}$$

$$\left.-\frac{1}{5}(1280-512\eta)\frac{(-2E)^{3/2}}{h}\right\}, \tag{4.12b}$$

$$\beta_4 = -\frac{5632}{15}Eh^2+\frac{1}{c^2}\left\{\frac{1}{7}(1024-3072\eta)E+\frac{512}{105}(1729\right.$$

$$-930\eta)\,E^2\,h^2\right\}+\frac{1}{c^4}\left\{\left(\frac{46840064}{2835}+\frac{3537664}{945}\eta\right.\right.$$

$$\left.-\frac{2315648}{315}\eta^2\right)E^2-\frac{128}{105}(86403-89968\eta$$

$$+20923\eta^2)\,E^3h^2-\frac{256}{15}(17-11\eta)\frac{E}{h^2}-\frac{1}{5}(7040$$

$$\left.-2816\eta)(-2E)^{5/2}h\right\}, \tag{4.12c}$$

$$\beta_5 = -\frac{512}{105c^2}(3232-1395\eta)\,E^2h^2+\frac{1}{c^4}\left\{-\left[\frac{14200576}{2835}\right.\right.$$

$$\left.-\frac{38656}{189}\eta-\frac{219904}{63}\eta^2\right]E^2+\frac{256}{945}[148\,648\,8$$

$$\left.-1545569\eta+343813\eta^2]\,E^3h^2\right\}, \tag{4.12d}$$

$$\beta_6 = -\frac{512}{35c^2}(687-620\eta)E^3h^4-\frac{1}{c^4}\left\{\frac{256}{945}[1221526\right.$$

$$-1333624\eta+319739\eta^2]\,E^3h^2-\frac{512}{105}[51396$$

$$\left.-91541\eta+27508\eta^2]\,E^4h^4\right\}, \tag{4.12e}$$

$$\beta_7 = -\frac{512}{945c^4}\{748\,032-1385005\eta+387911\eta^2\}\,E^4h^4, \tag{4.12f}$$

$$\beta_8 = -\frac{4096}{315c^4}\{2501-202\,34\eta+8404\eta^2\}\,E^5h^6. \tag{4.12g}$$

To 1PN order Eqs. (4.12) agree with Eqs. (4.15) of [46]. The far-zone energy flux $(d\mathcal{E}/dt)$ is a periodic function of time with period $P=2\pi/n$. Averaging $(d\mathcal{E}/dt)$, given by Eqs. (4.9), (4.11), and (4.12) over one time period P, we obtain

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle = \frac{1}{P}\int_0^P\frac{d\mathcal{E}}{dt}(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{ndt}{du}\right)\frac{d\mathcal{E}}{dt}(u)du. \tag{4.13}$$

The integrals in Eq. (4.13) are the Laplace second integrals for the Legendre polynomials [63] which yield

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{\{1-e_r\cos u\}^{N+1}}$$

$$= \frac{1}{(1-e_r^2)^{(N+1)/2}}P_N\left(\frac{1}{\sqrt{(1-e_r^2)}}\right), \tag{4.14}$$

where P_N is the Legendre polynomial. Using Eq. (4.14) in Eq. (4.13), we obtain $\langle d\mathcal{E}/dt\rangle$ in terms of E and e_r:

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle=\frac{1024}{5}\frac{\mu\eta}{Gmc^5}\frac{(-E)^5}{(1-e_r^2)^{7/2}}\left\{1+\frac{73}{24}e_r^2+\frac{37}{96}e_r^4+\frac{1}{168}\frac{(-E)}{c^2(1-e_r^2)}\left[13-6414e_r^2-\frac{274\,05}{4}e_r^4-\frac{537\,7}{16}e_r^6\right.\right.$$

$$\left.-\left(840+\frac{6419}{2}e_r^2+\frac{5103}{8}e_r^4-\frac{259}{8}e_r^6\right)\eta\right]-\frac{(-E)^2}{c^4}\left[\frac{1}{16\,(1-e_r^2)^{5/2}}[(480-192\eta)+(500-200\eta)e_r^2\right.$$

$$-(2255-902\eta)e_r^4+(1090-436\eta)e_r^6+(185-74\eta)e_r^8]-\frac{1}{(1-e_r^2)^2}\left(\frac{253\,937}{4536}-\frac{180\,65}{504}\eta+10\eta^2\right.$$

$$-\left(\frac{879\,749}{4536}-\frac{301\,37}{72}\eta-\frac{1877}{48}\eta^2\right)e_r^2-\left(\frac{513\,337}{6048}-\frac{531\,871}{672}\eta+\frac{1139}{192}\eta^2\right)e_r^4+\left(\frac{249\,479\,5}{8064}+\frac{4823}{128}\eta-\frac{383}{96}\eta^2\right)e_r^6$$

$$\left.\left.\left.+\left(\frac{283\,685}{16128}-\frac{131\,47}{2688}\eta+\frac{37}{192}\eta^2\right)e_r^8\right]\right]\right\}.\tag{4.15}$$

Following exactly a similar procedure, we obtain the 2PN correction to $\langle d\mathcal{J}/dt\rangle$. The final result we obtain is

$$\left\langle\frac{d\mathcal{J}}{dt}\right\rangle=\frac{4}{5}\frac{\mu\eta}{c^5}\frac{(-2E)^{7/2}}{(1-e_r^2)^3}\left\{8-e_r^2-7e_r^4-\frac{(-E)}{168c^2}[(2920+7056\eta)+(197\,38+144\,34\ \eta)e_r^2+(127+1330\eta)e_r^4]\right.$$

$$-\frac{(-E)^2}{c^4}\left[\frac{1}{(1-e_r^2)^{1/2}}(240-96\eta-(30-12\eta)e_r^2-(210-84\eta)e_r^4)-\frac{1}{(1-e_r^2)}\left(\frac{299\,623}{1134}-\frac{220\,25}{252}\eta+\frac{351}{4}\eta^2\right.\right.$$

$$\left.-\left(\frac{131\,627\,3}{864}-\frac{815\,597}{336}\eta-\frac{292\,07}{96}\eta^2\right)e_r^2-\left(\frac{290\,113\,3}{6048}-\frac{124\,403}{96}\eta-\frac{7187}{48}\eta^2\right)e_r^4\right.$$

$$\left.\left.\left.+\left(\frac{7526}{63}-\frac{2869}{224}\eta+\frac{1435}{96}\eta^2\right)e_r^6\right)\right]\right\}.\tag{4.16}$$

To 1PN order, Eqs. (4.15) and (4.16) agree with [46,48] as required. For the special case of circular orbits, $e_r=0$ and we observe that, $\langle d\mathcal{E}/dt\rangle=\omega\langle d\mathcal{J}/dt\rangle$ to 2PN order, where ω, the mean angular frequency of the relative motion, defined by $\omega=n(1+k)$ is given by

$$\omega=\frac{(-2\,E)^{3/2}}{Gm}\left\{1-\frac{1}{4\,c^2}(9+\eta)E+\frac{1}{32\,c^4}(2811-1170\eta+11\eta^2)E^2\right\}.\tag{4.17}$$

It is not very difficult to trace the origin of the two types of terms in Eqs. (4.15) and (4.16) at the 2PN order. It is related to the fact that the ''corrections'' in Eqs. (4.7), arising from the transformation equations connecting the harmonic and the ADM coordinates have a different functional form than the 2PN contributions to the corresponding far-zone fluxes in the harmonic coordinates. For example, in the far-zone energy flux, corrections contain a common factor (G^4m^3/r^5), unlike the 2PN contributions in harmonic coordinates which have only (G^3m^2/r^4) as the common factor [cf. Eqs. (3.5) and (4.7)]. These different functional forms, after the averaging procedure give rise to the two different types of terms in Eqs. (4.15) and (4.16).

We display below $\langle d\mathcal{E}/dt\rangle$ and $\langle d\mathcal{J}/dt\rangle$ in terms of Gm/a_r and e_r, which can easily be obtained from Eqs. (4.15) and (4.16), using E written in terms of Gm/a_r and e_r to the 2PN order. The required equation for E is obtained from Eqs. (4.3) for a_r and e_r by inverting them for E and h^2, respectively, order by order. Eliminating h^2 from the expression for E we finally get

$$E=-\frac{c^2}{2}\ \zeta\left\{1-\frac{1}{4}(7-\eta)\zeta+\frac{1}{8}\left[(25-2\eta+\eta^2)-2\frac{(17-11\eta)}{(1-e_r^2)}\right]\ \zeta^2\right\},\tag{4.18}$$

where $\zeta=G\ m/c^2a_r$. Using the above expression for E, Eq. (4.15) becomes

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle=\frac{1}{15}\frac{c^5}{G}\ \eta^2\frac{\zeta^5}{(1-e_r^2)^{13/2}}\left\{[(96+292e_r^2+37e_r^4)(1-e_r^2)^3]-\frac{1}{56}\zeta(1-e_r^2)^2[(468\,32+6720\eta)+(198\,664+376\,32\eta)e_r^2\right.$$

$$-(153\,30-280\,56\eta)e_r^4-(127\,53-2072\eta)e_r^6]+\zeta^2\left[\frac{1}{6048}(1-e_r^2)[(224\,053\,12+122\,492\,16\eta)+(912\,416\,00\right.$$

$$+973\,409\,76\eta+290\,304\eta^2)e_r^2-(977\,677\,44-731\,619\,00\eta-239\,500\,8\ \eta^2)e_r^4-(757\,105\,2+606\,592\,8\ \eta$$

$$\left.\left.-280\,627\,2\ \eta^2)e_r^6+(680\,528\,7-148\,921\,2\ \eta+223\,776\eta^2)e_r^8]-\frac{3}{2}(1-e_r^2)^{5/2}[(96+292e_r^2+37e_r^4)(5-2\ \eta)]\right]\right\}\tag{4.19}$$

while Eq. (4.16) gets transformed to

$$\left\langle \frac{d\mathcal{J}}{dt} \right\rangle = \frac{4}{5}\mu\eta c^2 \frac{\zeta^{7/2}}{(1-e_r^2)^4}\bigg\{ (8+7e_r^2)(1-e_r^2)^2 - \frac{1}{336}\zeta(1-e_r^2)[(193\,84+4704\,\eta)+(176\,80+147\,28\,\eta)e_r^2-(142\,79$$

$$-3388\,\eta)e_r^4] + \zeta^2\bigg[\frac{1}{181\,44}((381\,349\,6+314\,114\,4\,\eta+725\,76\,\eta^2)-(346\,264\,8-137\,197\,26\,\eta-815\,724\,\eta^2)e_r^2$$

$$-(112\,754\,91-786\,483\,\eta-139\,784\,4\,\eta^2)e_r^4+(357\,872\,4-121\,329\,9\,\eta+238\,896\,\eta^2)e_r^6)$$

$$-\frac{3}{2}(1-e_r^2)^{3/2}(5-2\,\eta)\,(8+7e_r^2)\bigg]\bigg\}. \tag{4.20}$$

We observe that in the test particle limit ($\eta\to 0$) and for small radial eccentricities, Eqs. (4.19) and (4.20) become

