Inhomogeneous string cosmologies

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We present exact inhomogeneous and anisotropic cosmological solutions of low-energy string theory containing dilaton and axion fields. The spacetime metric possesses cylindrical symmetry. The solutions describe ever-expanding universes with an initial curvature singularity and contain known homogeneous solutions as subcases. The asymptotic form of the solution near the initial singularity has a spatially varying Kasner-like form. The inhomogeneous axion and dilaton fields are found to evolve quasihomogeneously on scales larger than the particle horizon. When the inhomogeneities enter the horizon they oscillate as nonlinear waves and the inhomogeneities attentuate. When the inhomogeneities are small they behave as small perturbations of homogeneous universes. The manifestation of duality and the asymptotic behavior of the solutions are investigated. [S0556-2821(97)00114-8]

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I. INTRODUCTION

The low-energy effective action of the bosonic sector of string theory provides a gravitation theory containing dilaton and axion fields that possess cosmological solutions. These solutions provide models for the behavior of the universe near the Planck (or string) energy scale [1]. They allow us to investigate a number of long-standing questions regarding the occurrence of singularities, the behavior of the general solution of the theory in the vicinity of a singularity, and the likelihood of our Universe arising from generic initial data. They also provide a basis for investigation of higher-order corrections to low-energy cosmological string theory. Several studies have recently been made of string cosmologies in order to ascertain the behavior of simple isotropic and anisotropic universes, investigate the implications of duality, and search for inflationary solutions [2–8]. Since one of the prime reasons for studying such solutions is to shed light on the behavior of the universe at very high energies, where our knowledge of its material content, geometrical and topological properties, or its anisotropies and inhomogeneities, is necessarily incomplete, it is unwise to make special assumptions about the form of the cosmological solutions. Indeed, any dimensional reduction process could be viewed as an extreme form of anisotropic evolution in more than three dimensions in which three spatial dimensions expand while the rest remain static. A number of studies have focused on obtaining particular solutions for (3+1)-dimensional spacetimes in cases where spatial homogeneity (and sometimes also isotropy) is assumed for the metric of spacetime, where the H field is set to zero [4], or where the H field is included by assuming that it takes a particular form which satisfies its constraints and its equation of motion [5]. For example, Copeland *et al.* [2] discussed Friedmann and Bianchi type-I universes, allowing $^*H$ to be time dependent or space dependent, respectively. In a second paper [3] they discussed Bianchi I solutions with a homogeneous antisymmetric tensor field. In [6] (see also [5]) Batakis presented an overview of all possible configurations of a (spatially) homogeneous H field in diagonal Bianchi models. Whereas, in Ref. [7], we investigated the case for a (spatially) homogeneous tensor potential $B_{\mu\nu}$ in Bianchi metrics that are not necessarily diagonal. We also gave a classification of all the degrees of freedom permitted for the H field in spatially homogeneous universes possessing a three-parameter group of motions. The only spatially homogeneous universe excluded from this study is the (closed) $S^2\times S^1$ Kantowski-Sachs universe. A detailed study of this universe was made by Barrow and Dabrowski [8].

In this paper, we take one further step upwards in generality and consider a wide class of inhomogeneous and anisotropic string cosmologies. These possess cylindrical symmetry and contain homogeneous Bianchi and Kantowski-Sachs universes as special cases [9]. They allow us to investigate the propagation of nonlinear inhomogeneities in the axion and dilaton fields. On scales larger than the particle horizon inhomogeneities in the axion and dilaton fields evolve quasihomogeneously but when the inhomogeneities enter the horizon they undergo oscillations and attentuate. In the limit that the amplitude of the inhomogeneities is small we will recover the results of perturbation studies of homogeneous string cosmologies in an appropriate gauge. Besides providing exact descriptions of the gravitational self-interaction of strongly inhomogeneous axion and dilaton fields, these solutions allow us to investigate the impact of duality upon the form of the solution in a situation where there exist characteristic spatial scales.

The string world-sheet action for a closed bosonic string in a background field including all the massless states of the string as part of the background is given by [1]

$$S=-\frac{1}{4\pi\alpha'}\int d^2\sigma\{\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X^\rho)+\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X^\rho)+\alpha'\sqrt{h}\phi(X^\rho)R^{(2)}\},\quad(1)$$

where $h^{\alpha\beta}$ is the two-dimensional world sheet metric, $R^{(2)}$ the world sheet Ricci scalar, $\epsilon^{\alpha\beta}$ the world sheet antisymmetric tensor, $B_{\mu\nu}(X^\rho)$ the antisymmetric tensor field, $g_{\mu\nu}(X^\rho)$ the background spacetime metric (graviton), $\phi(X^\rho)$ the dilaton, α' is the inverse string tension, and the functions $X^\rho(\sigma)$ map the string world sheet into the physical D-dimensional spacetime manifold.

For the consistency of string theory it is essential that local scale invariance holds. Imposing this condition results in equations of motion for the fields $g_{\mu\nu}$, $B_{\mu\nu}$, and ϕ which can be derived to lowest order in α' from the low-energy effective action for a vanishing cosmological constant:

$$S=\int d^Dx\sqrt{-g}e^{-\phi}\left(R+g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-\frac{1}{12}H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}\right), \tag{2}$$

where the antisymmetric tensor field strength $H_{\alpha\beta\gamma}=\partial_{[\alpha}B_{\beta\gamma]}$ is introduced.

In a cosmological context it is generally assumed that by some means all but four of the 10 or 26 dimensions of spacetime are compactified, leaving an expanding $(3+1)$-dimensional spacetime $(D=4)$. Since we are interested in cosmological solutions of the field equations derived from the variation of this action, we adopt the Einstein frame by making the conformal transformation

$$g_{\alpha\beta}\rightarrow e^{-\Phi}g_{\alpha\beta}. \tag{3}$$

In this frame the four-dimensional string field equations and the equations of motion are given by (Greek indices run $0\leqslant\alpha,\ \beta\leqslant 3$).

The low-energy effective action in the Einstein frame yields the following set of equations ($\kappa^2\equiv 8\pi G,\ c\equiv 1$):

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa^2({}^{(\Phi)}T_{\mu\nu}+{}^{(H)}T_{\mu\nu}), \tag{4}$$

$$\nabla_\mu(e^{-2\Phi}H^{\mu\nu\lambda})=0, \tag{5}$$

$$\Box\Phi+\frac{1}{6}e^{-2\Phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}=0, \tag{6}$$

where

$${}^{(\Phi)}T_{\mu\nu}=\frac{1}{2}\left(\Phi_{,\mu}\Phi_{,\nu}-\frac{1}{2}g_{\mu\nu}(\partial\Phi)^2\right), \tag{7}$$

$${}^{(H)}T_{\mu\nu}=\frac{1}{12}e^{-2\Phi}\left(3H_{\mu\lambda\kappa}H_\nu^{\lambda\kappa}-\frac{1}{2}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}\right). \tag{8}$$

Thus, in this frame, the problem reduces to the study of inhomogeneous general relativistic cosmologies containing two gravitationally interacting matter fields. In the next section we shall manipulate these equations into a soluble system by introducing a particular inhomogeneous spacetime metric with cylindrical symmetry with a particular topology. In Sec. III we give exact solutions in cases where one (or both) of the axion and dilaton fields depends only upon the time variable. In Sec. IV we consider the case where both fields depend upon time and space coordinates. In Sec. V we investigate the asymptotic behaviors of these fully inhomogeneous solutions on scales large and smaller than the horizon. In Sec. VI we study the nature of duality in these solutions and the results are discussed in Sec. VII.

