Techniques of amplitude analysis for two-pseudoscalar systems

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The analytical tools needed for an amplitude analysis of two-pseudoscalar systems are described in some detail. Analyses involving two identical spinless particles require a new type of polynomial; the general form of such a polynomial is given for the first time. [S0556-2821(97)02721-5]

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I. INTRODUCTION

This paper describes in some detail the formalism necessary for exploring partial-wave amplitudes in a system involving two spinless particles. Although the results of this paper can be applied to a wide class of production of resonances, emphasis is given to a peripheral production of states in exclusive reactions initiated by π or K beams. As such, it is addressed to experimentalists engaged in searches for new states in the classic channels for resonance production [1–8]. The main purpose to this paper is to present a general method of dealing with the ambiguity problem endemic to this type of system. This paper follows closely the notation and conventions spelled out previously by Chung [9] and Chung and Trueman [10].

The most familiar example of a system consisting of two pseudoscalar particles is that of $\pi\pi$, for which one must have $I+\ell=$even, where I is the isotopic spin and ℓ is the spin—equal to the orbital angular momentum in this case. If one restricts oneself to the states with $\ell<3$, then only a P wave is allowed for $I=1$, whereas both S and D waves are possible if $I=0$. Similarly, both S and D waves are allowed for an $\eta\eta$ system since $I=0$ in this case. But for a system with two dissimilar spinless particles, e.g., $\pi\eta$ or $\eta\eta'$, all three possible states S, P, and D should be present. It is for this reason that the $\pi\eta$ system has been chosen as an example in this paper.

If a system contains two identical spinless particles, then only even waves are allowed because of Bose symmetrization. A classic example of such a system would be the $I=0$ $\pi^0\pi^0$ system. Analysis of $\pi\eta$ and $\pi^0\pi^0$ systems requires very different techniques, and this paper shows how to treat such a system containing a set of partial waves with $0\leqslant\ell\leqslant\ell_m$, where ℓ_m is an arbitrary maximum integer spin in the set. In particular, the general treatment of a $\pi^0\pi^0$ system requires the introduction of a new type of polynomial in a single variable $v=2\cot\theta$, where θ is the scattering angle of the $\pi^0\pi^0$ system in its rest frame. To the best of this author's knowledge, this polynomial has never been encountered in physics so far, and one of the main objectives of this paper is to give a general form of the polynomial (see derivation of the ε function in Appendix B).

In Secs. II and III are given the angular distributions and the amplitudes in the reflectivity basis. Section IV is devoted to the ambiguities in the partial waves for the $\pi\eta$ system. The method detailed here for searching for ambiguities in the amplitudes follows and expands on that given by Sadovsky [11]. Treatment of the $\pi^0\pi^0$ system, or a system consisting of two identical particles in general, is given in Sec. V. In Secs. VI–IX are given several examples of practical importance, i.e., the systems consisting of $\pi\eta$ or $\pi^0\pi^0$ with partial waves up to $\ell=4$.

The angular distribution is specified uniquely, once a set of moments $\{H\}$ is given. For measurements of these moments from experimental data, the technique of the extended maximum-likelihood analysis [12] is often used, and a brief introduction to this topic is given in Sec. X. Also given in this section is a treatment of the method of finding a set of "true" moments, given the experimental moments measured in the face of the finite acceptance of the apparatus. The d functions, as well as the e functions derived from them, are given explicitly for ℓ up to 4 in Appendix A. Finally, the polynomials ε, needed for treating $\pi^0\pi^0$ systems, are worked out in Appendix B.

It should be emphasized that the amplitude analysis on two-body spinless particles necessarily entails simplifying assumptions. These assumptions are not needed when a partial-wave analysis is carried out on three- and four-body final states. One fundamental reason for this is that the dimension of the decay space (which includes all the independent variables consisting of appropriate momenta, energies, and angles) expands from two for two-body to five and eight for three- and four-body systems. A thorough spin-parity analysis of a resonance, therefore, must include—where possible—a study of its three- and four-body decay modes, but the formalism needed for such an analysis is very different from that outlined in this paper [13,14]. Another independent check of a state decaying into two spinless particles would be to study its production in a multiparticle final state from a known initial system, e.g., $p\bar{p}$ annihilations at rest. In this case, interference effects in the multiparticle final state allow for a relaxation of the simplifying assumptions. In particular, the ambiguity problem, the main focus of this paper, can be avoided.

II. GENERAL ANGULAR DISTRIBUTIONS

Consider the following reaction:

$$\pi^- p\to\pi^0\eta n. \tag{1}$$

In the Jackson frame[1] the amplitudes may be expanded in terms of the partial waves for the $\pi\eta$ system:

$$U_k(\Omega)=\sum_{\ell m} V_{\ell m k} A_{\ell m}(\Omega), \tag{2}$$

where $V_{\ell m k}$ stands for the production amplitude for a state $|\ell m\rangle$ and k represents the spin degrees of freedom for the initial and final nucleons ($k=1,2$ for spin-flip and spin-nonflip amplitudes). $A_{\ell m}(\Omega)$ is the decay amplitude given by

$$A_{\ell m}(\Omega)=\sqrt{\frac{2\ell+1}{4\pi}}\,D^{\ell *}_{m0}(\phi,\theta,0)=Y^m_\ell(\Omega), \tag{3}$$

where the angles $\Omega=(\theta,\phi)$ describe the direction of the η in the Jackson frame. It is noted, in passing, that the small d function implicit in Eq. (3) is related to the associated Legendre polynomial via

$$d^\ell_{m0}(\theta)=(-)^m\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\,P^m_\ell(\cos\theta). \tag{4}$$

The angular distribution is defined by

$$I(\Omega)=\sum_k |U_k(\Omega)|^2. \tag{5}$$

It should be emphasized that the nucleon helicities are external entities, and the summation on k is therefore applied to the absolute square of the amplitudes. A complete study of the $\pi\eta$ system requires four variables: $m(\pi\eta)$, $-t$, and the two angles in Ω. The distribution (5) is therefore to be applied to a given bin of $m(\pi\eta)$ and of $-t$.

The angular distribution may be expanded in terms of the moments $H(LM)$ via

$$I(\Omega)=\sum_{LM}\left(\frac{2L+1}{4}\right)H(LM)D^{L*}_{M0}(\phi,\theta,0), \tag{6}$$

with

$$H(LM)=\sum_{\substack{\ell m\\ \ell' m'}}\left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2}\rho^{\ell\ell'}_{mm'}(\ell' m' LM|\ell m)\times(\ell' 0 L 0|\ell 0), \tag{7}$$

where ρ is the spin-density matrix given by

$$\rho^{\ell\ell'}_{mm'}=\sum_k V_{\ell m k}V^*_{\ell' m' k}. \tag{8}$$

It is seen that the moments $H(LM)$ are measurable quantities since

$$H(LM)=\int d\Omega\, I(\Omega) D^L_{M0}(\phi,\theta,0). \tag{9}$$

The normalization integral is

$$H(00)=\int d\Omega\, I(\Omega). \tag{10}$$

The symmetry relations for H's are well known. From the hermiticity of ρ, one gets

$$H^*(LM)=(-)^M H(L-M), \tag{11}$$

and from parity conservation [9,10] in the production process, one finds

$$H(LM)=(-)^M H(L-M). \tag{12}$$

These show that H's are real. The angular distribution can now be recast into

$$I(\Omega)=\sum_{LM}\left(\frac{2L+1}{4\pi}\right)\tau(M)H(LM)d^L_{M0}(\theta)\cos M\phi, \tag{13}$$

where

$$\begin{aligned}\tau(M)&=2\quad(M>0)\\ &=1\quad(M=0)\\ &=0\quad(M<0).\end{aligned} \tag{14}$$

Note that all the terms of Eq. (13) are now real. Since the D functions form a complete orthonormal set in the space $\Omega=(\theta,\phi)$, one merely needs to specify a set of the H's to uniquely define an angular distribution.

