Rotating boson stars in general relativity

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We have succeeded in obtaining highly relativistic structures of stationary axisymmetric configurations consisting of massive complex scalar fields, i.e., rotating boson stars. Scalar fields are assumed to have harmonic azimuthal angular dependence, i.e., $\phi=\phi_0(t,r,\theta)\ e^{im\varphi}$, where m is an integer. Equilibrium configurations are characterized by values of m so that the total angular momentum of the boson star becomes discrete. We have solved sequences of equilibrium states with $m=1$ and $m=2$ by changing one parameter which characterizes the model. The maximum mass for $m=1$ models is $1.314M_{\rm Pl}^2/\mu$, where $M_{\rm Pl}$ and μ are the Planck mass and the mass of the scalar field, respectively. It is interesting that properly defined specific angular momentum for rotating boson stars is constant in space. [S0556-2821(97)04714-0]

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I. INTRODUCTION

Gravitational bound states of scalar fields were first obtained by Kaup [1] and by Ruffini and Bonazzola [2]. Such objects have recently been considered important in the context of cosmology and have been called boson stars (see, e.g., Refs. [3–5] for recent reviews). The self-gravity of such an object is balanced by the dispersive effect due to the wave character of the scalar field. It is important to note that there is a critical mass for boson stars which is of the order of $M_{\rm Pl}^2/\mu$, where $M_{\rm Pl}$ and μ are the Planck mass and the mass of the scalar field, respectively. There can be no equilibrium states for boson stars with larger masses and furthermore boson stars with higher densities are unstable against gravitational collapse in the radial direction [6–11].

Until recently, people have mainly studied spherical configurations for boson stars [1,2,12,13], because it has been uncertain whether axisymmetric solutions of boson stars exist or not. Only a few years ago, however, Silveira and de Sousa [14] and Schupp and van der Bij [15] obtained axisymmetric equilibrium configurations of massive scalar fields in the framework of *Newtonian gravity* (see, also, Ref. [16]). Since scalar fields can be either spherical or nonspherical, there arises a possibility that the source of gravity is nonspherical. Consequently axisymmetric equilibrium configurations can exist for massive scalar fields without employing unnatural assumptions.

In the framework of general relativity, on the other hand, it is very recently that two different kinds of axisymmetric solutions have been obtained. The first solutions are *static* ones obtained by Yoshida and Eriguchi [17]. These solutions are an extension of the Newtonian solutions obtained by Schupp and van der Bij [15], and the distributions of scalar fields are equatorially antisymmetric. The second solutions found by Schunck and Mielke [18] belong to axisymmetric and stationary ones, i.e., rotating solutions. They obtained their solutions by assuming that scalar fields have azimuthal angular dependence of $\phi=\phi_0(t,r,\theta)e^{im\varphi}$, where m is any integer. It implies that scalar fields in their equilibrium solutions are not axially symmetric, although the spacetime is stationary and axisymmetric.

It is remarkable that there exist *stationary* solutions for scalar fields. However, there are several unsatisfactory points about their solutions. First, Schunck and Mielke [18] did not solve *sequences* of equilibrium configurations but obtained only one or two configurations for different values of m. Second, they solved models with very weak gravity so that most of their solutions were nearly Newtonian. Third, there is a curious behavior in the distribution of the energy density; i.e., there are ''spikelike'' structures in the energy density contour near the ''rotation'' axis in their solutions.

In this paper, we will obtain *equilibrium sequences* of rotating boson stars even for *strong gravity*. Equilibrium sequences can be computed by changing model parameters which characterize equilibrium configurations. By using those equilibrium sequences we will be able to know the maximum mass model beyond which equilibrium states become unstable against gravitational collapse. Furthermore, from newly obtained solutions we will be able to show whether or not the peculiar behavior of the solutions of Schunck and Mielke [18] appear.

The plan of this paper is as follows. In Sec. II, we derive the basic equations for stationary axisymmetric equilibrium configurations of boson stars. The basic equations are converted into an integral representation, because boundary conditions can be easily taken into account. In Sec. III, we describe our numerical method and present numerical results. In the final section we summarize and discuss our results.

II. ROTATING BOSON STARS AND THE SPACETIME

A. General framework

The basic equations for the complex massive scalar field coupled with the Einstein gravity are well known (for details see, e.g., Refs. [3–5]). Here we will summarize them briefly without going into details. The scalar field is assumed to obey the equation derived from the Lagrangian density

$$\mathcal{L}_M=\sqrt{-g}(-g^{\mu\nu}\phi^*_{;\mu}\phi_{;\nu}-\mu^2|\phi|^2), \qquad (2.1)$$

where $g_{\mu\nu}$ and g are the metric and its determinant, respectively. Here the asterisk denotes complex conjugate and the

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semicolon is used for covariant derivative with respect to the quantity followed. Throughout this paper we will use units in which $c=\hbar=1$.

Since the Lagrangian (2.1) is invariant under the U(1) global gauge transformation, i.e., $\phi\to e^{i\lambda}\phi$, where λ is an arbitrary real constant, we obtain the continuity equation as follows:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}j^{\mu})_{,\mu}=0, \tag{2.2}$$

where the comma denotes usual differentiation with respect to the quantity followed and j^{μ} is the conserved current four-vector defined by

$$j^{\mu}=-ig^{\mu\nu}(\phi_{,\nu}\phi^{*}-\phi^{*}_{,\nu}\phi). \tag{2.3}$$

The conserved Noether charge N can be expressed as

$$N=\int d^3x\sqrt{-g}j^0. \tag{2.4}$$

From the Lagrangian (2.1), we obtain the scalar field equation

$$g^{\mu\nu}\phi_{;\mu\nu}-\mu^2\phi=0 \tag{2.5}$$

and the Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi GT_{\mu\nu}, \tag{2.6}$$

where the energy-momentum tensor $T_{\mu\nu}$ is defined by

$$T_{\mu\nu}=\phi^{*}_{,\mu}\phi_{,\nu}+\phi^{*}_{,\nu}\phi_{,\mu}-g_{\mu\nu}[g^{\alpha\beta}\phi^{*}_{,\alpha}\phi_{,\beta}+\mu^2|\phi|^2]. \tag{2.7}$$

Here $R_{\mu\nu}$, R, and G are the Ricci tensor, the scalar curvature, and the gravitational constant, respectively.

