Perturbations of Einstein-Maxwell-dilaton fields

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Expressions for the complete perturbations of the solution of the Einstein-Maxwell-dilaton equations which represents the spacetime for gravitational waves, possibly coupled with electromagnetic waves and with dilaton fields, bound to collision are determined in terms of complex scalar potentials. These expressions are obtained using Wald's method of adjoint operators without imposing any gauge condition on the perturbed tetrad. The complex scalar potentials satisfy a system of five second-order linear partial differential equations. [S0556-2821(97)07022-7]

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I. INTRODUCTION

Recently there has been considerable interest in the study of exact solutions of the Einstein-Maxwell-dilaton (EMD) equations which are motivated by higher-dimensional unified theories, such as string theory or Kaluza-Klein theory, where a scalar dilaton field appears naturally as an essential ingredient [1–5]. The presence of the dilatonic field changes radically certain features of the solutions in comparison with those that appear in ordinary Einstein-Maxwell (EM) theory, for example, changes in the casual structure [3], implications on the black-hole thermodynamics [2], and other questions. The special nature of the dilaton comes from the fact that this scalar field is nonminimally coupled to the tensor fields, unlike the scalar fields considered in the ordinary general relativity, where they are weakly coupled to the EM fields [6].

The four-dimensional action describing the Einstein-Maxwell fields interacting with the dilaton is [7]

$$S=\int d^4x\sqrt{-g}[-R+2(\nabla\phi)^2+\xi F^2],$$

where $\xi\equiv e^{-2a\phi}$, $F^2\equiv F_{\mu\nu}F^{\mu\nu}$, ϕ is the dilaton field, $F_{\mu\nu}$ is the electromagnetic field, R is the scalar curvature, $g=\det(g_{\mu\nu})$, and $\mu,\nu=0,1,2,3$. The constant a, called the dilaton coupling constant, is a parameter that governs the coupling of the dilaton to the electromagnetic field. Extremizing the action as usual, the EMD fields satisfy the following field equations:

$$\nabla_\mu(\xi F^{\mu\nu})=0,\quad \nabla_{[\mu}F_{\nu\lambda]}=0 \quad \text{(Maxwell)}, \tag{1}$$

$$\nabla_\mu\nabla^\mu\phi+\frac{1}{2}a\xi F^2=0 \quad \text{(dilaton)}, \tag{2}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu} \quad \text{(Einstein)}, \tag{3}$$

with the energy-momentum tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu}=2(\partial_\mu\phi)(\partial_\nu\phi) -g_{\mu\nu}(\partial^\alpha\phi)(\partial_\alpha\phi)+2\xi\left(F_{\mu\lambda}F^\lambda_\nu-\frac{1}{4}g_{\mu\nu}F^2\right). \tag{4}$$

One way of understanding the attributes of any exact solution of some set of field equations such as the EMD equations (1)–(4), is to investigate how it reacts under external perturbations and, in the first instance, infinitesimal perturbations and how these are affected. To this purpose, in this paper we are especially interested in the linearized versions of Eqs. (1)–(4), and for that reason we consider small perturbation fields (metric, electromagnetic field, and dilaton field perturbations) around a general background solution in order to obtain the equations governing the perturbations; the explicit form and the discussions about these linearized equations are given in Sec. II.

In this work we shall study the perturbations of the plane wave geometries, which are not only important in ordinary general relativity but also in string theory, since these geometries correspond to exact solutions of the string theory at all orders of the string tension parameter [1,2]. In the framework of classical EM theory, Chandrasekhar and Xanthopoulos have studied the coupled perturbations of the Bell-Szekeres solution, which represents the collision of two plane electromagnetic waves, in the regions containing the incoming waves [8], and their results were that there not exist nontrivial incoming perturbations; however, in a recent paper [9] Torres del Castillo has demonstrated that in those regions there exist nontrivial purely incoming perturbations, contrary to the claim made in Ref. [8]. Besides, Xanthopoulos [10] considered also the coupled perturbations for the general metric representing plane waves bound for a collision in the same framework of EM theory and he obtained similar results to those of Ref. [8], however, also in this case it has been demonstrated that actually there exist nontrivial purely incoming perturbations [11], the reasons of these discrepancies are widely discussed in Refs. [9,11]. The approach followed in Refs. [9,11] to solve the equations for the perturbations (which is very different from that used in Refs. [8,10]), is Wald's method of adjoint operators, which applies when we can obtain a decoupled set of equations from the original equations for the perturbations [12–16]. In Sec. III, we discuss the self-adjoint character of the operators that govern the perturbations of the EMD fields, which is necessary in order to find the perturbations in terms of scalar potentials.

In Sec. IV, the perturbations of the spacetime corresponding to gravitational plane waves possibly coupled to electromagnetic waves and dilaton fields bound to a collision are studied in the framework of the EMD theory. Using the Newman-Penrose formalism, we find a decoupled system of equations and then we generate the master equations for the scalar potentials which determine the complete perturbations to the metric, electromagnetic potential and dilaton field. These expressions are used in Sec. V to demonstrate that the existence of the purely incoming perturbations is a property that persists in the more general framework of the EMD theory. In the Appendix we write Eqs. (1)–(4) in the Newman-Penrose formalism, which are useful in Sec. IV and for future reference.

II. LINEARIZED EINSTEIN-MAXWELL-DILATON EQUATIONS

In order to find the linearized equations from the field equations (1)–(4), we consider linear small perturbation fields around a general background solution and, first of all, we determine the linear perturbations of the various quantities appearing in those equations; in the following expressions and throughout the superscript B denotes the corresponding perturbations. In particular, the metric, vector potential, and dilaton perturbations are represented by $h_{\mu\nu}$, b_{μ}, and $\phi^{\rm B}$, respectively.