Let ℓ_m be the maximum spin present in a given $\pi\eta$ mass bin. It is easy to show that the number of independent H's are

$$N_0=(\ell_m+1)(2\ell_m+1). \tag{15}$$

N_0 as a function of ℓ_m is given below as a table:

ℓ_m	0	1	2	3	4
N_0	1	6	15	28	45

One makes a crucial assumption for amplitude analyses—the z component m of spin ℓ can take on the values 0 or 1 only: i.e., the production amplitudes V are zero[2] if $m>1$. This implies that the H's are zero if $M>2$. For $\ell_m>0$ the number of zero H's is

$$N_1=(\ell_m-1)(2\ell_m-1), \tag{16}$$

so that the number of nonzero moments is

[1]The z axis is defined to be along the beam direction in the $\pi\eta$ rest frame, whereas the y axis is chosen to be along the production normal in the overall center-of-mass frame.

[2]In a peripheral production of meson resonances from πp and Kp quasi-two-body processes, one does not expect the amplitudes with $m>1$ to be important, because the initial and final baryon helicities can impart at most $m=1$ amplitudes to the meson resonance.

$$N_2 = N_0 - N_1 = 6\ell_m, \tag{17}$$

valid for $\ell_m > 0$. Now the number of nonzero H's is linear in ℓ_m. This allows one to determine amplitudes, as shown in the next section.

N_2 as a function of ℓ_m is tabulated below:

ℓ_m	0	1	2	3	4
N_2	1	6	12	18	24

III. AMPLITUDE ANALYSIS

The parity conservation in the two-body→two-body production process can be treated by introducing reflection operators [10] which preserve all the relevant momenta in the S matrix and act directly on the rest states of the particles involved. The coordinate system is always defined with the y axis along the production normal, so that the reflection operator is simply the parity operator followed by a rotation by π around the y axis.

The eigenstates of this reflection operator are

$$|\epsilon\ell m\rangle = \theta(m)\{|\ell m\rangle - \epsilon(-)^m|\ell -m\rangle\}, \tag{18}$$

where

$$\theta(m) = \frac{1}{\sqrt{2}} \quad (m>0)$$
$$= \frac{1}{2} \quad (m=0)$$
$$= 0 \quad (m<0). \tag{19}$$

One sees that $\tau(m) = 4\theta^2(m)$; see Eq. (14).

For a positive reflectivity, the $m=0$ states are not allowed: i.e.,

$$|\epsilon\ell 0\rangle = 0 \quad \text{if} \quad \epsilon = +. \tag{20}$$

The reflectivity quantum number ϵ has been defined so that it coincides with the naturality of the exchanged particle in reaction (1). One can prove this by noting that the meson production vertex is in reality a time-reversed process in which a state of arbitrary spin parity J^{η_J} decays into a pion (the beam) and a particle of a given naturality (the exchanged particle):

$$J^{\eta_J} \to s^{\eta_s} + \pi, \tag{21}$$

where η's stand for intrinsic parities. The helicity-coupling amplitude F^J for this decay [9] is

$$A_p^J(M) \propto F_\lambda^J D_{M\lambda}^{J*}(\phi_p, \theta_p, 0), \tag{22}$$

where λ is the helicity of the exchanged particle and the subscript p stands for the ''production'' variables. M is the z component of spin J in the rest frame. From the parity conservation in the decay, one finds

$$F_\lambda^J = -F_{-\lambda}^J, \tag{23}$$

where one has used the relationships $\eta_J = (-)^J$ (true for two-pseudoscalar systems) and $\eta_s = (-)^s$ (natural-parity exchange). This formula shows that the helicity-coupling amplitude F^J is zero if λ is zero. Since angular momentum is conserved, its decay into two spinless particles cannot have $M=0$ along the beam direction (the Jackson rest system); i.e., the D^J function is zero unless $M=\lambda$, if $\theta_p = \phi_p = 0$. Finally, one may identify J with ℓ and M with m, which proves Eq. (20).

The modified D functions in the reflectivity basis are given by

$${}^{\epsilon}D_{m0}^{\ell *}(\phi,\theta,0) = \theta(m)[D_{m0}^{\ell *}(\phi,\theta,0) - \epsilon(-)^m D_{-m0}^{\ell *}(\phi,\theta,0)]. \tag{24}$$

It is seen that they are real if $\epsilon = -1$ and imaginary if $\epsilon = +1$:

$${}^{(-)}D_{m0}^{\ell *}(\phi,\theta,0) = 2\theta(m) d_{m0}^{\ell}(\theta)\cos m\phi,$$
$${}^{(+)}D_{m0}^{\ell *}(\phi,\theta,0) = 2i\theta(m) d_{m0}^{\ell}(\theta)\sin m\phi. \tag{25}$$

The overall amplitude in the reflectivity basis is now

$${}^{\epsilon}U_k(\Omega) = \sum_{\ell m} {}^{\epsilon}V_{\ell mk}\, {}^{\epsilon}A_{\ell m}(\Omega), \tag{26}$$

where

$${}^{\epsilon}A_{\ell m}(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}}\, {}^{\epsilon}D_{m0}^{\ell *}(\phi,\theta,0) \tag{27}$$

and the resulting angular distribution is

$$I(\Omega) = \sum_{\epsilon k} |{}^{\epsilon}U_k(\Omega)|^2. \tag{28}$$

It is seen that the sum involves four noninterfering terms for $\epsilon = \pm$ and $k=1,2$. The absence of the interfering terms of different reflectivities is a direct consequence of parity conservation in the production process.

The moments $H(LM)$ can be expressed in terms of the amplitudes in the reflectivity basis. From the definition (9), one finds

$$H(LM) = \sum_{\epsilon} \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} {}^{\epsilon}\rho_{mm'}^{\ell\ell'}\, {}^{\epsilon}b(\ell' m' LM \ell m) \times (\ell' 0 L 0|\ell 0), \tag{29}$$

where ${}^{\epsilon}\rho$ is the spin-density matrix in the reflectivity basis given by

$${}^{\epsilon}\rho_{mm'}^{\ell\ell'} = \sum_k {}^{\epsilon}V_{\ell mk}\, {}^{\epsilon}V_{\ell' m' k}^{*} \tag{30}$$

and a new function ${}^{\epsilon}b$ is a sum of Clebsch-Gordan coefficients:

$$
\begin{aligned}
{}^{\epsilon}b(\ell' m' LM \ell m) = \theta(m')\,\theta(m)[&(\ell' m' LM|\ell m)\\
&+(-)^M(\ell' m' L-M|\ell m)\\
&-\epsilon(-)^{m'}(\ell' -m' LM|\ell m)\\
&-\epsilon(-)^{m}(\ell' m' LM|\ell -m)].
\end{aligned}
\tag{31}
$$

Formula (29) is essential in checking that a given set of the partial waves satisfies the experimentally measured moments $H(LM)$.