Once appropriate initial conditions and/or boundary conditions are specified, the scalar field ϕ and the metric up to certain coordinate choices can be determined from Eqs. (2.5)–(2.7). It should be noted that we do not need to specify an equation of state for this system, because full information about the matter is contained in the scalar field ϕ.

B. Stationary axisymmetric configurations

Very recently, a prescription for relativistic rotating boson stars has been proposed by Schunck and Mielke [18]. Their remarkable solutions can be obtained by allowing the harmonic azimuthal angle dependence of scalar fields. As is shown below, it implies that the scalar field is no longer axisymmetric but that the energy-momentum tensor can be axisymmetric. Consequently the spacetime around a boson star can be stationary and axisymmetric; i.e., the dragging of the inertial frame will not vanish.

However, as discussed in the Introduction, their solutions seem to behave unnaturally and cannot be regarded as satisfactory ones. Therefore we will develop a new scheme to compute sequences of equilibrium configurations of rotating boson stars by adopting the clever assumption about the scalar fields proposed by Schunck and Mielke [18]. It should be noted that our notation is somewhat different from that of Schunck and Mielke [18], in particular for the metric functions.

Since we want to investigate a stationary and axisymmetric spacetime, the metric can be written as

$$ds^2=-e^{2\nu}dt^2+e^{2\alpha}(dr^2+r^2d\theta^2)+e^{2\beta}r^2\sin^2\theta(d\varphi-\omega dt)^2, \tag{2.8}$$

where ν, α, β, and ω are functions of r and θ. Here the spherical coordinates (r,θ,φ) are used. In this paper we follow the sign convention of Misner, Thorne, and Wheeler [19].

We assume the following time dependence and φ dependence for the scalar fields just as Schunck and Mielke adopted [18]:

$$\phi=\phi_0(r,\theta)e^{-i(\sigma t-k\varphi)}, \tag{2.9}$$

where ϕ_0 is a real function of r and θ, and σ and k are two real constants. Since scalar fields must be single-valued functions with respect to φ, the scalar fields must obey the periodic condition

$$\phi(t,r,\theta,\varphi)=\phi(t,r,\theta,\varphi+2\pi). \tag{2.10}$$

From this periodicity, values of k must be integer. Thus we will denote m instead of k hereafter, i.e., $k=m=0,\pm1,\pm2,\ldots$. Models with $m=0$ correspond to spherical and static axisymmetric configurations which have been extensively investigated thus far. Therefore we will concentrate our attention on models whose values of m are nonzero in the following of this paper.

By writing down the energy-momentum tensor explicitly, we can easily see that the energy-momentum tensor is axisymmetric:

$$T^0{}_0=-(\sigma^2-m^2\omega^2)e^{-2\nu}\phi_0{}^2-e^{-2\alpha}(\phi_{0,r})^2-e^{-2\alpha}\frac{1}{r^2}(\phi_{0,\theta})^2-e^{-2\beta}\frac{m^2}{r^2\sin^2\theta}\phi_0{}^2-\mu^2\phi_0{}^2, \tag{2.11}$$

$$T^1{}_1=(\sigma-m\omega)^2e^{-2\nu}\phi_0{}^2+e^{-2\alpha}(\phi_{0,r})^2-e^{-2\alpha}\frac{1}{r^2}(\phi_{0,\theta})^2-e^{-2\beta}\frac{m^2}{r^2\sin^2\theta}\phi_0{}^2-\mu^2\phi_0{}^2, \tag{2.12}$$

$$T^2{}_2=(\sigma-m\omega)^2e^{-2\nu}\phi_0{}^2-e^{-2\alpha}(\phi_{0,r})^2+e^{-2\alpha}\frac{1}{r^2}(\phi_{0,\theta})^2-e^{-2\beta}\frac{m^2}{r^2\sin^2\theta}\phi_0{}^2-\mu^2\phi_0{}^2, \tag{2.13}$$

$$T^3{}_3=(\sigma^2-m^2\omega^2)e^{-2\nu}\phi_0{}^2-e^{-2\alpha}(\phi_{0,r})^2-e^{-2\alpha}\frac{1}{r^2}(\phi_{0,\theta})^2+e^{-2\beta}\frac{m^2}{r^2\sin^2\theta}\phi_0{}^2-\mu^2\phi_0{}^2, \tag{2.14}$$

$$T^0{}_3=2m(\sigma-m\omega)e^{-2\nu}\phi_0{}^2, \tag{2.15}$$

$$T^3{}_0=-2\sigma\left[\omega(\sigma-m\omega)e^{-2\nu}+e^{-2\beta}\frac{m}{r^2\sin^2\theta}\right]\phi_0{}^2, \tag{2.16}$$

$$T^1{}_2=2e^{-2\alpha}\phi_{0,r}\phi_{0,\theta}, \tag{2.17}$$

and

$$T^2{}_1=2e^{-2\alpha}\frac{1}{r^2}\phi_{0,r}\phi_{0,\theta}. \tag{2.18}$$

From this expression it is clearly seen that if m is not zero, $T^0{}_3$ does not vanish. Nonvanishing T^0_3 allows the spacetime to be stationary and axisymmetric. It is also noted that the stress-energy tensor of the scalar field is not isotropic as seen from the above expression. Consequently the scalar field can behave differently from the perfect fluid.

Since it is convenient to use nondimensional quantities in numerical computations, we introduce the following variables and use them hereafter:

$$r\rightarrow\mu\widetilde{r},\quad M\rightarrow G\mu\widetilde{M},\quad \phi_0\rightarrow(4\pi G)^{1/2}\widetilde{\phi_0},$$

$$\sigma\rightarrow\widetilde{\sigma}/\mu,\quad \omega\rightarrow\widetilde{\omega}/\mu. \tag{2.19}$$

Here, for the sake of simplicity, quantities with a tilde are used for real physical, i.e., dimensional, quantities and those without a tilde are regarded as nondimensional ones.