II. EINSTEIN-ROSEN METRIC

Consider the anisotropic and inhomogeneous spacetime metric [10,11]

$$ds^2=-e^{2(\chi-\psi)}(d\tau^2-dr^2)+R(e^{2\psi}dz^2+e^{-2\psi}d\phi^2), \tag{9}$$

where χ,ψ,R are unknown functions of τ and r. Thus $\partial/\partial z$ and $\partial/\partial\phi$ are Killing vectors. Without loss of generality it can be assumed that $0\leqslant z\leqslant 1$ and $0\leqslant\phi\leqslant 1$. When $\psi=0$ and $R=e^{2\chi}$, with $R\equiv R(\tau)$ and $\chi\equiv\chi(\tau)$, we recover an isotropic Friedmann universe. Other homogeneous specializations of the metric reduce it to one of the Bianchi-type homogeneous universes [9]. Properties of the metric (9) depend on whether $B_\mu\equiv R_{,\mu}$ is spacelike, timelike, or null (Greek indices run $0\rightarrow 3$). The cases with a globally null or spacelike B_μ correspond to plane or cylindrical gravitational waves, respectively [12]. Metrics where the sign of $B_\mu B^\mu$ varies throughout the spacetime describe colliding gravitational waves [13] or cosmologies with timelike and spacelike singularities [14]. Metrics with a globally timelike B_μ describe cosmological models with spacelike singularities. If the spacelike hypersurfaces are compact then the allowed spatial topologies, [15], are a three-torus, $S^1\otimes S^1\otimes S^1$, for $R=(\det g_{ab})^{1/2}=\tau$ with $0\leqslant\tau<\infty$ and $0\leqslant r<\infty$; a hypertorus, $S^1\otimes S^2$, or a three-sphere, S^3, for $R=(\det g_{ab})^{1/2}=\sin r\sin\tau$ with $0\leqslant r\leqslant\pi$ and $0\leqslant\tau\leqslant\pi$. We shall present solutions for the globally timelike case $R=\tau$. These correspond to ever-expanding cosmological models with an initial curvature singularity at $\tau=0$. Note that the behavior of the closed S^3 models approaches that of the three-torus universes as the singularities are approached because $\sin\tau\rightarrow\tau$ as $\tau\rightarrow 0$ and π, and so the role played by the duality invariance of these models can be investigated along with the implications for the ''pre big bang'' scenario of Gasperini *et al.* [16]. The homogeneous models of the $S^1\otimes S^2$ case will be the Kantowski-Sachs universes studied in Ref. [8]. For further results about the singularity structure and global existence of these metrics (the strong cosmic censorship hypothesis holds) see the paper by Chrusciel *et al.* [17].

Rewriting Eq. (5) as

$$d(\star H)-2(d\Phi)\wedge(\star H)=0 \tag{10}$$

and using

$$dH=0, \tag{11}$$

we can determine the general form of H that is compatible with the Einstein-Rosen spacetime geometry. Denoting $x^0=\tau$, $x^1=r$, $x^2=z$, and $x^3=\phi$ we require

$$H=6A(\tau,r)dx^0\wedge dx^1\wedge dx^2+6B(\tau,r)dx^0\wedge dx^1\wedge dx^3+6C(\tau,r)dx^0\wedge dx^2\wedge dx^3+6D(\tau,r)dx^1\wedge dx^2\wedge dx^3. \tag{12}$$

The quantities H and Φ can be functions only of r and τ here since the energy-momentum tensor is allowed to depend only on these variables. Hence Eq. (11) implies

$$\partial_1 C - \partial_0 D = 0, \tag{13}$$

while $\star H$ is given by

$$\star H = \frac{1}{6} \epsilon_{\mu\nu\lambda\alpha} H^{\mu\nu\lambda} dx^\alpha \equiv F_\alpha dx^\alpha. \tag{14}$$

Since $d\Phi = \partial_0 \Phi dx^0 + \partial_1 \Phi dx^1$, Eq. (10) reduces to

$$[\partial_0 F_1 - \partial_1 F_0 - 2(F_1 \partial_0 \Phi - F_0 \partial_1 \Phi)] dx^0 \wedge dx^1 + [\partial_0 F_2 - 2F_2 \partial_0 \Phi] dx^0 \wedge dx^2 + [\partial_1 F_2 - 2F_2 \partial_1 \Phi] dx^1 \wedge dx^2 + [\partial_0 F_3 - 2F_3 \partial_0 \Phi] dx^0 \wedge dx^3 + [\partial_1 F_3 - 2F_3 \partial_1 \Phi] dx^1 \wedge dx^3 = 0. \tag{15}$$

This implies

$$\partial_0 F_1 - 2F_1 \partial_0 \Phi - \partial_1 F_0 + 2F_0 \partial_1 \Phi = 0, \tag{16}$$

$$d(e^{-2\Phi} F_2) = 0, \tag{17}$$

$$d(e^{-2\Phi} F_3) = 0, \tag{18}$$

so Eqs. (17) and (18) yield

$$F_2 = e^{2\Phi} A_2, \tag{19}$$

$$F_3 = e^{2\Phi} A_3, \tag{20}$$

where A_2 and A_3 are constants.

Using the fact that $C(r,\tau) = g_{00} g_{22} g_{33} H^{023} = g_{00} g_{22} g_{33} \epsilon^{0231} F_1 = -RF_1$ and, similarly, that $D(r,\tau) = -RF_0$, Eq. (13) becomes

$$\partial_1(RF_1) - \partial_0(RF_0) = 0 \tag{21}$$

and Eq. (16) implies that

$$\partial_0(e^{-2\Phi} F_1) = \partial_1(e^{-2\Phi} F_0). \tag{22}$$

In order to solve the system of differential equations (21) and (22) there are two obvious choices: (i) $e^{-2\Phi} F_1 = \partial_1 b$ and $e^{-2\Phi} F_0 = \partial_0 b$, (ii) $RF_1 = \partial_0 h$ and $RF_0 = \partial_1 h$. The latter choice corresponds to taking B_{23} to be the only nonvanishing component of the antisymmetric tensor potential defined by $H = dB$ and depending only on τ and r. The choice (i) reduces Eq. (21) to

$$\Box b + 2\nabla^\mu b \nabla_\mu \Phi = 0, \tag{23}$$

while choice (ii) produces another coupled wave equation

$$\ddot{h} - h'' - \frac{\dot{R}}{R}\dot{h} + \frac{R'}{R} h' - 2(\dot{\Phi}\dot{h} - \Phi' h') = 0 \tag{24}$$

where the overdot is equivalent to $\partial/\partial\tau$ and the prime is equivalent to $\partial/\partial r$.

The off-diagonal components of the Einstein tensor G_{02}, G_{03}, G_{12}, G_{13}, and G_{23}, are zero in the spacetime (9). The corresponding components of ${}^{(\Phi)}T_{\mu\nu}$ all vanish so we only need to ensure that all the corresponding off-diagonal components of ${}^{(H)}T_{\mu\nu}$ are also zero. Since we have

$${}^{(H)}T_{02} = \frac{1}{2} F_2 F_0 e^{-2\Phi} = \frac{1}{2} A_2 F_0, \tag{25}$$

we must therefore set $A_2 = 0$, and hence $H^{013} = B = 0$. For the (03) component we have

$${}^{(H)}T_{03} = \frac{1}{2} F_3 F_0 e^{-2\Phi} = \frac{1}{2} A_3 F_0, \tag{26}$$

so we must set $A_3 = 0$, and hence $H^{120} = A = 0$. With these choices, the components ${}^{(H)}T_{12}$, ${}^{(H)}T_{13}$, and ${}^{(H)}T_{23}$ also all vanish. Therefore, the equations governing the dilaton and antisymmetric tensor field for the two choices are given by Eq. (6) together with the following coupled propagation equations (27)–(28) or (29)–(30), in the cases (i) and (ii), respectively:

(i)

$$\frac{1}{R}(R\Phi')' - \frac{1}{R}(R\dot{\Phi})^{\cdot} - e^{2\Phi}[b'^2 - \dot{b}^2] = 0, \tag{27}$$

$$\frac{1}{R}(Rb')' - \frac{1}{R}(R\dot{b})^{\cdot} + 2[\Phi' b' - \dot{\Phi}\,\dot{b}] = 0. \tag{28}$$

(ii)

$$R^{-1}(R\Phi')' - R^{-1}(R\dot{\Phi})^{\cdot} + R^{-2} e^{-2\Phi}[h'^2 - \dot{h}^2] = 0, \tag{29}$$

$$\ddot{h} - h'' - \frac{\dot{R}}{R}\dot{h} + \frac{R'}{R} h' = -2(\Phi' h' - \dot{\Phi}\dot{h}). \tag{30}$$

Since both choices involve the same number of independent functions they are equivalent; here, choice (i) is taken.