At this point one makes the second crucial assumption—necessary for carrying out the amplitude analysis: the production amplitudes ${}^{\epsilon}V$ do not depend on k. The distribution function is then given by a sum of two terms

$$
I(\Omega) = |{}^{(+)}U(\Omega)|^2 + |{}^{(-)}U(\Omega)|^2. \tag{32}
$$

The first assumption demands that the production amplitudes ${}^{\epsilon}V$ should be zero if $m>1$. It is therefore convenient to separate out the θ dependence from that of ϕ, as follows:

$$
\begin{aligned}
{}^{(-)}U(\Omega) &= \frac{1}{\sqrt{4\pi}}[h_0(\theta)+\sqrt{2}h_-(\theta)\cos\phi],\\
{}^{(+)}U(\Omega) &= \frac{1}{\sqrt{4\pi}}[\sqrt{2}h_+(\theta)\sin\phi],
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
h_0(\theta) &= \sum_{\ell=0}^{\ell_m}\sqrt{2\ell+1}[\ell]_0 d^{\ell}_{00}(\theta),\\
h_-(\theta) &= \sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}[\ell]_- d^{\ell}_{10}(\theta),\\
h_+(\theta) &= \sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}[\ell]_+ d^{\ell}_{10}(\theta).
\end{aligned}
\tag{34}
$$

Note that

$$
h_0(-\theta) = +h_0(\theta) \quad \text{and} \quad h_{\pm}(-\theta) = -h_{\pm}(\theta), \tag{35}
$$

because of a symmetry relation for the d functions [see Eq. (A3) of Appendix A]. Following convention [1–8], one has introduced a notation for partial amplitudes via

$$
[\ell]_0 = {}^{(-)}V_{\ell 0}, \quad [\ell]_- = {}^{(-)}V_{\ell 1}, \quad [\ell]_+ = {}^{(+)}V_{\ell 1}, \tag{36}
$$

where $[\ell]$ stands for the partial waves S, P, D, F, and G corresponding to $\ell=0$, 1, 2, 3, and 4.

It is useful to write down explicitly the formulas for $H(LM)$ in terms of the partial waves. From Eq. (29), one finds

$$
\begin{aligned}
H(L0) &= \sum_{\ell\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\{[\ell]_0[\ell']_0^*(\ell' 0 L 0|\ell 0)+[\ell]_-[\ell']_-^*(\ell' 1 L 0|\ell 1)+[\ell]_+[\ell']_+^*(\ell' 1 L 0|\ell 1)\}(\ell' 0 L 0|\ell 0),\\
H(L1) &= \frac{1}{\sqrt{2}}\sum_{\ell\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\{[\ell]_-[\ell']_0^*(\ell' 0 L 1|\ell 1)-[\ell]_0[\ell']_-^*(\ell' -1 L 1|\ell 0)\}(\ell' 0 L 0|\ell 0),\\
H(L2) &= \frac{1}{2}\sum_{\ell\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\{-[\ell]_-[\ell']_-^*(\ell' -1 L 2|\ell 1)+[\ell]_+[\ell']_+^*(\ell' -1 L 2|\ell 1)\}(\ell' 0 L 0|\ell 0).
\end{aligned}
\tag{37}
$$

These equations can be transformed further as follows:

$$
\begin{aligned}
H(L0) &= \sum_{\ell}\{[\ell]_0[\ell]_0^*(\ell 0 L 0|\ell 0)+[\ell]_-[\ell]_-^*(\ell 1 L 0|\ell 1)+[\ell]_+[\ell]_+^*(\ell 1 L 0|\ell 1)\}(\ell 0 L 0|\ell 0)\\
&\quad +2\sum_{\ell<\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\mathrm{Re}\{[\ell]_0[\ell']_0^*(\ell' 0 L 0|\ell 0)+[\ell]_-[\ell']_-^*(\ell' 1 L 0|\ell 1)+[\ell]_+[\ell']_+^*(\ell' 1 L 0|\ell 1)\}\\
&\quad \times(\ell' 0 L 0|\ell 0),\\
H(L1) &= \sqrt{2}\sum_{\ell}\mathrm{Re}\{[\ell]_-[\ell]_0^*\}(\ell 0 L 1|\ell 1)(\ell 0 L 0|\ell 0)+\sqrt{2}\sum_{\ell<\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\mathrm{Re}\{[\ell]_-[\ell']_0^*(\ell' 0 L 1|\ell 1)-[\ell]_0[\ell']_-^*\\
&\quad \times(\ell' -1 L 1|\ell 0)\}(\ell' 0 L 0|\ell 0),\\
H(L2) &= \frac{1}{2}\sum_{\ell}\{-[\ell]_-[\ell]_-^*(\ell -1 L 2|\ell 1)+[\ell]_+[\ell]_+^*(\ell -1 L 2|\ell 1)\}(\ell 0 L 0|\ell 0)+\sum_{\ell<\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\\
&\quad \times\mathrm{Re}\{-[\ell]_-[\ell']_-^*(\ell' -1 L 2|\ell 1)+[\ell]_+[\ell']_+^*(\ell' -1 L 2|\ell 1)\}(\ell' 0 L 0|\ell 0).
\end{aligned}
\tag{38}
$$

One sees that only the real part of the interference terms contribute to the moments, so that the moments themselves are now explicitly real. It is worth mentioning that the moments $H(L1)$ have contributions only from the waves with unnatural-parity exchange. Note also that the moments are zero identically unless $\ell'+\ell+L=$ even. The first formula above shows that $H(00)$ is simply a sum of all the waves in the problem:

$$H(00)=\sum_{\ell}\{|[\ell]_0|^2+|[\ell]_-|^2+|[\ell]_+|^2\}. \tag{39}$$

There are three amplitudes for each ℓ, except for the S wave which comes with only one amplitude. The angular distribution is a sum of two noninterfering terms as given in Eq. (32), so that one amplitude each for $\epsilon=\pm$ can be set to be real. There are therefore $6\ell_m$ real parameters to be determined—exactly equal to the number N_2 of H's as given in the previous section. This allows one to solve in principle for the partial waves, given a set of the moments $\{H\}$. For example, one finds that, if $\ell_m=2$, there are 12 nonzero moments

$$H(00),\ H(10),\ H(11),\ H(20),\ H(21),\ H(22),$$
$$H(30),\ H(31),\ H(32),\ H(40),\ H(41),\ H(42), \tag{40}$$

while the partial waves $[\ell]$ are, for unnatural-parity exchange,

$$S_0,\quad P_0,\quad P_-,\quad D_0,\quad D_- \tag{41}$$

and, for natural-parity exchange,

$$P_+,\quad D_+. \tag{42}$$

One wave in each naturality can be set to be real (S_0 and P_+, for example), so that there are again 12 real parameters to be determined.

IV. AMBIGUITIES IN THE PARTIAL WAVES

It is instructive to rewrite the angular distribution as follows:

$$I(\Omega)=\frac{1}{4\pi}[f_0(\theta)+2f_1(\theta)\cos\phi+2f_2(\theta)\cos 2\phi]. \tag{43}$$

The f functions are experimentally measurable, as they are completely determined given a set of moments $\{H\}$. Indeed one finds, from Eq. (13),

$$f_M(\theta)=\sum_{L=0}^{2\ell_m}(2L+1)H(LM)d_{M0}^L(\theta), \tag{44}$$

where ℓ_m is again the maximum ℓ in the problem. An alternative expression for $I(\Omega)$ as a function of the partial waves $[\ell]$ is, from Eqs. (32) and (33),

$$I(\Omega)=\frac{1}{4\pi}\{|h_0(\theta)+\sqrt{2}h_-(\theta)\cos\phi|^2+|\sqrt{2}h_+(\theta)\sin\phi|^2\}. \tag{45}$$

Comparing the two expressions for $I(\Omega)$, one finds

$$f_0(\theta)=|h_0(\theta)|^2+|h_-(\theta)|^2+|h_+(\theta)|^2,$$
$$f_1(\theta)=\sqrt{2}\ \mathrm{Re}\{h_0(\theta)h_-^*(\theta)\},$$
$$f_2(\theta)=\frac{1}{2}\{|h_-(\theta)|^2-|h_+(\theta)|^2\}. \tag{46}$$

These equations succinctly summarize the problem at hand; on the left-hand side are the functions involving measured H's, and on the right-hand side are the functions containing partial waves $[\ell]$ to be determined.