One easily can demonstrate that

$$(g^{\mu\nu})^{\rm B}=-h^{\mu\nu},$$

$$F^{\rm B}_{\mu\nu}=\partial_{\mu}b_{\nu}-\partial_{\nu}b_{\mu},$$

$$\xi^{\rm B}\equiv(e^{-2a\phi})^{\rm B}=-2a\xi\phi^{\rm B}, \tag{5}$$

$$(\Gamma^{\lambda}_{\mu\nu})^{\rm B}=\frac{1}{2}g^{\lambda\rho}[\nabla_{\mu}h_{\nu\rho}+\nabla_{\nu}h_{\mu\rho}-\nabla_{\rho}h_{\mu\nu}],$$

$$R^{\rm B}=g^{\mu\nu}R^{\rm B}_{\mu\nu}-R_{\mu\nu}h^{\mu\nu},$$

we do not require the explicit form of $R^{\rm B}_{\mu\nu}$ in terms of $h_{\mu\nu}$, because this perturbed quantity appears in other references [12] in the framework of the EM theory and then its features are well known. The indices are raised and lowered by means of the background metric $g_{\mu\nu}$, for example $h^{\mu\nu}=g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta}$, which will be useful below. Furthermore, using symmetry properties of $F_{\mu\nu}$ we can find that

$$(F^2)^{\rm B}=2F^{\lambda\mu}F^{\alpha}{}_{\lambda}h_{\mu\alpha}+2F^{\mu\nu}F^{\rm B}_{\mu\nu},$$

$$(F_{\mu\lambda}F_{\nu}{}^{\lambda})^{\rm B}=-F_{\mu}{}^{\lambda}F_{\nu}{}^{\gamma}h_{\gamma\lambda}-2F^{\lambda}{}_{(\mu}F^{\rm B}_{\nu)\lambda}. \tag{6}$$

Let us consider first the linearized Maxwell equation; for that purpose it is suitable to write Eq. (1) in the form

$$g^{\mu\alpha}\nabla_{\alpha}(\xi F_{\mu\nu})=g^{\mu\alpha}[\partial_{\alpha}(\xi F_{\mu\nu})-\Gamma^{\lambda}_{\alpha\nu}\xi F_{\mu\lambda}-\Gamma^{\lambda}_{\alpha\mu}\xi F_{\lambda\nu}]=0.$$

Now we consider linear perturbations around a background solution; using Eqs. (5) and (6) and grouping suitably, the preceding equation takes the form

$$8a\xi F^{\alpha}{}_{\nu}\nabla_{\alpha}\phi^{\rm B}-4\{g_{\nu}{}^{\mu}\nabla^{\rho}(\xi\nabla_{\rho})-\nabla^{\mu}(\xi\nabla_{\nu})\}b_{\mu}$$
$$+4\left\{[\nabla^{\alpha}\xi F^{\mu}{}_{\nu}]+\xi\left[F^{\alpha}{}_{\nu}\nabla^{\mu}+g^{\mu}{}_{\nu}F^{\rho\alpha}\nabla_{\rho}\right.\right.$$
$$\left.\left.-\frac{1}{2}g^{\mu\alpha}F^{\rho}{}_{\nu}\nabla_{\rho}\right]\right\}h_{\mu\alpha}=0, \tag{7}$$

where we have multiplied by a factor -4 for future convenience [12].

Similarly, we can write the dilaton equation (2) as

$$g^{\mu\alpha}[\partial_{\alpha}(\partial_{\mu}\phi)-\Gamma^{\lambda}_{\alpha\mu}\partial_{\lambda}\phi]+\frac{1}{2}a\xi F^2=0,$$

now taking linear perturbations and multiplying by a factor -4, the linearized dilaton equation is

$$4(a^2\xi F^2-\nabla^{\mu}\nabla_{\mu})\phi^{\rm B}-8a\xi F^{\mu\nu}\nabla_{\mu}b_{\nu}+4\left\{(\nabla^{\alpha}\nabla^{\mu}\phi)\right.$$
$$\left.+a\xi F^{\mu\lambda}F^{\alpha}{}_{\lambda}+(\nabla^{\alpha}\phi)\nabla^{\mu}-\frac{1}{2}g^{\mu\alpha}(\nabla^{\rho}\phi)\nabla_{\rho}\right\}h_{\mu\alpha}=0. \tag{8}$$

In order to find the linearized version of Eqs. (3), we first linearize the various terms appearing in the energy-momentum tensor given in Eq. (4). It is not difficult to demonstrate that the linearized first term is given by

$$2(\partial_{\mu}\phi\partial_{\nu}\phi)^{\rm B}=4\partial_{(\mu}\phi\partial_{\nu)}\phi^{\rm B}. \tag{9}$$

The second term can be suitably written as

$$-g_{\mu\nu}(\partial^{\alpha}\phi)(\partial_{\alpha}\phi)=-g_{\mu\nu}g^{\lambda\alpha}(\partial_{\lambda}\phi)(\partial_{\alpha}\phi),$$

then

$$-[g_{\mu\nu}(\partial^{\alpha}\phi)\partial_{\alpha}\phi]^{\rm B}=\{g_{\mu\nu}(\partial^{\lambda}\phi)(\partial^{\alpha}\phi)-(\partial^{\rho}\phi)$$
$$\times(\partial_{\rho}\phi)g_{\nu}{}^{\lambda}g_{\mu}{}^{\alpha}\}h_{\lambda\alpha}$$
$$-2g_{\mu\nu}(\partial^{\alpha}\phi)\partial_{\alpha}\phi^{\rm B}. \tag{10}$$

The linearized third term can be expressed as

$$(\xi T^{\rm M}_{\mu\nu})^{\rm B}=T^{\rm M}_{\mu\nu}\xi^{\rm B}+\xi(T^{\rm M}_{\mu\nu})^{\rm B},$$

where

$$T^{\rm M}_{\mu\nu}=2\left[F_{\mu\lambda}F_{\nu}{}^{\lambda}-\frac{1}{4}g_{\mu\nu}F^2\right], \tag{11}$$

is the usual energy-momentum tensor of the electromagnetic field. The last expression can be written using the formulas (5) and (6), as follows:

$$(\xi T^{\rm M}_{\mu\nu})^{\rm B}=-2a\xi T^{\rm M}_{\mu\nu}\phi^{\rm B}-2\xi\left[F_{\mu}{}^{\alpha}F_{\nu}{}^{\gamma}+\tfrac{1}{4}F^2g_{\mu}{}^{\alpha}g_{\nu}{}^{\gamma}\right.$$
$$\left.+\frac{1}{2}g_{\mu\nu}F^{\lambda\gamma}F^{\alpha}{}_{\lambda}\right]h_{\alpha\gamma}-2\xi\left[2F^{\gamma}{}_{(\mu}F^{\rm B}_{\nu)\gamma}\right.$$
$$\left.+\frac{1}{2}g_{\mu\nu}F^{\alpha\gamma}F^{\rm B}_{\alpha\gamma}\right]. \tag{12}$$

As we can see from this last expression, the metric perturbations coming from this linearized third term are the same that appear in the case when the only matter field present is the electromagnetic field [12], except for the phase factor ξ, which does not change the multiplicative character of the overall factor acting on $h_{\alpha\gamma}$; this point will be important in the next section where we will discuss the self-adjointness of this factor.