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{\eta=0} = \frac{32}{5}\frac{c^5}{G}\frac{\mu^2}{m^2}\zeta^5\bigg\{ 1 - \frac{292\,7}{336}\zeta + \frac{282\,043}{907\,2}\zeta^2$$

$$+\bigg[\frac{157}{24} - \frac{639\,7}{84}\zeta + \frac{273\,523}{864}\zeta^2\bigg]e_r^2\bigg\}, \tag{4.21a}$$

$$\left\langle \frac{d\mathcal{J}}{dt} \right\rangle_{\eta=0} = \frac{32}{5}\frac{\mu^2}{m}c^2\,\zeta^{\frac{7}{2}}\bigg\{ 1 - \frac{2423}{336}\zeta + \frac{340\,607}{181\,44}\zeta^2 + \bigg[\frac{23}{8}$$

$$- \frac{947\,9}{336}\zeta + \frac{101\,464\,7}{181\,44}\zeta^2\bigg]e_r^2\bigg\}. \tag{4.21b}$$

Such expressions for average energy and angular momentum fluxes for a test particle moving in a slightly eccentric orbit around a Schwarzschild black hole have been obtained by Tagoshi [24], using black hole perturbation methods: Eqs. (4.9) and (4.12) of [24] (with $q=0$). They are given by

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \frac{32}{5}\frac{\mu^2}{Gm^2c^5}v^{10}\bigg\{ 1 - \frac{1247}{336}\frac{v^2}{c^2} - \frac{447\,11}{9072}\frac{v^4}{c^4}$$

$$+\bigg[\frac{37}{24} - \frac{65}{21}\frac{v^2}{c^2} - \frac{474\,409}{9072}\frac{v^4}{c^4}\bigg]e^2\bigg\}, \tag{4.22a}$$

$$\left\langle \frac{d\mathcal{J}}{dt} \right\rangle = \frac{32}{5}\frac{\mu^2}{m\,c^5}v^7\bigg\{ 1 - \frac{1247}{336}\frac{v^2}{c^2} - \frac{447\,11}{9072}\frac{v^4}{c^4} + \bigg[-\frac{5}{8}$$

$$+\frac{749}{96}\frac{v^2}{c^2} - \frac{238\,229}{6048}\frac{v^4}{c^4}\bigg]e^2\bigg\}, \tag{4.22b}$$

where v and e refer to the radial velocity and the eccentricity in Schwarzschild coordinates. Equations (4.21) and (4.22) are consistent, if the ADM variables a_r and e_r are related to the Schwarzschild variables v and e by

$$\frac{Gm}{a_r} = v^2\bigg\{ 1 + \frac{v^2}{c^2} + \frac{5}{4}\frac{v^4}{c^4} - \bigg[1 + \frac{v^2}{c^2} - \frac{3329}{818}\frac{v^4}{c^4}\bigg]e^2\bigg\}, \tag{4.23a}$$

$$e_r^2 = e^2\bigg\{ 1 + 2\frac{v^2}{c^2} + \frac{1708}{409}\frac{v^4}{c^4}\bigg\}. \tag{4.23b}$$

As stressed by Tagoshi, the fluxes reveal the more familiar coefficients in terms of a parameter v', related to the angular frequency in the ϕ coordinates rather than v, which is adapted to the radial coordinate r. For slightly eccentric orbits, v and v' are related by

$$v = v'\bigg\{ 1 + \frac{1}{2}\bigg[1 - 3\frac{v'^2}{c^2} - 12\frac{v'^4}{c^4}\bigg]e^2\bigg\}. \tag{4.24}$$

In terms of v' the far-zone fluxes for a test particle in Schwarzschild geometry, Eqs. (4.22) may be written as

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \frac{32}{5}\frac{\mu^2}{Gm^2c^5}v'^{10}\bigg\{ 1 - \frac{124\,7}{336}\frac{v'^2}{c^2} - \frac{447\,11}{9072}\frac{v'^4}{c^4}$$

$$+e^2\bigg[\frac{157}{24} - \frac{678\,1}{168}\frac{v'^2}{c^2} - \frac{151\,18}{189}\frac{v'^4}{c^4}\bigg]\bigg\}, \tag{4.25a}$$

$$\left\langle \frac{d\mathcal{J}}{dt} \right\rangle = \frac{32}{5}\frac{\mu^2}{m\,c^5}v'^7\bigg\{ 1 - \frac{1247}{336}\frac{v'^2}{c^2} - \frac{447\,11}{9072}\frac{v'^4}{c^4}$$

$$+e^2\bigg[\frac{23}{8} - \frac{3259}{168}\frac{v'^2}{c^2} - \frac{105\,949\,3}{181\,44}\frac{v'^4}{c^4}\bigg]\bigg\}. \tag{4.25b}$$

In this form at the Newtonian order, one recovers the results of Peters and Mathews [42]. The quantities a_r and e_r in the ADM coordinates are related to v' and e by the relations

$$\frac{Gm}{a_r} = v'^2\bigg\{ 1 + \frac{1}{c^2}(1-2e^2)\,v'^2$$

$$+\frac{1}{4\,c^4}\bigg(5 - \frac{166\,55}{409}e^2\bigg)v'^4\bigg\}, \tag{4.26a}$$

$$e_r^2 = e^2\bigg\{ 1 + 2\frac{v'^2}{c^2} + \frac{1708}{409}\frac{v'^4}{c^4}\bigg\}. \tag{4.26b}$$

The above relations may be rewritten, in terms of the conserved energy E using [43]

$$v^2 = -2E\bigg\{ 1 + e^2 - \frac{E}{2c^2}(3-e^2) + \frac{E^2}{c^4}(18+4e^2)\bigg\}, \tag{4.27a}$$

$$v'^2 = -2E\left\{1 - \frac{E}{2c^2}(3+8e^2) + \frac{E^2}{c^4}(18+52e^2)\right\}.$$
(4.27b)

We obtain

$$\frac{Gm}{a_r} = v'^2\left\{1 - \frac{E}{c^2}(2-4e^2) + \frac{E^2}{c^4}\left(8 - \frac{15837}{409}e^2\right)\right\},$$
(4.28a)

$$e_r^2 = e^2\left\{1 - 4\frac{E}{c^2} + \frac{9286}{409}\frac{E^2}{c^4}\right\},$$
(4.28b)

which are the generalizations of similar 1PN relations in [48].

D. The evolution of the orbital elements

In this section, we compute the 2PN corrections to the evolution of orbital elements due to the emission of gravitational radiation. We describe the procedure to compute the rate of decrease of the orbital period of the binary in some detail and display the final expressions for the rate of decay of other elements; namely, $\langle da_r/dt\rangle$ and $\langle de_r/dt\rangle$. Employing the Heuristic argument, based on the energy and the angular momentum conservation to the 2PN order, the rate of

decrease of the orbital period, $\dot{P}$ of the two compact objects moving, in quasielliptical orbits is computed. The 2PN accurate orbital period, $P = 2\pi/n$ given in [34–36] reads as

$$P = \frac{2\pi Gm}{(-2E)^{3/2}}\left\{1 - \frac{1}{4c^2}(15-\eta)E - \frac{3}{32c^4}\left[(35+30\eta + 3\eta^2)E^2 - 16(5-2\eta)\frac{(-2E)^{3/2}}{h}\right]\right\}.$$
(4.29)

Differentiating Eq. (4.29) with respect to t and equating dE/dt to $(-\langle d\mathcal{E}/dt\rangle/\mu)$ and dh/dt to $(-\langle d\mathcal{J}/dt\rangle/Gm\mu)$ we find

$$\dot{P} = \frac{6\pi Gm}{(-2E)^{5/2}}\left\{1 - \frac{1}{12c^2}(15-\eta)E + \frac{1}{32c^4}(35+30\eta + 3\eta^2)E^2\right\}\left\langle\frac{d\mathcal{E}}{dt}\right\rangle - \frac{3\pi}{c^4 h^2}(5-2\eta)\left\langle\frac{d\mathcal{J}}{dt}\right\rangle.$$
(4.30)