One may eliminate h_+ by combining $f_0(\theta)$ and $f_2(\theta)$ and modify $f_1(\theta)$ to obtain

$$f_a(\theta)\equiv f_0(\theta)+2f_2(\theta)=|h_0(\theta)|^2+|\sqrt{2}h_-(\theta)|^2,$$
$$f_b(\theta)\equiv 2f_1(\theta)=2\ \mathrm{Re}\{h_0(\theta)\sqrt{2}h_-^*(\theta)\}. \tag{47}$$

The form of f_a and f_b suggests that one can define, from Eqs. (35),

$$g(\theta)=\frac{1}{\sqrt{2}}[h_0(\theta)+\sqrt{2}h_-(\theta)],$$
$$g(-\theta)=\frac{1}{\sqrt{2}}[h_0(\theta)-\sqrt{2}h_-(\theta)], \tag{48}$$

and find

$$f_a(\theta)=|g(\theta)|^2+|g(-\theta)|^2,$$
$$f_b(\theta)=|g(\theta)|^2-|g(-\theta)|^2. \tag{49}$$

In order to examine the ambiguities in the problem, it is necessary to express the h functions as ratios of polynomials in a single variable. This is accomplished by introducing a variable $u=\tan(\theta/2)$ and the functions $e_{m'm}^{\ell}(u)$, as shown in Appendix A. One finds that the h functions assume the form

$$h_0(u)=\frac{\sum_{\ell=0}^{\ell_m}\sqrt{2\ell+1}[\ell]_0(1+u^2)^{\ell_m-\ell}e_{00}^{\ell}(u)}{(1+u^2)^{\ell_m}},$$
$$h_-(u)=\frac{\sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}[\ell]_-(1+u^2)^{\ell_m-\ell}e_{10}^{\ell}(u)}{(1+u^2)^{\ell_m}},$$
$$h_+(u)=\frac{\sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}[\ell]_+(1+u^2)^{\ell_m-\ell}e_{10}^{\ell}(u)}{(1+u^2)^{\ell_m}}. \tag{50}$$

The numerator of $h_0(u)$ is a polynomial in u^2 of order ℓ_m [see Eq. (A8) of Appendix A], and the numerator of $h_\pm(u)$ is u times a polynomial in u^2 of order ℓ_m-1. Consequently, the function $g(u)$ has a numerator expressed as a polynomial in u of order $2\ell_m$. The functions f_M may also be given in terms of the e functions:

$$f_M(u)=\frac{\Sigma_{L=0}^{2\ell_m}(2L+1)H(LM)(1+u^2)^{2\ell_m-L}e_{M0}^L(u)}{(1+u^2)^{2\ell_m}}. \tag{51}$$

Suppose now that a set of $[\ell]$ has been found satisfying Eqs. (46). One can then find $2\ell_m$ roots of the function

$$(1+u^2)^{\ell_m}g(u)=c_0\prod_{k=1}^{2\ell_m}(u-u_k), \tag{52}$$

where u_k's are complex roots—these are the so-called ''Barrelet'' zeros [15]—and c_0 is a complex constant. Since $g(u)$ and $g(-u)$ enter as absolute squares in the expression for the f functions [see Eqs. (49)], the complex conjugate of a root u_k is an equally valid solution; i.e., the f functions are not perturbed. One concludes immediately that, for the partial waves corresponding to unnatural-parity exchange ($\epsilon=-$), there are in general $2^{2\ell_m-1}$ solutions to the problem, after eliminating those which may be obtained by taking complex conjugation of the entire function $g(u)$. For each new $g(u)$, one may calculate

$$h_0(\theta)=\frac{1}{\sqrt{2}}[g(\theta)+g(-\theta)],$$

$$h_-(\theta)=\frac{1}{2}[g(\theta)-g(-\theta)], \tag{53}$$

to search for a new set of partial waves $[\ell]_0$ and $[\ell]_-$.

The third equation of Eqs. (46) is used to calculate

$$|h_+(\theta)|^2=|h_-(\theta)|^2-2f_2(\theta). \tag{54}$$

Note that all the allowed set of partial waves, $[\ell]_0$ and $[\ell]_-$, must satisfy the condition that the right-hand side of this equation remain non-negative [the first equation of Eqs. (46), of course, does not constitute a new constraint]. The relationship (54), in addition, indicates that the ambiguity problem for $[\ell]_+$ can be dealt with by setting

$$(1+u^2)^{\ell_m}h_+(u)=c_+u\prod_{k=1}^{\ell_m-1}(u^2-r_k), \tag{55}$$

where r_k's are the complex roots in u^2 and c_+ is a complex constant. For $\ell_m>1$, there must be in general 2^{ℓ_m-2} solutions for the partial waves with natural-parity exchange, i.e., $[\ell]_+$.

For each of the $2^{2\ell_m-1}$ solutions involving $[\ell]_0$ and $[\ell]_-$, there must be a total of 2^{ℓ_m-2} solutions for $[\ell]_+$. Therefore, a system containing the partial waves up to $\ell=\ell_m\geqslant 2$ has a total of $N_a=2^{2\ell_m-1}\times 2^{\ell_m-2}$ ambiguous solutions.

N_a is given below for $\ell_m\leqslant 4$:

ℓ_m	0	1	2	3	4
N_a	1	2	8	64	512

V. SYSTEM OF TWO IDENTICAL PARTICLES

Consider the following reaction, as an example:

$$\pi^- p\rightarrow\pi^0\pi^0 n \tag{56}$$

for production of a dipion system $\pi^0\pi^0$ in the forward direction, i.e., approximately along the beam line. In the Jackson frame, the decay amplitudes are again given by Eq. (26) and the resulting angular distribution is given by Eq. (28). Because of the identity of the two final-state particles, the odd ℓ's are absent and therefore $H(LM)=0$ if $L=$odd. It should be emphasized, once again, that $H(LM)$'s are always real from parity conservation in the production process.

Let ℓ_m be the maximum spin present in a given $\pi\pi$ mass bin. It is easy to show that the number of independent nonzero H's is

$$N^{\rm e}=3\ell_m+1. \tag{57}$$

$N^{\rm e}$ as a function of ℓ_m is given below as a table:

ℓ_m	0	2	4	6
$N^{\rm e}$	1	7	13	19

The ambiguity problem in the amplitude analysis can be dealt with in exactly the same way as in the case of two dissimilar particles, except that all the odd waves should be set to zero, i.e., $[\ell]=0$ for $\ell=$odd. There are $\ell_m/2$ partial waves greater than the S wave (which could be set to be real). Under the assumption that the z component of spin is either zero or ±1, each wave greater than zero requires three complex numbers, but one wave of natural-parity exchange could be set to be real. One concludes therefore that there are $3\ell_m$ real parameters to be determined. But the number of nonzero moments was shown to be $3\ell_m+1$ in the previous section—indicating that there must exist one linear relationship among the moments. Such relationships are given explicitly in the two examples worked out in this paper.

Consider once again a system in which the highest allowed partial wave is given by $\ell_m=$even. It is shown in Appendix B that the ambiguities among the partial waves with unnatural-parity exchange are determined by an examination of the ℓ_m complex roots v_k of the function

$$\mathcal{G}_-(v)=\frac{1}{u^{\ell_m}}G_-(u)=\frac{1}{u^{\ell_m}}(1+u^2)^{\ell_m}[h_0(u)+\sqrt{2}h_-(u)]$$

$$=a_{\ell_m}\prod_{k=1}^{\ell_m}(v-v_k), \tag{58}$$

which is a polynomial in v of order ℓ_m and a_{ℓ_m} is the coefficient of v^{ℓ_m}. The new variable v is related to $u=\tan(\theta/2)$ via

$$v=\frac{1}{u}-u=2\cot\theta. \tag{59}$$

Recall that the complex conjugate of a root v_k does not perturb $H(LM)$ and hence leaves the angular distribution invariant but could alter the partial waves. Since taking the complex conjugate of *all* the roots does not lead to a new

TABLE I. Moments in terms of partial waves.