The energy-momentum tensor in Eq. (4) reads

$$\kappa^{2(\Phi)} T^\lambda_\mu = \frac{1}{2}\left(g^{\nu\lambda} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \delta^\lambda_\mu (\partial\Phi)^2 \right), \tag{31}$$

$$\kappa^{2(H)} T^\lambda_\mu = \frac{1}{2} e^{2\Phi}\left(g^{\nu\lambda} \partial_\mu b \partial_\nu b - \frac{1}{2} \delta^\lambda_\mu (\partial b)^2 \right), \tag{32}$$

so the nonvanishing components of the energy-momentum tensor are

$$\kappa^2 T^0_0 = -\frac{1}{4} e^{-2(\chi-\psi)}[\dot{\Phi}^2 + \Phi'^2 + (\dot{b}^2 + b'^2) e^{2\Phi}] = -\kappa^2 T^1_1,$$

$$\kappa^2 T^1_0 = \frac{1}{2} e^{-2(\chi-\psi)}[\dot{\Phi}\Phi' + \dot{b} b' e^{2\Phi}] = -\kappa^2 T^0_1,$$

$$\kappa^2 T^2_2 = \frac{1}{4} e^{-2(\chi-\psi)}[\dot{\Phi}^2 - \Phi'^2 + (\dot{b}^2 - b'^2) e^{2\Phi}] = \kappa^2 T^3_3, \tag{33}$$

The energy-momentum tensor can be interpreted as describing two stiff perfect fluids where the energy density for the dilaton fluid is found to be

$$p_\Phi=\rho_\Phi=\frac{1}{4}e^{-2(\chi-\psi)}[\dot{\Phi}^2-\Phi'^2]$$

and its four-velocity is given by

$$u^\alpha=e^{-(\chi-\psi)}[\dot{\Phi}^2-\Phi'^2]^{-1/2}(-\dot{\Phi},\Phi',0,0)$$

and for the axion fluid we have

$$p_H=\rho_H=\frac{1}{4}e^{-2(\chi-\psi)}[\dot{b}^2-b'^2]e^{2\Phi}$$

and its four-velocity is

$$v^\alpha=e^{-(\chi-\psi)}[\dot{b}^2-b'^2]^{-1/2}(-b,b',0,0).$$

Furthermore, $T_0^0+T_1^1=0$ and $T_2^2-T_3^3=0$ as the cylindrical symmetry of the metric demands, and Einstein's equations for $R(\tau,r)$ and $\psi(\tau,r)$ are given by the linear wave equations [12]

$$\ddot{R}-R''=0, \tag{34}$$

$$\ddot{\psi}+\frac{\dot{R}}{R}\dot{\psi}-\psi''-\frac{R'}{R}\psi'=0. \tag{35}$$

The remaining metric function, $\chi(\tau,r)$, is determined by the two Einstein constraint equations

$$\chi'=\psi'-\frac{1}{4}\frac{R'}{R}-(\dot{R}^2-R'^2)^{-1}[RR'(\dot{\psi}^2+\psi'^2)+R'R'' -2\dot{R}R\dot{\psi}\psi'-\dot{R}\dot{R}'-\kappa^2Re^{2(\chi-\psi)}(T_0^0R'+T_0^1\dot{R})], \tag{36}$$

$$\dot{\chi}=\dot{\psi}-\frac{1}{4}\frac{\dot{R}}{R}-(\dot{R}^2-R'^2)^{-1}[2RR'\dot{\psi}\psi'-R\dot{R}(\dot{\psi}^2+\psi'^2) -\dot{R}R''+R'\dot{R}'+\kappa^2e^{2(\chi-\psi)}R(T_0^0\dot{R}+T_0^1R')]. \tag{37}$$

Since cosmological solutions are of interest to us here, we consider only the timelike solution of Eq. (34). Using the general coordinate invariances $[\tau\pm r\rightarrow f(\tau\pm r)]$ of the metric this may be taken without loss of generality to be

$$R=R(\tau)=\tau. \tag{38}$$

Then, Eq. (35) reduces to

$$\ddot{\psi}+\frac{1}{\tau}\dot{\psi}-\psi''=0 \tag{39}$$

which is solved by

$$\psi(\tau,r)={}^0\psi_0+{}^0\psi_1\ln\tau+\sum_{n=1}^{\infty}\cos[n(r-r_n)] \times[{}^A\Psi_nJ_0(n\tau)+{}^B\Psi_nN_0(n\tau)], \tag{40}$$

where ${}^0\psi_i$, ${}^A\Psi_n$, ${}^B\Psi_n$, r_n are constants and $J_0(x)$ and $N_0(x)$ denote the zeroth-order Bessel and Neumann functions, respectively.

Equations (27)–(28) read

$$\Phi''-\frac{\dot{\Phi}}{\tau}-\ddot{\Phi}-e^{2\Phi}[b'^2-\dot{b}^2]=0, \tag{41}$$

$$b''-\frac{\dot{b}}{\tau}-\ddot{b}+2[\Phi'b'-\dot{\Phi}\dot{b}]=0. \tag{42}$$

In the next section several solutions will be found.

III. SOLUTIONS OF VARYING GENERALITY

Before explicit solutions are given, we make some remarks about the procedure for solving the system of partial differential equations for the metric function $\chi(\tau,r)$. For $R(\tau)=\tau$, Eqs. (36) and (37) reduce to

$$\chi'=\psi'+2\tau\dot{\psi}\psi'+\kappa^2\tau e^{2(\chi-\psi)}T_0^1, \tag{43}$$

$$\dot{\chi}=\dot{\psi}-\frac{1}{4\tau}+\tau(\dot{\psi}^2+\psi'^2)-\kappa^2\tau e^{2(\chi-\psi)}T_0^0. \tag{44}$$

Generally speaking, the most difficult step is to find the integral for the part coupled to $\psi(\tau,r)$. However, this problem was solved by Charach [18]. Define a function $G(\psi;\tau,r)$ by

$$G'=\tau\dot{\psi}\psi', \tag{45}$$

$$\dot{G}=\frac{1}{2}\tau(\dot{\psi}^2+\psi'^2). \tag{46}$$

Note that ψ satisfying

$$\ddot{\psi}+\frac{1}{\tau}\dot{\psi}-\psi''=0$$

is kept as a functional dependence in G. The explicit dependence on τ and r might sometimes be suppressed and we write $G(\psi)$ as

$$G(\psi;\tau,r)={}^0\psi_0+\frac{1}{2}({}^0\psi_1)^2\ln\tau+{}^0\psi_1\sum_{n=1}^{\infty}\cos[n(r-r_n)][{}^A\Psi_nJ_0(n\tau)+{}^B\Psi_nN_0(n\tau)]$$
$$+\frac{1}{4}\tau^2\sum_{n=1}^{\infty}n^2\{[{}^A\Psi_nJ_0(n\tau)+{}^B\Psi_nN_0(n\tau)]^2+[{}^A\Psi_nJ_1(n\tau)+{}^B\Psi_nN_1(n\tau)]^2\}-\frac{1}{2}\tau\sum_{n=1}^{\infty}n\cos^2[n(r-r_n)]$$
$$\times\{({}^A\Psi_n)^2J_0(n\tau)J_1(n\tau)+{}^A\Psi_n^B\Psi_n[N_0(n\tau)J_1(n\tau)+J_0(n\tau)N_1(n\tau)]+({}^B\Psi_n)^2N_0(n\tau)N_1(n\tau)\}$$
$$+\frac{1}{2}\tau\sum_{n=1}^{\infty}\sum_{m=1,n\neq m}^{\infty}\frac{nm}{n^2-m^2}\{\sin[n(r-r_n)]\sin[m(r-r_m)][nU_{nm}^{(0)}(\tau)-mU_{nm}^{(1)}(\tau)]$$
$$+\cos[n(r-r_n)]\cos[m(r-r_m)][mU_{nm}^{(0)}(\tau)-nU_{nm}^{(1)}(\tau)]\},\tag{47}$$

where

$$U_{nm}^{(0)}(\tau)\equiv{}^A\Psi_n^A\Psi_mJ_1(n\tau)J_0(n\tau)+{}^B\Psi_n^B\Psi_mN_0(m\tau)N_1(n\tau)+2{}^A\Psi_n^B\Psi_mJ_1(n\tau)N_0(m\tau),$$

$$U_{nm}^{(1)}(\tau)\equiv{}^A\Psi_n^A\Psi_mJ_0(n\tau)J_0(m\tau)+{}^B\Psi_n^B\Psi_mN_0(n\tau)N_1(m\tau)+2{}^A\Psi_n^B\Psi_mJ_0(n\tau)N_1(m\tau).$$

We now consider classes of solutions in which one (or both) of the Φ and b fields depend on only one of the coordinates r and t.