Note that, in the above equation we need $\langle d\mathcal{J}/dt\rangle$ to the Newtonian accuracy only. Using in Eq. (4.30), $\langle d\mathcal{E}/dt\rangle$ given by Eq. (4.15) and the Newtonian part of Eq. (4.16) for $\langle d\mathcal{J}/dt\rangle$, we get

$$\dot{P} = -\frac{192}{5}\pi\eta\frac{\zeta^{5/2}}{(1-e_r^2)^{7/2}}\left\{1 + \frac{73}{24}e_r^2 + \frac{37}{96}e_r^4 - \frac{1}{161\,28}\zeta\frac{1}{(1-e_r^2)}[(598\,56+309\,12\eta) + (431\,352+134\,848\eta)e_r^2 + (168\,210\right.$$

$$+556\,08\eta)e_r^4 - (7179-207\,2\eta)e_r^6] + \zeta^2\frac{1}{(1-e_r^2)^2}\left[\frac{1}{580\,608}[(763\,955\,2+607\,737\,6\,\eta + 483\,840\,\eta^2) + (263\,832\,80\right.$$

$$+814\,273\,20\eta + 251\,596\,8\,\eta^2)e_r^2 - (190\,546\,44-825\,636\,06\eta - 170\,553\,6\,\eta^2)e_r^4 - (145\,177\,2-532\,202\,4\,\eta$$

$$\left.\left.-935\,424\eta^2)e_r^6 + (159\,698\,7-193\,374\eta + 745\,92\,\eta^2)e_r^8] - \frac{1}{64}(5-2\eta)(1-e_r^2)^{3/2}(64+296e_r^2+65e_r^4)\right]\right\}.$$
(4.31)

Finally inserting the expressions for e_r^2 and Gm/a_r in terms of E and h^2 in Eq. (4.31) we obtain

$$\dot{P} = -\frac{\pi\eta}{5c^5}\frac{1}{(-E)h^7}\left\{425 + 732\,Eh^2 + 148E^2h^4 + \frac{1}{c^2h^2}\left[\frac{403\,41}{8} + \frac{381\,35}{4}Eh^2 + \frac{722\,37}{14}E^2h^4 + \frac{4983}{7}E^3h^6\right.\right.$$

$$\left.-\left(\frac{5635}{2} + \frac{481\,25}{6}Eh^2 + 535\,4\,E^2h^4 + \frac{140\,6}{3}E^3h^6\right)\eta\right] + \frac{1}{c^4}\left[\frac{1}{672}(291\,982\,55-309\,096\,90\eta + 690\,606\,0\,\eta^2)\frac{1}{h^4}\right.$$

$$+\frac{1}{432}(293\,418\,53-505\,570\,59\eta + 187\,777\,80\eta^2)\frac{E}{h^2} + \frac{1}{2}(6375-2550\eta)\frac{(-2E)^{3/2}}{h}$$

$$+\frac{1}{252}(864\,965\,0-219\,467\,70\,\eta + 137\,502\,75\,\eta^2)\,E^2 - (3195-1278\eta)(-2E)^{5/2}h + \frac{1}{84}(166\,451\,5-206\,289\,3\,\eta$$

$$\left.\left.+171\,217\,2\,\eta^2)\,E^3h^2 + \frac{1}{2}(975-390\eta)(-2E)^{7/2}h^3 + \frac{1}{42}(163\,085-693\,68\,\eta + 445\,48\eta^2)\,E^4h^4\right]\right\}.$$
(4.32)

In the expression above, $\dot{P}$ is given as a function of the masses and of the 2PN-conserved energy and angular momentum. This expression for $\dot{P}$ is independent of the coordinate system used to derive it. Since P is a measurable quantity, one would have

liked to express $\dot{P}$ in terms of other directly observable parameters such as the orbital period and some convenient eccentricity as in the 1PN case [46]. However, at present, to 2PN accuracy we do not have any such suitable and convenient choice and therefore we leave the expression for $\dot{P}$ in terms of the 2PN accurate E and h^2.

Similarly, using the definition of a_r and e_r in terms of E and h^2 and following the method described above, we obtain after a rather long but straightforward calculation

$$\left\langle \frac{da_r}{dt} \right\rangle = -\frac{2}{15}\eta c \frac{\zeta^3}{(1-e_r^2)^{11/2}} \left\{ (1-e_r^2)^2(96+292e_r^2+37e_r^4) - \frac{1}{56}\zeta(1-e_r^2)[(280\,16+9408\,\eta)+(160\,248+431\,20\,\eta)e_r^2 \right.$$

$$+ (346\,50+209\,16\,\eta)e_r^4 - (5501-1036\,\eta)e_r^6] + \zeta^2 \frac{1}{(1-e_r^2)^{11/2}} \left[\frac{1}{6048}[(137\,748\,16+585\,129\,6\,\eta+290\,304\,\eta^2) \right.$$

$$+ (428\,878\,40+874\,684\,80\,\eta+188\,395\,2\,\eta^2)e_r^2 - (396\,797\,28-824\,068\,08\,\eta-221\,886\,0\,\eta^2)e_r^4 - (449\,753\,4$$

$$- 103\,086\,\eta-123\,832\,8\,\eta^2)e_r^6 + (262\,800\,9-632\,718\,\eta+839\,16\,\eta^2)e_r^8] - \frac{3}{2}(1-e_r^2)^{3/2}[(5-2\,\eta)(96+292e_r^2$$

$$\left. \left. +37e_r^4)] \right\}, \tag{4.33}$$

$$\left\langle \frac{de_r}{dt} \right\rangle = -\frac{1}{15}\frac{c^3}{G}\frac{\eta}{m}\frac{\zeta^4 e_r}{(1-e_r^2)^{9/2}} \left\{ (304+121e_r^2)(1-e_r^2)^2 - \frac{1}{56}\zeta(1-e_r^2)[(133\,640+374\,08\,\eta)+(108\,984+336\,84\,\eta)e_r^2 \right.$$

$$- (252\,11-338\,8\,\eta)e_r^4] + \zeta^2 \left[\frac{1}{201\,6}((174\,096\,16+170\,583\,84\,\eta+491\,904\,\eta^2)-(120\,536\,4-397\,143\,72\,\eta \right.$$

$$- 760\,788\,\eta^2)e_r^2 - (150\,068\,86-224\,584\,2\,\eta-560\,952\,\eta^2)e_r^4 + (384\,043\,5-619\,614\,\eta+914\,76\,\eta^2)e_r^6)$$

$$\left. \left. -\frac{3}{2}(1-e_r^2)^6\,(304+121e_r^2)\,(5-2\,\eta) \right] \right\}. \tag{4.34}$$

To 1PN accuracy we recover the results of [48]. For the special case of circular orbits $\langle da_r/dt \rangle$ takes the simple form

$$\left\langle \frac{da_r}{dt} \right\rangle = -\frac{64}{5}\zeta^3\eta c \left\{ 1-\zeta\left[\frac{1751}{336}+\frac{7}{4}\eta \right] + \zeta^2\left[\frac{294\,383}{181\,44} \right. \right.$$

$$\left. \left. +\frac{263\,65}{2016}\eta+\frac{1}{2}\eta^2 \right] \right\}. \tag{4.35}$$

Equation (4.35) is consistent with the expression for $\dot{r}$ given in [55], after taking due account of the coordinate transformations required to relate the ADM and the harmonic gauges for the circular orbits.

V. THE 2PN CONTRIBUTION TO THE WAVEFORM

In this section, we compute the instantaneous part of the 2PN accurate gravitational waveform, i.e., the transverse-traceless (TT) part of the 2PN accurate far-zone field for two compact objects of arbitrary mass ratio, moving in a general orbit. It is given by [16]

$$(h_{km}^{TT})_{inst} = \frac{2G}{c^4 R}\mathcal{P}_{ijkm}\left\{ I_{ij}^{(2)} + \frac{1}{c}\left[\frac{1}{3}N_a I_{ija}^{(3)} + \frac{4}{3}\varepsilon_{ab(i}J_{j)a}^{(2)}N_b \right] \right.$$

$$+ \frac{1}{c^2}\left[\frac{1}{12}N_{ab}I_{ijab}^{(4)} + \frac{1}{2}\varepsilon_{ab(i}J_{j)ac}^{(3)}N_{bc} \right]$$

$$+ \frac{1}{c^3}\left[\frac{1}{60}N_{abc}I_{ijabc}^{(5)} + \frac{2}{15}\varepsilon_{ab(i}J_{j)acd}^{(4)}N_{bcd} \right]$$

$$\left. + \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}I_{ijabcd}^{(6)} + \frac{1}{36}\varepsilon_{ab(i}J_{j)acde}^{(5)}N_{bcde} \right] \right\}, \tag{5.1}$$

where R is the Cartesian observer-source distance and N_a's are the components of $\mathbf{N}=\mathbf{X}/R$, the unit normal in the direction of the vector $\mathbf{X}$, pointing from the source to the observer. The transverse traceless projection operator projecting orthogonal to $\mathbf{X}$, is given by

$$\mathcal{P}_{ijkm}(\mathbf{N}) = (\delta_{ik}-N_iN_k)(\delta_{jm}-N_jN_m)$$

$$- \frac{1}{2}(\delta_{ij}-N_iN_j)(\delta_{km}-N_kN_m). \tag{5.2}$$

Evaluating the appropriate time derivatives of the multipole moments and performing the relevant contractions with $\mathbf{N}$ as required by Eq. (5.1), some details of which are given in Appendix A, we obtain

$$(h_{km}^{TT})_{\text{inst}} = \frac{2G\mu}{c^4 R} P_{ijkm}\left\{\xi_{ij}^{(0)} + \frac{1}{c}\frac{\delta m}{m}\xi_{ij}^{(0.5)} + \frac{1}{c^2}\xi_{ij}^{(1)} + \frac{1}{c^3}\frac{\delta m}{m}\xi_{ij}^{(1.5)} + \frac{1}{c^4}\xi_{ij}^{(2)}\right\},$$
(5.3)

where the various ξ_{ij}'s are given by

$$\xi_{ij}^{(0)} = 2\left(v_{ij} - \frac{Gm}{r}n_{ij}\right),$$
(5.4a)