Finally, from Eqs. (3), (9)–(12) the linearized Einstein equations are given by

$$2\{a\xi T^{\mathrm{M}}_{\mu\nu}+g_{\mu\nu}(\partial^{\alpha}\phi)\partial_{\alpha}-2(\partial_{(\mu}\phi)\partial_{\nu)}\}\phi^{\mathrm{B}}-2\xi[2g^{\alpha}{}_{(\mu}F_{\nu)}{}^{\gamma}-2g^{\gamma}{}_{(\mu}F_{\nu)}{}^{\alpha}-g_{\mu\nu}F^{\alpha\gamma}]\nabla_{\alpha}b_{\gamma}+\{\mathcal{E}_{\mathrm{G}}{}'+(\partial^{\rho}\phi)\times(\partial_{\rho}\phi)g_{\mu}{}^{\alpha}g_{\nu}{}^{\lambda}-g_{\mu\nu}(\partial^{\lambda}\phi)(\partial^{\alpha}\phi)\}h_{\alpha\lambda}=0, \tag{13}$$

where the operators acting on the metric perturbations $h_{\mu\nu}$ coming from the linearization of the first member of Eq. (3) $R^{\mathrm{B}}_{\mu\nu}-\frac{1}{2}(g_{\mu\nu}R)^{\mathrm{B}}$, and those coming from Eq. (12) have been represented by the operator $\mathcal{E}_{\mathrm{G}}{}'$, whose explicit form is not important, because it is essentially the same appearing in the framework of the EM theory [see the paragraph after Eqs. (5) and (12)] and then it is well known [12].

The linearized EMD equations (7), (8), and (13) can be expressed in the following form for future convenience:

$$\begin{bmatrix} \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG} \\ \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG} \\ \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G \end{bmatrix}\begin{bmatrix} \phi^B \\ (b_\mu) \\ (h_{\mu\nu}) \end{bmatrix}=0, \tag{14}$$

where $\mathcal{E}_D$, $\mathcal{E}_{DE}$, $\mathcal{E}_{DG}$, $\mathcal{E}_{ED}$, $\mathcal{E}_E$, $\mathcal{E}_{EG}$, $\mathcal{E}_{GD}$, $\mathcal{E}_{GE}$, and $\mathcal{E}_G$ are linear partial differential operators involving the background fields, whose explicit forms can be read from Eqs. (7), (8), and (13), which correspond to the second, first, and third rows, respectively.

III. WALD'S METHOD

In order to find expressions for the complete solutions of systems of linear partial differential equations in terms of scalar potentials, Wald introduced a method which makes use of the concept of the adjoint of a linear operator [12]. If $\mathcal{E}$ corresponds to a linear partial differential operator which maps m-index tensor fields into n-index tensor fields then, the adjoint operator of $\mathcal{E}$, denoted by $\mathcal{E}^{\dagger}$, is that linear partial differential operator mapping n-index tensor fields into m-index tensor fields such that

$$t^{\rho\sigma\cdots}[\mathcal{E}(f_{\mu\nu\ldots})]_{\rho\sigma\ldots}=[\mathcal{E}^{\dagger}(t^{\rho\sigma\cdots})]^{\mu\nu\cdots}f_{\mu\nu\ldots}+\nabla_{\mu}v^{\mu}, \tag{15}$$

where v^{μ} is some vector field, and similarly for any other operator. For example, in the Newman-Penrose formalism we have that

$$D^{\dagger}=-(D+\varepsilon+\overline{\varepsilon}-\rho-\overline{\rho}),\quad \Delta^{\dagger}=-(\Delta-\gamma-\overline{\gamma}+\mu+\overline{\mu}),$$
$$\delta^{\dagger}=-(\delta+\beta-\overline{\alpha}-\tau+\overline{\pi}),\quad \overline{\delta}^{\dagger}=-(\overline{\delta}+\overline{\beta}-\alpha-\overline{\tau}+\pi), \tag{16}$$

which will be useful below.

On the other hand, from Eqs. (7), (8), and (13), the definition (15) and assuming that the background fields satisfy Eqs. (1)–(4) one can demonstrate that

$$\mathcal{E}^{\dagger}_D=\mathcal{E}_D,\quad \mathcal{E}^{\dagger}_E=\mathcal{E}_E,\quad \mathcal{E}^{\dagger}_G=\mathcal{E}_G,$$
$$\mathcal{E}^{\dagger}_{DE}=\mathcal{E}_{ED},\quad \mathcal{E}^{\dagger}_{DG}=\mathcal{E}_{GD},\quad \mathcal{E}^{\dagger}_{GE}=\mathcal{E}_{EG}, \tag{17}$$

in this manner, from Eqs. (17) we have found that the adjoint of the matrix operator appearing in Eq. (14) is [16]

$$\begin{bmatrix} \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG} \\ \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG} \\ \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G \end{bmatrix}^{\dagger}=\begin{bmatrix} \mathcal{E}_D & \mathcal{E}_{DE} & \mathcal{E}_{DG} \\ \mathcal{E}_{ED} & \mathcal{E}_E & \mathcal{E}_{EG} \\ \mathcal{E}_{GD} & \mathcal{E}_{GE} & \mathcal{E}_G \end{bmatrix}, \tag{18}$$

which means that this operator is self-adjoint. It is important to point out that, on a curved background, the operators corresponding to the Weyl neutrino equation and the linearized Yang-Mills are also self-adjoint. On the other hand, the operator corresponding to the usual free massless field equations of spin greater that one is not self-adjoint on a curved spacetime.

Besides, as we shall see in the next section, when the background solution is the spacetime corresponding to plane waves bound to collision, a decoupled system of equations can be obtained from Eqs. (14), which can also be written in matrix form [see Eq. (43)].