A. Solutions homogeneous in τ: $\Phi=\Phi(\tau)$, $b=b(\tau)$

The well-known solution to Eqs. (41) and (42) [6] in this subcase is

$$e^{\Phi}=\cosh(N\zeta)+\sqrt{1-(B^2/N^2)}\sinh(N\zeta),\tag{48}$$

$$b(\zeta)=\frac{N}{B}\frac{\sinh(N\zeta)+\sqrt{1-(B^2/N^2)}\cosh(N\zeta)}{\cosh(N\zeta)+\sqrt{1-B^2/N^2}\sinh(N\zeta)},\tag{49}$$

where N and B are constants and $d\tau=\tau d\zeta$. Using this in the expression for the components of the energy-momentum tensor gives an expression for $\chi(\tau,r)$,

$$\chi(\tau,r)=\psi(\tau,r)+2G(\psi;\tau,r)+\frac{N^2-1}{4}\ln\tau+M,\tag{50}$$

where M is a constant. Hence, the metric function $\exp[\chi-\psi]$ is given by

$$\exp[\chi-\psi]=e^{2G(\psi)}e^{M}\tau^{(N^2-1)/4}.\tag{51}$$

B. Solutions homogeneous in r: $\Phi=\Phi(r)$, $b=b(r)$

The solution in this subcase is given by

$$e^{\Phi}=\cosh(Nr)+\sqrt{1-\frac{B^2}{N^2}}\sinh(Nr),\tag{52}$$

$$b(r)=\frac{N}{B}\frac{\sinh(Nr)+\sqrt{1-(B^2/N^2)}\cosh(Nr)}{\cosh(Nr)+\sqrt{1-(B^2/N^2)}\sinh(Nr)},\tag{53}$$

where N, B are constants. From these expressions, $\chi(\tau,r)$ is found to be

$$\chi(\tau,r)=\psi(\tau,r)+2G(\psi;\tau,r)-\frac{1}{4}\ln\tau+\frac{N^2}{8}\tau^2+M,\tag{54}$$

which gives the remaining metric component

$$\exp[\chi-\psi]=e^{2G(\psi)}e^{2M}\tau^{-1/4}e^{(N^2/8)\tau^2}.\tag{55}$$

C. Solutions with an oscillatory axion: $\Phi=\Phi(\tau)$, $b=b(\tau,r)$

If we rewrite Eq. (42) as

$$\frac{\partial^2 b}{\partial r^2}-\frac{\partial^2 b}{\partial\tau^2}-\frac{\partial b}{\partial\tau}\left(2\frac{\partial\Phi}{\partial\tau}+\frac{1}{\tau}\right)+2\frac{\partial\Phi}{\partial r}\frac{\partial b}{\partial r}=0\tag{56}$$

and take a solution $2\Phi(\tau)=-\ln(\tau/\tau_0)$, then the axion field $b(r,\tau)$ also satisfies the wave equation

$$\frac{\partial^2 b}{\partial r^2}-\frac{\partial^2 b}{\partial\tau^2}=0,\tag{57}$$

which has the general solution

$$b(r,\tau)=\alpha b_1(r+\tau)+\beta b_2(r-\tau),\tag{58}$$

where α, β are constants and b_i are arbitrary functions of their arguments. Equation (41) is satisfied if

$$\left(\frac{\partial b}{\partial r}\right)^2-\left(\frac{\partial b}{\partial\tau}\right)^2=0.\tag{59}$$

This implies

$$\left(\frac{\partial b}{\partial r}+\frac{\partial b}{\partial\tau}\right)\left(\frac{\partial b}{\partial r}-\frac{\partial b}{\partial\tau}\right)=0\tag{60}$$

so that either α or β must vanish. Thus we obtain the solution

$$\Phi(\tau)=-\frac{1}{2}\ln\frac{\tau}{\tau_0},\tag{61}$$

$$b(\tau,r)=\Theta(u)b_1(r+\tau)+[1-\Theta(u)]b_2(r-\tau),\tag{62}$$

where $\Theta(u)$ is the step function $[\Theta(u)=0$ for $u\leqslant 0$; $\Theta(u)=1$ for $u>0]$ and u an arbitrary real parameter.

It is interesting to have a solution with a homogeneous dilaton and an *in*homogeneous axion. Note, that in this case the axion behaves quite differently from the dilaton.

The function $\chi(\tau,r)$ is determined by

$$\chi' = \psi' + 2\tau\dot{\psi}\psi' + \frac{\tau_0}{2}\dot{b}b', \tag{63}$$

$$\dot{\chi} = \dot{\psi} - \frac{3}{16\tau} + \tau(\dot{\psi}^2 + \psi'^2) + \frac{\tau_0}{4}(\dot{b}^2 + b'^2). \tag{64}$$

To solve this system of equations we define a new function $B(\tau,r)$ satisfying

$$B' = \dot{b}b' \tag{65}$$

$$\dot{B} = \frac{1}{2}(\dot{b}^2 + b'^2). \tag{66}$$

Changing to new variables,

$$X = r + \tau, \quad Y = r - \tau,$$

we find

$$\frac{\partial B}{\partial X} = \left(\frac{\partial b}{\partial X}\right)^2, \quad \frac{\partial B}{\partial Y} = -\left(\frac{\partial b}{\partial Y}\right)^2.$$

This implies

$$\frac{\partial^2 B}{\partial X \partial Y} = 0 \tag{67}$$

which is generally solved by

$$B(X,Y) = B_1(X) + B_2(Y) \tag{68}$$

with B_i arbitrary functions of their arguments. Using the general solution for b in terms of X and Y, $B(X,Y)$ is given by

$$B(X,Y) = \Theta(u)\int dX\left(\frac{db_1}{dX}\right)^2 - [1-\Theta(u)]\int dY\left(\frac{db_2}{dY}\right)^2. \tag{69}$$

Finally, an expression for $\chi(\tau,r)$ is obtained,

$$\chi(\tau,r) = \psi(\tau,r) + 2G(\psi) + \frac{\tau_0}{2}B(\tau,r) - \frac{3}{16}\ln\tau + M, \tag{70}$$

which results in

$$\exp[\chi - \psi] = e^{2G(\psi)}e^{M}\tau^{-3/16}e^{(\tau_0/2)B(\tau,r)}. \tag{71}$$

D. Solutions with $\Phi=\Phi(\tau,r)$, $b=b(\tau)$

If we take $b(\tau)=A\tau^2/2$, A constant, and $\Phi(\tau,r)=-\ln\tau+S(r)$, then Eq. (41) requires $S(r)$ to satisfy

$$\frac{d^2S}{dr^2} + A^2e^{2S(r)} = 0.$$

Hence,

$$e^{-\Phi(r,\tau)} = \tau\left[\cosh(Nr) + \sqrt{1-\frac{A^2}{N^2}}\sinh(Nr)\right], \tag{72}$$

$$b(\tau) = \frac{A}{2}\tau^2 \tag{73}$$

is a solution of Eqs. (41),(42).