Our basic equations consist of four Einstein equations and the scalar field equation as follows:

$$\Delta_3[\rho e^{\gamma/2}]=S_\rho(r,\theta), \tag{2.20}$$

$$\Delta_2[r\sin\theta\ \gamma e^{\gamma/2}]=r\sin\theta\ S_\gamma(r,\theta), \tag{2.21}$$

$$\left(\Delta_3+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)[r\sin\theta\cos\varphi\ \omega e^{(\gamma-2\rho)/2}]$$
$$=r\sin\theta\cos\varphi\ S_\omega(r,\theta), \tag{2.22}$$

$$\alpha_{,\theta}=-\nu_{,\theta}+S_\alpha(r,\theta), \tag{2.23}$$

and

$$\left(\Delta_3+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\kappa^2\right)[\phi_0 e^{im\varphi}]=S_\phi(r,\theta)e^{im\varphi}, \tag{2.24}$$

where

$$\rho=\nu-\beta, \tag{2.25}$$

$$\gamma=\nu+\beta, \tag{2.26}$$

$$\kappa^2=1-\sigma^2, \tag{2.27}$$

$$\Delta_3=\frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right), \tag{2.28}$$

$$\Delta_2=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2.29}$$

$$S_\rho(r,\theta)=e^{\gamma/2}\left[2e^{2\alpha}(T^3{}_3-T^0{}_0-2\omega T^0{}_3)+\frac{1}{r}\left(\gamma_{,r}+\frac{1}{r}\cot\theta\gamma_{,\theta}\right)+\frac{\rho}{2}\left(2e^{2\alpha}(T^1{}_1+T^2{}_2)-\frac{\gamma_{,r}}{2r}(2+r\gamma_{,r})-\frac{\gamma_{,\theta}}{2r^2}(2\cot\theta+\gamma_{,\theta})\right)\right.$$
$$\left.+e^{-2\rho}\sin^2\theta(r^2\omega_{,r}^2+\omega_{,\theta}^2)\right], \tag{2.30}$$

$$S_\gamma(r,\theta)=e^{\gamma/2}\left[2e^{2\alpha}(T^1{}_1+T^2{}_2)+\frac{\gamma}{2}\left(2e^{2\alpha}(T^1{}_1+T^2{}_2)-\frac{1}{2}\gamma_{,r}{}^2-\frac{1}{2r^2}\gamma_{,\theta}{}^2\right)\right], \tag{2.31}$$

$$S_\omega(r,\theta)=e^{(\gamma-2\rho)/2}\left\{-\frac{4}{r^2\sin^2\theta}e^{2(\alpha+\rho)}T^0{}_3+\omega\left[e^{2\alpha}(T^1{}_1+T^2{}_2-2T^3{}_3+2T^0{}_0+4\omega T^0{}_3)-\frac{1}{r}\left(2\rho_{,r}+\frac{1}{2}\gamma_{,r}\right)\right.\right.$$
$$\left.\left.-\frac{1}{r^2}\cot\theta\left(2\rho_{,\theta}+\frac{1}{2}\gamma_{,\theta}\right)+\frac{1}{4}(4\rho_{,r}^2-\gamma_{,r}^2)+\frac{1}{4r^2}(4\rho_{,\theta}^2-\gamma_{,\theta}^2)-e^{-2\rho}\sin^2\theta(r^2\omega_{,r}^2+\omega_{,\theta}^2)\right]\right\}, \tag{2.32}$$

$$S_\alpha(r,\theta)=\left[B\left(\cos\theta+\frac{1}{2}\sin\theta\gamma_{,\theta}\right)\gamma_{,\theta}+\sin\theta B\left(\frac{1}{2}A(A-1)+\nu_{,\theta}^2-r^2\nu_{,r}^2+\frac{1}{2}(\gamma_{,\theta\theta}-r^2\gamma_{,rr})\right)+\sin^2\theta rA(2\nu_{,r}\nu_{,\theta}+\gamma_{,r\theta})\right.$$
$$\left.+\frac{1}{4}\sin^3\theta r^2Be^{-2\rho}(r^2\omega_{,r}^2-\omega_{,\theta}^2)-\frac{1}{2}\sin^4\theta r^3Ae^{-2\rho}\omega_{,r}\omega_{,\theta}+\sin\theta e^{2\alpha}[Br^2(T^2{}_2-T^1{}_1)+2\ \sin\theta rAT^1{}_2]\right]\Big/[\sin^2\theta A^2$$
$$+B^2], \tag{2.33}$$

$$A=1+r\gamma_{,r}, \tag{2.34}$$

$$B=\cos\theta+\sin\theta\gamma_{,\theta}, \tag{2.35}$$

and

$$S_\phi(r,\theta)=-e^{2\alpha}[\kappa^2e^{-2\alpha}+(\sigma-m\omega)^2e^{-2\nu}-1]\phi_0+\frac{m^2}{r^2\sin^2\theta}(e^{2(\alpha-\beta)}-1)\phi_0-\gamma_{,r}\phi_{0,r}-\frac{1}{r^2}\gamma_{,\theta}\phi_{0,\theta}. \tag{2.36}$$

C. Boundary conditions and integral representation of basic equations

If appropriate boundary conditions are imposed, the system of the partial differential equations mentioned in the previous subsection can be regarded as a set of equations for an eigenvalue problem.

Concerning the scalar field, we investigate boson stars, i.e., the matter confined within a finite region by its self-gravity. This implies that the scalar field must tend to vanish very rapidly as a function of the distance from the origin. The compactness of the matter distribution necessarily requires that the metric must satisfy the asymptotically flat condition. These boundary conditions at infinity cannot be easily treated in the usual approach in which partial differential equations are solved directly.

There is a simple way to include the boundary conditions mentioned above. If we can find proper Green's functions which satisfy the required boundary conditions, we transform the basic equation into an integral representation. Thus the integral representation has the advantage that we need not worry about the boundary conditions any more. Therefore, in this subsection, we will derive integral equations which are equivalent to our basic equations (2.20)–(2.24) supplemented by proper boundary conditions. This procedure is essentially the same as the methods used in computations of static axisymmetric configurations of boson stars [17] and rapidly rotating relativistic stars [20–23].