$$\xi_{ij}^{(0.5)} = \left\{3(\mathbf{N}\cdot\mathbf{n})\frac{Gm}{r}[2n_{(i}v_{j)} - \dot{r}n_{ij}] + (\mathbf{N}\cdot\mathbf{v})\left[\frac{Gm}{r}n_{ij} - 2v_{ij}\right]\right\},$$
(5.4b)

$$\xi_{ij}^{(1)} = \frac{1}{3}\left\{(1-3\eta)\left[(\mathbf{N}\cdot\mathbf{n})^2\frac{Gm}{r}\left(\left(3v^2 - 15\dot{r}^2 + 7\frac{Gm}{r}\right)n_{ij} + 30\dot{r}n_{(i}v_{j)} - 14v_{ij}\right) + (\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})\frac{Gm}{r}[12\dot{r}n_{ij} - 32n_{(i}v_{j)}]\right.\right.$$
$$+ (\mathbf{N}\cdot\mathbf{v})^2\left(6v_{ij} - 2\frac{Gm}{r}n_{ij}\right)\right] + \left[3(1-3\eta)v^2 - 2(2-3\eta)\frac{Gm}{r}\right]v_{ij} + 4\frac{Gm}{r}\dot{r}(5+3\eta)n_{(i}v_{j)} + \frac{Gm}{r}\left[3(1-3\eta)\dot{r}^2 - (10\right.$$
$$\left.\left. + 3\eta)v^2 + 29\frac{Gm}{r}\right]n_{ij}\right\},$$
(5.4c)

$$\xi_{ij}^{(1.5)} = \frac{1}{12}(1-2\eta)\left\{(\mathbf{N}\cdot\mathbf{n})^3\frac{Gm}{r}\left[\left(45v^2 - 105b^2 + 90\frac{Gm}{r}\right)\dot{r}n_{ij} - 96\dot{r}v_{ij} - \left(42v^2 - 210\dot{r}^2 + 88\frac{Gm}{r}\right)n_{(i}v_{j)}\right]\right.$$
$$- (\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})\frac{Gm}{r}\left[\left(27v^2 - 135\dot{r}^2 + 84\frac{Gm}{r}\right)n_{ij} + 336\dot{r}n_{(i}v_{j)} - 172v_{ij}\right] - (\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})^2\frac{Gm}{r}[48\dot{r}n_{ij} - 184n_{(i}v_{j)}]$$
$$+ (\mathbf{N}\cdot\mathbf{v})^3\left[4\frac{Gm}{r}n_{ij} - 24v_{ij}\right]\right\} - \frac{1}{12}(\mathbf{N}\cdot\mathbf{n})\frac{Gm}{r}\left\{\left[(69-66\eta)v^2 - (15-90\eta)\dot{r}^2 - (242-24\eta)\frac{Gm}{r}\right]\dot{r}n_{ij} - \left[(66-36\eta)v^2\right.\right.$$
$$\left.\left. + (138+84\eta)\dot{r}^2 - (256-72\eta)\frac{Gm}{r}\right]n_{(i}v_{j)} + (192+12\eta)\dot{r}v_{ij}\right\} + \frac{1}{12}(\mathbf{N}\cdot\mathbf{v})\left\{\left[(23-10\eta)v^2 - (9-18\eta)\dot{r}^2 - (104\right.\right.$$
$$\left.\left. - 12\eta)\frac{Gm}{r}\right]\frac{Gm}{r}n_{ij} - (88+40\eta)\frac{Gm}{r}\dot{r}n_{(i}v_{j)} - \left[(12-60\eta)v^2 - (20-52\eta)\frac{Gm}{r}\right]v_{ij)}\right\},$$
(5.4d)

$$\xi_{ij}^{(2)} = \frac{1}{120}(1-5\eta+5\eta^2)\left\{240(\mathbf{N}\cdot\mathbf{v})^4v_{ij} - (\mathbf{N}\cdot\mathbf{n})^4\frac{Gm}{r}\left[\left(90v^4 + \left(318\frac{Gm}{r} - 1260\dot{r}^2\right)v^2 + 344\frac{G^2 m^2}{r^2} + 1890\dot{r}^4\right.\right.\right.$$
$$\left.\left. - 2310\frac{Gm}{r}\dot{r}^2\right)n_{ij} + \left(1620v^2 + 3000\frac{Gm}{r} - 3780\dot{r}^2\right)\dot{r}n_{(i}v_{j)} - \left(336v^2 - 1680\dot{r}^2 + 688\frac{Gm}{r}\right)v_{ij}\right]$$
$$- (\mathbf{N}\cdot\mathbf{n})^3(\mathbf{N}\cdot\mathbf{v})\frac{Gm}{r}\left[\left(1440v^2 - 3360\dot{r}^2 + 3600\frac{Gm}{r}\right)\dot{r}n_{ij} - \left(1608v^2 - 8040\dot{r}^2 + 3864\frac{Gm}{r}\right)n_{(i}v_{j)} - 3960\dot{r}v_{ij}\right]$$
$$+ 120(\mathbf{N}\cdot\mathbf{v})^3(\mathbf{N}\cdot\mathbf{n})\frac{Gm}{r}(3\dot{r}n_{ij} - 20n_{(i}v_{j)}) + (\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})^2\frac{Gm}{r}\left[\left(396v^2 - 1980\dot{r}^2 + 1668\frac{Gm}{r}\right)n_{ij} + 6480\dot{r}n_{(i}v_{j)}\right.$$
$$\left.\left. - 3600v_{(ij)}\right]\right\} - \frac{1}{30}(\mathbf{N}\cdot\mathbf{v})^2\left\{\left[(87-315\eta+145\eta^2)v^2 - (135-465\eta+75\eta^2)\dot{r}^2 - (289-905\eta+115\eta^2)\frac{Gm}{r}\right]\frac{Gm}{r}n_{ij}\right.$$
$$- (240-660\eta-240\eta^2)\dot{r}n_{(i}v_{j)} - \left[(30-270\eta+630\eta^2)v^2 - 60(1-6\eta+10\eta^2)\frac{Gm}{r}\right]v_{(ij)}\right\} + \frac{1}{30}(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})\frac{Gm}{r}\left\{\left[(270\right.\right.$$
$$\left. - 1140\eta+1170\eta^2)v^2 - (60-450\eta+900\eta^2)\dot{r}^2 - (1270-3920\eta+360\eta^2)\frac{Gm}{r}\right]\dot{r}n_{ij} - \left[(186-810\eta+1450\eta^2)v^2 + (990\right.$$
$$\left.\left. - 2910\eta-930\eta^2)\dot{r}^2 - (1242-4170\eta+1930\eta^2)\frac{Gm}{r}\right]n_{(i}v_{j)} + [1230-3810\eta-90\eta^2]\dot{r}v_{ij}\right\} + \frac{1}{60}(\mathbf{N}\cdot\mathbf{n})^2\frac{Gm}{r}\left\{\left[(117\right.\right.$$
$$\left. - 480\eta+540\eta^2)v^4 - (630-2850\eta+4050\eta^2)v^2\dot{r}^2 - (125-740\eta+900\eta^2)\frac{Gm}{r}v^2 + (105-1050\eta+3150\eta^2)\dot{r}^4\right.$$

$$+(2715-8580\eta+1260\eta^2)\frac{Gm}{r}\,\dot{r}^2-(1048-3120\eta+240\eta^2)\frac{G^2m^2}{r^2}\Big]n_{ij}+\Big[(216-1380\eta+4320\eta^2)v^2+(1260-3300\eta$$

$$-3600\eta^2)\dot{r}^2-(3952-12860\eta+3660\eta^2)\frac{Gm}{r}\Big]\dot{r}\,n_{(i}v_{j)}-\Big[(12-180\eta+1160\eta^2)v^2+(1260-3840\eta-780\eta^2)\dot{r}^2-(664$$

$$-2360\eta+1700\eta^2)\frac{Gm}{r}\Big]v_{ij}\Big\}-\frac{1}{60}\Big\{\Big[(66-15\eta-125\eta^2)v^4+(90-180\eta-480\eta^2)v^2\dot{r}^2-(389+1030\eta-110\eta^2)\frac{Gm}{r}\,v^2$$

$$+(45-225\eta+225\eta^2)\dot{r}^4+(915-1440\eta+720\eta^2)\frac{Gm}{r}\,\dot{r}^2+(1284+1090\eta)\frac{G^2m^2}{r^2}\Big]\frac{Gm}{r}\,n_{ij}-\Big[(132+540\eta-580\eta^2)v^2$$

$$+(300-1140\eta+300\eta^2)\dot{r}^2+(856+400\eta+700\eta^2)\frac{Gm}{r}\Big]\frac{Gm}{r}\,\dot{r}\,n_{(i}v_{j)}-\Big[(45-315\eta+585\eta^2)v^4+(354-210\eta$$

$$-550\eta^2)\frac{Gm}{r}\,v^2-(270-30\eta+270\eta^2)\frac{Gm}{r}\,\dot{r}^2-(638+1400\eta-130\eta^2)\frac{G^2m^2}{r^2}\Big]v_{(ij)}\Big\}. \tag{5.4e}$$

The "tail" contribution reads

$$(h_{km}^{TT})_{\text{tail}}=\frac{2G}{c^4R}\frac{2Gm}{c^3}\mathcal{P}_{ijkm}\int_0^{+\infty}d\tau\Big\{\ln\Big(\frac{\tau}{2b_1}\Big)I_{ij}^{(4)}(T_R-\tau)$$

$$+\frac{1}{3c}\ln\Big(\frac{\tau}{2b_2}\Big)N_aI_{ija}^{(5)}(T_R-\tau)$$

$$+\frac{4}{3c}\ln\Big(\frac{\tau}{2b_3}\Big)\varepsilon_{ab(i}N_bJ_{j)a}^{(4)}(T_R-\tau)\Big\}\ , \tag{5.5}$$

where we have used for simplicity the notation

$$b_1\equiv b\ e^{-11/12},\quad b_2\equiv b\ e^{-97/60},\quad b_3\equiv b\ e^{-7/6}. \tag{5.6}$$

We do not discuss the "tail" terms in this paper. Some details of these tail terms may be found in [16,18].

