

Chaos in anisotropic preinflationary universes

H. P. de Oliveira*

*Universidade do Estado do Rio de Janeiro, Instituto de Física, R. São Francisco Xavier, 524,
CEP 20550-013, Rio de Janeiro, RJ, Brazil*

I. Damião Soares†

Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro, RJ, Brazil

T. J. Stuchi‡

*Universidade Federal do Rio de Janeiro, Instituto de Física, Caixa Postal 68528, CEP 21945-970, Rio de Janeiro,
RJ, Brazil*

(Received 27 December 1996)

We study the dynamics of anisotropic Bianchi type-IX models with matter and a cosmological constant. The models can be thought of as describing the role of anisotropy in the early stages of inflation, where the cosmological constant Λ plays the role of the vacuum energy of the inflaton field. The concurrence of the cosmological constant and anisotropy are sufficient to produce a chaotic dynamics in the gravitational degrees of freedom, connected to the presence of a critical point of saddle-center-type in the phase space of the system. In the neighborhood of the saddle center, the phase space presents the structure of cylinders emanating from unstable periodic orbits. The nonintegrability of the system implies that the extension of the cylinders away from this neighborhood has a complicated structure arising from their transversal crossings, resulting in a chaotic dynamics. The invariant character of chaos is guaranteed by the topology of cylinders. The model also presents a strong asymptotic de Sitter attractor but the way out from the initial singularity to the inflationary phase is completely chaotic. For a large set of initial conditions, even with very small anisotropy, the gravitational degrees of freedom oscillate a long time in the neighborhood of the saddle center before recollapsing or escaping to the de Sitter phase. These oscillations may provide a resonance mechanism for amplification of specific wavelengths of inhomogeneous fluctuations in the models. A geometrical interpretation is given for Wald's inequality in terms of invariant tori and their destruction by increasing values of the cosmological constant. [S0556-2821(97)02914-7]

PACS number(s): 98.80.Hw, 95.10.Fh, 98.80.Cq

I. INTRODUCTION

One of the cornerstones in the paradigm of inflation [1] is the presence of the cosmological constant, arising as the vacuum energy of the inflaton field. The cosmological constant plays a fundamental role in the gravitational dynamics of the inflationary model, by inducing an exponential expansion of the scales of the model toward the de Sitter configuration. This asymptotic approach to the de Sitter solution constitutes the basis of the so-called cosmic no-hair conjecture. In the realm of homogeneous cosmologies, Wald [2] showed that all initially expanding Bianchi cosmological models with a positive cosmological constant, except Bianchi type IX, evolve towards the de Sitter configuration. Bianchi type-IX models demand further that the absolute value of the cosmological constant be sufficiently large compared with spatial curvature terms. For the more general case of inhomogeneous models, Starobinskii [3] showed that they do inflate if a positive cosmological constant is present. This crucial aspect of the cosmological constant has been sufficiently emphasized in the literature, and many authors have

examined its role in producing nontrivial dynamics in the early stages of inflaton. In particular, Calzetta and El Hasi [4], and Cornish and Levin [5] exhibited chaotic behavior in the dynamics of Friedmann-Robertson-Walker models with a cosmological constant term and scalar fields conformally and/or minimally coupled to the curvature. Due to this feature of the dynamics, small fluctuations in initial conditions of the model preclude or induce the Universe to inflate. The Hamiltonian dynamical system originating from the field equations is complex. In Ref. [4], the effective degrees of freedom of the model are the scale factor and a conformally coupled scalar field. Such a scalar field is interpreted as a radiation field, which is later assumed to gain mass by the presence of the inflaton field. In Cornish and Levin's work, the degrees of freedom are the scale factor and two minimally and/or conformally coupled scalar fields. In both cases, the cosmological constant is responsible for the existence of a critical point S in the finite region of the phase space of the model. S is a pure saddle point [8] and the separatrices emanating from S are wholly contained in an invariant plane of the dynamical system, corresponding to the gravitational degree of freedom only. The separatrices connect S to other critical points, and are denoted heteroclinic [10]. These connections are known to be highly unstable, and their breaking, due to the perturbations, come from the coupling of the gravitational variable with the scalar

*Electronic address: OLIVEIRA@SYMBCOMP.UERJ.BR

†Electronic address: IVANO@LCA1.DRP.CBPF.BR

‡Electronic address: TSTUCHI@IF.UFRJ.BR

fields, are basically responsible for the chaotic dynamics referred to above. This chaotic behavior is a consequence of the so-called Poincaré homoclinic phenomena in dynamical systems [11].

There is, however, another important feature in the preinflationary dynamics, arising from the presence of a positive cosmological constant, whenever anisotropy is also present even in the form of small perturbations. This point has not been emphasized yet in the literature of inflation and will be object of our paper. The degrees of freedom of the system are taken basically in the gravitational sector and, for simplicity, we restrict ourselves to two distinct scale factors. The conjunction of the cosmological constant and anisotropy in Bianchi IX models with perfect fluid implies the existence of a critical point E in phase space, identified as a saddle center [8]. As a consequence, we have a wealthy dynamics based on structures as homoclinic cylinders which emanate from unstable periodic orbits that exist in a neighborhood of E . Analogous to the breaking and crossing of heteroclinic curves, for instance, these cylinders will cross each other in nonintegrable cases producing a chaotic dynamics. As we shall discuss, these structures will constitute an invariant characterization of chaos [12] in the models. We will also discuss their implications for the occurrence or not of inflation, as well as for the physics in the early stages of inflation. The paper is organized as follows. In Sec. II we establish the Hamiltonian and the basic characteristics of the model. Section III is devoted to present the cylindrical topology near the saddle center which will be very useful to understand the dynamics of orbits on the phase space. The numerical evidence for a physically relevant chaotic behavior is showed in Sec. IV, whereas in Sec. V an interesting connection between the break up of torus and the Wald's analysis is discussed. Finally, in Sec. VI, we conclude and trace some perspectives of the present work.

II. THE DYNAMICS OF THE MODEL

We consider anisotropic Bianchi type-IX cosmological models characterized by two scale functions $A(t)$ and $B(t)$ with the line element

$$ds^2 = dt^2 - A^2(t)(w^1)^2 - B^2(t)[(w^2)^2 + (w^3)^2]. \quad (1)$$

Here t is the cosmological time and (w^1, w^2, w^3) are invariant one-forms for Bianchi type-IX models [13]. The matter content is assumed to be a perfect fluid with velocity field δ_0^μ in the comoving coordinate system used plus a cosmological constant Λ . The cosmological constant term is interpreted as arising from the vacuum energy of the inflaton field such that our models may provide a description for a preinflationary anisotropic stage of the Universe. The energy-momentum tensor of the fluid is described by

$$T^{\mu\nu} = (\rho + p)\delta_0^\mu \delta_0^\nu - p g^{\mu\nu}, \quad (2)$$

where ρ and p are the energy density and pressure, respectively. We assume the equation of state $p = \gamma\rho$, $0 \leq \gamma \leq 1$. For sake of simplicity, we restrict ourselves to the case of dust ($\gamma = 0$). Distinct features arising in the cases of $\gamma \neq 0$ will be discussed in the conclusions.