Three metric functions are expressed in the integral form by using three-dimensional and two-dimensional Green's functions for the Laplacian in the *flat* space as

$$\rho=-\frac{1}{4\pi}e^{\gamma/2}\int_0^\infty dr'\int_0^\pi d\theta'\int_0^{2\pi}d\varphi' r'^2\sin\theta'\frac{S_\rho(r',\theta')}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.37}$$

$$r\sin\theta\gamma=\frac{1}{2\pi}e^{\gamma/2}\int_0^\infty dr'\int_0^{2\pi}d\theta' r'^2\sin\theta' S_\gamma(r',\theta')\times\ln|\mathbf{r}-\mathbf{r}'|, \tag{2.38}$$

and

$$r\ \sin\theta\ \cos\varphi\omega=-\frac{1}{4\pi}e^{(2\rho-\gamma)/2}\int_0^\infty dr'\int_0^\pi d\theta'\int_0^{2\pi}d\varphi' r'^3\times\sin^2\theta'\cos\varphi'\frac{S_\omega(r',\theta')}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.39}$$

where $S_\gamma(r,\theta)$ is the analytically continued source term into the range $\pi<\theta\leqslant 2\pi$ by defining [20]

$$S_\gamma(r,\theta)=S_\gamma(r,\theta-\pi). \tag{2.40}$$

As for the scalar field, since the equation is a partial differential equation of the Helmholtz type, we can transform it into the integral form by using other three-dimensional Green's function as

$$\phi_0e^{im\varphi}=-\frac{1}{4\pi}\int_0^\infty dr'\int_0^\pi d\theta'\int_0^{2\pi}d\varphi' r'^2\times\sin\theta'\frac{S_\phi(r',\theta')e^{im\varphi'}}{|\mathbf{r}-\mathbf{r}'|}e^{-\kappa|\mathbf{r}-\mathbf{r}'|}. \tag{2.41}$$

It should be noted that the quantity κ is real and positive definite, because we consider gravitational bound states of scalar fields. Otherwise the wave field extends to infinity.

In the integral representations described above, if the source terms are appropriate, we can easily see that the asymptotic flatness of spacetime is automatically satisfied and that the scalar fields are gravitationally bound.

D. Equatorially symmetric distributions of scalar fields and metric functions

As far as the scalar field is either equatorially symmetric or equatorially antisymmetric, metric functions are equatorially symmetric as can be seen from the Einstein equations. Thus, we will consider only equatorially symmetric scalar fields.

By using a series expansion of Green's functions and assumptions both for the spacetime and for the scalar field, we can rewrite our basic equations as

$$\rho=-e^{-\gamma/2}\sum_{n=0}^{\infty}\int_0^{\infty}dr'\int_0^{\pi/2}d\theta' r'^2\sin\theta' \times {}_1f_{2n}(r,r')P_{2n}(\theta)P_{2n}(\theta')S_\rho(r',\theta'), \quad (2.42)$$

$$r\sin\theta\gamma=-\frac{2}{\pi}e^{-\gamma/2}\sum_{n=0}^{\infty}\int_0^{\infty}dr'\int_0^{\pi/2}d\theta' r'^2\sin\theta' \times {}_2f_{2n+1}(r,r')\frac{1}{2n+1}\sin[(2n+1)\theta] \times\sin[(2n+1)\theta']S_\gamma(r',\theta'), \quad (2.43)$$

$$r\sin\theta\omega=-e^{(2\rho-\gamma)/2}\sum_{n=0}^{\infty}\int_0^{\infty}dr'\int_0^{\pi/2}d\theta' r'^3\sin^2\theta' \times {}_1f_{2n+1}(r,r')\frac{1}{(2n+2)(2n+1)} \times P^1_{2n+1}(\theta)P^1_{2n+1}(\theta')S_\omega(r',\theta'), \quad (2.44)$$

and

$$\phi_0=\kappa\sum_{n=0}^{\infty}(4n+2|m|+1)\int_0^{\infty}dr'\int_0^{\pi/2}d\theta' r'^2\sin\theta' \times {}_3f_{2n+|m|}(r,r')P^m_{2n+|m|}(\theta)P^m_{2n+|m|}(\theta')S_\phi(r',\theta'), \quad (2.45)$$

where

$${}_1f_n(r,r')=\begin{cases}(1/r)(r'/r)^n & \text{for } r'\leq r,\\ (1/r')(r/r')^n & \text{for } r'>r,\end{cases} \quad (2.46)$$

$${}_2f_n(r,r')=\begin{cases}(r'/r)^n & \text{for } r'\leq r,\\ (r/r')^n & \text{for } r'>r,\end{cases} \quad (2.47)$$

and

$${}_3f_n(r,r')=\begin{cases}j_n(i\kappa r')h^{(1)}_n(i\kappa r) & \text{for } r'\leq r,\\ j_n(i\kappa r)h^{(1)}_n(i\kappa r') & \text{for } r'>r.\end{cases} \quad (2.48)$$

Here, P_n, P_n^m, j_n, and $h_n^{(1)}$ are the Legendre polynomial, the associated Legendre function, the spherical Bessel function, and the spherical Hankel function of the first kind, respectively.

The metric coefficient α is obtained from other metric functions by using the equation

$$\alpha=\beta(r,\theta=0)-\nu(r,\theta)+\nu(r,\theta=0)+\int_0^{\theta}d\theta' S_\alpha(r,\theta'). \quad (2.49)$$

In the above integral representation, the local flatness condition on the symmetry axis is imposed, that is,

$$\alpha(r,\theta=0)=\beta(r,\theta=0). \quad (2.50)$$

If metric coefficients except α and the scalar field satisfy appropriate boundary conditions at spatial infinity, α calculated from Eq. (2.49) automatically satisfies its boundary condition.