(a)

	U[a] 00×00	U 00×10	U 00×11	U 00×20	U 00×21	U 10×10	U 10×11	U 10×20	U 10×21
H(00)	1					1			
H(10)		1/3[b]						4/15	
H(11)			1/6						1/10
H(20)				1/5		4/25			
H(21)					1/10		3/50		
H(30)								27/245	
H(31)									12/245

(b)

	U[c] 11×11	U 11×20	U 11×21	U 20×20	U 20×21	U 21×21	N[d] 11×11	N 11×21	N 21×21
H(00)	1			1		1	1		1
H(10)			1/5[b]					1/5	
H(11)		−1/30							
H(20)	−1/25			4/49		1/49	−1/25		1/49
H(21)					1/98				
H(22)	3/50					3/98	−3/50		−3/98
H(30)			−9/245					−9/245	
H(31)		9/245							
H(32)			3/98					−3/98	
H(40)				4/49		−16/441			−16/441
H(41)					5/147				
H(42)						10/441			−10/441

[a]The waves with unnatural-parity exchange. The notations are, e.g., $(00)(00)=S_0S_0^*=|S_0|^2$ and $(00)(11)=S_0P_-^*+P_-S_0^*=2\,\mathrm{Re}\{S_0P_-^*\}$.

[b]A square root is understood for both numerators and denominators. An example: $-2/3$ stands for $-\sqrt{2/3}$.

[c]The waves with unnatural-parity exchange. The notations are, e.g., $(20)(20)=D_0D_0^*=|D_0|^2$ and $(11)(21)=P_-D_-^*+D_-P_-^*=2\,\mathrm{Re}\{P_-D_-^*\}$.

[d]The waves with natural-parity exchange. The notations are, e.g., $(11)(11)=P_+P_+^*=|P_+|^2$ and $(11)(21)=P_+D_+^*+D_+P_+^*=2\,\mathrm{Re}\{P_+D_+^*\}$.

solution, there are in general 2^{ℓ_m-1} solutions. The h functions appearing in the function are given in terms of the partial waves $[\ell]_0$ and $[\ell]_\pm$:

$$\frac{1}{u^{\ell_m}}(1+u^2)^{\ell_m}h_0(u)=\sum_{\ell=0}^{\ell_m}\sqrt{2\ell+1}[\ell]_0 \times\left(\frac{1}{u}+u\right)^{\ell_m-\ell}\varepsilon_0^\ell(u),$$

$$\frac{1}{u^{\ell_m}}(1+u^2)^{\ell_m}h_\pm(u)=\sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}[\ell]_\pm \times\left(\frac{1}{u}+u\right)^{\ell_m-\ell}\varepsilon_1^\ell(u), \tag{60}$$

where a new function ε is defined by

$$\varepsilon_m^\ell(u)=\frac{1}{u^\ell}e_{m0}^\ell(u). \tag{61}$$

For the case $\ell=m=0$, one sees that

$$\varepsilon_0^0(u)=1. \tag{62}$$

It is shown in Appendix B that both the ε function and

$$\left(\frac{1}{u}+u\right)^n=2^n\csc^n\theta \tag{63}$$

for *even* n can be expressed as polynomials in a single variable v.

The ambiguity among the partial waves $[\ell]_+$ can be treated by examining the function

$$\mathcal{G}_+(v)=\frac{1}{u^{\ell_m}}G_+(u)=\frac{1}{u^{\ell_m}}(1+u^2)^{\ell_m}h_+(u) =c_+v\prod_{k=1}^{\ell_m/2-1}(v^2-r_k), \tag{64}$$

where c_+ is the coefficient of v^{ℓ_m-1} and comes with $\ell_m/2-1$ complex roots r_k. This means that there exists a total of $2^{\ell_m/2-2}$ ambiguous solutions involving $[\ell]_+$. Combining the two ambiguities, one concludes that a system containing partial waves for $\ell \leqslant \ell_m$ has a total of $N_a^e = 2^{\ell_m - 1} 2^{\ell_m/2 - 2}$ ambiguous solutions.[3]

N_a^e is given below for $\ell_m \leqslant 6$:

ℓ_m	0	2	4	6
N_a^e	1	2	8	64

VI. EXAMPLE WITH *S*, *P*, AND *D* WAVES

Consider an example of the $\pi\eta$ system with $\ell_m = 2$, produced in reaction (1). For the sake of completeness, all the relevant moments are given here in terms of the partial waves in the problem; see Eqs. (38). The results are tabulated in Table I, and are given explicitly in the next set of equations:

$$H(00) = S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2,$$

$$H(10) = \frac{1}{\sqrt{3}} S_0 P_0 + \frac{2}{\sqrt{15}} P_0 D_0 + \frac{1}{\sqrt{5}} (P_- D_- + P_+ D_+),$$

$$H(11) = \frac{1}{\sqrt{6}} S_0 P_- + \frac{1}{\sqrt{10}} P_0 D_- - \frac{1}{\sqrt{30}} P_- D_0,$$

$$H(20) = \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{5} P_0^2 - \frac{1}{5} (P_-^2 + P_+^2) + \frac{2}{7} D_0^2 + \frac{1}{7} (D_-^2 + D_+^2),$$

$$H(21) = \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{5} \sqrt{\frac{3}{2}}\, P_0 P_- + \frac{1}{7\sqrt{2}} D_0 D_-,$$

$$H(22) = \frac{1}{5} \sqrt{\frac{3}{2}}\, (P_-^2 - P_+^2) + \frac{1}{7} \sqrt{\frac{3}{2}}\, (D_-^2 - D_+^2),$$

$$H(30) = \frac{3}{7\sqrt{5}} (\sqrt{3} P_0 D_0 - P_- D_- - P_+ D_+),$$

$$H(31) = \frac{1}{7} \sqrt{\frac{3}{5}}\, (2 P_0 D_- + \sqrt{3} P_- D_0),$$

$$H(32) = \frac{1}{7} \sqrt{\frac{3}{2}}\, (P_- D_- - P_+ D_+),$$

$$H(40) = \frac{2}{7} D_0^2 - \frac{4}{21} (D_-^2 + D_+^2),$$

$$H(41) = \frac{1}{7} \sqrt{\frac{5}{3}}\, D_0 D_-,$$

$$H(42) = \frac{\sqrt{10}}{21} (D_-^2 - D_+^2). \tag{65}$$

One should note that the moments $H(4M)$ have contributions from the D wave only, while the moments $H(3M)$ result from interference between P and D waves.

In terms of the experimentally measured moments $\{H\}$'s, the f functions are, from Eq. (44),

$$(1+u^2)^4 f_M(u) = \sum_{L=0}^{4} (2L+1) H(LM) (1+u^2)^{4-L} e_{M0}^L(u) \tag{66}$$

or, explicitly,

$$\begin{aligned}(1+u^2)^4 f_0(u) = {} & H(00)(1+u^2)^4 + 3H(10)(1+u^2)^3(1-u^2) \\ & + 5H(20)(1+u^2)^2(1-4u^2+u^4) \\ & + 7H(30)(1+u^2)(1-9u^2+9u^4-u^6) \\ & + 9H(40)(1-16u^2+36u^4-16u^6+u^8),\end{aligned}$$

$$\begin{aligned}(1+u^2)^4 f_1(u) = {} & -3\sqrt{2}H(11)(1+u^2)^3 u - 5\sqrt{6}H(21) \\ & \times(1+u^2)^2 u(1-u^2) - 14\sqrt{3}H(31)(1+u^2) \\ & \times u(1-3u^2+u^4) - 18\sqrt{5}H(41)u(1-6u^2 \\ & + 6u^4 - u^6),\end{aligned}$$

$$\begin{aligned}(1+u^2)^4 f_2(u) = {} & 5\sqrt{6}H(22)(1+u^2)^2 u^2 + 7\sqrt{30}H(32) \\ & \times(1+u^2)u^2(1-u^2) + 9\sqrt{10}H(42)u^2 \\ & \times(3-8u^2+3u^4).\end{aligned} \tag{67}$$