Einstein's field equations¹ [14]

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = T^{\mu\nu} \quad (3)$$

for Eqs. (1) and (2) can be obtained from the Hamiltonian constraint

$$\begin{aligned} H(A, B, P_A, P_B) &= \frac{P_A P_B}{4B} - \frac{A P_A^2}{8B^2} + 2A - \frac{A^3}{2B^2} - 2\Lambda A B^2 - E_0 \\ &= 0, \end{aligned} \quad (4)$$

where P_A and P_B are the momenta canonically conjugated to A and B , respectively, and E_0 is a constant proportional to the total energy of the models. It also occurs as the first integral of the Bianchi identities, $2\rho A B^2 = E_0$. The full dynamics is governed by the Hamilton's equations

$$\begin{aligned} \dot{A} &= \frac{\partial H}{\partial P_A} = \frac{P_B}{4B} - \frac{A P_A}{4B^2}, \\ \dot{B} &= \frac{\partial H}{\partial P_B} = \frac{P_A}{4B}, \end{aligned} \quad (5)$$

$$\dot{P}_A = -\frac{\partial H}{\partial A} = \frac{P_A^2}{8B^2} - 2 + \frac{3A^2}{2B^2} + 2\Lambda B^2,$$

$$\dot{P}_B = -\frac{\partial H}{\partial B} = \frac{P_A P_B}{4B^2} - \frac{A P_A^2}{4B^3} - \frac{A^3}{B^3} + 4\Lambda A B.$$

The dynamical system (5) has one critical point E in the finite region of the phase space whose coordinates are

$$E: \quad A_0 = B_0 = \frac{1}{\sqrt{4\Lambda}}, \quad P_A = P_B = 0, \quad (6)$$

with associated energy $E_0 = E_{\text{cr}} = 1/\sqrt{4\Lambda}$. This critical point represents the static Einstein universe. Linearizing Eqs. (5) about the critical point E , we can show that the constant matrix determining the linear system about E has the four eigenvalues

$$\lambda_{1,2} = \pm \frac{1}{2E_{\text{cr}}}, \quad \lambda_{3,4} = \pm \frac{i\sqrt{2}}{E_{\text{cr}}}, \quad (7)$$

which characterizes E as a saddle center. The system (5) has also a degenerate critical point [15] at $A = B = 0$, $P_A = P_B = 0$. A straightforward analysis of the infinity of the phase space under consideration shows that it has two critical points at this region, corresponding to the de Sitter solution, one acting as an attractor (stable de Sitter configuration) and the other as a repeller (unstable de Sitter configuration). The scale factors A and B approach the de Sitter attractor as $A = B \sim e^{\sqrt{\Lambda/3}t}$ and $P_B = 2P_A \sim e^{2\sqrt{\Lambda/3}t}$. One of the questions to be examined in this paper is the characterization of sets of initial conditions for which this asymptotic de Sitter attractor is attained.

¹We assume here $8\pi G = c = 1$.

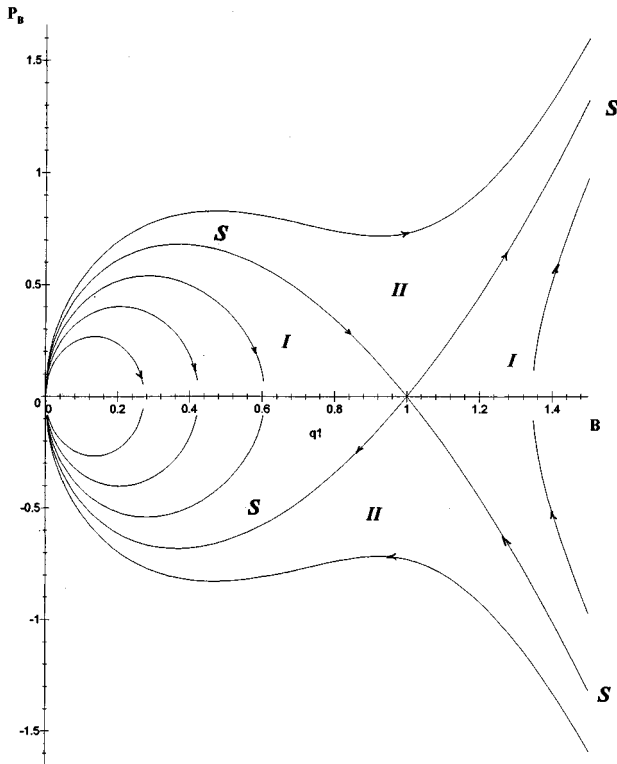


FIG. 1. Integral curves on the invariant manifold $A=B$, $P_A = P_B/2$. The orbits on regions (I) have $E_0 < 1/\sqrt{4\Lambda}$, whereas for those orbits on (II), $E_0 > 1/\sqrt{4\Lambda}$. The separatrices are characterized by the energy $E_0 = E_{cr} = 1/\sqrt{4\Lambda}$.

Another important feature of the dynamical system (5) is to admit an invariant manifold \mathcal{M} defined by

$$A=B, \quad P_A = P_B/2. \quad (8)$$

On \mathcal{M} the dynamics is governed by the two-dimensional system

$$\begin{aligned} \dot{B} &= \frac{P_B}{8B}, \\ \dot{P}_B &= \frac{P_B^2}{16B^2} - 1 + 4\Lambda B^2. \end{aligned} \quad (9)$$

The system (9) is integrable. Its phase portrait is depicted Fig. 1, where the integral curves represent homogeneous and isotropic universes with Bianchi type-IX compact spatial sections, with dust and a cosmological constant.

We remark that the critical point E belongs to \mathcal{M} . Contrary to the models examined in Refs. [4] and [5], the separatrices S are present as solutions of the full dynamical equations. As we will discuss in Secs. III and IV, the behavior of the separatrices found in the previous studies will be played by the cylinders emanating from unstable periodic orbits associated to the saddle center of E .

III. CYLINDRICAL TOPOLOGY NEAR THE CRITICAL POINT E

As a consequence of the saddle-center nature of the critical point E , the dynamics of the models described by Eq. (5)

exhibits completely new features, which will be analyzed now. Part of this analysis is based on Refs. [6–8]. The overall scenario originates from both the anisotropy of the model and the cosmological constant. To discuss the topology of the phase space in the neighborhood of E , we make use of a theorem by Moser [16] which states that it is always possible to find a set of canonical variables such that, in a small neighborhood of a saddle center at the origin, the Hamiltonian is expressed as

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= \frac{\sqrt{\Lambda}}{2} (p_2^2 - q_2^2) - \sqrt{2\Lambda} (p_1^2 + q_1^2) + O(3) \\ &\quad - E_0 + E_{cr} = 0. \end{aligned} \quad (10)$$

The critical point is located at the origin $q_1 = q_2 = p_1 = p_2 = 0$, with $E_0 = E_{cr}$. Here $O(3)$ denotes nonquadratic terms of the expansion and $\pm\sqrt{\Lambda}$, $\pm 2i\sqrt{2\Lambda}$ are the eigenvalues of the linearized system about E .

Let us now restrict ourselves to a small neighborhood of E such that we can neglect $O(3)$ and the quantity $\mathcal{E} = E_0 - E_{cr}$ is small. The Hamiltonian reduces to

$$H(q_1, q_2, p_1, p_2) \sim \frac{\sqrt{\Lambda}}{2} (p_2^2 - q_2^2) - \sqrt{2\Lambda} (p_1^2 + q_1^2) - \mathcal{E} = 0. \quad (11)$$

In this approximation, H is separable with two approximate constants of motion given by the partial energies

$$E_2 = \frac{\sqrt{\Lambda}}{2} (p_2^2 - q_2^2), \quad E_1 = \sqrt{2\Lambda} (p_1^2 + q_1^2). \quad (12)$$

The energies E_2 and E_1 will be referred [17] to as the hyperbolic motion energy and the rotational motion energy of the system about E , respectively. Note that E_1 is always positive. To describe all possible motions, the following situations must be taken into account.