III. METHOD OF SOLUTIONS AND NUMERICAL RESULTS

A. Method of solutions

Rotating boson stars can be specified by one parameter if both the number of nodes and the azimuthal quantum number m are prescribed. In this paper, we choose the following quantity P as the parameter which characterizes the model:

$$P\equiv\max\{\phi_0(r,\theta)\}. \quad (3.1)$$

This quantity will be used to show the behavior of equilibrium sequences.

Once the value of P is specified, we can, in principle, solve the equations which govern the spacetime and the scalar field. However, since basic equations are nonlinear, we need some iteration procedure to get solutions numerically. In this paper, we will follow a procedure similar to that adopted by Yoshida and Eriguchi [17]. This scheme is basically one of the self-consistent-field methods. In the actual computations it is easier to specify the value of the scalar field at a certain point rather at point of the maximum value of the field. Thus the value of the scalar field at a certain but fixed point is chosen to specify a model for our numerical computations. The detailed procedure for numerical computations can be found in Yoshida and Eriguchi [17].

In actual computations, we employ equi-distantly spaced discrete meshes both in the r direction ($0\leq r\leq r_{\max}$) and in the θ direction ($0\leq\theta\leq\pi/2$). Values of $r_{\max}$ are chosen appropriately for each boson star. In practice, the value of $r_{\max}$ is chosen to be 5 times larger than the radius of boson stars defined below.

Basic equations are discretized on these mesh points. The number of mesh points is 21×301 ($\theta\times r$). The functions of the metric, ρ, γ, α, and ω, and the scalar field ϕ_0 are expanded up to $n=5$ in Eqs. (2.42)–(2.45). The iteration for the metric functions and the scalar field is pursued until the relative changes of the functions between two iteration cycles become small enough, i.e., 10^{-4} in this paper.

B. Physical quantities of equilibrium configurations

Once a consistent solution is obtained, we can compute conserved quantities characterizing the configuration. The total gravitational mass of a boson star M is defined as

$$M=-\frac{1}{4\pi}\int R^\alpha{}_\beta\xi^\beta_{(t)}d^3\Sigma_\alpha \quad (3.2)$$

$$=\int(-2T^0{}_0+T^\alpha{}_\alpha)\sqrt{-g}\,dr\,d\theta\,d\varphi \quad (3.3)$$

$$=\frac{1}{2\pi}\int\{2\sigma(\sigma-m\omega)e^{-2\nu}-1\}\phi_0^2e^{\gamma+2\alpha}dr\,d\theta\,d\varphi, \quad (3.4)$$

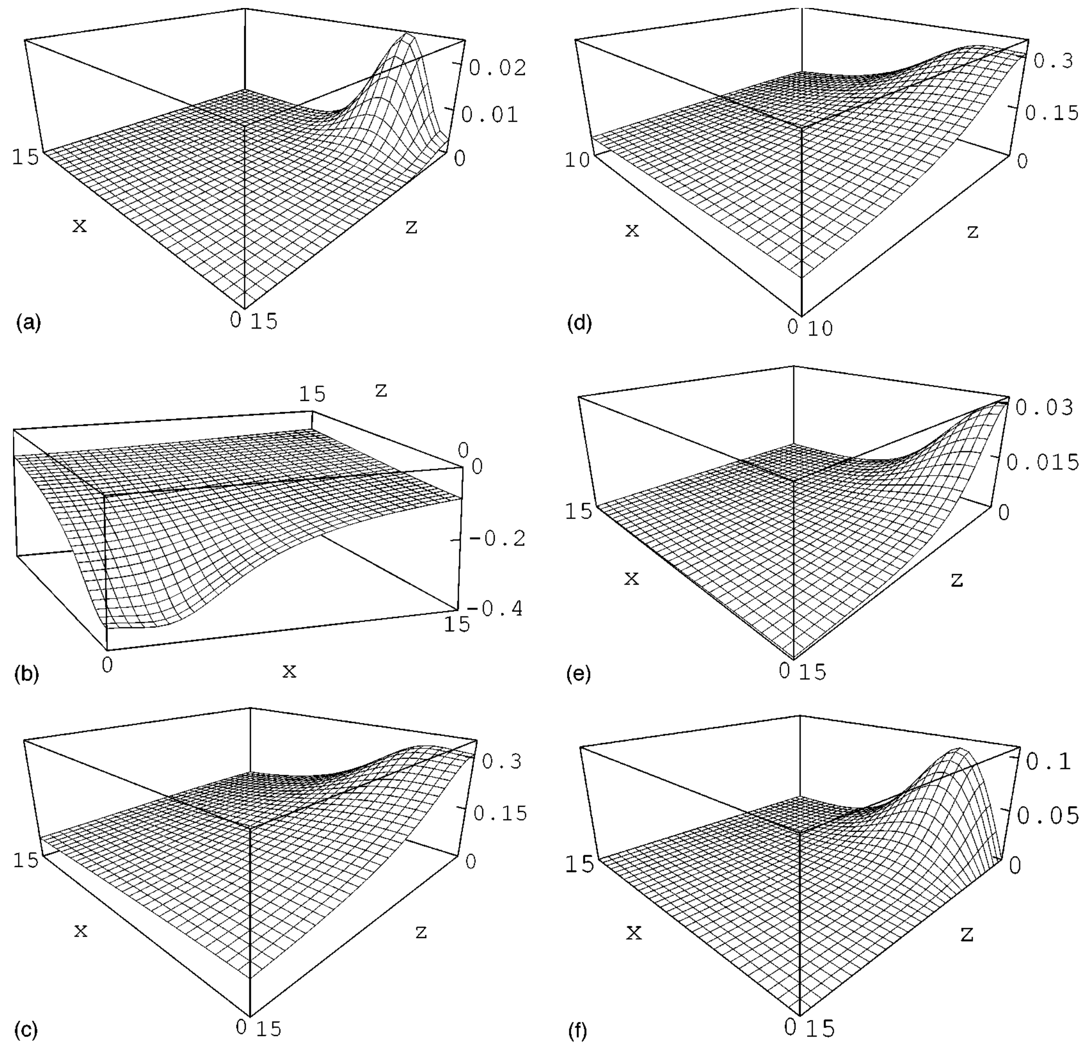

FIG. 1. The distributions of the mass-energy density, the metric coefficients, and the scalar field in the meridional plane are shown for the model with $P=0.100[(4\pi)^{-1/2}M_{\rm Pl}]$ and $m=1$. Here variables x and z are defined as $x=r\sin\theta\cos\varphi$ and $z=r\cos\theta$, respectively. (a) The mass-energy density $-T^0{}_0$ in units of $\mu^2 M_{\rm Pl}^2/(4\pi)$. (b) The same as (a), but for the metric function ν. (c) The same as (a), but for the metric function α. (d) The same as (a), but for the metric function β. (e) The same as (a), but for the metric function ω. (f) The same as (a), but for the scalar field ϕ_0 in units of $(4\pi)^{-1/2}M_{\rm Pl}$.