IV. COLLIDING WAVES AND THEIR PERTURBATIONS

A. Solution to be perturbed

The spacetime corresponding to the colliding plane waves can be specified by the line element [4]

$$ds^2=2e^{-M}du\,dv+e^{-U}[e^{-V}(dx^2)^2+e^{V}(dx^1)^2], \tag{19}$$

or in equivalent form, by the null tetrad

$$D=\frac{\sqrt{2}}{N}\partial_u,\quad \Delta=\frac{\sqrt{2}}{N}\partial_v,$$

$$\delta=\frac{1}{\sqrt{2H}}(\chi^{-1/2}\partial_{x^1}+i\chi^{1/2}\partial_{x^2}),$$

$$\overline{\delta}=\frac{1}{\sqrt{2H}}(\chi^{-1/2}\partial_{x^1}-i\chi^{1/2}\partial_{x^2}), \tag{20}$$

where u, v, x^1, and x^2 are real coordinates and for simplicity we have defined

$$N^2\equiv 2e^{-M},\quad H\equiv e^{-U},\quad \chi\equiv e^{V}.$$

The tetrad (20) applies to the entire spacetime, with the metric functions $N=N(u,v)$, $H=H(u,v)$, $\chi=\chi(u,v)$, potential

$A_\mu = A_\mu(u,v)$, and dilaton field $\phi=\phi(u,v)$ taking different forms in the four regions of the usual framework of colliding waves.

Specifically in the region prior to the collision, which contains one of the approaching waves, we have that the metric components $g_{\mu\nu}$, ϕ, and A_μ fields depend only on v [4]:

$$U(v), \quad V(v), \quad M(v), \quad \phi(v), \quad A_\mu(v); \tag{21}$$

besides, the only nonvanishing spin coefficients are

$$\gamma(v) = -\frac{1}{\sqrt{2}N}\frac{d}{dv}\ln N, \quad \mu(v) = -\frac{1}{\sqrt{2}N}\frac{d}{dv}\ln H,$$

$$\lambda(v) = \frac{1}{\sqrt{2}N}\frac{d}{dv}\ln\chi, \tag{22}$$

and the only nonvanishing component of the spinor Weyl is

$$\Psi_4 = -\frac{1}{2}\left[\frac{d^2V}{dv^2} - \frac{dV}{dv}\left(\frac{dU}{dv} - \frac{dM}{dv}\right)\right]. \tag{23}$$

Region II supports null electromagnetic fields, therefore, if we take the tetrad vector l^μ along the principal null direction of the background electromagnetic field, then we have that

$$\varphi_0 = 0 = \varphi_1, \tag{24}$$

$\varphi_2(v)$ being the only nonvanishing component. Using Eq. (24), from Eqs. (A8) the Einstein field equations reduce to

$$\Phi_{22} = -(\Delta\phi)^2 + 2\xi\varphi_2\overline{\varphi_2}, \tag{25}$$

since $\Delta\phi$ is the only nonvanishing derivative of ϕ.

B. Decoupled equations for the perturbations and master equations

From the Maxwell equations (A1) and (A3) and from Eqs. (21) and (24) we obtain

$$\bar{\delta}\varphi_0^{\mathrm{B}} - D\varphi_1^{\mathrm{B}} - \varphi_2\kappa^{\mathrm{B}} = 2\pi\xi^{-1}l^\mu j_\mu, \tag{26}$$

$$(\Delta - 2\gamma + \mu - a\Delta\phi)\varphi_0^{\mathrm{B}} - \delta\varphi_1^{\mathrm{B}} - \varphi_2\sigma^{\mathrm{B}} + a\overline{\varphi_2}(D\phi)^{\mathrm{B}}$$
$$= 2\pi\xi^{-1}m^\mu j_\mu, \tag{27}$$

where it has been included a source j_μ for the electromagnetic perturbations (see Ref. [9] and references cited therein). One of the reasons to choose Eqs. (A1) and (A3) is that the components of the perturbed tetrad acting on φ_0 and φ_1 do not appear, because these last quantities vanish in the background in accordance with Eq. (24). Moreover, from the Ricci identities and by considering that the only nonvanishing spin coefficients are given in Eq. (22), one finds that

$$D\sigma^{\mathrm{B}} - \delta\kappa^{\mathrm{B}} = \Psi_0^{\mathrm{B}}. \tag{28}$$

Applying δ to Eq. (26) and D to Eq. (27), subtracting and considering Eq. (28) and that $[D,\delta]=0$, we have that

$$\mathcal{O}_E\varphi_0^{\mathrm{B}} - \varphi_2\Psi_0^{\mathrm{B}} + a\overline{\varphi_2}D(D\phi)^{\mathrm{B}} = 2\pi\mathcal{S}_E(j_\mu), \tag{29}$$

with

$$\mathcal{O}_E = D(\Delta - 2\gamma + \mu - a\Delta\phi) - \delta\bar{\delta},$$

$$\mathcal{S}_E(j_\mu) = \xi^{-1}[D(m^\mu j_\mu) - \delta(l^\mu j_\mu)]. \tag{30}$$

On the other hand, from the Bianchi identities and by considering that the only nonvanishing spinor Weyl component is given in Eq. (23) we obtain that

$$\bar{\delta}\Psi_0^{\mathrm{B}} - D\Psi_1^{\mathrm{B}} = 4\pi[\delta(l^\mu l^\nu T_{\mu\nu}) - D(l^\mu m^\nu T_{\mu\nu})], \tag{31}$$

$$(\Delta - 4\gamma + \mu)\Psi_0^{\mathrm{B}} - \delta\Psi_1^{\mathrm{B}} + 2\xi\overline{\varphi_2}D\varphi_0^{\mathrm{B}} = 4\pi[\delta(l^\mu m^\nu T_{\mu\nu})$$
$$- \overline{\lambda}\, l^\mu l^\nu T_{\mu\nu} - D(m^\mu m^\nu T_{\mu\nu})], \tag{32}$$

where we also have included an additional source for the gravitational perturbations, $T_{\mu\nu}$ [9]. Applying the same procedure used in Eqs. (26) and (27) to eliminate φ_1^{B}, we can cancel the terms with Ψ_1^{B} of Eqs. (31) and (32) and we get

$$\mathcal{O}_G\Psi_0^{\mathrm{B}} + 2\xi\overline{\varphi_2}D^2\varphi_0^{\mathrm{B}} = 4\pi\mathcal{S}_G(T_{\mu\nu}), \tag{33}$$

where

$$\mathcal{S}_G(T_{\mu\nu}) = \{D[\delta(l^\mu m^\nu T_{\mu\nu}) - D(m^\mu m^\nu T_{\mu\nu}) - \overline{\lambda}\, l^\mu l^\nu T_{\mu\nu}]$$
$$+ \delta[D(l^\mu m^\nu T_{\mu\nu}) - \delta(l^\mu l^\nu T_{\mu\nu})]\},$$

$$\mathcal{O}_G = D(\Delta - 4\gamma + \mu) - \delta\bar{\delta}. \tag{34}$$

With the purpose to complete the system of Eqs. (29) and (33) and to avoid the appearance of undesirable perturbed quantities [17], before considering the perturbations, we apply D to Eq. (A6) and we obtain

$$D(\Delta + \mu - \gamma - \bar{\gamma})D\phi - \bar{\rho}D\Delta\phi - (\Delta\phi)D\bar{\rho} + D(\bar{\tau}\delta\phi)$$
$$+ D(-\delta + \bar{\alpha} - \beta + \tau)\bar{\delta}\phi + \frac{1}{4}aD(\xi F^2)$$
$$= 0. \tag{35}$$