Calculating the appropriate components of the energy-momentum tensor yields

$$\chi(\tau,r) = \psi(\tau,r) + 2G(\psi;\tau,r) + \frac{1}{2}\ln\left[\cosh(Nr) + \sqrt{1-\frac{A^2}{N^2}}\sinh(Nr)\right] + \frac{N^2}{8}\tau^2 + M, \tag{74}$$

hence

$$\exp[\chi - \psi] = e^{2G(\psi)}e^{M}\left[\cosh(Nr) + \sqrt{1-\frac{A^2}{N^2}}\sinh(Nr)\right]^{1/2} e^{(N^2/8)\tau^2}. \tag{75}$$

E. Discussion

Apart from case III C the solutions presented so far describe nonoscillatory axion-dilaton systems on an oscillatory cosmological background. In case III C the axion field is allowed to oscillate which couples the dilatonic and gravitational waves. However, because of condition (59), only travelling wave solutions in $b(\tau,r)$ are described in this case.

IV. CHARACH SOLUTIONS

The system of equations (41),(42) is very similar to equations determining the components of the electromagnetic potential in the electromagnetic Gowdy universe [18,19]. It was stated in [18] (and references therein) that the geometric requirements of the Einstein-Rosen spacetimes allow four independent components of the six possible components of the Maxwell tensor which can be derived from two nonvanishing components of the electromagnetic potential. In Sec. II, we found that only two of the four possible components of the antisymmetric tensor field strength can be nonvanishing, which can then be accordingly derived from the potential-like function $b(\tau,r)$ or $h(\tau,r)$. In the latter case there is a direct connection to the antisymmetric tensor field potential $B_{\mu\nu}$, where $H=dB$. In order to obtain an exact solution of Eqs. (41),(42), where Φ and b are dependent on r and τ we employ a procedure introduced by Charach [18].

Assume that

$$\Phi(r,\tau) = -\frac{1}{2}\ln v[b(r,\tau)], \tag{76}$$

where $v(b)$ is a function yet to be determined. Since

$$\dot{\Phi} = -\frac{1}{2}\dot{b}\frac{d\ln v}{db}, \quad \Phi' = -\frac{1}{2}b'\frac{d\ln v}{db},$$

$$\ddot{\Phi} = -\frac{1}{2}\left[\ddot{b}\frac{d\ln v}{db} + \dot{b}^2\frac{d^2\ln v}{db^2}\right],$$

$$\Phi'' = -\frac{1}{2}\left[b''\frac{d\ln v}{db} + b'^2\frac{d^2\ln v}{db^2}\right],$$

we can use Eq. (42) to transform Eq. (41) into

$$\left(\frac{d^2 v}{db^2} + 2\right)(b'^2 - \dot{b}^2) = 0, \tag{77}$$

while Eq. (42) becomes

$$b'' - \ddot{b} - \frac{1}{\tau}\dot{b} = [b'^2 - \dot{b}^2]\frac{d\ln v}{db}. \tag{78}$$

Assuming $b'^2 - \dot{b}^2 \neq 0$, Eq. (77) implies

$$v(b) = -b^2 + c_1 b + c_2, \tag{79}$$

where the logarithm in Eq. (76) requires that the constants c_1, c_2 satisfy the inequality

$$c_1^2 + 4c_2 > 0. \tag{80}$$

Equation (78) becomes

$$b'' - \ddot{b} - \frac{1}{\tau}\dot{b} = \frac{c_1 - 2b}{c_2 + c_1 b - b^2}(b'^2 - \dot{b}^2). \tag{81}$$

If we make the substitution

$$b = b_0 + M\tanh(M\omega), \tag{82}$$

Eq. (81) becomes

$$\frac{M^2}{\cosh^2(M\omega)}\left[\omega'' - \ddot{\omega} - \frac{1}{\tau}\dot{\omega}\right] + 2\frac{M^3\sinh(M\omega)}{\cosh^3(M\omega)}[\dot{\omega}^2 - \omega'^2]$$
$$= 2\frac{M^3\sinh(M\omega)}{\cosh^3(M\omega)}[\dot{\omega}^2 - \omega'^2], \tag{83}$$

where $b_0 = \frac{1}{2}c_1$ and $M^2 = c_2 + \frac{1}{4}c_1^2 > 0$. Hence, $\omega(r,\tau)$ satisfies the linear wave equation

$$\omega'' - \ddot{\omega} - \frac{1}{\tau}\dot{\omega} = 0. \tag{84}$$

The wave-packet solution to Eq. (84) is given by

$$\omega(\tau,r) = {}^0\omega_0 + {}^0\omega_1\ln\tau + \sum_{n=1}^{\infty}\cos[n(r - r_n)]$$
$$\times[{}^A\Omega_n J_0(n\tau) + {}^B\Omega_n N_0(n\tau)], \tag{85}$$

where ${}^0\omega_i$, ${}^A\Omega_n$, ${}^B\Omega_n$, and r_n are constants.

In summary, Eqs. (41) and (42) admit the inhomogeneous solution

$$\Phi(\tau,r) = \ln\frac{\cosh(M\omega)}{M}, \tag{86}$$

$$b(\tau,r) = \frac{1}{2}c_1 + M\tanh(M\omega). \tag{87}$$

Rewriting the components T_0^0 and T_0^1 of the energy-momentum tensor in terms of $\omega(\tau,r)$ gives

$$\kappa^2 e^{2(\chi-\psi)}T_0^0 = -\frac{M^2}{4}(\dot{\omega}^2 + \omega'^2), \tag{88}$$

$$\kappa^2 e^{2(\chi-\psi)}T_0^1 = \frac{M^2}{2}\dot{\omega}\omega', \tag{89}$$

and Eqs. (43) and (44), which determine $\chi(\tau,r)$, reduce to

$$\chi' = \psi' + 2\tau\dot{\psi}\psi' + \frac{M^2}{2}\tau\dot{\omega}\omega', \tag{90}$$

$$\dot{\chi} = \dot{\psi} - \frac{1}{4\tau} + \tau(\dot{\psi}^2 + \psi'^2) + \frac{M^2}{4}\tau(\dot{\omega}^2 + \omega'^2). \tag{91}$$

Using the function $G(f;\tau,r)$, where $\ddot{f} + \tau^{-1}\dot{f} - f'' = 0$ and $G(f;\tau,r)$ is given by Eq. (47) and Eqs. (90),(91) lead to

$$d\chi = d\psi - \frac{1}{4}d\ln\tau + 2dG(\psi;\tau,r) + \frac{M^2}{2}dG(\omega;\tau,r) \tag{92}$$

which yields

$$\chi(\tau,r) = \psi(\tau,r) - \frac{1}{4}\ln\tau + 2G(\psi;\tau,r) + \frac{M^2}{2}G(\omega;\tau,r) + L, \tag{93}$$

where L is some constant. So that the metric function $\exp[\chi - \psi]$ is given by

$$\exp[\chi - \psi] = e^L\tau^{-1/4}\exp\left(2G(\psi) + \frac{M^2}{2}G(\omega)\right). \tag{94}$$

V. ASYMPTOTIC BEHAVIOR

The existence of inhomogeneity in the solutions found in Sec. IV introduces characteristic length scales and the gravitational self-interaction of the dilatonic and axionic waves will differ over scales according as they are causally coherent or not. The horizon distance in the r direction is defined by $ds^2|_{z,\phi} = 0$; hence,

$$\triangle r = \int_0^\tau d\tau = \tau. \tag{95}$$

Therefore, the combination $n\tau$ in the solutions above can be interpreted as the ratio of the radial horizon distance to the coordinate wavelength λ since $n \propto 1/\lambda(n)$. There are two limiting cases to be considered: the case $n\tau \ll 1$, when the comoving wavelength is much larger than the radial horizon scale, and the case $n\tau\gg 1$, when the wavelength of the inhomogeneities is well within the horizon scale. We consider these two cases separately.