where $\xi^\alpha_{(t)}$ is the time Killing vector. The total particle number can be computed as

$$N=\int j^0\sqrt{-g}\,dr\,d\theta\,d\varphi \tag{3.5}$$

$$=\frac{1}{2\pi}\int(\sigma-m\omega)\phi_0^2 e^{2\alpha-\rho}dr\,d\theta\,d\varphi. \tag{3.6}$$

Finally, the total angular momentum of a boson star is calculated by

$$J=\frac{1}{8\pi}\int R^\alpha{}_\beta\xi^\beta_{(\varphi)}d^3\Sigma_\alpha \tag{3.7}$$

$$=\int T^0{}_3\sqrt{-g}\,dr\,d\theta\,d\varphi \tag{3.8}$$

$$=\frac{m}{2\pi}\int(\sigma-m\omega)\phi_0^2 e^{2\alpha-\rho}dr\,d\theta\,d\varphi, \tag{3.9}$$

where $\xi^\alpha_{(\varphi)}$ is the rotational Killing vector.

From Eqs. (3.6) and (3.9), we obtain the important relation of a rotating boson star which was found by Schunck and Mielke [18] as follows:

$$J=mN. \tag{3.10}$$

As a result of this relation, if equilibrium configurations with the same total particle number N are considered, the total angular momentum has to be discrete, i.e., quantized. This property contrasts clearly with that of rotating perfect fluid stars. The angular momentum of rotating stars with the same baryon number can be any value, as far as equilibrium states exist.

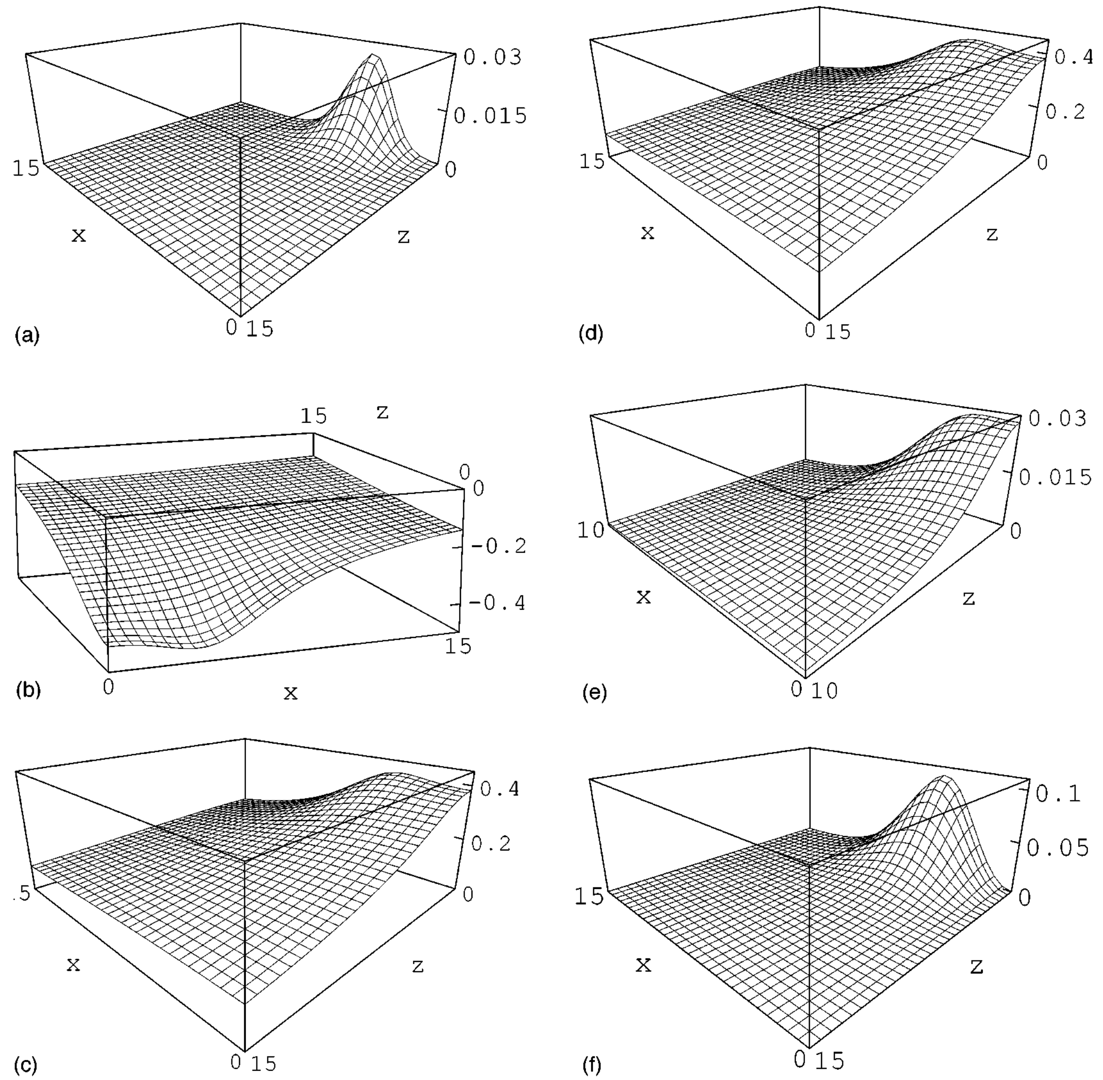

FIG. 2. The same as Fig. 1 but for the model with $m=2$.