Suppose now that one has found a set of solutions $\{S_0, P_0, P_-, D_0, D_-\}$ for unnatural-parity exchange and $\{P_+, D_+\}$ for natural-parity exchange. It is helpful to write down the h's explicitly:

$$\begin{aligned}(1+u^2)^2 h_0(u) = {} & S_0(1+u^2)^2 + \sqrt{3}P_0(1-u^4) \\ & + \sqrt{5}D_0(1-4u^2+u^4),\end{aligned}$$

$$\begin{aligned}\sqrt{2}(1+u^2)^2 h_-(u) = {} & -2u[\sqrt{3}P_-(1+u^2) \\ & + \sqrt{15}D_-(1-u^2)],\end{aligned}$$

$$\begin{aligned}\sqrt{2}(1+u^2)^2 h_+(u) = {} & -2u[\sqrt{3}P_+(1+u^2) \\ & + \sqrt{15}D_+(1-u^2)].\end{aligned} \tag{68}$$

The last equation above shows that there are no ambiguities for the partial waves P_+ and D_+, since the expression inside the square brackets is linear in u^2. On the other hand, from the first two equations, one finds that the function $g(u)$ is given by

$$\begin{aligned}G(u) \equiv \sqrt{2}(1+u^2)^2 g(u) = {} & S_0(1+u^2)^2 + \sqrt{3}P_0(1-u^4) \\ & + \sqrt{5}D_0(1-4u^2+u^4) - 2\sqrt{3}P_-(u+u^3) \\ & - 2\sqrt{15}D_-(u-u^3),\end{aligned} \tag{69}$$

which is a polynomial of order 4 in u and thus gives rise to the ambiguities in the unnatural-parity partial waves through the Barrelet zeros.

[3]This formula is valid only if $\ell_m \geqslant 4$.

One may write

$$G(u)=a_4u^4-a_3u^3+a_2u^2-a_1u+a_0, \qquad (70)$$

with

$$a_4=S_0-\sqrt{3}P_0+\sqrt{5}D_0,$$
$$a_3=2\sqrt{3}(P_- -\sqrt{5}D_-),$$
$$a_2=2S_0-4\sqrt{5}D_0,$$
$$a_1=2\sqrt{3}(P_- +\sqrt{5}D_-),$$
$$a_0=S_0+\sqrt{3}P_0+\sqrt{5}D_0. \qquad (71)$$

The inverse is

$$6S_0=2a_0+a_2+2a_4,$$
$$2\sqrt{3}P_0=a_0-a_4,$$
$$6\sqrt{5}D_0=a_0-a_2+a_4,$$
$$4\sqrt{3}P_-=a_1+a_3,$$
$$4\sqrt{15}D_-=a_1-a_3. \qquad (72)$$

Since $G(u)$ is a fourth-order polynomial in u with four complex roots $\{u_1,u_2,u_3,u_4\}$, it is given by

$$G(u)=a_4(u-u_1)(u-u_2)(u-u_3)(u-u_4), \qquad (73)$$

so that

$$a_3=a_4(u_1+u_2+u_3+u_4),$$
$$a_2=a_4(u_1u_2+u_1u_3+u_1u_4+u_2u_3+u_2u_4+u_3u_4),$$
$$a_1=a_4(u_1u_2u_3+u_2u_3u_4+u_3u_4u_1+u_4u_1u_2),$$
$$a_0=a_4(u_1u_2u_3u_4). \qquad (74)$$

Finally, substituting these into Eqs. (72), the partial waves can be expressed in terms of the roots or the Barrelet zeros:

$$6S_0=a_4(2u_1u_2u_3u_4+u_1u_2+u_1u_3+u_1u_4+u_2u_3+u_2u_4+u_3u_4+2),$$
$$2\sqrt{3}P_0=a_4(u_1u_2u_3u_4-1),$$
$$6\sqrt{5}D_0=a_4(u_1u_2u_3u_4-u_1u_2-u_1u_3-u_1u_4-u_2u_3-u_2u_4-u_3u_4+1),$$
$$4\sqrt{3}P_-=a_4(u_1u_2u_3+u_2u_3u_4+u_3u_4u_1+u_4u_1u_2+u_1+u_2+u_3+u_4),$$
$$4\sqrt{15}D_-=a_4(u_1u_2u_3+u_2u_3u_4+u_3u_4u_1+u_4u_1u_2-u_1-u_2-u_3-u_4). \qquad (75)$$

These expressions show how the ambiguities in the partial waves can be explored through complex conjugation of one or more of the Barrelet zeros. It should be borne in mind, however, that a_4 above is fixed and is given by the original set of the partial waves; see Eqs. (71). More precisely, its magnitude $|a_4|$ must remain invariant.

There should be in general eight ambiguous solutions involving the partial waves S_0, P_0, P_-, D_0, and D_-. The eight solutions are enumerated below in two columns:

$$\{u_1,u_2,u_3^*,u_4^*\} \quad \{u_1,u_2,u_3^*,u_4\},$$
$$\{u_1,u_2,u_3,u_4\} \quad \{u_1,u_2,u_3,u_4\},$$
$$\{u_1,u_2,u_3,u_4^*\} \quad \{u_1^*,u_2,u_3,u_4\},$$
$$\{u_1,u_2,u_3^*,u_4\} \quad \{u_1,u_2^*,u_3,u_4\},$$
$$\{u_1,u_2,u_3^*,u_4^*\} \quad \{u_1,u_2,u_3^*,u_4\},$$
$$\{u_1,u_2^*,u_3,u_4\} \quad \{u_1,u_2,u_3,u_4^*\},$$
$$\{u_1,u_2^*,u_3,u_4^*\} \quad \{u_1^*,u_2^*,u_3,u_4\},$$
$$\{u_1,u_2^*,u_3^*,u_4\} \quad \{u_1^*,u_2,u_3^*,u_4\},$$
$$\{u_1,u_2^*,u_3^*,u_4^*\} \quad \{u_1^*,u_2,u_3,u_4^*\}. \qquad (76)$$

The first column results from a procedure in which u_1 is left invariant and the remaining three roots u_2, u_3, and u_4 are allowed to undergo complex conjugation—one sees that there are $2^3=8$ ways of doing this. The second column represents an alternative method of ennumerating the eight ambiguous solutions. For each solution, a new $G(u)$ is given by Eq. (73), and hence one obtains new h's via

$$h_0(u)=\frac{1}{\sqrt{2}}[g(u)+g(-u)],$$
$$h_-(u)=\frac{1}{2}[g(u)-g(-u)], \qquad (77)$$

and the new partial waves $\{S_0,P_0,P_-,D_0,D_-\}$ are given by Eqs. (75). Note that S_0 may become complex in the process, but it can be made real again by dividing each wave in the set by the phase of S_0.

One can find the corresponding P_+ and D_+ from

$$|h_+(u)|^2=|h_-(u)|^2-2f_2(u). \qquad (78)$$

Formulas (67) and (68) show that, if the factor u^2 is taken out from both sides of the equation above, then one is left with a quadratic function in u^2. One can, therefore, find P_+, D_+ and the phase difference between the two, by setting the three coefficients to zero. But a more direct way of finding them is contained in Tables I(a) and I(b). One sees that

$$|D_+|^2 = |D_-|^2 - \frac{21}{\sqrt{10}} H(42),$$

$$|P_+|^2 = |P_-|^2 + 3\sqrt{\frac{5}{2}}\, H(42) - 5\sqrt{\frac{2}{3}}\, H(22),$$

$$2\,\mathrm{Re}\{P_+ D_+^*\} = 2\,\mathrm{Re}\{P_- D_-^*\} - 7\sqrt{\frac{2}{3}}\, H(32). \quad (79)$$

It is clear that any solution resulting in negative values for the first two equations above is not allowed. In addition, any solution which makes the cosine of the phase difference greater than 1 is clearly not allowed.