If $E_2 = 0$ two possibilities arise. First, we have $p_2 = q_2 = 0$ meaning that the motions are unstable periodic orbits τ_{E_0} in the plane (p_1, q_1) . Such orbits depend continuously on the parameter $E_1 \sim -\mathcal{E}$ [cf. Fig. 2(a)]. The second possibility will be $p_2 = \pm q_2$, which defines the linear stable V_s and unstable V_u one-dimensional manifolds of Fig. 2(b). These manifolds are tangent at E to the separatrices S of the invariant manifold (described as $q_1 = 0, p_1 = 0$ in the new variables) of Fig. 1. The separatrices are actually the nonlinear extension of V_u and V_s . The direct product of the periodic orbit τ_{E_0} with V_s and V_u generates, in the linear neighborhood of E , the structure of stable $(\tau_{E_0} \times V_s)$ and unstable cylinders $(\tau_{E_0} \times V_u)$, which coalesce into the orbit τ_{E_0} for times going to $+\infty$ or $-\infty$, respectively [cf. Figs. 2(c) and 2(d)]. The energy of any orbit on these cylinders is the same as that of the periodic orbit τ_{E_0} . It can be showed that, in the nonlinear regime (when nonquadratic terms of the Hamiltonian must be taken into account), the plane (q_1, p_1) of the rotational motion in the linear regime extends to a two-dimensional manifold, the center manifold [9], of unstable periodic orbits of the system. The intersection of the center manifold with the energy surface

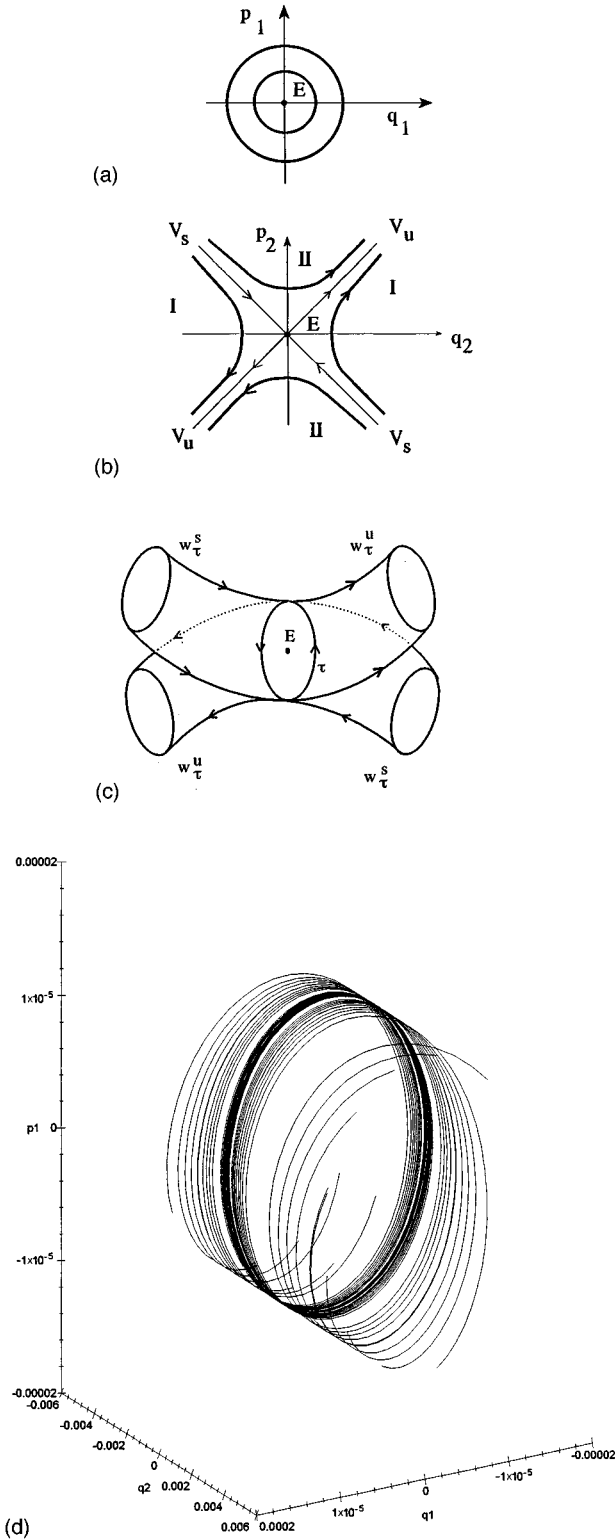


FIG. 2. (a) Periodic orbit of the Hamiltonian system (5) in the linear approximation and projected onto plane (q_1, p_1) of the normal variables. (b) The linear unstable V_u and stable V_s one-dimensional manifolds. The hyperbolae are the linearized solutions in the plane (B, P_B) of the saddle for $E_2 < 0$ (region I) and $E_2 > 0$ (region II). (c) Stable and unstable cylinder manifolds emanating from the periodic orbit τ . They are the nonlinear extension of the linearized cylinders $\tau \times V_u$ and $\tau \times V_s$ in the neighborhood of E . (d) Numerical illustration of the linear stable and unstable cylinders for $\Lambda = 0.25$ and $E_0 = 0.999\ 999\ 900$ in the neighborhood of E .

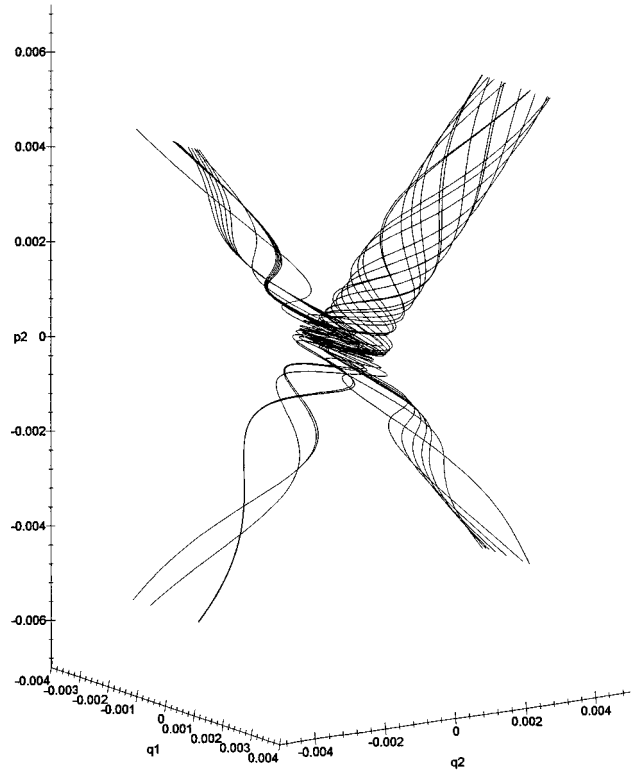


FIG. 3. General orbits with oscillatory approach to the cylinders on the neighborhood of E corresponding to $E_0 = 0.999\ 999\ 990$.

$$H(\mathcal{E}) = 0 \tag{13}$$

[cf. Eq. (11)] is a periodic orbit parametrized with E_0 , from which a pair of cylinders emanates. We can see easily from Eq. (11) that the intersection of the central manifold with the energy surface $\mathcal{E} = 0$ is just the point E . For $\mathcal{E} > 0$ the energy surface does not intersect the central manifold, the occurrence of the structure of cylinders being, therefore, restricted to the energy surfaces in which $\mathcal{E} < 0$.

For the case $E_2 \neq 0$ and $\mathcal{E} < 0$, the motion is restricted on infinite cylinders resulting from the direct product of the hyperbolae lying in the regions I and II of Fig. 2(b), with periodic orbits of the central manifold in a small neighborhood of E . A general orbit which visits the neighborhood of E belongs to the general case $E_1 \neq 0, E_2 \neq 0$. In this region the orbit has an oscillatory approach to the linear cylinders (cf. Fig. 3), the closer as $E_2 \rightarrow 0$. For instance, the outcome of this oscillatory regime will collapse if $E_2 < 0$ or escape to de Sitter attractor if $E_2 > 0$ for initial conditions taken in the quadrant $q_2 < 0, p_2 > 0$. This is for the linear regime. In general, for orbits which visit a neighborhood of E , the nonintegrability of the Hamiltonian system (5) induces that the partition of the energy \mathcal{E} into the rotational mode energy E_1 and the hyperbolic mode energy E_2 is chaotic. In other words, given a general initial condition of energy \mathcal{E} , we are no longer able to foretell which amount of \mathcal{E} goes to each mode, namely, in what of the regions I or II [Fig. 2(b)] the orbit will land when it approaches E . This manifestation of the nonintegrability will be physically meaningful to characterizing a chaotic exit to inflation, as we will show in the next section.