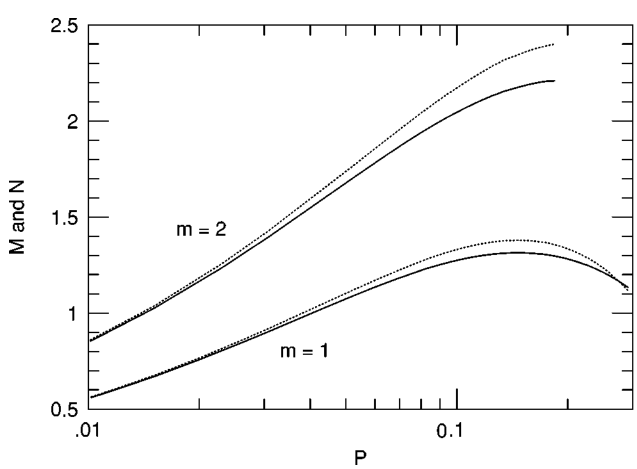

FIG. 3. The gravitational mass of the boson star in units of M_{Pl}^2/μ (solid line) and the particle number in units of M_{Pl}^2/μ^2 (dotted line) are plotted as a function of P in units of $(4\pi)^{-1/2}M_{\mathrm{Pl}}$. Attached labels $m=1,2$ denote the azimuthal quantum numbers.

In order to understand the behavior of equilibrium sequences easily, it is convenient to define the mean radius of boson stars as follows, although such a radius is not an invariant quantity:

$$R \equiv \frac{\int r j^0 \sqrt{-g}\, d^3x}{\int j^0 \sqrt{-g}\, d^3x}. \tag{3.11}$$

C. Numerical results

In this paper we consider scalar fields without nodes for simplicity. Therefore equilibrium configurations of boson stars can be characterized by two parameters P and m. Since the main purpose of this paper is to show existence of rotating boson stars in *highly relativistic* regions, we only compute two sequences corresponding to different values of m, i.e., $m=1$ and $m=2$.

Our procedure to solve eigenvalue problems explained in the previous section works nicely and we have succeeded in obtaining sequences of rotating boson stars. The distributions of the mass-energy density, the metric potentials, and the

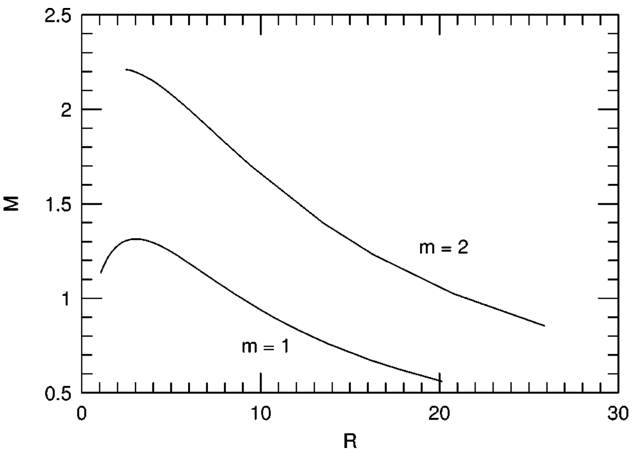

FIG. 4. The gravitational mass of the boson star in units of $M_{\rm Pl}^2/\mu$ is plotted as a function of the mean radius R in units of $1/\mu$. Attached labels $m=1,2$ denote the azimuthal quantum numbers.

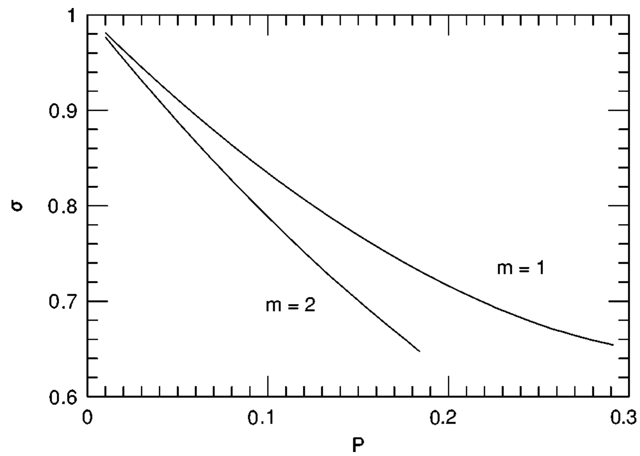

FIG. 5. The eigenvalue in units of μ is plotted as a function of P in units of $(4\pi)^{-1/2}M_{\rm Pl}$. Attached labels $m=1,2$ denote the azimuthal quantum numbers.

scalar fields of two typical solutions with different values of m but the same values of the parameter P are shown in Figs. 1 ($m=1$) and 2 ($m=2$). The selected value of P is

$$P=0.100[(4\pi)^{-1/2}M_{\rm Pl}].$$

As seen from these figures, distributions of the mass-energy density $-T^0{}_0$ are very different from those of spherical configurations, i.e., almost toroidal distributions. As a result of these distributions of the mass-energy density, the spacetime structure of our solutions is considerably nonspherical, i.e., strongly axisymmetric. The main difference in $-T^0{}_0$ distributions between the two models with $m=1$ and $m=2$ can be seen on the symmetry axis. For $m=1$ models values of $-T^0{}_0$ on the symmetry axis near the origin are nonzero. On the other hand, values of $-T^0{}_0$ for solutions having $m=2$ (and $|m|\geqslant 2$) must vanish on the symmetry axis. This property is obvious from Eqs. (2.11) and (2.45) because $\phi_{0,\theta}/r$ does not vanish for $m=1$.