Using the commutation relations the fifth term can be expressed as

$$D(-\delta + \bar{\alpha} - \beta + \tau)\bar{\delta}\phi$$
$$= -(\delta - \bar{\alpha} - \beta + \bar{\pi})(\bar{\delta} - \alpha - \bar{\beta} + \pi)D\phi$$
$$+ (\delta - \bar{\alpha} - \beta + \bar{\pi})\bar{\kappa}\Delta\phi - (\delta - \bar{\alpha} - \beta + \bar{\pi})$$
$$\times[(\rho + \bar{\varepsilon} - \varepsilon)\bar{\delta}\phi + \bar{\sigma}\delta\phi] + \kappa\Delta\bar{\delta}\phi$$
$$-(\bar{\rho} + \bar{\varepsilon} - \varepsilon)\delta\bar{\delta}\phi - \sigma\bar{\delta}^2\phi$$
$$+ D[(\bar{\alpha} - \beta + \tau)\bar{\delta}\phi] \tag{36}$$

and using the fact that the only nonvanishing spin coefficients are those given in Eq. (22), the linearization of Eq. (36) is given by

$$[D(-\delta+\overline{\alpha}-\beta+\tau)\overline{\delta\phi}]^{\mathrm{B}}=-\delta\overline{\delta(D\phi)}^{\mathrm{B}}+\Delta\phi\delta\overline{\kappa}^{\mathrm{B}}. \tag{37}$$

Furthermore, the linearization of nonvanishing remaining terms of Eq. (35) is given by

$$[D(\Delta+\mu-\gamma-\overline{\gamma})D\phi]^{\mathrm{B}}=D(\Delta+\mu-\gamma-\overline{\gamma})(D\phi)^{\mathrm{B}},$$

$$-[(\Delta\phi)(D\overline{\rho})]^{\mathrm{B}}=-(\Delta\phi)D\overline{\rho}^{\mathrm{B}}, \tag{38}$$

$$[D(\xi F^2)]^{\mathrm{B}}=4\xi[\varphi_2 D\varphi_0^{\mathrm{B}}+\overline{\varphi_2}D\overline{\varphi_0}^{\mathrm{B}}],$$

and from the Ricci identities we can find that

$$D\overline{\rho}^{\mathrm{B}}-\delta\overline{\kappa}^{\mathrm{B}}=0. \tag{39}$$

Then, from Eqs. (37)–(39) the linearization of Eq. (35) finally takes the form

$$\mathcal{O}_D(D\phi)^{\mathrm{B}}+a\xi[\varphi_2 D\varphi_0^{\mathrm{B}}+\overline{\varphi_2}D\overline{\varphi_0}^{\mathrm{B}}]=4\pi[\mathcal{S}_D\phi_s+\Delta\phi l^{\mu}l^{\nu}T_{\mu\nu}], \tag{40}$$

where

$$\mathcal{O}_D=D(\Delta+\mu-\gamma-\overline{\gamma})-\delta\overline{\delta},$$

$$\mathcal{S}_D=\frac{1}{2}D, \tag{41}$$

and ϕ_s represents a source for the dilaton field perturbations, additional to those considered in Eqs. (29) and (33).

The system of three equations given by Eqs. (29), (33), and (40) actually does not represent a linear system for the three unknowns $(D\phi)^{\mathrm{B}}$, φ_0^{B}, and Ψ_0^{B}, because in Eq. (40) the quantity $\overline{\varphi_0}^{\mathrm{B}}$ is present, which must be considered as a new unknown. To rectify this situation, we need to consider the complex conjugates of Eqs. (29) and (33) to obtain two additional equations:

$$\overline{\mathcal{O}}_E\overline{\varphi_0}^{\mathrm{B}}-\overline{\varphi_2}\overline{\Psi}_0^{\mathrm{B}}+a\varphi_2 D(D\phi)^{\mathrm{B}}=2\pi\overline{\mathcal{S}_E}(j_\mu),$$

$$\overline{\mathcal{O}_G}\overline{\Psi}_0^{\mathrm{B}}+2\xi\varphi_2 D^2\overline{\varphi_0}^{\mathrm{B}}=4\pi\overline{\mathcal{S}_G}(T_{\mu\nu}), \tag{42}$$

where the operators $\overline{\mathcal{S}_E}$ and $\overline{\mathcal{S}_G}$ correspond to the complex conjugates of those of Eqs. (30) and (34), and we have considered that $(D\phi)^{\mathrm{B}}=(\overline{D\phi})^{\mathrm{B}}$, because $D\phi$ is real. Hence the complete system of five equations (30), (33), (40), and (42) now represents a system for five unknowns $(D\phi)^{\mathrm{B}}$, φ_0^{B}, Ψ_0^{B}, $\overline{\varphi_0}^{\mathrm{B}}$, and $\overline{\Psi}_0^{\mathrm{B}}$. Note that the complex conjugate of Eq. (40) is itself, since $\overline{\mathcal{O}_D}=\mathcal{O}_D$, which follows from the fact that D, Δ, μ, and γ are real and $[\delta,\overline{\delta}]=0$ [see Eqs. (20)–(22)]. This system of five equations can be expressed in the following matrix form:

$$\begin{pmatrix} \mathcal{O}_G & 2\xi\overline{\varphi_2}D^2 & 0 & 0 & 0 \\ -\varphi_2 & \mathcal{O}_E & 0 & 0 & a\overline{\varphi_2 D} \\ 0 & 0 & \overline{\mathcal{O}_G} & 2\xi\varphi_2 D^2 & 0 \\ 0 & 0 & -\overline{\varphi_2} & \overline{\mathcal{O}_E} & a\varphi_2 D \\ 0 & a\xi\varphi_2 D & 0 & a\xi\overline{\varphi_2 D} & \mathcal{O}_D \end{pmatrix} \times \begin{pmatrix} \Psi_0^{\mathrm{B}} \\ \varphi_0^{\mathrm{B}} \\ \overline{\Psi}_0^{\mathrm{B}} \\ \overline{\varphi_0}^{\mathrm{B}} \\ (D\phi)^{\mathrm{B}} \end{pmatrix} = 4\pi \begin{pmatrix} \mathcal{S}_G T_{\mu\nu} \\ \frac{1}{2}\mathcal{S}_E j_\mu \\ \overline{\mathcal{S}_G} T_{\mu\nu} \\ \frac{1}{2}\overline{\mathcal{S}_E} j_\mu \\ \mathcal{S}_D\phi_s+\Delta\phi l^{\mu}l^{\nu}T_{\mu\nu} \end{pmatrix} \tag{43}$$

The right-hand side corresponds to

$$\mathcal{S}\begin{bmatrix} (T_{\mu\nu}) \\ (j_\mu) \\ \phi_s \end{bmatrix}, \tag{44}$$

where $\mathcal{S}$ is given by the following 5×3 matrix:

$$\mathcal{S}=\begin{pmatrix} \mathcal{S}_G & 0 & 0 \\ 0 & \frac{1}{2}\mathcal{S}_E & 0 \\ \overline{\mathcal{S}_G} & 0 & 0 \\ 0 & \frac{1}{2}\overline{\mathcal{S}_E} & 0 \\ \Delta\phi l^{\mu}l^{\nu} & 0 & \mathcal{S}_D \end{pmatrix}. \tag{45}$$

With $\mathcal{O}$ being the matrix operator appearing on the left-hand and using Eqs. (16), (30), (34), and (41), we find that

$$\mathcal{O}^{\dagger}=\begin{pmatrix} \mathcal{O}_G^{\dagger} & -\varphi_2 & 0 & 0 & 0 \\ 2\xi\overline{\varphi_2 D}^2 & \mathcal{O}_E^{\dagger} & 0 & 0 & -a\xi\varphi_2 D \\ 0 & 0 & \overline{\mathcal{O}_G^{\dagger}} & -\overline{\varphi_2} & 0 \\ 0 & 0 & 2\xi\varphi_2 D^2 & \overline{\mathcal{O}_E^{\dagger}} & -a\xi\overline{\varphi_2 D} \\ 0 & -a\overline{\varphi_2 D} & 0 & -a\varphi_2 D & \mathcal{O}_D^{\dagger} \end{pmatrix}, \tag{46}$$

where

$$\mathcal{O}_G^\dagger = (\Delta + 2\gamma + \mu)D - \overline{\delta}\delta,$$
$$\mathcal{O}_E^\dagger = (\Delta + \mu + a\Delta\phi)D - \overline{\delta}\delta, \tag{47}$$
$$\mathcal{O}_D^\dagger = (\Delta + \mu)D - \overline{\delta}\delta,$$

and

$$\mathcal{S}^\dagger = \begin{pmatrix} \mathcal{S}_G^\dagger & 0 & \overline{\mathcal{S}_G^\dagger} & 0 & \Delta\phi l^\mu l^\nu \\ 0 & \frac{1}{2}\mathcal{S}_E^\dagger & 0 & \frac{1}{2}\overline{\mathcal{S}_E^\dagger} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{S}_D^\dagger \end{pmatrix}, \tag{48}$$

where

$$\mathcal{S}_G^\dagger = -l^\mu l^\nu[\delta^2 - \overline{\lambda D}] - m^\mu m^\nu D^2 + 2l^{(\mu}m^{\nu)}\delta D,$$
$$\mathcal{S}_E^\dagger = \xi^{-1}[l^\mu\delta - m^\mu D], \tag{49}$$
$$\mathcal{S}_D^\dagger = -\frac{1}{2}D,$$

where we have used the fact that ξ^{-1} and φ_2 depend only on v and $[\delta, D] = 0$.

In this manner, if the matrix potential (ψ) satisfies $\mathcal{O}^\dagger(\psi) = 0$ with

$$(\psi) = \begin{pmatrix} \psi_G \\ \psi_E \\ \overline{\psi_G} \\ \overline{\psi_E} \\ \psi_D \end{pmatrix}, \tag{50}$$

it means, using Eq. (46), that

$$\mathcal{O}_G^\dagger\psi_G - \varphi_2\psi_E = 0,$$
$$2\xi\overline{\varphi_2 D}^2\psi_G + \mathcal{O}_E^\dagger\psi_E - a\xi\varphi_2 D\psi_D = 0,$$
$$-a\overline{\varphi_2 D}\psi_E - a\varphi_2 D\overline{\psi_E} + \mathcal{O}_D^\dagger\psi_D = 0, \tag{51}$$
$$\overline{\mathcal{O}_G^\dagger}\,\overline{\psi_G} - \overline{\varphi_2\psi_E} = 0,$$
$$2\xi\varphi_2 D^2\overline{\psi_G} + \overline{\mathcal{O}_E^\dagger}\,\overline{\psi_E} - a\xi\overline{\varphi_2 D}\psi_D = 0,$$

then the metric, vector potential, and dilaton field perturbations are given by [12]

$$\begin{pmatrix} h_{\mu\nu} \\ b_\mu \\ \phi^{\rm B} \end{pmatrix} = \mathcal{S}^\dagger(\psi) = \begin{pmatrix} \mathcal{S}_G^\dagger\psi_G + \overline{\mathcal{S}_G^\dagger}\,\overline{\psi_G} + \Delta\phi l_\mu l_\nu\psi_D \\ \frac{1}{2}\mathcal{S}_E^\dagger\psi_E + \frac{1}{2}\overline{\mathcal{S}_E^\dagger}\,\overline{\psi_E} \\ \mathcal{S}_D^\dagger\psi_D \end{pmatrix}, \tag{52}$$

where the last equality follows from Eqs. (48) and (50). Using Eqs. (49), we have finally that the *real* perturbations are

$$h_{\mu\nu} = -2\{l_\mu l_\nu[\delta^2 - \overline{\lambda D}] + m_\mu m_\nu D^2 - 2l_{(\mu}m_{\nu)}\delta D\}\psi_G + \text{c.c.} + \Delta\phi l_\mu l_\nu\psi_D,$$
$$b_\mu = \frac{1}{2}\xi^{-1}(l_\mu\delta - m_\mu D)\psi_E + \text{c.c.}, \tag{53}$$
$$\phi^{\rm B} = -\frac{1}{2}D\psi_D.$$