We must finally discuss the extension of the structure of cylinders outside the neighborhood of E . Let us consider a linearized cylinder associated to a periodic orbit with energy E_0 such that $\mathcal{E} = E_0 - E_{\text{cr}}$ is very small. One possible way to examine the extension of this cylinder is to consider the linearization of the dynamical system (5) about the separatrix S . The equations for the separatrix are denoted by

$$(a_S(t), p_S(t)), \quad (14)$$

and consider the expansion about S :

$$\begin{aligned} A &= a_S + X, \\ P_A &= p_S + Z, \\ B &= a_S + Y, \\ P_B &= 2p_S + W. \end{aligned} \quad (15)$$

To the first order in (X, Y, Z, W) , the system (5) results

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \\ \dot{W} \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}, \quad (16)$$

where $\mathcal{A}(t)$ is a 4×4 time-dependent matrix, the entries of which are simple functions of a_S and p_S . We give its expression in the Appendix, and remark that $\mathcal{A}(t)$ is bounded and Lebesgue integrable, except for $a_S \rightarrow 0, \infty$. A matrix is defined to be bounded if all its entries as well as its determinant are bounded.²

Clearly, for a small neighborhood of E , the separatrix is approximated by V_s and V_u , and the linear cylinder based on τ_{E_0} will be a solution of Eq. (16) by construction [18]. Furthermore, it follows from the system (16) that its extension will be contained in a four-dimensional small tube about the separatrix, as far as $a_S(t)$ does not go to zero. When $a_S(t) \rightarrow 0$ the above linear approximation is no longer valid. Higher-order terms become important for the dynamics, and the nonintegrability of the system results in the distortion and twisting of the cylinders. The stable cylinder and the unstable one will cross each other transversally, producing chaotic sets [6,8] and consequent ‘‘destroyed’’ regions of the Poincaré maps of the system, as showed in Sec. V. This also occurs for cylinders emanating from periodic orbits of the center manifold which are not in a small neighborhood of E (periodic orbits in the nonlinear regime).

Finally, with view to the next section, let us select a four-dimensional small sphere of initial conditions about one point S_0 of the separatrix with radius $R(S_0)$ of the order of the linear perturbation in Eq. (15), namely, $R(S_0) \sim (X^2$

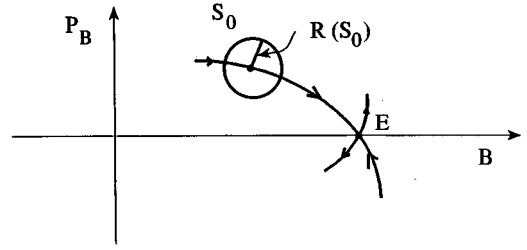


FIG. 4. Projection of the four-dimensional sphere of initial conditions about the point S_0 of the separatrix S .

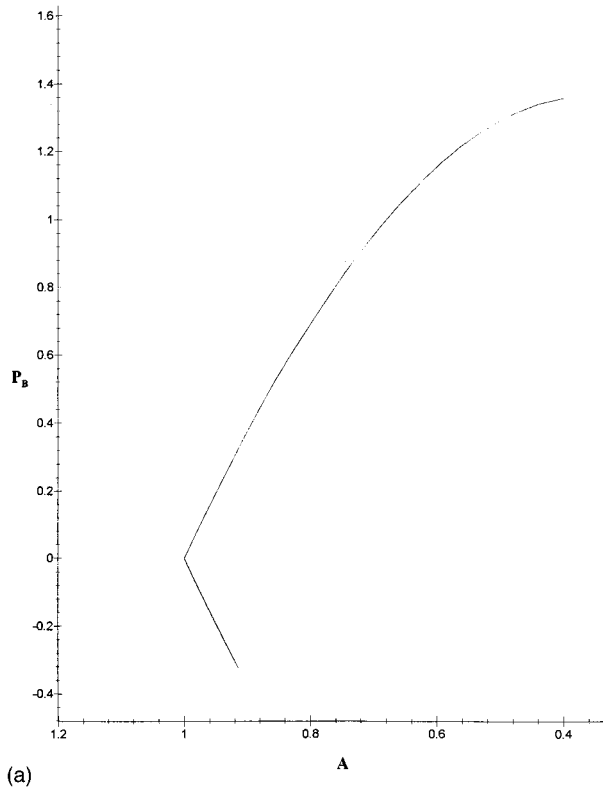
$+ Y^2 + Z^2 + W^2)^{1/2}$ (cf. Fig. 4). The energy surfaces which intersect the sphere are those with E_0 in the domain $|E_0 - E_{\text{cr}}| \leq R(S_0)$. The cylinders (associated to the periodic orbits τ_{E_0}) in these surfaces will obviously intersect the sphere. As we discussed before, if we evolve the sphere back $a_S \rightarrow 0$ the small tube spreads and twists, as well as the cylinder associated to τ_{E_0} . This stable cylinder and the unstable one will cross each other and return eventually several times to the sphere. This geometry will be the basis of the numerical experiments of the next section. By evolving dynamically the sphere towards the neighborhood of E , we show that for a band of energy, the sphere contains a chaotic set which induces a chaotic exit to inflation.

IV. CHAOTIC EXIT TO INFLATION

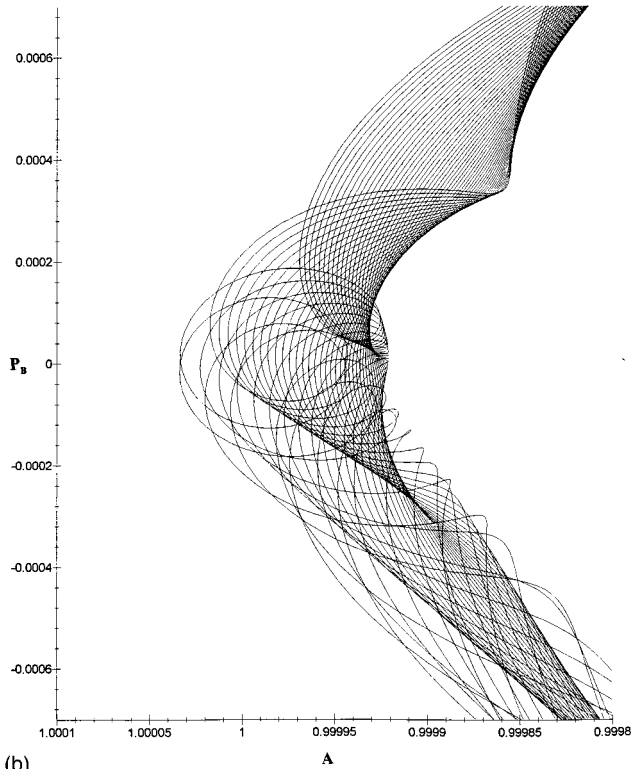
In the numerical experiments performed here, we use the variables (A, B, P_A, P_B) . All calculations were made using the package POINCARÉ [19] in an IBM compatible PC Pentium 133 with 64 MB of RAM memory, and a FORTRAN program to construct Poincaré surface of sections, where we enforce that the error of the Hamiltonian never exceeds a given threshold of 10^{-10} . Henceforth, we assume $\Lambda = 0.25$, so that the critical point E is characterized by $A = B = 1.0$, $P_A = P_B = 0$, and $E_{\text{cr}} = 1$.