The Charach solutions discussed in the last section are the most general ones of those given. Apart from solutions of Sec. III C the limiting properties of the other solutions are included in those of the Charach-type solutions. Therefore in this section only the asymptotes of these solutions, of Sec. IV, are discussed. Explicit formulas for the functions involved are given in the Appendix along with some useful definitions. It is convenient to define metric functions

$$A_1(\tau,r)\equiv\exp[\chi(\tau,r)-\psi(\tau,r)], \tag{96}$$

$$A_2(\tau,r)\equiv\tau^{1/2}e^{\psi(\tau,r)}, \tag{97}$$

$$A_3(\tau,r)\equiv\tau^{1/2}e^{-\psi(\tau,r)}. \tag{98}$$

A. The limit $n\tau\ll 1$

In this case

$$A_1(\tau,r)\sim e^{L}\tau^{-1/4+2\gamma_2(\psi;r)+(M^2/2)\gamma_2(\omega;r)}\times\exp\left[2\gamma_1(\psi;r)+\frac{M^2}{2}\gamma_1(\omega;r)\right], \tag{99}$$

$$A_2(\tau,r)\sim e^{\alpha_1(\psi;r)}\tau^{1/2+\alpha_2(\psi;r)}, \tag{100}$$

$$A_3(\tau,r)\sim e^{-\alpha_1(\psi;r)}\tau^{1/2-\alpha_2(\psi;r)}. \tag{101}$$

This limit corresponds to the case where the comoving wavelength is much larger than the (radial) horizon size, or in other words the universe consists of causally disconnected regions. In this case one would not expect to have any oscillatory behavior in $n\tau$.

Concentrating on the homogeneous limit for τ approaching zero allows to discuss cosmological solutions near the singularity. The metric functions are found to approach

$$A_1(\tau)\sim\tau^{-1/4+({}^0\psi_1)^2+(M^2/4)({}^0\omega_1)^2}, \tag{102}$$

$$A_2(\tau)\sim\tau^{1/2+{}^0\psi_1}, \tag{103}$$

$$A_3(\tau)\sim\tau^{1/2-{}^0\psi_1}. \tag{104}$$

Changing to proper time using, in the homogeneous limit, the relation

$$t=\int d\tau A_1(\tau),$$

$\tau(t)$ is found to be

$$\tau\propto t^{1/[3/4+({}^0\psi_1)^2+(M^2/4)({}^0\omega_1)^2]}. \tag{105}$$

Defining the Kasner exponents p_i, $i=1, 2, 3$, by

$$g_{\mu\nu}\sim\mathrm{diag}(-1,t^{2p_1},t^{2p_2},t^{2p_3})$$

they are found to be slowly spatially varying:

$$p_1\equiv\frac{-1/4+({}^0\psi_1)^2+(M^2/4)({}^0\omega_1)^2}{3/4+({}^0\psi_1)^2+(M^2/4)({}^0\omega_1)^2}, \tag{106}$$

$$p_2\equiv\frac{1/2+{}^0\psi_1}{3/4+({}^0\psi_1)^2+(M^2/4)({}^0\omega_1)^2}, \tag{107}$$

$$p_3\equiv\frac{1/2-{}^0\psi_1}{3/4+({}^0\psi_1)^2+(M^2/4)({}^0\omega_1)^2}. \tag{108}$$

They satisfy the algebraic constraints

$$\sum_{i=1}^{3}p_i=1, \tag{109}$$

$$\sum_{i=1}^{3}p_i^2=1-\frac{M^2}{2}\frac{({}^0\omega_1)^2}{[3/4+({}^0\psi_1)^2+M^2/4({}^0\omega_1)^2]^2}. \tag{110}$$

The fact that $\Sigma_{i=1}^3 p_i^2\leq 1$, where the equality holds in the vacuum case ($M=0$), shows immediately that there are isotropic solutions. This feature is present in the matter-filled Gowdy solutions [18,19] and in the spatially homogeneous Kasner universes containing a stiff fluid.

The axion-dilaton system is independent of the gravitational background in the sense that its determining equations [see Eqs. (41) and (42)] do not involve any of the metric functions apart from $R(\tau,r)$. However, due to the general structure of the equations the solutions for Φ and b are very similar to those of the metric functions. As $\tau\to 0$, the dilaton and axion fields approach

$$\Phi(\tau,r)\sim\ln[e^{M\alpha_1(\omega;r)}\tau^{M\alpha_2(\omega;r)}+e^{-M\alpha_1(\omega;r)}\tau^{-M\alpha_2(\omega;r)}]-\ln 2M, \tag{111}$$

$$b(\tau,r)\sim\frac{1}{2}c_1+M\frac{e^{2M\alpha_1(\omega;r)}\tau^{2M\alpha_2(\omega;r)}-1}{e^{2M\alpha_1(\omega;r)}\tau^{2M\alpha_2(\omega;r)}+1}. \tag{112}$$

We note that the early-time behavior of these solutions falls under the category of ‘‘velocity-dominated’’ solutions used in studies of general relativistic cosmologies [17]. As the singularity is approached the spatial gradients become negligible with respect to the time derivatives, three-curvature anisotropies are ignored, and velocities are assumed to be less than the speed of light. This approximation does not encompass the most general known behavior in general relativity, with the metric undergoing chaotic oscillations on approach to the singularity [20]. Chaos in string cosmologies will the subject of a separate study [21].

B. The limit $n\tau\gg 1$

In this case the comoving wavelength is smaller than the (radial) horizon size allowing interaction between different modes and hence an oscillatory behavior of the metric components. From the limits of $\psi(\tau,r)$ and $G(\psi;\tau,r)$ given in the Appendix it can be seen that ψ displays an oscillatory behavior while the oscillations in G are damped out:

$$A_1(\tau,r)\sim e^L\tau^{-1/4+2\gamma_4(\psi)+(M^2/2)\gamma_4(\omega)}\times\exp\left[2\gamma_3(\psi)+\frac{M^2}{2}\gamma_3(\omega)\right]\times\exp\left\{\left[2\gamma_5(\psi)+\frac{M^2}{2}\gamma_5(\omega)\right]\tau\right\}, \quad (113)$$

$$A_2(\tau,r)\sim\tau^{1/2+\beta_2(\psi)}e^{\beta_1(\psi)}\exp[\tau^{-1/2}h(\psi;\tau,r)], \quad (114)$$

$$A_3(\tau,r)\sim\tau^{1/2-\beta_2(\psi)}e^{-\beta_1(\psi)}\exp[-\tau^{-1/2}h(\psi;\tau,r)]. \quad (115)$$

As can be easily seen from the definition of $h(\psi;\tau,r)$ given in the Appendix it satisfies the wave equation (in Minkowski space)

$$\ddot{h}-h''=0. \quad (116)$$

The exponential in A_1 ensures that the homogeneous limit is approached at large τ, and is an anisotropic universe which can be at most axisymmetric $[\beta_2(\psi;t,r)=0]$. Since $g_{\mu\nu}\sim\mathrm{diag}(-A_1^2,A_1^2,A_2^2,A_3^2)$ and, for large values of τ, we have

$$\exp[2\tau^{-1/2}h(\psi;\tau,r)]\sim 1+2\tau^{-1/2}h[\psi;\tau,r]$$

and so $g_{\mu\nu}$ can be written as the sum of a background part $\eta_{\mu\nu}$ and a "wave" part $h_{\mu\nu}$,

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},$$

which are found to be

$$\eta_{\mu\nu}\equiv\mathrm{diag}(-A_1^2,A_1^2,\tau^{1+2\beta_2(\psi)}e^{2\beta_1(\psi)},\tau^{1-2\beta_2(\psi)}e^{-2\beta_1(\psi)}), \quad (117)$$

$$h_{\mu\nu}\equiv\mathrm{diag}[0,0,2\tau^{1/2+2\beta_2(\psi)}e^{2\beta_1(\psi)}h(\psi;\tau,r),\ -2\tau^{1/2-2\beta_2(\psi)}e^{-2\beta_1(\psi;t,r)}h(\psi;\tau,r)]. \quad (118)$$

The dilaton-axion system displays an oscillatory behavior as well, although as emphasized before, there is no interaction between gravitational and axion-dilaton waves.