Hence the perturbations given in Eq. (53) are defined completely by the five scalar potentials ψ_G, $\overline{\psi_G}$, ψ_E, $\overline{\psi_E}$, and ψ_D which satisfy the five coupled equations (51), called *the master equations* [10]. It should be emphasized that since no gauge condition has been imposed on the perturbed null tetrad, then our results are independent on the six degrees of perturbed tetrad gauge freedom, contrary to other approaches which make use of this gauge freedom in order to simplify the equations for the perturbations [8,10]. In summary, the linearized EMD equations (14) have been solved in a gauge invariant way in the special case when the background solution is the spacetime corresponding to colliding waves in the incoming regions.

C. Expressions for the field perturbations

The components of the electromagnetic field perturbations can be obtained from Eqs. (53), using the formula $F^{\rm B}_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ and the following definitions:

$$\overline{\varphi_0{}^{\rm B}} \equiv l^\mu\overline{m^\nu}F^{\rm B}_{\mu\nu} = \frac{1}{2}\xi^{-1}D^2\psi_{\rm E},$$
$$\overline{\varphi_1{}^{\rm B}} \equiv \frac{1}{2}(l^\mu n^\nu + m^\mu\overline{m^\nu})F^{\rm B}_{\mu\nu} = \frac{1}{2}\xi^{-1}\delta D\psi_{\rm E}, \tag{54}$$
$$\overline{\varphi_2{}^{\rm B}} \equiv m^\mu n^\nu F^{\rm B}_{\mu\nu} = \frac{1}{2}\xi^{-1}(\delta^2 - \overline{\lambda D})\psi_{\rm E} + \varphi_2 D^2\overline{\psi_{\rm G}} - \frac{1}{2}a\xi^{-1}(\Delta\phi)D\overline{\psi_{\rm E}} - \frac{1}{2}a\overline{\varphi_2 D}\psi_{\rm D}.$$

Similarly, the components of the Weyl spinor perturbations can be obtained from Eq. (53) making use of the formula

$$\Psi^{\rm B}_{ACDE} = \frac{1}{2}\nabla^{R'}{}_{(A}\nabla^{S'}{}_{C}h_{DE)R'S'} + \frac{1}{2}h_{(AC}{}^{R'S'}\Phi_{DE)R'S'}.$$

Then, we find that

$$\overline{\Psi^{\rm B}_0} = -D^4\psi_{\rm G},$$
$$\overline{\Psi^{\rm B}_1} = -\delta D^3\psi_{\rm G},$$
$$\overline{\Psi^{\rm B}_2} = -(\delta^2 - \overline{\lambda D})D^2\psi_{\rm G} + \tfrac{1}{6}\Delta\phi D^2\psi_D, \tag{55}$$
$$\overline{\Psi^{\rm B}_3} = -(\delta^2 - 3\overline{\lambda D})\delta D\psi_{\rm G} + \tfrac{1}{4}\Delta\phi\,\delta D\psi_D,$$

$$\overline{\Psi_4^{\rm B}}=-(\delta^2-3\overline{\lambda D})(\delta^2-\overline{\lambda D})\psi_{\rm G}-\delta^2(\overline{\delta^2}-\lambda D)\overline{\psi_{\rm G}}$$
$$+\left[\frac{1}{2}\Phi_{22}-(\Delta+2\gamma+\mu)(\Delta+\mu)\right]D^2\overline{\psi_{\rm G}}$$
$$+[2(\Delta+2\gamma+2\mu)\delta+\overline{\lambda\,\delta}]\overline{\delta D}\,\overline{\psi_{\rm G}}+\Delta\phi(\overline{\delta^2}-\overline{\lambda D})\psi_D,$$

where Φ_{22} is given by Eq. (25). As we can see from Eqs. (54) and (55), the perturbations of the electromagnetic field and of the spinor Weyl are completely determined by the quantities given in Eqs. (53), and in this manner they have the same gauge independence [see the paragraph after Eq. (53)], which allows us, for example, to appropriately define fluxes of energy [18], and to study the junction conditions between the different regions occurring in the collision of plane waves [19].

V. EXISTENCE OF PURELY INCOMING PERTURBATIONS

The purely incoming perturbations correspond to the u-independent perturbations; these special perturbations were studied by Xanthopoulos [10] in the framework of the EM theory and he found that there exist no nontrivial u-independent perturbations, contrary to the recent results presented in Refs. [9,11] within the same scheme of the EM theory. We will demonstrate that the existence of purely incoming perturbations found in Refs. [9,11] is a property that persists in the more general framework of the EMD theory and, for this end, it is convenient to define the complex variable

$$z\equiv\frac{1}{\sqrt{2}}[\chi^{1/2}x^1+i\chi^{-1/2}x^2], \tag{56}$$

and its complex conjugate to replace the real coordinates x^1 and x^2. With this definition, the relevant components of the null tetrad (20) can be rewritten as $\overline{\delta}=(1/\sqrt{H})\partial_z$ and $\delta=(1/\sqrt{H})\partial_{\overline{z}}$. A direct way to obtain u-independent field perturbations is to assume that the potentials ψ_E, ψ_G, and ψ_D do not depend on u; then the master equations (51) reduce to

$$\frac{1}{H}\partial_z\partial_{\overline{z}}\psi_G+\varphi_2\psi_E=0,\quad \partial_z\partial_{\overline{z}}\psi_E=0,\quad \partial_z\partial_{\overline{z}}\psi_D=0, \tag{57}$$

whose solutions can be written as

$$\psi_E=-\frac{1}{\varphi_2}H\partial_zF(v,z),$$
$$\psi_G=H^2[\overline{z}\,\overline{F}(v,z)+G(v,z)], \tag{58}$$
$$\psi_D=J(v,z),$$

where $F(v,z)$, $G(v,z)$, and $J(v,z)$ are arbitrary functions and the factor $(1/\varphi_2)H$ and H are introduced for convenience. The only nonvanishing components of the electromagnetic field perturbations and of the Weyl field perturbations can be obtained from Eqs. (54) and (55),