The phase space under consideration is not compact, and we will actually identify a chaotic behavior associated to the possible asymptotic outcomes of the orbits in this phase space, namely, escape to de Sitter state attractor at infinity (inflationary regime) or collapse after a burst of initial expansion. To begin with our numerical experiments, we consider the invariant manifold \mathcal{M} ($A = B$, $P_A = P_B/2$). The basic characteristic of the curves in \mathcal{M} have been discussed already (cf. Fig. 1). We remark, however, that the separatrices define the regions of collapse and expansion in \mathcal{M} , but not in the full four-dimensional phase space. Following the theoretical backgrounds presented in Sec. III, we investigate numerically the behavior of orbits in a domain near the separatrices. In essence, the sets of initial conditions are constructed in the following way. Let S_0 be a point belonging to the separatrix ($E_0 = 1.0$), with coordinates $A = B = 0.4$, $P_B = 2P_A = 1.357\ 645\ 019\ 8$. We are interested, therefore, in those orbits representing expanding models after the initial singularity. Around this point, we construct a four-dimensional sphere in the phase space with arbitrary small radius R , for instance $R = 10^{-2}$, 10^{-3} , 10^{-4} , etc. (cf. Fig. 4). The values of A , B , P_A , and P_B are taken in energy surfaces which have a nonempty intersection with this sphere as

²A similar analysis can be made of the motion around other integral curves of the invariant manifold depicted in Fig. 1. In Eqs. (14) and (15) it is sufficient to substitute $a_S(t)$ and $p_S(t)$ by the solution of the integral curve $(a_I(t), p_I(t))$ associated to a given energy E_I in the invariant manifold.

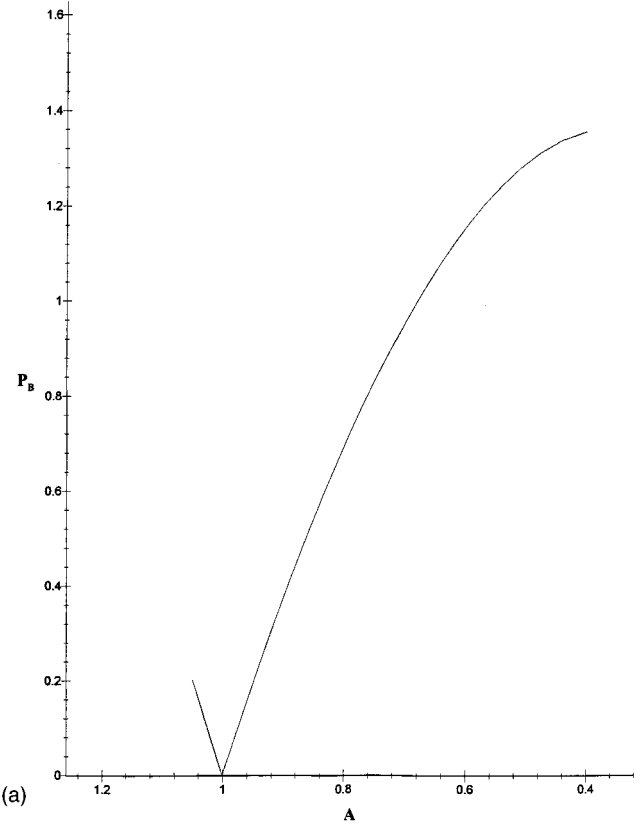


(a)

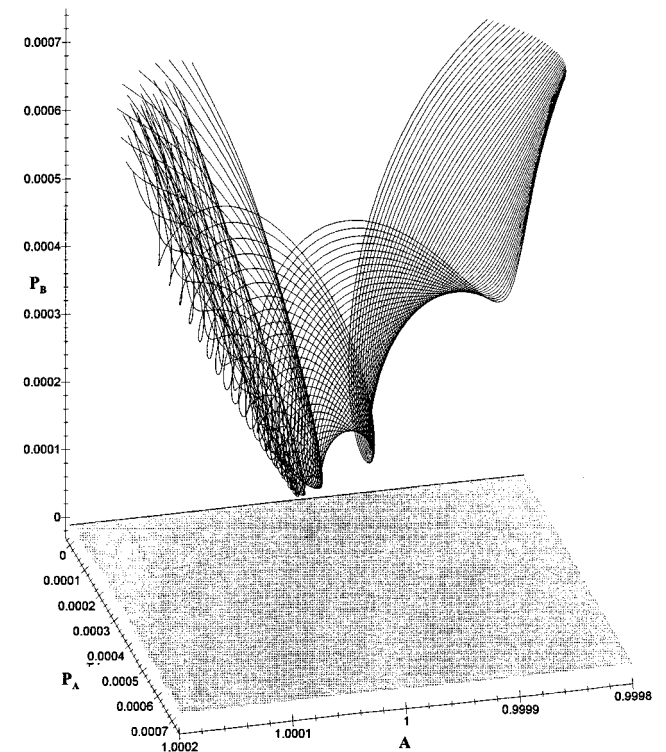


(b)

FIG. 5. Collapse of 50 orbits initially inside a sphere of initial conditions of radius $R=10^{-4}$ for $E_0=0.999\,999\,993\,7$ and $\Lambda=0.25$. (a) View of orbits projected onto the plane (A, P_A) . (b) Zoom of the region near the critical point. Note the oscillations around the separatrix as well as the critical point E .



(a)



(b)

FIG. 6. Escape of 50 orbits to the de Sitter configuration. We consider a sphere of initial conditions of radius $R=10^{-4}$ within an energy surface $E_0=0.999\,999\,999\,9$. (a) View of orbits projected onto the plane (A, P_A) . (b) Zoom of the three-dimensional region near the critical point.