VII. SYSTEM WITH *S*, *P*, *D*, AND *F* WAVES

In this section, a brief description is given of a system containing the partial waves $\ell = 0$, 1, 2, and 3. There are 13 parameters to be determined describing the seven amplitudes S_0, P_0, D_0, F_0, P_-, D_-, and F_- produced via unnatural-parity exchange, and there are 5 parameters for three amplitudes P_+, D_+, and F_+ with natural-parity exchange. There are, on the other hand, 18 real moments $H(LM)$ with $0 \leq L \leq 6$ and $M = 0$, 1, and 2, which specify the angular distribution. Therefore, it is in general possible to find a set of amplitudes satisfying the moments, but the process is not unique, and there are in fact a total of 64 ambiguous solutions.

The h functions are given by, from Appendix A,

$$\begin{aligned}(1+u^2)^3 h_0(u) = {} & S_0(1+u^2)^3 + \sqrt{3}P_0(1+u^2)^2(1-u^2) \\ & + \sqrt{5}D_0(1+u^2)(1-4u^2+u^4) \\ & + \sqrt{7}F_0(1-9u^2+9u^4-u^6),\end{aligned}$$

$$\begin{aligned}\sqrt{2}(1+u^2)^3 h_-(u) = {} & -2u[\sqrt{3}P_-(1+u^2)^2 + \sqrt{15}D_-(1-u^4) \\ & + \sqrt{42}F_-(1-3u^2+u^4)],\end{aligned}$$

$$\begin{aligned}\sqrt{2}(1+u^2)^3 h_+(u) = {} & -2u[\sqrt{3}P_+(1+u^2)^2 + \sqrt{15}D_+(1-u^4) \\ & + \sqrt{42}F_+(1-3u^2+u^4)]. \quad (80)\end{aligned}$$

The function $g(u)$ is given by

$$\begin{aligned}G(u) \equiv \sqrt{2}(1+u^2)^3 g(u) = {} & a_6u^6 - a_5u^5 + a_4u^4 - a_3u^3 + a_2u^2 \\ & - a_1u + a_0, \quad (81)\end{aligned}$$

which is a polynomial of order 6 in u and the ambiguities in the unnatural-parity partial waves can be found through the Barrelet zeros, as before. From Eqs. (80), one sees that

$$\begin{aligned}a_6 &= S_0 - \sqrt{3}P_0 + \sqrt{5}D_0 - \sqrt{7}F_0, \\ a_0 &= S_0 + \sqrt{3}P_0 + \sqrt{5}D_0 + \sqrt{7}F_0, \\ a_4 &= 3S_0 - \sqrt{3}P_0 - 3\sqrt{5}D_0 + 9\sqrt{7}F_0, \\ a_2 &= 3S_0 + \sqrt{3}P_0 - 3\sqrt{5}D_0 - 9\sqrt{7}F_0, \\ a_5 &= 2\sqrt{3}P_- - 2\sqrt{15}D_- + 2\sqrt{42}F_-, \\ a_1 &= 2\sqrt{3}P_- + 2\sqrt{15}D_- + 2\sqrt{42}F_-, \\ a_3 &= 4\sqrt{3}P_- - 6\sqrt{42}F_-. \quad (82)\end{aligned}$$

The inverse is

$$\begin{aligned}12S_0 &= 3(a_6+a_0) + (a_4+a_2), \\ 12\sqrt{5}D_0 &= 3(a_6+a_0) - (a_4+a_2), \\ 20\sqrt{3}P_0 &= 9(a_0-a_6) + (a_2-a_4), \\ 20\sqrt{7}F_0 &= (a_0-a_6) - (a_2-a_4), \\ 20\sqrt{3}P_- &= 3(a_5+a_1) + 2a_3, \\ 10\sqrt{42}F_- &= (a_5+a_1) - a_3, \\ 4\sqrt{15}D_- &= a_1 - a_5. \quad (83)\end{aligned}$$

Since $G(u)$ is a sixth-order polynomial in u with six complex roots $\{u_i\}$, $i = 1-6$, it can be written

$$G(u) = a_6(u-u_1)(u-u_2)(u-u_3)(u-u_4)(u-u_5)(u-u_6), \quad (84)$$

so that

$$\begin{aligned}a_5 &= a_6(u_1+u_2+u_3+u_4+u_5+u_6), \\ a_4 &= a_6 \sum_P u_i u_j, \\ a_3 &= a_6 \sum_P u_i u_j u_k, \\ a_2 &= a_6 \sum_P u_i u_j u_k u_\ell, \\ a_1 &= a_6 \sum_P u_i u_j u_k u_\ell u_m, \\ a_0 &= a_6(u_1u_2u_3u_4u_5u_6). \quad (85)\end{aligned}$$

The sums are over all permutations with the condition $i < j < k < \ell < m$. Note that there are 15 terms in a_4, 20 in a_3, 15 in a_2, and 6 in a_1. Finally, substituting these into Eqs. (83), the partial waves can be expressed in terms of the Barrelet zeros. An operation $u_i \to u_i^*$ for any i does not perturb $H(LM)$ and therefore results in an ambiguous solution. Since taking the complex conjugate of all six roots is not a new solution, one can leave u_1 invariant, and take complex conjugates of all the rest—there are thus $2^5=32$ ways of doing this, and this is the number of ambiguous solutions involving the waves with unnatural-parity exchange.

An examination of $h_+(u)$ in Eqs. (80) reveals that it is proportional to a polynomial of order 2 in u^2 involving the partial waves with natural-parity exchange. Define $Q(u)$ by

$$\begin{aligned} Q(u) &= \sqrt{3}P_+(1+u^2)^2+\sqrt{15}D_+(1-u^4) \\ &\quad +\sqrt{42}F_+(1-3u^2+u^4) \\ &= b_2u^4-b_1u^2+b_0, \end{aligned} \tag{86}$$

so that

$$\begin{aligned} b_2 &= \sqrt{3}P_+-\sqrt{15}D_++\sqrt{42}F_+, \\ b_0 &= \sqrt{3}P_++\sqrt{15}D_++\sqrt{42}F_+, \\ b_1 &= -2\sqrt{3}P_++3\sqrt{42}F_+. \end{aligned} \tag{87}$$

The inverse is

$$\begin{aligned} 5\sqrt{42}F_+ &= b_0+b_1+b_2, \\ 10\sqrt{3}P_+ &= 3b_0-2b_1+3b_2, \\ 2\sqrt{15}D_+ &= b_0-b_2. \end{aligned} \tag{88}$$

Let $r_1$and r_2 be two roots given by

$$Q(u)=b_2(u^2-r_1)(u^2-r_2). \tag{89}$$

Then one finds

$$\begin{aligned} b_1 &= b_2(r_1+r_2), \\ b_0 &= b_2r_1r_2. \end{aligned} \tag{90}$$

Finally, substituting these into Eqs. (88) with $r_2\to r_2^*$, one can obtain an alternate solution involving the partial waves with natural-parity exchange.