The first check on the above waveform is its circular limit, which matches with the waveform computed earlier in [16]. The next check of the waveform in the general case is performed by computing the far-zone energy flux using

$$\frac{d\mathcal{E}}{dt}=\frac{c^3\,R^2}{32\pi G}\int(\dot{h}_{km}^{TT}\dot{h}_{km}^{TT})d\Omega(\mathbf{N}). \tag{5.7}$$

The expression for $d\mathcal{E}/dt$ thus obtained is identical to the far-zone energy flux directly obtained from multipole moments Eq. (3.5). Of course, these checks do not uniquely fix the expressions in Eq. (5.4) and equivalent expressions are possible leading to the same transverse traceless parts as discussed below.

The above expressions for the waveform, computed using STF multipole moments differ from the corresponding expressions obtained by Will and Wiseman (Eqs. (6.10), (6.11) of [18]), using the Epstein-Wagoner multipole moments at 1.5PN and 2PN orders. Though the two expressions are totally different looking at these orders, even in the circular limit, it is possible to show that they are equivalent. The equivalence is established by showing that the difference between the two expressions, at 1.5PN and 2PN orders has a vanishing transverse-traceless (TT) part. The easiest

way of verifying this is to show that the "plus" and "cross" polarizations of the difference in the two expressions vanish at 1.5PN and 2PN orders [64]. In Appendix B, we present the difference—at 1.5PN and 2PN orders—between our wave form expression computed directly using the STF multipoles and the Will-Wiseman one computed using the EW multipoles and verify their equivalence. Finally we note that the statement in Appendix E of [18] should more precisely read that, STF multipole moments presented there yield an expression for the waveform *equivalent* to Eqs. (6.10) and (6.11) of [18], and not *identical* to it [64].

VI. CONCLUSION

In this paper using the BDI approach, we have computed the 2PN contributions to the mass quadrupole moment for two compat objects of arbitrary mass ratio moving in general orbits. Using this moment we have computed the 2PN contributions to the gravitational waveform and the associated energy and angular momentum fluxes. These expressions have already proved useful in the computation of the 2PN radiation reaction, i.e., the 4.5PN terms in the equations of motion [55], using the refined balance method proposed by Iyer and Will [56,57]. Work is in progress [65] to obtain the higher order corrections to the far-zone linear momentum flux from the gravitational waveform presented here, extending the treatment of Wiseman [66]. It should be noted that 2PN corrections to the linear momentum flux can be computed only if one knows h_{jk}^{TT} to 2.5PN accuracy. Using the 2PN accurate generalized quasi-Keplerian representation for elliptic orbits, we have computed here the instantaneous 2PN contributions to $\langle d\mathcal{E}/dt\rangle$ and $\langle d\mathcal{J}/dt\rangle$, the fluxes averaged over one orbital time scale. This is used to compute the evolution of the orbital elements, in particular $\dot{P}$, $\dot{e}_r$ and $\dot{a}_r$. The method employed to compute $\langle d\mathcal{E}/dt\rangle$ and $\langle d\mathcal{J}/dt\rangle$ could also be adapted to the case of hyperbolic orbits to generalize the work of Simone, Poisson and Will on the head-on collision [60].

tions. We also thank Clifford Will and Alan Wiseman for discussions on the waveform equivalence.

APPENDIX A: STF TENSORS AND FORMULAS FOR THE WAVEFORM COMPUTATIONS

We present details of the scheme, employed to compute the contributions to h_{jk} from various multipole moments, as required by Eqs. (5.1), (5.3), and (5.4). Our scheme proceeds in steps. In the first step, we write down schematically, the form of the desired time derivative of the STF multipole moment, using the compact notation $\{\}$, introduced by Blanchet and Damour [12]. Here $\{\}$ denotes an *unnormal-*

ized minimum number of terms, required to make the expression symmetric in all the indicated indices. The second step involves *peeling*, where by observation and counting, we rewrite the expression obtained in step 1, as STF on the free indices—i and j in our case. In step 3, we contract, the final expression of step 2 with appropriate number of **N**'s as required by Eq. (5.1). The actual evaluation of the result of step 3 is performed on Maple [53]. In all the formulas, S_L denotes the symmetric version of the object under consideration; e.g., $S_L = I_{(L)}^{(n)}$ if the object is $I_L^{(n)}$ and $S_L = J_{(L)}^{(m)}$ if the object is $J_L^{(m)}$—the object in the formula is obvious from the context.

The unnormalized symmetric blocks.

$$\delta_{\{ij}S_{a\}} = \delta_{ij}\,S_a + \delta_{ia}\,S_j + \delta_{ja}\,S_i, \tag{A1a}$$

$$\delta_{\{ij}S_{ab\}} = \delta_{ij}S_{ab} + \delta_{ia}S_{jb} + \delta_{ib}S_{aj} + \delta_{ja}S_{ib} + \delta_{jb}S_{ai} + \delta_{ab}S_{ij}, \tag{A1b}$$

$$\delta_{\{ij}\delta_{ab\}} = \delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{aj}, \tag{A1c}$$

$$\delta_{\{ij}S_{abc\}} = \delta_{ij}S_{abc} + \delta_{ia}S_{jbc} + \delta_{ib}S_{ajc} + \delta_{ic}S_{abj} + \delta_{ja}S_{ibc} + \delta_{jb}S_{aic} + \delta_{jc}S_{abi} + \delta_{ab}S_{ijc} + \delta_{ac}S_{ibj} + \delta_{bc}S_{aij}, \tag{A1d}$$

$$\delta_{\{ij}\delta_{ab}S_{c\}} = \{[\delta_{ja}\delta_{bc} + \delta_{jb}\delta_{ac} + \delta_{jc}\delta_{ab}]S_i + [\delta_{ia}\delta_{bc} + \delta_{ib}\delta_{ac} + \delta_{ic}\delta_{ab}]S_j + [\delta_{ij}\delta_{bc} + \delta_{ib}\delta_{jc} + \delta_{ic}\delta_{bj}]S_a + [\delta_{ij}\delta_{ac} + \delta_{ia}\delta_{jc}$$
$$+ \delta_{ic}\delta_{ja}]S_b + [\delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}]S_c\}, \tag{A1e}$$

$$\delta_{\{ij}S_{abcd\}} = \{\delta_{ij}S_{abcd} + \delta_{ia}S_{jbcd} + \delta_{ib}S_{ajcd} + \delta_{ic}S_{abjd} + \delta_{id}S_{abcj} + \delta_{ja}S_{ibcd} + \delta_{jb}S_{aicd} + \delta_{jc}S_{abid} + \delta_{jd}S_{abci} + \delta_{ab}S_{ijcd} + \delta_{ac}S_{bdij}$$
$$+ \delta_{ad}S_{bcij} + \delta_{bc}S_{adij} + \delta_{bd}S_{acij} + \delta_{cd}S_{abij}\}, \tag{A1f}$$

$$\delta_{\{ij}\delta_{ab}S_{cd\}} = \{[\delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{aj}]S_{cd} + [\delta_{ij}\delta_{ac} + \delta_{ia}\delta_{jc} + \delta_{ic}\delta_{aj}]S_{bd} + [\delta_{ij}\delta_{bc} + \delta_{ic}\delta_{jb} + \delta_{ib}\delta_{jc}]S_{ad} + [\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb}$$
$$+ \delta_{ib}\delta_{ac}]S_{jd} + [\delta_{cj}\delta_{ab} + \delta_{ca}\delta_{jb} + \delta_{cb}\delta_{aj}]S_{id} + [\delta_{ij}\delta_{ad} + \delta_{ia}\delta_{jd} + \delta_{id}\delta_{aj}]S_{cb} + [\delta_{ij}\delta_{db} + \delta_{id}\delta_{jb} + \delta_{ib}\delta_{dj}]S_{ca}$$
$$+ [\delta_{id}\delta_{ab} + \delta_{ia}\delta_{db} + \delta_{ib}\delta_{ad}]S_{cj} + [\delta_{dj}\delta_{ab} + \delta_{da}\delta_{jb} + \delta_{db}\delta_{aj}]S_{ci} + [\delta_{ij}\delta_{cd} + \delta_{ic}\delta_{jd} + \delta_{id}\delta_{cj}]S_{ab} + [\delta_{ai}\delta_{cd} + \delta_{ic}\delta_{ad}$$
$$+ \delta_{id}\delta_{ac}]S_{jb} + [\delta_{aj}\delta_{cd} + \delta_{ca}\delta_{jd} + \delta_{ad}\delta_{cj}]S_{ib} + [\delta_{ib}\delta_{cd} + \delta_{ic}\delta_{db} + \delta_{id}\delta_{bc}]S_{aj} + [\delta_{bj}\delta_{cd} + \delta_{bd}\delta_{jc} + \delta_{jd}\delta_{bc}]S_{ai}$$
$$+ [\delta_{cd}\delta_{ab} + \delta_{ca}\delta_{db} + \delta_{cb}\delta_{ad}]S_{ji}\}, \tag{A1g}$$

$$\delta_{\{ij}\delta_{ab}\delta_{cd\}} = \{[\delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}]\delta_{cd} + [\delta_{ij}\delta_{ac} + \delta_{ia}\delta_{jc} + \delta_{ic}\delta_{ja}]\delta_{bd} + [\delta_{ij}\delta_{cb} + \delta_{ic}\delta_{jb} + \delta_{ib}\delta_{jc}]\delta_{ad} + [\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb}$$
$$+ \delta_{ib}\delta_{ca}]\delta_{jd} + [\delta_{cj}\delta_{ab} + \delta_{ca}\delta_{jb} + \delta_{cb}\delta_{ja}]\delta_{id}\}. \tag{A1h}$$