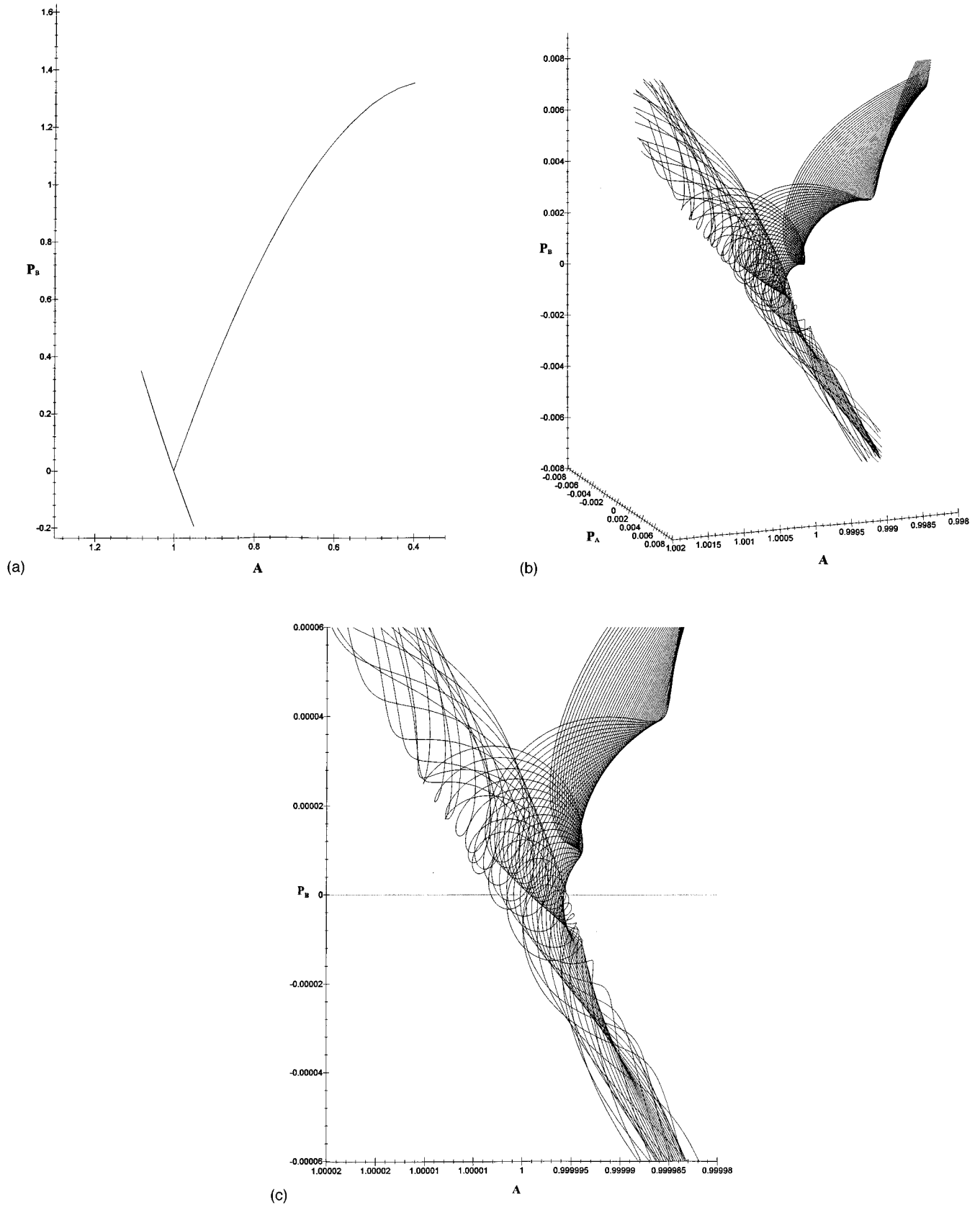


FIG. 7. Chaotic exit to inflation: (a) Outcome of 50 orbits showed in the plane (A, P_B) for $E_0 = 0.999\,999\,997\,7$ and the sphere of initial conditions with radius $R = 10^{-4}$. (b) Three-dimensional view of the region close to the critical point E for $R = 10^{-3}$. (c) View of the plane (A, P_B) for the case $R = 10^{-5}$ and energy $E_0 = 0.999\,999\,999\,9$. Note that the orbits remain in a neighborhood of E , which is of same order of the initial sphere.

TABLE I. Relation between the radius R and the gap of energy δE_0^* for the case $\gamma=0$ (dust).

R	δE_0^*
10^{-2}	0.928×10^{-4}
10^{-3}	0.618×10^{-6}
10^{-4}	0.617×10^{-8}
10^{-5}	0.619×10^{-10}
10^{-6}	0.630×10^{-12}

evaluated from the Hamiltonian constraint. As we have seen, such energy surfaces are those for which the range of energy, ΔE_0 , about $E_0 = E_{\text{cr}} = 1.0$ is of the order of, or smaller than the radius R . On physical grounds, we are considering the evolution of cosmological models with small anisotropic perturbations.

After several numerical experiments with the above sets of initial conditions, we note that, as expected, two possible outcomes arise: collapse or expansion into de Sitter configuration. The energy E_0 that varies from $1.0 - \Delta E_0$ to $1.0 + \Delta E_0$, with $\Delta E_0 \sim R$, determines the long-time behavior of the orbits. In Figs. 5 and 6, collapse and escape of 50 orbits are displayed for $R = 10^{-4}$. The orbits oscillate around the separatrix and approach the critical point E inside a sphere about this point with radius of the same order of R . In this region, the linear approximation is valid and the local separation of the dynamics into rotational motion and hyperbolic motion [cf. Eq. (10)] can be used to understand the results. The final state of the orbits depends crucially on the partition of the total energy $\mathcal{E} = E_0 - E_{\text{cr}}$ into the rotational motion mode and the hyperbolic motion mode. Hence, if $E_2 > 0$ the orbits escape, whereas collapse is characterized by $E_2 < 0$. The rotational motion (energy E_1) describes the oscillatory character of the orbits around the critical point, indicating that some orbits can spend more time around E than others.

The main result of this section is to show that, for a determined interval of energy δE^* contained in the domain $\mathcal{D} = [1.0 - \Delta E_0, 1.0 + \Delta E_0]$, this partition is chaotic. Indeed, let us try to determine in \mathcal{D} the values of energy $E_0 = E_{\text{min}}$ for which all orbits escape and $E_0 = E_{\text{max}}$ for which all orbits collapse. According to our numerical work, we find out that for each sphere of initial conditions, there always exists a non-null interval $\delta E^* = |E_{\text{max}} - E_{\text{min}}|$, where part of the orbits escapes and another part collapses, resulting in an indeterminate outcome. In Fig. 7, this behavior is showed for spheres of initial conditions of radius $R = 10^{-3}$, 10^{-4} , and 10^{-5} . An empirical relation between the gap δE^* and the radius R is obtained and showed in Table I. For spheres with R

TABLE II. Relation between the radius R and the gap of energy δE_0^* for the case $\gamma = 1/3$ (radiation).

R	δE_0^*
10^{-2}	2.874×10^{-4}
10^{-3}	2.890×10^{-6}
10^{-4}	2.855×10^{-8}
10^{-5}	2.880×10^{-10}
10^{-6}	2.830×10^{-12}

$\leq 10^{-2}$, we have $\delta E^* \propto R^2$. Indeterminate outcome due to $\delta E^* \neq 0$ occurs only for $\mathcal{E} = E_0 - E_{\text{cr}} < 0$, as expected [indeed, if $\mathcal{E} \geq 0$ all orbits escape since, from Eq. (11), $E_2 > 0$]. The orbits collapse or escape, depending on the partition of the energy \mathcal{E} into the modes E_1 and E_2 [cf. Eqs. (11) and (12)], such that the latter assumes negative or positive values, respectively. The way in which this partition works, once a set of orbits approaches E , is completely unknown for energies in within the gap δE^* . As a consequence, any infinitesimal fluctuation from a given initial condition inside the sphere can lead to an indeterminate outcome, that is, collapse or escape. This is the evidence of chaos in a physically relevant context. In other words, we may state that the boundaries of initial conditions for collapse and inflation are mixed as a consequence of the crossing of cylinders.

The presence of chaos in the system is a consequence of the crossing of stable and unstable cylinders, emanating from the unstable periodic orbits of the center manifold, as discussed before. This topological structure is actually an invariant characterization of chaos. Finally, we remark that the above behavior is not restricted to initial conditions taken in small neighborhoods of points of the separatrix. Any sets of initial conditions taken in an arbitrary neighborhood of the invariant manifold \mathcal{M} which results in orbits that visit a small neighborhood of E , display the above chaotic behavior.

In a less simple model, radiation could be taken into account as a more plausible matter field emerging after the Planck era [20] instead of dust. The effect of radiation is considered if we set the equation of state as $p = \frac{1}{3}\rho$. The saddle-center critical point and the invariant manifold are also present, and the same type of behavior occurs if we construct sets of initial conditions as before. There is also a similar relation between the gap δE^* and the radius R showed in the Table II. A more complete and detailed analysis will be subject of a forthcoming paper.

V. FURTHER NUMERICAL RESULTS AND WALD'S PHENOMENON

We consider here the evolution of completely anisotropic models taking initial conditions far from the invariant manifold \mathcal{M} . It is a remarkable fact that the Hamiltonian (4) is regular at the plane $A=0$ where the curvature is singular. Therefore, from the point of view of the Hamiltonian dynamics, orbits can be analytically extended beyond the singularity to the domain $A < 0$. Although these orbits are not physically meaningful in this domain, they are nevertheless essential in the description of the underlying geometrical structure of the full Hamiltonian system and its nonintegrability. In this context we will give a geometrical picture for Wald's inequality [2].