The asymptotes are given by

$$\Phi(\tau,r)\sim\ln\{e^{M\beta_1(\omega)}\tau^{M\beta_2(\omega)}+e^{-M\beta_1(\omega)}\tau^{-M\beta_2(\omega)}+M[e^{M\beta_1(\omega)}\tau^{M\beta_2(\omega)-1/2}-e^{-M\beta_1(\omega)}\tau^{-M\beta_2(\omega)-1/2}]h(\omega;\tau,r)\}-\ln(2M), \quad (119)$$

$$b(\tau,r)\sim\frac{1}{2}c_1+M\times\frac{e^{2M\beta_1(\omega)}\tau^{2M\beta_2(\omega)}[1+2M\tau^{-1/2}h(\omega;\tau,r)]-1}{e^{2M\beta_1(\omega)}\tau^{2M\beta_2(\omega)}[1+2M\tau^{-1/2}h(\omega;\tau,r)]+1}. \quad (120)$$

Furthermore, the string coupling constant

$$g_s^2(\tau,r)=e^{\Phi}$$

is a function of τ and r and is given by

$$g_s^2(\tau,r)=\frac{1}{M}\cosh(M\omega).$$

So that taking the limit $\tau\to\infty$ at constant r results in a diverging g_s^2, hence the string coupling is driven towards the strong-coupling regime. This indicates the limited physical interpretation of this model. The coupling goes off to infinity since there are no ways of stabilizing the dilaton, for example by a potential. Futhermore, in the next section we give a simple example of an O(2,2) transformation relating a strong and weak coupling solution.

In summary, these solutions highlight some new considerations for string cosmology. In the past the string cosmologies that have been studied have all possessed simple geometrical structures [2–8]: isotropic universes have a single scale factor and the anisotropic models have been spatially uniform. This ensures that the effects of duality are very simple. However, in inhomogeneous cosmologies the situation becomes more unusual. It is possible for the universe to display quite different behavior from place to place and for the universe to be expanding or contracting in different places. Under these conditions the simple "pre-big-bang picture" [16] that has been investigated in the context of scale factor duality becomes more complex. Our solutions do not contain trapped surfaces and so there are no gravitationally bound collapsing regions. However, if we had taken $R=\sin r\sin t$ for the solution of Eq. (34) then the S^3 topology would permit local regions to collapse prematurely to singularities. In the solutions found above, the oscillatory behavior arises because the gravitational force created by the inhomogeneities in the ϕ and b fields is balanced by the pressure forces once they enter the horizon. Once inside the horizon the fluctuations are within their Jeans length and behave as acoustic waves. The inhomogeneities do not collapse to form black holes because the pressure forces are able to support them inside the horizon: in effect, they never fall within their Schwarzschild radii.

VI. DUALITY

By means of dimensional reduction we can show that the low-energy effective action (2) is invariant under global $O(d,d)$ transformations, where $d\leqslant D$ refers to the number of coordinates it does not depend on [22]. So, if one assumes a spacetime of the form $N\times K$, where N is a $(D-d)$-dimensional spacetime with coordinates x^μ (μ

$=0,1,\ldots D-d-1$), and K a d-dimensional compact space with coordinates y^{α} ($\alpha=1,\ldots,d$), and furthermore that all fields are assumed to be independent of the y coordinates of the ''internal'' space K, then using the notation of [22] we can rewrite Eq. (2) as

$$S=\int_N dx\int_K dy\sqrt{-\hat{g}}e^{-\hat{\phi}}\Big(\hat{R}(\hat{g})+\hat{g}^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi}-\frac{1}{12}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\Big). \tag{121}$$

The hatted quantities now refer to the D-dimensional spacetime. Using the vielbein formalism, $\hat{g}_{\hat{\mu}\hat{\nu}}$ is written as

$$\hat{g}_{\hat{\mu}\hat{\nu}}=\begin{pmatrix} g_{\mu\nu}+A^{(1)\gamma}_{\mu}A^{(1)}_{\nu\gamma} & A^{(1)}_{\mu\beta}\\ A^{(1)}_{\nu\alpha} & G_{\alpha\beta}\end{pmatrix}, \tag{122}$$

where $g_{\mu\nu}$ is the metric on N and $G_{\alpha\beta}$ the metric on K.

Define a shifted dilaton

$$\hat{\Phi}\equiv\hat{\phi}-\frac{1}{2}\ln\det G_{\alpha\beta}, \tag{123}$$

and a $2d\times 2d$ matrix Q, written in $d\times d$ blocks,

$$Q\equiv\begin{pmatrix} G^{-1} & -G^{-1}B\\ BG^{-1} & G-BG^{-1}B\end{pmatrix}. \tag{124}$$

It can be shown that Eq. (121) is invariant under global $O(d,d)$ transformations

$$\hat{\Phi}\to\hat{\Phi},\quad Q\to\Omega Q\Omega^{T}, \tag{125}$$

where $\Omega\in O(d,d)$, that is, $\Omega^{T}\eta\Omega=\eta$, where

$$\eta=\begin{pmatrix} 0 & \mathbb{I}_d\\ \mathbb{I}_d & 0\end{pmatrix}$$

and $\mathbb{I}_d$ is the d-dimensional unity matrix.

In the case of a diagonal $\hat{g}_{\hat{\mu}\hat{\nu}}$ and a vanishing B field, with the choices $d=D-1$ and $\Omega=\eta$, the scale-factor duality is recovered. This was first discussed by Veneziano [23]. In this case the duality transformation results in an inversion of the scale factors in the string frame. For a comprehensive discussion of target-space duality see Ref. [24].

In the case of the Einstein-Rosen metric (9), considered here, the low-energy effective action (2) is invariant under O(2,2) transformations for $D=4$. In this section, the antisymmetric tensor potential $B_{\mu\nu}$ is assumed to be vanishing. Transforming the metric (9) to the string frame, the ''internal'' metric $G_{\alpha\beta}$ is found to be

$$G_{\alpha\beta}=\begin{pmatrix} Re^{2\psi}e^{\Phi} & 0\\ 0 & Re^{-2\psi}e^{\Phi}\end{pmatrix}. \tag{126}$$

The shifted dilaton defined above remains invariant under $O(d,d)$ transformations and this implies that the dilaton itself transforms as

$$\Phi\to\Phi-\frac{1}{2}\ln\frac{\det G_{\alpha\beta}}{\det G^{\mathrm{dual}}_{\alpha\beta}}, \tag{127}$$

where the index ''dual'' indicates an $O(d,d)$ transformed quantity.

A simple example. To illustrate the effects of an O(2,2) transformation we make a simple choice for Ω, namely, $\Omega=\eta$.

Transforming Q according to Eq. (125) results in

$$G\to G^{-1}, \tag{128}$$

so that

$$R^{\mathrm{dual}}e^{2\psi^{\mathrm{dual}}}e^{\Phi^{\mathrm{dual}}}=R^{-1}e^{-2\psi}e^{-\Phi}, \tag{129}$$

$$R^{\mathrm{dual}}e^{-2\psi^{\mathrm{dual}}}e^{\Phi^{\mathrm{dual}}}=R^{-1}e^{2\psi}e^{-\Phi}, \tag{130}$$

which implies

$$\psi^{\mathrm{dual}}=-\psi \tag{131}$$

$$R^{\mathrm{dual}}=R^{-1}\exp(-\Phi-\Phi^{\mathrm{dual}}). \tag{132}$$

Using Eq. (127) to find the transformed dilaton gives

$$\Phi^{\mathrm{dual}}=-\Phi-2\ln R, \tag{133}$$

and hence

$$R^{\mathrm{dual}}=R. \tag{134}$$

It can be explicitly checked that Eqs. (27), (34), and (35) are invariant under changes to the dual quantities. Equations (27), (28), (34), and (35) provide the integrability conditions for Eqs. (36) and (37) which in turn determine the function χ. Since the integrability conditions are invariant under the above transformation, the equations remain integrable and by substituting the dual quantities into Eqs. (36) and (37) χ^{dual} is found to be

$$\chi^{\mathrm{dual}}=\chi-2\psi+\Phi+\ln R+C, \tag{135}$$

where C is a constant. The dual metric functions (96)–(98) are found as follows:

$$A_1^{\mathrm{dual}}=e^{C}Re^{\Phi}A_1, \tag{136}$$

$$A_2^{\mathrm{dual}}=A_3, \tag{137}$$

$$A_3^{\mathrm{dual}}=A_2. \tag{138}$$

Since $b(\tau,r)=0$ in this section, $\Phi(\tau,r)$ satisfies an equation similar to that for ψ. To find the corresponding Kasner exponents (see Sec. V A) of the dual model it is necessary to set $M=1$ and ${}^{0}\omega_1={}^{0}\phi_1$ in the equations of Sec. V A. This reduces to

$$A_1^{\mathrm{dual}}\sim\tau^{3/4+{}^{0}\Phi_1+(1/4)({}^{0}\Phi_1)^2+({}^{0}\psi_1)^2}, \tag{139}$$

$$A_2^{\mathrm{dual}}\sim\tau^{1/2-{}^{0}\psi_1}, \tag{140}$$

$$A_3^{\mathrm{dual}}\sim\tau^{1/2+{}^{0}\psi_1}. \tag{141}$$

Changing to proper time and reading off the Kasner exponents as described in Sec. V A results in

$$\sum_{i=1}^{3} p_i = 1 \tag{142}$$

$$\sum_{i=1}^{3} p_i^2 = 1 - 2\frac{1+{}^0\Phi_1+(1/4)({}^0\Phi_1)^2}{[7/4+{}^0\Phi_1+(1/4)({}^0\Phi_1)^2+({}^0\psi_1)^2]^2}. \tag{143}$$