The STF tensors.

$$\mathrm{STF}_{ija}(I_{ija}) = S_{ija} - \frac{1}{5}\delta_{\{ij}S_{a\}tt}, \tag{A2a}$$

$$\mathrm{STF}_{ijab}(I_{ijab}) = S_{ijab} - \frac{1}{7}\delta_{\{ij}S_{ab\}tt} + \frac{1}{35}\delta_{\{ij}\delta_{ab\}}S_{sstt}, \tag{A2b}$$

$$\mathrm{STF}_{ijabc}(I_{ijabc}) = S_{ijabc} - \frac{1}{9}\delta_{\{ij}S_{abc\}tt} + \frac{1}{63}\delta_{\{ij}\delta_{ab}S_{c\}sstt}, \tag{A2c}$$

$$\mathrm{STF}_{ijabcd}(I_{ijabcd}) = S_{ijabcd} - \frac{1}{11}\delta_{\{ij}S_{abcd\}pp} + \frac{1}{99}\delta_{\{ij}\delta_{ab}S_{cd\}ttqq} - \frac{1}{693}\delta_{\{ij}\delta_{ab}\delta_{cd\}}S_{ppqqtt}. \tag{A2d}$$

The peeling.

$$\text{STF}_{ija}(I_{ija}) = \text{STF}_{ij}\left\{ S_{ija} - \frac{2}{5}\delta_{ia}S_{jtt}\right\}, \tag{A3a}$$

$$\text{STF}_{ijab}(I_{ijab}) = \text{STF}_{ij}\left\{ S_{ijab} - \frac{1}{7}[2\delta_{ia}S_{jbtt} + 2\delta_{ib}S_{jatt} + \delta_{ba}S_{jitt}] + \frac{2}{35}[\delta_{ia}\delta_{jb}S_{ttss}]\right\}, \tag{A3b}$$

$$\text{STF}_{ijabc}(I_{ijabc}) = \text{STF}_{ij}\left\{ S_{ijabc} - \frac{1}{9}[2(\delta_{ia}S_{jbcpp} + \delta_{ib}S_{jacpp} + \delta_{ic}S_{abjpp}) + (\delta_{ab}S_{ijcpp} + \delta_{ac}S_{ijbpp} + \delta_{bc}S_{ijapp})]\right.$$
$$\left. + \frac{2}{63}[(\delta_{ja}\delta_{bc} + \delta_{jb}\delta_{ac} + \delta_{jc}\delta_{ab})S_{ippqq} + (\delta_{ib}\delta_{jc}S_{appqq} + \delta_{ia}\delta_{jc}S_{bppqq} + \delta_{ia}\delta_{jb}S_{cppqq})]\right\}, \tag{A3c}$$

$$\text{STF}_{ijabcd}(I_{ijabcd}) = \text{STF}_{ij}\left\{ S_{ijabcd} - \frac{1}{11}[2(\delta_{ia}S_{jbcdpp} + \delta_{ib}S_{jacdpp} + \delta_{ic}S_{abjdpp} + \delta_{id}S_{abcjpp}) + (\delta_{ad}S_{bcijpp} + \delta_{ab}S_{dcijpp}\right.$$
$$+ \delta_{ac}S_{bdijpp} + \delta_{bc}S_{adijpp} + \delta_{bd}S_{acijpp} + \delta_{cd}S_{baijpp})] + \frac{1}{99}[2(\delta_{ia}\delta_{jb}S_{cdppqq} + \delta_{ia}\delta_{jc}S_{bdppqq}$$
$$+ \delta_{ic}\delta_{jb}S_{adppqq} + \delta_{ia}\delta_{jd}S_{cbppqq} + \delta_{id}\delta_{jb}S_{acppqq} + \delta_{ic}\delta_{jd}S_{bappqq}) + 2(\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb} + \delta_{ib}\delta_{ca})S_{jdppqq}$$
$$+ 2(\delta_{id}\delta_{ab} + \delta_{ia}\delta_{db} + \delta_{ib}\delta_{da})S_{jcppqq} + 2(\delta_{ia}\delta_{cd} + \delta_{ic}\delta_{da} + \delta_{id}\delta_{ca})S_{jbppqq} + 2(\delta_{ib}\delta_{cd} + \delta_{ic}\delta_{db}$$
$$+ \delta_{id}\delta_{bc})S_{jappqq} + (\delta_{cd}\delta_{ab} + \delta_{ca}\delta_{db} + \delta_{da}\delta_{cb})S_{jippqq}] - \frac{2}{693}[(\delta_{ia}\delta_{jb}\delta_{cd} + \delta_{ia}\delta_{jc}\delta_{bd} + \delta_{ic}\delta_{jb}\delta_{ad})$$
$$\left. + (\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb} + \delta_{ib}\delta_{ac})\delta_{jd}]S_{ppqqtt}\right\}. \tag{A3d}$$

The contractions with N_L.

$$\text{STF}_{ija}(I_{ija}^{(3)})\,N_a = \text{STF}_{ij}\left\{ S_{ija}^{(3)}\,N_a - \frac{2}{5}\,N_i S_{jtt}^{(3)}\right\}, \tag{A4a}$$

$$\text{STF}_{ijab}(I_{ijab}^{(4)})N_{ab} = \text{STF}_{ij}\left\{ S_{ijab}^{(4)}N_{ab} - \frac{1}{7}[4N_{ia}S_{jatt}^{(4)} + S_{ijtt}^{(4)}] + \frac{2}{35}N_{ij}S_{ttss}^{(4)}\right\}, \tag{A4b}$$

$$\text{STF}_{ijabc}(I_{ijabc}^{(5)})N_{abc} = \text{STF}_{ij}\left\{ S_{ijabc}^{(5)}N_{abc} - \frac{6}{9}N_{ibc}S_{jbcpp}^{(5)} - \frac{1}{3}S_{ijcpp}^{(5)}N_c + \frac{6}{63}S_{ippqq}^{(5)}N_j + \frac{6}{63}N_{ija}S_{appqq}^{(5)}\right\}, \tag{A4c}$$

$$\text{STF}_{ijabcd}(I_{ijabcd}^{(6)})N_{abcd} = \text{STF}_{ij}\left\{ S_{ijabcd}^{(6)}\,N_{abcd} - \frac{8}{11}N_{ibcd}\,S_{jbcdpp}^{(6)} - \frac{6}{11}S_{ijcdpp}^{(6)}\,N_{cd} + \frac{12}{99}N_{ijcd}\,S_{cdppqq}^{(6)} + \frac{24}{99}N_{id}\,S_{jdppqq}^{(6)}\right.$$
$$\left. + \frac{3}{99}S_{ijppqq}^{(6)} - \frac{12}{693}N_{ij}\,S_{ppqqtt}^{(6)}\right\}. \tag{A4d}$$

The current multipole moments. $\epsilon_{pq(i}\hat{J}_{j)pL} = \text{STF}_{ij}\{\epsilon_{pqi}\hat{J}_{jpL}\}$,

$$\epsilon_{pq(i}\hat{J}_{j)p}^{(2)}\,N_q = \text{STF}_{ij}\{\epsilon_{pqi}S_{jp}^{(2)}\,N_q\}, \tag{A5a}$$

$$\epsilon_{pq(i}\hat{J}_{j)pa}^{(3)}\,N_{qa} = \text{STF}_{ij}\left\{\epsilon_{pqi}\left[S_{jpa}^{(3)}\,N_{qa} - \frac{1}{5}S_{ptt}^{(3)}\,N_{qj}\right]\right\}, \tag{A5b}$$

$$\epsilon_{pq(i}\hat{J}_{j)pab}^{(4)}\,N_{qab} = \text{STF}_{ij}\left\{\epsilon_{pqi}\left[S_{jpab}^{(4)}\,N_{qab} - \frac{1}{7}(2\,S_{pbtt}^{(4)}\,N_{qjb} + S_{pjtt}^{(4)}\,N_q)\right]\right\}, \tag{A5c}$$

$$\epsilon_{pq(i}\hat{J}_{j)pabc}^{(5)}\,N_{qabc} = \text{STF}_{ij}\left\{\epsilon_{pqi}\left[S_{jpabc}^{(5)}\,N_{qabc} - \frac{1}{3}(S_{pbctt}^{(5)}\,N_{qjbc} + S_{jpctt}^{(5)}\,N_{qc}) + \frac{1}{21}(S_{pttvv}^{(5)}N_{qj})\right]\right\}. \tag{A5d}$$

The explicit computations of the above equations require the following identities, which are easily derived, using the rules governing the product of ϵ's. The identities are