In Fig. 8(a), we exhibit the Poincaré map of the system for ($E_0 = 0.925\,048\,311\,6$, $\Lambda = 0.25$) with surface of section ($P_B = 0$, $\dot{P}_B > 0$). This map makes explicit the coexistence of KAM tori and destroyed regions in the phase space, as a consequence of the nonintegrability of the system. We were not able to find a torus totally contained in the domain $A > 0$. Thus, orbits on, or inside, the tori (with initial conditions taken on $A > 0$) necessarily collapse. This can be seen from the double Poincaré map with surface of section P_B

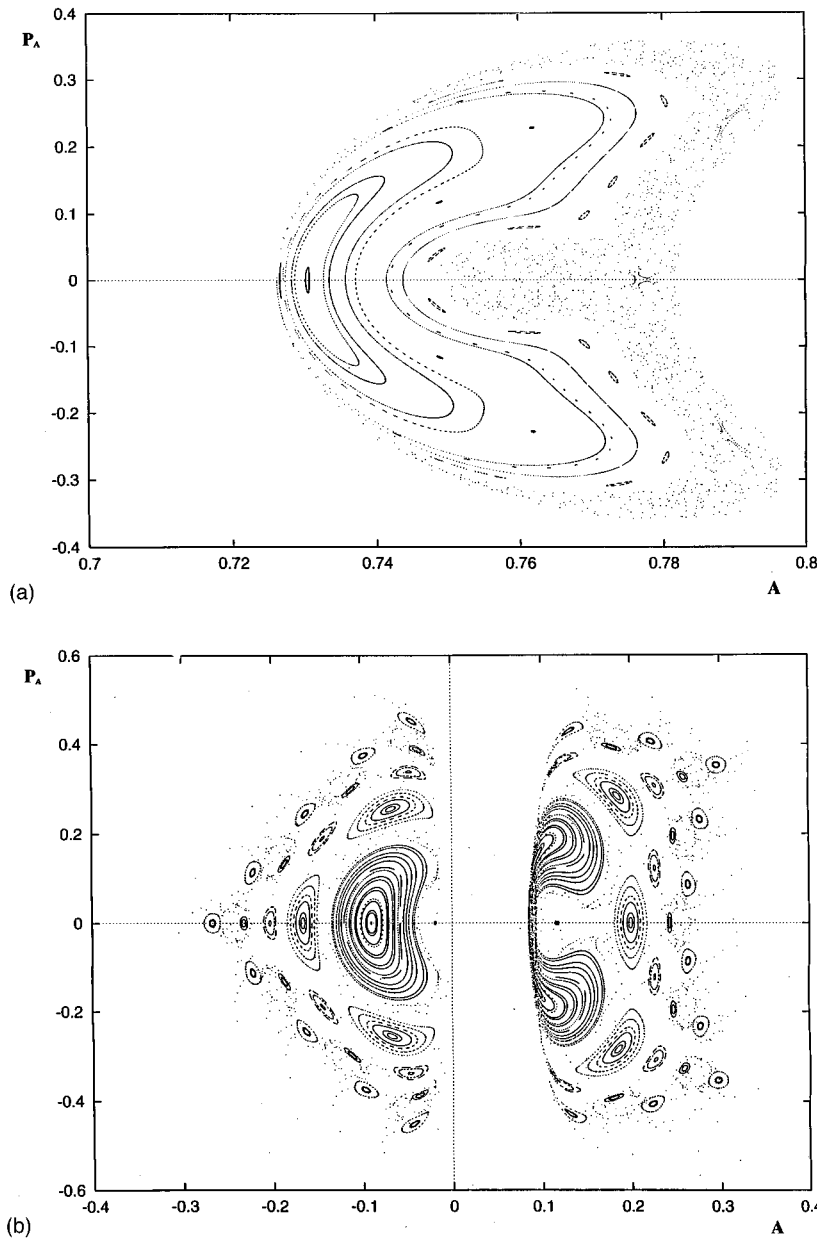


FIG. 8. (a) Poincaré surface of section $P_B=0$, $\dot{P}_B>0$ for $\Lambda=0.25$ and $E_0=0.925\ 048\ 311\ 634\ 35$. (b) Double Poincaré surface of section with the condition $\dot{P}_B>0$ relaxed with $E_0=0.15$ and $\Lambda=2.0$. It is clear the presence of destroyed regions of the phase space together with KAM tori structure. The tori cross the plane $A=0$ performing the so-called cosmic cycles.

$=0$ as showed in Fig. 8(b), where ($E_0=0.15$, $\Lambda=2.0$). On the contrary, orbits in the destroyed region are free to escape. Orbits on a torus with analytical continuation to a domain of negative A are said to perform cosmic cycles (according to Ref. [4]).

Our next step is to examine the effect of varying the value of the cosmological constant Λ on the tori structure in the same region of phase space. As can be verified numerically, the effective result of increasing Λ is the destruction of the tori. We showed this in Fig. 9, where orbits are drawn in the same region of initial conditions but with distinct values of Λ . If we change Λ from 0.25 to 0.30 (the energy changes from $E_0=0.8$, taken initially, to $E_0=0.800\ 005\ 000\ 1$), orbits initially in a torus escape to the de Sitter attractor after some cycles; if we further increase Λ , no cosmic cycles take place before escape. The

phenomenon of destruction of tori, by increasing the value of the cosmological constant, is a geometrical picture for Wald's inequality. We nevertheless remark that, for any value of the cosmological constant (however large, except infinity), there will always be collapsing orbits.

VI. FINAL REMARKS AND CONCLUSIONS

In this paper we have discussed the dynamics of Bianchi type-IX models with perfect fluid, which may provide a description of preinflationary stages of the Universe. The main ingredients of the models are a cosmological constant, arising as the vacuum of the inflaton field, and anisotropy. The degrees of freedom are taken in the gravitational sector, and

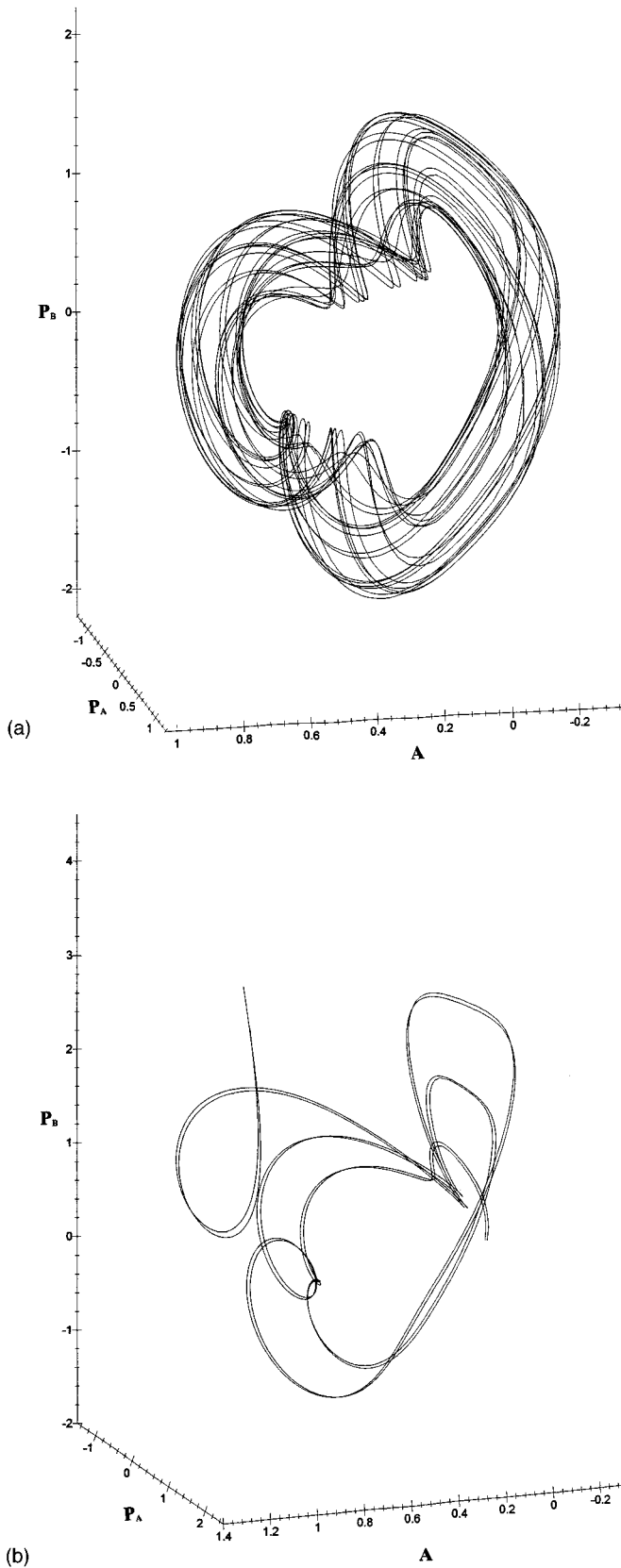


FIG. 9. (a) Two collapsing orbits inside a torus with $\Lambda=0.25$ and $E_0=0.8$. (b) The result of increasing the cosmological constant to $\Lambda=0.30$, keeping the same region of the phase space and assuming $E_0=0.800\,005\,000\,1$, is the destruction of the torus: after some cycles, both orbits escape to the de Sitter configuration.