This shows that the behaviors of the gravitational backgrounds in the original and dual model are very similar. This is expected since only A_1 really changes, while A_2 and A_3 are just interchanged. However, the evolution of the string coupling is changed significantly. In the original background the general solution for the dilaton is given by the wave packet solution in terms of Bessel and Neumann functions [cf. Eq. (40)]. In the limit $\tau \rightarrow \infty$, at constant r, the string coupling $g_s^2(\tau,r)$ diverges. However, in the dual background the string coupling evolves according to

$$(g_s^{\text{dual}})^2 = R^{-2} g_s^{-2}(\tau,r).$$

Assuming $R=\tau$, this shows that the string coupling in the dual model evolves towards the weak-coupling regime in the limit $\tau \rightarrow \infty$. Hence, picking $\Omega = \eta$ provides a simple example of the interconnection of backgrounds with strong and weak couplings via O(2,2) transformations.

Finally, it should be mentioned that the usual general-relativistic constraint on the sum of the Kasner exponents defining the quasi-Kasner behavior is recovered [cf. Eqs. (109) and (142)] since we are working in the Einstein frame. Assuming a Bianchi I background in the Einstein frame, and transforming the Kasner solutions from the Einstein to the string frame, results in a constraint on the sum of the squares of the Kasner exponents being unity. This behavior is characteristic for Kasner-like solutions in the string frame [25]. This, in a way, is more illuminating, since it reflects directly the invariance under scale factor duality which implies the (discrete) transformation of a Kasner exponent to its negative $(p_i \rightarrow -p_i)$.

VII. DISCUSSION

We have shown that it is possible to find exact inhomogeneous cosmological solutions of low-energy string cosmology. These solutions are cylindrically symmetric and represent cylindrical axionic, dilatonic, and gravitational waves propagating inhomogeneously on a flat anisotropic background. When the inhomogeneities are of small amplitude these solutions will approach the behavior of small perturbations of isotropic and homogeneous anisotropic string models. These solutions also allow us to study the evolution of the universe in two physically distinct limits: when the inhomogeneities are larger or smaller than the particle horizon. The behavior found has a simple physical interpretation. When inhomogeneities are larger than the horizon they evolve quasihomogeneously but when they enter the horizon there is time for self-interaction to occur and the inhomogeneities oscillate as waves. The axion and dilaton fields behave similar to two fluids in which the sound speed equals the speed of light and so shock waves do not form even when the nonlinearities are of large amplitude. The global structure of our solutions prevents the formation of gravitationally trapped regions and so there is no primordial black hole formation. (If the S^3 topology had been chosen, with the associated choice $R=\sin\tau$, then this would have been possible).

Solutions of varying degree of generality to (3+1)-dimensional string cosmology with dilaton and axion in a spacetime of cylindrical symmetry have been discussed. We found that, in general, the axion-dilaton system is decoupled from the gravitational background by the cylindrical symmetry. However, the solutions of Sec. III C are special in the sense that they describe a universe at large τ which contains scalar and gravitational waves that are coupled by the wavelike solutions in the axion field. The most general Charach-type solutions describe at large values of τ an anisotropic universe filled with gravitational and scalar waves caused by the dynamics of the axion and dilaton. These two regimes also allow us to find the asymptotic behavior of the universe as $\tau \rightarrow 0$ and $\tau \rightarrow \infty$. There is an initial curvature singularity where the density of the dilaton and axion fields is formally infinite (hence we venture outside of the low-energy string theory regime assumed here). The early-time behavior resembles the Kasner singularity of general relativity with spatially varying indices and is analogous to that observed on scales larger than the horizon at later times. The late-time evolution cannot straightforwardly be compared with the present universe because of the absence of fermionic fields which provide the standard matter and radiation components of the big bang model. The impact of duality upon these solutions is more subtle than in the cosmological models that have been examined previously in string theory because of the presence of inhomogeneity. This was discussed in detail in Sec. VI together with the relationships between the results in the Einstein and string frames.

In summary, we have found exact inhomogeneous and anisotropic cosmological solutions of low-energy string theory with nonzero axion and dilaton stresses. These provide a new theoretical laboratory in which to explore the ramifications of low-energy string cosmology and to use as a basis for incorporating the effects of higher-order corrections.

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APPENDIX

Using the properties of the Bessel and Neumann functions [26] the limits for $\psi(\tau,r)$ [or $\omega(\tau,r)$] and $G(\tau,r)$ are found.

The limit $n\tau \ll 1$,

$$\psi(\tau,r) \sim \alpha_1(\psi;r) + \alpha_2(\psi;r)\ln\tau,$$

where

$$\alpha_1(\psi;r)\equiv{}^0\psi_0+\sum_{n=1}^{\infty}\cos[n(r-r_n)]\left[{}^A\Psi_n+\frac{2}{\pi}{}^B\Psi_n\ln n\right],$$

$$\alpha_2(\psi;r)\equiv{}^0\psi_1+\frac{2}{\pi}\sum_{n=1}^{\infty}\cos[n(r-r_n)]{}^B\Psi_n\,.$$

The function $G(\tau,r)$, of Eq. (47), is found to approach

$$G(\psi;\tau,r)\sim\gamma_1(\psi;r)+\gamma_2(\psi;r)\ln\tau,$$

where

$$\gamma_1(\psi;r)\equiv{}^0\psi_0+{}^0\psi_1\sum_{n=1}^{\infty}\cos[n(r-r_n)][{}^A\Psi_n+{}^B\Psi_n\ln n],$$

$$\gamma_2(\psi;r)\equiv\frac{1}{2}({}^0\psi_1)^2+{}^0\psi_1\sum_{n=1}^{\infty}\cos[n(r-r_n)]{}^B\Psi_n\,.$$

The limit $n\tau\gg1$,

$$\psi(\tau,r)\sim\beta_1(\psi)+\beta_2(\psi)\ln\tau+\tau^{-1/2}h(\psi;\tau,r),$$

where

$$\beta_1(\psi)\equiv{}^0\psi_0,$$

$$\beta_2(\psi)\equiv{}^0\psi_1,$$

$$h(\psi;\tau,r)\equiv\sum_{n=1}^{\infty}\left(\frac{2}{\pi n}\right)^{1/2}\cos[n(r-r_n)]\left[{}^A\Psi_n\cos\left(n\tau-\frac{\pi}{4}\right)+{}^B\Psi_n\sin\left(n\tau-\frac{\pi}{4}\right)\right].$$

For $G(\psi;\tau,r)$, the limiting behavior is found to be

$$G(\psi;\tau,r)\sim\gamma_3(\psi)+\gamma_4(\psi)\ln\tau+\gamma_5(\psi)\tau,$$

where

$$\gamma_3(\psi)\equiv{}^0\psi_0,$$

$$\gamma_4(\psi)\equiv\frac{1}{2}({}^0\psi_1)^2,$$

$$\gamma_5(\psi)\equiv\frac{1}{2\pi}\sum_{n=1}^{\infty}n[({}^A\Psi_n)^2+({}^B\Psi_n)^2].$$

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