$$\overline{\varphi_2{}^{\rm B}}=-\frac{1}{2\varphi_2}\xi^{-1}\partial_z^3F(v,z),$$
$$\overline{\Psi_4^{\rm B}}=-\partial_z^4\{\overline{z}\,\overline{F}(v,z)+G(v,z)\}+\frac{\Delta\phi}{H}\partial_z^2J(v,z), \tag{59}$$

and from Eqs. (53) we obtain that

$$\phi^{\rm B}=0. \tag{60}$$

Thus there exist nontrivial incoming perturbations. In the framework of the EM theory where, of course, one has only electromagnetic field and gravitational perturbations, if the electromagnetic field vanishes (i.e., $\partial_z^3F=0$) in Eq. (59), the remaining nonvanishing purely gravitational perturbations not only would correspond to a solution of the linearized EM equations but they also would correspond to an *exact* solution of the EM equations [20]. Since the dilaton field perturbations vanish in Eq. (60), does the nonvanishing purely gravitational perturbation (59) also correspond to an exact solution of the EMD equations? Because we do not have yet an analogous result in the EMD theory to that given in Ref. [20] for the EM theory, this is an open question.

VI. CONCLUDING REMARKS

The self-adjointness of the operators governing the linearized EMD equations has shown its usefulness in order to find particular solutions of these equations in terms of few scalar potentials in a direct way and without imposing any gauge condition on the perturbed tetrad. This property opens the possibility to find the linear perturbations of any exact solution of the EMD theory, with only the finding of the corresponding decoupled system remaining. One interesting case would be the spacetime for colliding plane waves in the interaction region, whose perturbations will allow us to study the stability of the singularities emerging in this region [4] and the matching of these perturbations with those found in this paper in the incoming region. Another case is the study of the charged black holes in string theory, which have new implications for black-hole thermodynamics [2] and for the possible violation of the cosmic censorship [3]. All this will be the subject of forthcoming communications.

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APPENDIX: EMD FIELD EQUATIONS IN THE NEWMAN-PENROSE FORMALISM

Projecting on the null tetrad l_ν, n_ν, m_ν, and $\overline{m_\nu}$ as usual, the Maxwell field equations (1) take the form

$$\xi\{(\overline{\delta}+\pi-2\alpha)\varphi_0-(D-2\rho)\varphi_1-\kappa\varphi_2$$
$$-a[\varphi_0\overline{\delta}+\overline{\varphi_0\delta}-(\varphi_1+\overline{\varphi_1})D]\phi\}$$
$$=0, \tag{A1}$$

$$\xi\{-(\delta+2\beta-\tau)\varphi_2+(\Delta+2\mu)\varphi_1-\nu\varphi_0 - a[(\varphi_1+\overline{\varphi_1})\Delta-\varphi_2\delta-\overline{\varphi_2\bar{\delta}}]\phi\}=0, \tag{A2}$$

$$\xi\{(\Delta-2\gamma+\mu)\varphi_0-(\delta-2\tau)\varphi_1-\sigma\varphi_2 - a[\varphi_0\Delta+(\overline{\varphi_1}-\varphi_1)\delta-\overline{\varphi_2 D}]\phi\}=0, \tag{A3}$$

$$\xi\{-(D-\rho+2\varepsilon)\varphi_2+(\bar{\delta}+2\pi)\varphi_1-\lambda\varphi_0 - a[\overline{\varphi_0\Delta}+(\varphi_1-\overline{\varphi_1})\bar{\delta}-\varphi_2 D]\phi\}=0. \tag{A4}$$

Writing the operator $\nabla^{\mu}\nabla_{\mu}$ in the Newman-Penrose formalism, the dilaton field equation (2) takes the form

$$2[\mu D+(D+\varepsilon+\bar{\varepsilon}-\bar{\rho})\Delta-\pi\delta+(-\delta+\bar{\alpha}-\beta-\bar{\pi})\bar{\delta}]\phi + \frac{1}{2}a\xi F^2=0, \tag{A5}$$

or, using the usual commutation relations

$$2[(\Delta+\mu-\gamma-\bar{\gamma})D-\bar{\rho}\Delta+\bar{\tau}\delta+(-\delta+\bar{\alpha}-\beta+\tau)\bar{\delta}]\phi + \frac{1}{2}a\xi F^2=0, \tag{A6}$$

where

$$F^2=4[\varphi_0\varphi_2+\overline{\varphi_0\varphi_2}-\varphi_1^2-\overline{\varphi_1}^2]. \tag{A7}$$

Moreover, from $\Phi_{\mu\nu}\equiv-\frac{1}{2}(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)$ and from the Ricci tensor given by

$$R_{\mu\nu}=2(\partial_\mu\phi)(\partial_\nu\phi)+2\xi\left(F_{\mu\lambda}F^{\lambda}_{\nu}-\frac{1}{4}g_{\mu\nu}F^2\right),$$

the Ricci scalars can be expressed in the form

$$\begin{aligned}
\Phi_{00}&\equiv l^\mu l^\nu\Phi_{\mu\nu}=-(D\phi)^2+2\xi\varphi_0\overline{\varphi_0},\\
\Phi_{11}&\equiv l^\mu n^\nu\Phi_{\mu\nu}=-\frac{1}{2}[(D\phi)(\Delta\phi)+(\delta\phi)(\bar{\delta}\phi)]+2\xi\varphi_1\overline{\varphi_1},\\
\Phi_{22}&\equiv n^\mu n^\nu\Phi_{\mu\nu}=-(\Delta\phi)^2+2\xi\varphi_2\overline{\varphi_2},\\
\Phi_{01}&\equiv l^\mu m^\nu\Phi_{\mu\nu}=-(D\phi)(\delta\phi)+2\xi\varphi_0\overline{\varphi_1},\\
\Phi_{02}&\equiv m^\mu m^\nu\Phi_{\mu\nu}=-(\delta\phi)^2+2\xi\varphi_0\overline{\varphi_2},\\
\Phi_{12}&\equiv m^\mu n^\nu\Phi_{\mu\nu}=-(\Delta\phi)(\delta\phi)+2\xi\varphi_1\overline{\varphi_2},\\
\Lambda&\equiv\tfrac{1}{24}R=\frac{1}{6}[(D\phi)(\Delta\phi)-(\delta\phi)(\bar{\delta}\phi)],
\end{aligned} \tag{A8}$$

with $\overline{\Phi_{ij}}=\Phi_{ji}$ $(i,j=0,1,2)$.

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