In Fig. 3 we plot the total gravitational mass and the particle number against P. Here we do not show the total angular momentum of boson stars, because the angular momentum is easily determined from Eq. (3.10). In Fig. 4 the total gravitational mass is plotted against the mean radius of the boson star. These curves are similar to those of spherically symmetric boson stars and also those of static axisymmetric boson stars. For $m=1$ equilibrium sequence it is clearly seen that there exists the maximum mass state. For smaller values of P the mass is smaller because gravity of the boson star is weak. On the other hand, the small values of the mass and the particle number for larger values of P are due to the compactness of the configurations. It should be noted that the maximum state of the mass and that of the particle number coincide each other just as the case for ordinary spherical stars.

From Fig. 3 the values of the maximum mass and the maximum particle number for $m=1$ are

$$M_{\rm max}=1.31[M_{\rm Pl}^2/\mu],$$

and

$$N_{\rm max}=1.38[M_{\rm Pl}^2/\mu^2],$$

respectively. These maximum states are realized when the value of the parameter P is

$$P=0.108[(4\pi)^{-1/2}M_{\rm Pl}].$$

For the $m=2$ equilibrium sequence we could not reach the maximum state because of numerical difficulties, although we think we have obtained almost the maximum states. From our results for $m=2$ equilibrium states, the lower limit of the maximum mass and the maximum particle number for $m=2$ can be estimated as

$$M_{\rm max}\geqslant 2.21[M_{\rm Pl}^2/\mu],$$

and

$$N_{\rm max}\geqslant 2.40[M_{\rm Pl}^2/\mu^2],$$

respectively. It is well known that the maximum mass of spherical boson stars with no nodes is $0.633[M_{\rm Pl}^2/\mu]$. Consequently, as expected, equilibrium solutions with larger values of m have lager maximum masses.

In Fig. 5 the eigenvalues σ are plotted against P. As seen from Eq. (2.9), the oscillation pattern of scalar fields ϕ turns out to be rotating around the symmetry axis with the phase velocity σ/m. Thus behavior of the phase velocity of the oscillation pattern can be evaluated from Fig. 5. For larger values of P the boson star has smaller phase velocities. As the tendency of the curve for $m=1$ shows, the eigenfrequency may come to the minimum at a certain value of P and begin to increase beyond that point, although we have not pursued it numerically in this paper.

IV. DISCUSSION AND CONCLUSIONS

A. Discussion

In order to see the rotational nature of boson stars, we will define the specific angular momentum $\tilde{j}$ by

$$\tilde{j}=T^0{}_3/j^0. \tag{4.1}$$

This definition is the natural extension of that for perfect fluid stars [24]. By applying the definition of the specific angular momentum above, we have the following relation for boson stars:

$$\tilde{j} = m. \tag{4.2}$$

Thus we may say that the rotation law of boson stars in the present investigations correspond to $\tilde{j}$ constant law of ordinary fluid stars. However, it should be reminded that for boson stars the value of $\tilde{j}$ must be integer due to Eq. (2.10). Therefore, similar to the total angular momentum J, the specific angular momentum is quantized. From Eq. (4.2), it can be said that boson stars with larger values of m have larger specific angular momentum.

We need to refer to previous works concerned with rotating boson stars in general relativity. First of all, the relation of our results to the investigation by Kobayashi, Kasai, and Futamase [25] has to be discussed. They tried to get slowly rotating boson stars by a perturbational approach and concluded that rotating boson stars could not exit as far as slow rotation approximation was used. However, we have to point out that they assumed not only slow rotation but also axisymmetry of scalar fields. Therefore their conclusion only applies for axisymmetric scalar fields but not for our models in which nonaxisymmetric scalar fields have been treated.

As discussed in the Introduction, concerning the work of Schunck and Mielke [18] who solved relativistic rotating boson stars, we are not satisfied with their results from several standpoints. Although they obtained equilibrium solutions for a wide range of azimuthal quantum numbers ($1 \leqslant m \leqslant 500$), they only calculated one or two solutions for each m, and obtained solutions were nearly Newtonian as seen from their figures, e.g., Fig. 6.6 in their paper. On the contrary, in the present investigation, we have obtained sequences of equilibrium solutions to highly relativistic regions for $m=1$ and $m=2$, although our basic assumption about the scalar field is exactly same as theirs.

For the case of $m=1$, we can compare our results with those of Schunck and Mielke for weakly relativistic models. The total particle number N seems to agree each other to within 2.7% for the model with $M=0.6166[M_{\rm Pl}^2/\mu]$. As seen from Fig. 3, however, gravity for this value is weak so that it is hard to identify corresponding models exactly because the difference between the mass and the particle number is rather small. Therefore we may not say definitely that these results are in good agreement.

Concerning the distribution of physical quantities of their results [18], as discussed before, it seems strange that the mass-energy density near the symmetry axis shows ''spikelike'' structures for models with $m=1$. Moreover, as shown in the previous section, for models with $m=1$, the mass-energy density must not vanish on this rotation axis. However, the mass-energy density of their solutions seems to vanish exactly there. The spikelike structure may be caused by this inappropriate boundary conditions. Furthermore, the slopes of the mass-energy density distribution of their solutions do not seem continuous in some regions. In our present results, as seen from Fig. 1(a), there are no such peculiar behaviors. The reason of these differences is not certain at the moment. However, since there is no reason for the energy-density distribution to behave nonsmoothly, our solutions can be said to represent real behaviors of rotating boson stars.

In this paper we could not obtain the maximum mass model with $m=2$, although equilibrium solutions with relatively strong gravity are obtained. The reason can be considered as follows. The convergence of iteration depends crucially on the choice of model parameters. Thus our model parameter for numerical computations may not be appropriate for highly compact models with $m=2$ cases. If one could find more proper model parameters, we would be able to obtain solutions with larger values of m and for much stronger gravity.

B. Conclusion

We have investigated rotating boson stars and have succeeded in showing numerically that there exist equilibrium configurations of rotating boson stars even *for highly relativistic regions*. We have also found that there is a maximum mass model along the equilibrium sequence which is characterized by one parameter. The obtained solutions show that configurations and the spacetime have toroidal topology. The important finding is that the specific angular momentum is constant in space for rotating boson stars.

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