for simplicity we restrict ourselves to two distinct scale factors only. The presence of anisotropy, even in the form of small perturbations, together with the cosmological constant, produces in the phase space of the system a critical point identified as a saddle center. Associated to the saddle center we have a two-dimensional manifold of unstable periodic orbits, and the structure of infinite unstable cylinders emanating from them. The stable and unstable cylinders coalesce into the corresponding periodic orbit for time going to ∞ and $-\infty$, respectively. The nonintegrability of the system results in the distortion and twisting of the cylinders, and the eventual intersection of one (stable) with the other (unstable) with a consequent chaotic behavior of the dynamics of the phase space. This behavior of the cylinders is analogous to the one played by the separatrices (heteroclinic curves connecting saddle points) of the gravitational sector in the models discussed in [4,5]. In these models the gravitational variable dynamics couples with the scalar fields giving rise in the heteroclinic breaking and tangle which is the origin of chaos in these models.

The results of Sec. IV, where we describe chaotic exit to inflation, extends the result of Cornish and Levin [5] for the case of two gravitational degrees of freedom. The breaking of the boundary between initial condition domains of collapse and escape to inflation, showed in [5], is indeed meaningful for the case of one gravitational degree of freedom only (the unperturbed separatrix is in fact a sharp boundary between the domains of initial conditions). In our case, the separatrices are present in the full dynamics and define sharply regions of collapse and inflation in the invariant manifold, but not in the full four-dimensional phase space of the gravitational dynamics. However, for each small sphere of initial conditions taken about one point of the separatrix, it is always possible to find a small domain (or gap) of energy such that the intersection of the sphere with an energy surface associated to the above domain is a chaotic set in the sense discussed in Sec. IV. Namely, a small perturbation in initial conditions taken in this set would change an orbit from collapse to escape to the de Sitter phase. Furthermore, we conjecture that it is feasible to select initial conditions in this set such that the scale factors oscillate an arbitrary fixed time T about E ($T=\infty$ included) before collapsing or escaping to a de Sitter phase. Putting in other words, chaos is established by the uncertainty in the partition of the energy E_0 into the rotational motion energy and hyperbolic motion energy about E .

The oscillations of the scale factors about E may have an important physical effect concerning inhomogeneous perturbations. Let them be scalar field perturbations and/or matter density perturbations in this gravitational background. In fact, the time-dependent amplitude of each Fourier component of the perturbation will satisfy a linear differential equation, the coefficients of which will be oscillatory functions about E . As we have seen, the latter may be approximated by a periodic function describing τ_{E_0} . Therefore, by a resonance mechanism, there will occur amplification of the particular Fourier components having period approximately equal to the period τ_{E_0} (we obviously assume here that no dissipation effects are included in the equations of motion of the perturbation in this stage). Even if the Universe inflates afterwards, the relative ratio of amplitudes produced after

this mechanism of amplification will be nevertheless maintained.

Finally, as discussed in Sec. IV, we have given a geometric interpretation for Wald's result in terms of invariant tori and their destruction by increasing the cosmological constant. For a given value of the cosmological constant, the phase space presents the structure of KAM tori, and orbits on, or inside, these tori collapse necessarily. However, if we increase the value of the cosmological constant, the tori are

destroyed and orbits previously on, or inside, them may eventually escape to de Sitter configuration.

ACKNOWLEDGMENT

The authors are grateful to CNPq for financial support.

APPENDIX

The matrix $\mathcal{A}(t)$ is given by

$$\mathcal{A} = \begin{pmatrix} -\frac{p_S}{4a_S^2} & 0 & -\frac{1}{4a_S} & \frac{1}{4a_S} \\ 0 & -\frac{p_S}{4a_S^2} & \frac{1}{4a_S} & 0 \\ \frac{3}{a_S} & -\left(\frac{3}{a_S} + \frac{p_S^2}{4a_S^3} + 4\Lambda a_S\right) & \frac{p_S}{4a_S^2} & 0 \\ -\left(\frac{3}{a_S} + \frac{p_S^2}{4a_S^3} + 4\Lambda a_S\right) & \left(\frac{3}{a_S} + \frac{p_S^2}{4a_S^3} - 4\Lambda a_S\right) & 0 & \frac{p_S}{4a_S^2} \end{pmatrix}.$$

-
- [1] Edward W. Kolb and Michael S. Turner, *The Early Universe*, Frontiers of Physics (Addison-Wesley, Reading, MA, 1990).
- [2] R. M. Wald, Phys. Rev. D **28**, 2118 (1983).
- [3] A. A. Starobinskii, JETP Lett. **37**, 66 (1983).
- [4] E. Calzetta and E. El Hasi, Class. Quantum Grav. **10**, 1825 (1993); Phys. Rev. D **51**, 2713 (1995).
- [5] N. J. Cornish and J. J. Levin, Phys. Rev. D **53**, 3022 (1996).
- [6] A. M. Osório de Almeida, N. De Leon, M. A. Metha, and C. C. Marston, Physica D **46**, 265 (1990).
- [7] W. M. Vieira, Ph.D. thesis, Universidade Estadual de Campinas, São Paulo, 1994.
- [8] S. Wiggins, *Global Bifurcations and Chaos* (Springer-Verlag, Berlin, 1988).
- [9] J. Llibre, R. Martinez, and C. Simó, J. Diff. Eqns. **58**, 104 (1985); C. Simó and T. J. Stuchi (unpublished).
- [10] M. V. Berry, in *Regular and Irregular Motion*, edited by S. Jorna, AIP Conf. Proc. No. 46 (AIP, New York, 1978).
- [11] J. Guckenheimer and P. Holmes, *Dynamical Systems and Bifurcations of Vector Fields*, Appl. Math. Sciences Vol. 42 (Springer-Verlag, New York, 1983); J. Koiller, J. R. T. M. Neto, and I. D. Soares, Phys. Lett. **110A**, 260 (1985).
- [12] G. Francisco and G. E. Matsas, Gen. Relativ. Gravit. **20**, 1047 (1988); A. Burd and R. Tavakol, Phys. Rev. D **47**, 5336 (1993).
- [13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [14] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).
- [15] O. I. Bogoyavlensky, *Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics* (Springer-Verlag, Berlin, 1985).
- [16] M. A. Moser, Commun. Pure Appl. Math. **11**, 257 (1958).
- [17] A. Mielke, P. Holmes, and O. O'Reilly, J. Diff. Eqns. **4**, 95 (1992).
- [18] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (MacGraw-Hill, New York, 1955).
- [19] E. S. Ceb-Terrab and H. P. de Oliveira, Comput. Phys. Commun. **95**, 171 (1996).
- [20] J. Halliwell and S. Hawking, Phys. Rev. D **31**, 1777 (1985).