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q y_j \widetilde{L}_p\} = \text{STF}_{ij}\{-(\mathbf{N}\cdot\mathbf{v})y_{ij}+(\mathbf{N}\cdot\mathbf{n})r\,\,y_i v_j\}, \tag{A6a}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q v_j \widetilde{L}_p\} = \text{STF}_{ij}\{-(\mathbf{N}\cdot\mathbf{v})y_i v_j+(\mathbf{N}\cdot\mathbf{n})\,\,r\,v_{ij}\}, \tag{A6b}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_{qj}\widetilde{L}_p\} = \text{STF}_{ij}\{(\mathbf{N}\cdot\mathbf{n})\,\,r\,N_i v_j-(\mathbf{N}\cdot\mathbf{v})\,\,y_i N_j\}, \tag{A6c}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q y_p \widetilde{L}_j\} = \text{STF}_{ij}\{-(\mathbf{N}\cdot\mathbf{v})\,\,y_{ij}+(\mathbf{N}\cdot\mathbf{n})\,\,r\,\,y_i v_j+(r\dot{r})N_i y_j-r^2 N_i v_j\}, \tag{A6d}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q v_p \widetilde{L}_j\} = \text{STF}_{ij}\{(\mathbf{N}\cdot\mathbf{n})\,\,r\,v_{ij}-(\mathbf{N}\cdot\mathbf{v})y_i v_j-r\dot{r}\,\,N_i v_j+v^2\,\,N_i y_j\}, \tag{A6e}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_{qj}v_p(\widetilde{\mathbf{L}}\cdot\mathbf{N})\} = \text{STF}_{ij}\{[v^2-(\mathbf{N}\cdot\mathbf{v})^2]N_i y_j+[(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})-\dot{r}]rN_i v_j+[\dot{r}(\mathbf{N}\cdot\mathbf{v})-v^2(\mathbf{N}\cdot\mathbf{n})]rN_{ij}\}, \tag{A6f}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_{qj}y_p(\widetilde{\mathbf{L}}\cdot\mathbf{N})\} = \text{STF}_{ij}\{[\dot{r}-(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})]rN_i y_j+[(\mathbf{N}\cdot\mathbf{n})^2-1]r^2 N_i v_j+[(\mathbf{N}\cdot\mathbf{v})-\dot{r}(\mathbf{N}\cdot\mathbf{n})]r^2 N_{ij}\}, \tag{A6g}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q y_{jp}(\widetilde{\mathbf{L}}\cdot\mathbf{N})\} = \text{STF}_{ij}\{[\dot{r}-(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})]ry_{ij}+[(\mathbf{N}\cdot\mathbf{n})^2-1]r^2 y_i v_j+[(\mathbf{N}\cdot\mathbf{v})-\dot{r}(\mathbf{N}\cdot\mathbf{n})]r^2 N_i y_j\}, \tag{A6h}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q y_j v_p(\widetilde{\mathbf{L}}\cdot\mathbf{N})\} = \text{STF}_{ij}\{[v^2-(\mathbf{N}\cdot\mathbf{v})^2]y_{ij}+[(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})-\dot{r}]ry_i v_j+[\dot{r}(\mathbf{N}\cdot\mathbf{v})-v^2\,(\mathbf{N}\cdot\mathbf{n})]rN_i y_j\}, \tag{A6i}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q y_p v_j(\widetilde{\mathbf{L}}\cdot\mathbf{N})\} = \text{STF}_{ij}\{[\dot{r}-(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})]\,\,r\,y_i v_j+[(\mathbf{N}\cdot\mathbf{n})^2-1]\,\,r^2\,v_{ij}+[(\mathbf{N}\cdot\mathbf{v})-\dot{r}\,(\mathbf{N}\cdot\mathbf{n})]r^2\,N_i v_j\}, \tag{A6j}$$

$$\text{STF}_{ij}\{\epsilon_{pqi}\,N_q v_{jp}(\widetilde{\mathbf{L}}\cdot\mathbf{N})\} = \text{STF}_{ij}\{[v^2-(\mathbf{N}\cdot\mathbf{v})^2]y_i v_j+[(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})-\dot{r}]rv_{ij}+[\dot{r}(\mathbf{N}\cdot\mathbf{v})-v^2(\mathbf{N}\cdot\mathbf{n})]rN_i v_j\}, \tag{A6k}$$

where $\widetilde{L}_p = \epsilon_{pkl}\,y_k\,v_l$.

APPENDIX B: THE EQUIVALENCE TO WILL-WISEMAN WAVEFORM

The expression for the gravitational waveform, obtained by Will and Wiseman [18] differs from our waveform expression at the 1.5PN and the 2PN orders. We give below the difference in the waveform expressions at these orders and show that the two polarization states h_+ and $h_\times$ of the difference are zero at 1.5PN and 2PN orders.

$$\{(h_{km}^{TT})_{\text{BDI}}^{(1.5)} - (h_{km}^{TT})_{WW}^{(1.5)}\} = \frac{1}{3c^3}\,\mathcal{P}_{ijkm}\,\frac{\delta m}{m}\,\frac{Gm}{r}\,(1-2\eta)\{3(\mathbf{N}\cdot\mathbf{n})^3\dot{r}v_{ij}-(\mathbf{N}\cdot\mathbf{v})(\mathbf{N}\cdot\mathbf{n})^2[v_{ij}+6\dot{r}n_{(i}v_{j)}]$$

$$+(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})^2\,[2n_{(i}v_{j)}+3\,\dot{r}\,n_{ij}]-(\mathbf{N}\cdot\mathbf{v})^3 n_{ij}+3\,(\mathbf{N}\cdot\mathbf{n})\dot{r}[v^2 n_{ij}+v_{ij}-2\dot{r}n_{(i}v_{j)}]$$

$$+(\mathbf{N}\cdot\mathbf{v})[v^2 n_{ij}+v_{ij}-2\dot{r}n_{(i}v_{j)}]\}, \tag{B1a}$$

$$\{(h_{km}^{TT})_{\text{BDI}}^{(2)} - (h_{km}^{TT})_{WW}^{(2)}\} = \frac{1}{15c^4}\,\mathcal{P}_{ijkm}\,\frac{Gm}{r}\Bigg\{(1-5\eta+5\eta^2)\Bigg[12(\mathbf{N}\cdot\mathbf{v})^4 n_{ij}-3\,(\mathbf{N}\cdot\mathbf{n})^4\Big(3v^2-15\dot{r}^2+\frac{Gm}{r}\Big)v_{ij}+6(\mathbf{N}\cdot\mathbf{n})^3(\mathbf{N}\cdot\mathbf{v})$$

$$\times\Big(\Big[3v^2-15\dot{r}^2+\frac{Gm}{r}\Big]n_{(i}v_{j)}-9\dot{r}v_{ij}\Big)-6(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})^3(9\dot{r}n_{ij}+4n_{(i}v_{j)})-3(\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})^2\Big(\Big[3v^2$$

$$-15\dot{r}^2+\frac{Gm}{r}\Big]n_{ij}-36\dot{r}n_{(i}v_{j)}-4v_{ij}\Big)\Big]-(\mathbf{N}\cdot\mathbf{v})^2\Big[\Big((51-185\eta+55\eta^2)v^2-(117-375\eta-15\eta^2)\dot{r}^2$$

$$-(39-125\eta-5\eta^2)\,\frac{Gm}{r}\Big)n_{ij}-24(1-5\eta+5\eta^2)\dot{r}n_{(i}v_{j)}+12(1-5\eta+5\eta^2)v_{ij}\Big]+2(\mathbf{N}\cdot\mathbf{v})(\mathbf{N}\cdot\mathbf{n})$$

$$\times\Big[27(1-5\eta+5\eta^2)\,\dot{r}\,v^2\,n_{ij}+\Big((39-125\eta-5\,\eta^2)\Big(v^2-\frac{Gm}{r}\Big)-(171-645\eta+255\eta^2)\dot{r}^2\Big)n_{(i}v_{j)}$$

$$+27(1-5\eta+5\eta^2)\dot{r}v_{ij}\Big]-(\mathbf{N}\cdot\mathbf{n})^2\Big[(1-5\eta+5\eta^2)\Big(-9\,v^4+45\dot{r}^2 v^2-3\,v^2\,\frac{Gm}{r}\Big)(n_{ij}-2\,\dot{r}n_{(i}v_{j)})+\Big((30$$

$$-80\eta-50\,\eta^2)v^2-(72-150\eta-240\,\eta^2)\dot{r}^2-(42-140\eta+10\,\eta^2)\frac{Gm}{r}\Big)\,v_{ij}\Big]$$

$$-\Big[\Big((39-125\eta-5\eta^2)\Big(\frac{Gm}{r}-v^2\Big)+(117-375\eta-15\eta^2)\dot{r}^2\Big)(v^2 n_{ij}-2\dot{r}n_{(i}v_{j)}+v_{ij})\Big]\Bigg\}\,. \tag{B1b}$$

The two independent polarization states of the gravitational wave h_+ and $h_\times$ are given by $h_+ = \frac{1}{2}(p_i\, p_j - q_i\, q_j)\, h_{ij}^{TT}$ and $h_\times = \frac{1}{2}(p_i\, q_j + p_j\, q_i)\, h_{ij}^{TT}$, where $\mathbf{p}$ and $\mathbf{q}$ are the two polarization vectors, forming along with $\mathbf{N}$ an orthogonal triad [16,67,18]. Note that there is no need to apply the TT projection before contracting on $\mathbf{p}$ and $\mathbf{q}$. Consequently, we write the difference in the waveform at the 1.5PN and the 2PN orders as

$$\{(h_{ij}^{TT})_{WW} - (h_{ij}^{TT})_{BDI}\} = \zeta_1 v_{ij} + \zeta_2 n_{ij} + \zeta_3 n_{(i} v_{j)}. \quad \text{(B2)}$$

The polarization states h_+ and $h_\times$, for Eqs. (B2) are given by

$$h_+ = \frac{1}{2}(p_i\, p_j - q_i\, q_j)\,(\zeta_1 v_{ij} + \zeta_2 n_{ij} + \zeta_3 n_{(i} v_{j)})$$

$$= \frac{\zeta_1}{2}[(\mathbf{p}\cdot\mathbf{v})^2 - (\mathbf{q}\cdot\mathbf{v})^2] + \frac{\zeta_2}{2}[(\mathbf{p}\cdot\mathbf{n})^2 - (\mathbf{q}\cdot\mathbf{n})^2]$$

$$+ \frac{\zeta_3}{2}[(\mathbf{p}\cdot\mathbf{n})(\mathbf{p}\cdot\mathbf{v}) - (\mathbf{q}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{v})], \quad \text{(B3a)}$$

$$h_\times = \frac{1}{2}(p_i q_j + p_j q_i)(\zeta_1 v_{ij} + \zeta_2 n_{ij} + \zeta_3 n_{(i} v_{j)})$$

$$= \zeta_1(\mathbf{p}\cdot\mathbf{v})(\mathbf{q}\cdot\mathbf{v}) + \zeta_2\, (\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})$$

$$+ \frac{\zeta_3}{2}[(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{v}) + (\mathbf{p}\cdot\mathbf{v})\,(\mathbf{q}\cdot\mathbf{n})]. \quad \text{(B3b)}$$

For the explicit computation of Eqs. (B3), we use the standard convention adopted in [16,67,18], which gives $\mathbf{p} = (0,1,0)$, $\mathbf{q} = (-\cos i, 0, \sin i)$, $\mathbf{N} = (\sin i, 0, \cos i)$, $\mathbf{n} = (\cos\phi, \sin\phi, 0)$, and $\mathbf{v} = (\dot{r}\cos\phi - r\,\omega\sin\phi, \dot{r}\sin\phi + r\,\omega\cos\phi, 0)$, where $\mathbf{n}$ and $\mathbf{v}$ are the unit separation vector, and the velocity vector respectively, ϕ is the orbital phase angle, such that the orbital angular velocity $\omega = d\phi/dt$, and i is the inclination angle of the source.

A straightforward but lengthy computation shows that h_+ and $h_\times$, given by Eqs. (B3) vanish, both at the 1.5PN and the 2PN orders. This establishes the equivalence of our waveform expression, Eqs. (5.3) and (5.4) with the WW one given by Eqs. (6.10) and (6.11) of [18].

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[1] A. Abramovici, W. E. Althouse, R. W. P. Drever, Y. Gürsel, S. Kawamura, F. J. Raab, D. Shoemaker, L. Sievers, R. E. Spero, K. S. Thorne, R. E. Vogt, R. Weiss, S. E. Whitcomb, and M. E. Zucker, Science **256**, 325 (1992).

[2] C. Bradaschia, E. Calloni, M. Cobal, R. Del Fasbro, A. Di Virgilio, A. Giazotto, L.E. Holloway, H. Kautzky, B. Michelozzi, V. Montelatici, D. Pascuello, and W. Velloso, in *Gravitation 1990*, Proceedings of the Banff Summer Institute, edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991).

[3] K. S. Thorne, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987), p. 330.

[4] B. F. Schutz, in *The Detection of Gravitational Waves*, edited by D. G. Blair (Cambridge University Press, Cambridge, England, 1991), p. 406.

[5] C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, É. E. Flanagan, D. Kennefick, D. M. Marković, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).

[6] L. S. Finn and D. F. Chernoff, Phys. Rev. D **47**, 2198 (1993).

[7] C. Cutler and É. E. Flanagan, Phys. Rev. D **49**, 2658 (1994).

[8] R. Balasubramanian, B. S. Sathyaprakash, and S. V. Dhurandhar, Phys. Rev. D **53**, 3033 (1996).

[9] A. Einstein, Preuss. Akad. Wiss. Berlin Sitzber. 688 (1916).

[10] V. Fock, *The Theory of Space, Time and Gravitation* (Macmillan, New York, 1964).

[11] L. D. Landau and E. M. Lifschitz, *Teoriya Polya* (Nauka, Moskow, 1941).

[12] L. Blanchet and T. Damour, Philos. Trans. R. Soc. London **A320**, 379 (1986); L. Blanchet, *ibid.* **A409**, 383 (1987); L. Blanchet and T. Damour, Phys. Rev. D **37**, 1410 (1988); Ann. Inst. Henri Poincaré, Phys. Theor. **50**, 377 (1989); T. Damour

and B. R. Iyer, *ibid.* **54**, 115 (1991); Phys. Rev. D **43**, 3259 (1991); L. Blanchet and T. Damour, *ibid.* **46**, 4304 (1992); T. Damour and B. R. Iyer, Class. Quantum Grav. **11**, 1353 (1994); L. Blanchet, Phys. Rev. D **51**, 2559 (1995); L. Blanchet, T. Damour, and B. R. Iyer, *ibid.* **51**, 5360 (1995); L. Blanchet, *ibid.* **54**, 1417 (1996).

[13] R. Epstein and R. V. Wagoner, Astrophys. J. **197**, 717 (1975); A. G. Wiseman, Phys. Rev. D **46**, 1517 (1992); C. M. Will and A. G. Wiseman, *ibid.* **54**, 4813 (1996).

[14] W. B. Bonnor, Philos. Trans. R. Soc. London **A251**, 233 (1959).

[15] L. Blanchet, Phys. Rev. D **51**, 2559 (1995).

[16] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995).

[17] L. Blanchet, Phys. Rev. D **54**, 1417 (1996).

[18] C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).

[19] E. Poisson, Phys. Rev. D **48**, 1860 (1993).

[20] C. Cutler, L. S. Finn, E. Poisson, and G. J. Sussmann, Phys. Rev. D **47**, 1151 (1993).

[21] H. Tagoshi and T. Nakamura, Phys. Rev. D **49**, 4016 (1994).

[22] H. Tagoshi and M. Sasaki, Prog. Theor. Phys. **92**, 745 (1994).

[23] M. Shibata, M. Sasaki, H. Tagoshi, and T. Tanaka, Phys. Rev. D **51**, 1646 (1995).

[24] H. Tagoshi, Prog. Theor. Phys. **93**, 307 (1995).

[25] H. Tagoshi, M. Shibata, T. Tanaka, and M. Sasaki, Phys. Rev. D **54**, 1439 (1996).

[26] T. Tanaka, H. Tagoshi, and M. Sasaki, Prog. Theor. Phys. **96**, 1087 (1996).

[27] S. L. Shapiro and S. A. Teukolsky, Astrophys. J. **292**, L41 (1985).

[28] G. D. Quinlan and S. L. Shapiro, Astrophys. J. **321**, 199 (1987).

[29] É. E. Flanagan and S. A. Hughes, gr-qc/9701039.

[30] D. Hills and P. L. Bender, Astrophys. J. **445**, L7 (1995).

[31] K. Danzmann *et al.*, Report No. MPQ-177, 1993 (unpublished).

[32] T. Damour and N. Deruelle, Ann. Inst. Henri Poincaré, Phys. Theor. **43**, 107 (1985).

[33] T. Damour and G. Schäfer, C. R. Acad. Sci., Ser. II: Mec. Phys., Chim., Sci. Terre Univers **305**, 839 (1987).

[34] T. Damour and G. Schäfer, Nuovo Cimento B **101**, 127 (1988).

[35] G. Schäfer and N. Wex, Phys. Lett. A **174**, 196 (1993); **177**, 461(E) (1993).

[36] N. Wex, Class. Quantum Grav. **12**, 983 (1995).

[37] R. A. Hulse and J. H. Taylor, Astrophys. J. **195**, L51 (1975).

[38] T. Damour and N. Deruelle, Phys. Lett. **87A**, 81 (1981).

[39] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), p. 59.

[40] T. Damour, Phys. Rev. Lett. **51**, 1019 (1983).

[41] A. Wolszczan, Class. Quantum Grav. **11**, A227 (1994).

[42] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).

[43] Note that the E denoting conserved energy in this paper differs from E in Ref. [24] by the rest mass $E_{\mathrm{Tagoshi}} - 1 = E_{\mathrm{here}}$.

[44] L. Esposito and E. R. Harrison, Astrophys. J. **196**, L1 (1975).

[45] R. V. Wagoner, Astrophys. J. **196**, L63 (1975).

[46] L. Blanchet and G. Schäfer, Mon. Not. R. Astron. Soc. **239**, 845 (1989).

[47] L. Blanchet and G. Schäfer, Class. Quantum Grav. **10**, 2699 (1993).

[48] W. Junker and G. Schäfer, Mon. Not. R. Astron. Soc. **254**, 146 (1992).

[49] T. Damour and N. Deruelle, C. R. Acad. Sci. Paris **293**, 537 (1981); **293**, 877 (1981).

[50] L. P. Grishchuk and S. M. Kopejkin, in *Relativity in Celestial Mechanics and Astrometry*, edited by J. Kovalevsky and V. A. Brumberg (Reidel, Dordrecht, 1986), p. 19.

[51] C. W. Lincoln and C. M. Will, Phys. Rev. D **42**, 1123 (1990).

[52] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).

[53] MAPLE, Waterloo Maple Software, Waterloo, Ontario, Canada.

[54] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995).

[55] A. Gopakumar, Bala R. Iyer, and Sai Iyer, Phys. Rev. D **55**, 6030 (1997).

[56] B. R. Iyer and C. M. Will, Phys. Rev. Lett. **70**, 113 (1993).

[57] B. R. Iyer and C. M. Will, Phys. Rev. D **52**, 6882 (1995).

[58] G. Schäfer and R. Rieth, Class. Quantum Grav. **14**, 2357 (1997).

[59] C. M. Will, in *Relativistic Cosmology*, Proceedings of the Eighth Nishinomiya-Yukawa Memorial Symposium, edited by M. Sasaki (Universal Academy Press, Kyoto, Japan, 1994), p. 83.

[60] L. E. Simone, E. Poisson, and C. M. Will, Phys. Rev. D **52**, 4481 (1995).

[61] T. Damour and G. Schäfer, Gen. Relativ. Gravit. **17**, 879 (1985).

[62] N. Wex and R. Rieth, in *Symposia Gaussiana*, edited by M. Behera, R. Fritsch, and R. G. Lintz (Gruyter, Berlin, 1995).

[63] W. H. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, England, 1927).

[64] C. M. Will and A. G. Wiseman (private communication).

[65] A. Gopakumar and Bala R. Iyer (in preparation).

[66] A. G. Wiseman, Phys. Rev. D **46**, 1517 (1992).

[67] L. Blanchet, B. R. Iyer, C. M. Will, and A. G. Wiseman, Class. Quantum Grav. **13**, 575 (1996).