There exist two solutions involving P_+, D_+, and F_+ for each of the 32 partial-wave sets with unnatural-parity exchange—one sees therefore that there must exist in general a total of 64 ambiguous solutions for a system containing spins up to 3. Because there should exist in general two distinct solutions for P_+, D_+, and F_+, it is not possible to invert the formulas for $H(L2)$ with $2\leqslant L\leqslant 6$ and solve for the amplitudes algebraically—the reader will recall that this was possible for the problem containing $\ell=0$, 1, and 2 (see Sec. V). Instead, one must resort to a function-minimization routine to find one set of solutions for P_+, D_+, and F_+ for each of the 32 partial-wave sets with unnatural-parity exchange; the second set can then be found through the technique of Barrelet zeros [see Eq. (89)]. Not all 64 solutions are ''correct'' in general; for each set one must calculate all the predicted $H(LM)$'s and check that they are identical to the experimental moments—at least within the error bars.

VIII. EXAMPLE WITH S AND D WAVES

Consider a case of two identical spinless particles with $\ell_m=2$. There are five parameters involving the partial waves with unnatural-parity exchange, i.e., S_0 (real), D_0 (complex), and D_- (complex), and only one parameter, D_+ (real), for the partial wave produced by natural-parity exchange. So one sees that a total of six parameters are required in this case.

The unnormalized moments are expressed in terms of the partial waves and are given in Eqs. (65) but with odd-L moments missing. Not all seven moments listed are independent; one finds, in fact,

$$2\sqrt{5}H(22)=3\sqrt{3}H(42). \tag{91}$$

Therefore, there are six independent moments, corresponding to six partial-wave parameters to be determined in the problem.

Recall that the ambiguities among the partial waves with unnatural-parity exchange are determined by the complex roots of the function

$$\begin{aligned} \mathcal{G}_-(v) &= S_0\left(\frac{1}{u}+u\right)^2+\sqrt{5}D_0\varepsilon_0^2(u)+\sqrt{10}D_-\varepsilon_1^2(u) \\ &= a_2v^2-a_1v+a_0, \end{aligned} \tag{92}$$

where

$$\begin{aligned} a_2 &= S_0+\sqrt{5}D_0, \\ a_1 &= 2\sqrt{15}D_-, \\ a_0 &= 4S_0-2\sqrt{5}D_0. \end{aligned} \tag{93}$$

Solving for S_0 and D_0, one finds

$$\begin{aligned} 6S_0 &= a_0+2a_2, \\ 6\sqrt{5}D_0 &= -a_0+4a_2. \end{aligned} \tag{94}$$

The Barrelet zeros are

$$\{v_1,v_2\}=\frac{a_1\pm\sqrt{a_1^2-4a_0a_2}}{2a_2}, \tag{95}$$

where

$$\mathcal{G}_-(v)=a_2(v-v_1)(v-v_2), \tag{96}$$

and one finds

$$\begin{aligned} a_1 &= a_2(v_1+v_2), \\ a_0 &= a_2(v_1v_2). \end{aligned} \tag{97}$$

Finally, the partial waves can be expressed in terms of the Barrelet zeros:

$$\begin{aligned} 6S_0 &= a_2(v_1v_2+2), \\ 6\sqrt{5}D_0 &= a_2(-v_1v_2+4), \\ 2\sqrt{15}D_- &= a_2(v_1+v_2). \end{aligned} \tag{98}$$

There exist two ambiguous solutions, corresponding to the sets $\{v_1, v_2\}$ and $\{v_1, v_2^*\}$.

The one wave with natural-parity exchange, D_+, can be set to be real, to be determined from any one of the moments $H(20)$, $H(22)$, $H(40)$, or $H(42)$. There are thus no ambiguities involving D_+.

IX. EXAMPLE WITH *S*, *D*, AND *G* WAVES

Consider a system which consists of two identical spinless particles with $\ell_m = 4$. This problem requires a total of 12 parameters to be fitted, consisting of S_0, D_0, D_-, G_0, G_-, D_+, and G_+ which are complex in general. As in previous examples, one may set one wave to be real in each group of a given naturality, e.g., S_0 and D_+.

The relevant moments are given below in terms of the partial waves:

$$H(00) = S_0^2 + D_0^2 + D_-^2 + G_0^2 + G_-^2 + D_+^2 + G_+^2,$$

$$H(20) = \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{7} D_0^2 + \frac{1}{7}(D_-^2 + D_+^2) + \frac{6}{7\sqrt{5}} D_0 G_0 + \frac{\sqrt{6}}{7}(D_- G_- + D_+ G_+) + \frac{20}{77} G_0^2 + \frac{17}{77}(G_-^2 + G_+^2),$$

$$H(21) = \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{7\sqrt{2}} D_0 D_- + \frac{\sqrt{3}}{7} D_0 G_- - \frac{2\sqrt{2}}{5\sqrt{5}} D_- G_0 + \frac{\sqrt{15}}{77} G_0 G_-,$$

$$H(22) = \frac{1}{7}\sqrt{\frac{3}{2}}(D_-^2 - D_+^2) - \frac{1}{14}(D_- G_- - D_+ G_+) + \frac{5\sqrt{6}}{77}(G_-^2 - G_+^2),$$

$$H(40) = \frac{2}{7} D_0^2 - \frac{4}{21}(D_-^2 + D_+^2) + \frac{1}{3} S_0 G_0 + \frac{20\sqrt{5}}{231} D_0 G_0 + \frac{162}{1001} G_0^2 + \frac{5\sqrt{2}}{77\sqrt{3}}(D_- G_- + D_+ G_+) + \frac{81}{1001}(G_-^2 + G_+^2),$$

$$H(41) = \frac{1}{7}\sqrt{\frac{5}{3}} D_0 D_- + \frac{1}{3\sqrt{2}} S_0 G_- + \frac{17\sqrt{5}}{231\sqrt{2}} D_0 G_- + \frac{5}{77\sqrt{3}} D_- G_0 + \frac{81}{1001\sqrt{2}} G_0 G_-,$$

$$H(42) = \frac{\sqrt{10}}{21}(D_-^2 - D_+^2) + \frac{3\sqrt{15}}{154}(D_- G_- - D_+ G_+) + \frac{27\sqrt{10}}{1001}(G_-^2 - G_+^2). \tag{99}$$

They are continued into the next set of formulas:

$$H(60) = \frac{15\sqrt{5}}{143} D_0 G_0 - \frac{10\sqrt{6}}{143}(D_- G_- + D_+ G_+) + \frac{20}{143} G_0^2 - \frac{1}{143}(G_-^2 + G_+^2),$$

$$H(61) = \frac{5\sqrt{21}}{143} D_0 G_- + \frac{5\sqrt{35}}{143\sqrt{2}} D_- G_0 + \frac{\sqrt{105}}{143} G_0 G_-,$$

$$H(62) = \frac{2\sqrt{70}}{143}(D_- G_- - D_+ G_+) + \frac{\sqrt{105}}{143}(G_-^2 - G_+^2),$$

$$H(80) = \frac{490}{2431} G_0^2 - \frac{392}{2431}(G_-^2 + G_+^2),$$

$$H(81) = \frac{147\sqrt{5}}{2431} G_0 G_-,$$

$$H(82) = \frac{42\sqrt{35}}{2431}(G_-^2 - G_+^2). \tag{100}$$

It is seen that there are 13 real moments, but the moments $H(LM)$ with $M = 2$ are not independent. It can be shown, in fact, that

$$6\sqrt{70} H(22) - 24\sqrt{42} H(42) + 182 H(62) - 119\sqrt{3} H(82) = 0. \tag{101}$$

This shows that there exist 12 independent moments corresponding to 12 parameters to be fitted for the partial waves in the problem.

One finds, from Eq. (58),

$$\begin{aligned}\mathcal{G}_-(v) &= S_0\left(\frac{1}{u}+u\right)^4 + \sqrt{5} D_0\left(\frac{1}{u}+u\right)^2 \varepsilon_0^2(u) + 3 G_0 \varepsilon_0^4(u) \\ &\quad + \sqrt{10} D_-\left(\frac{1}{u}+u\right)^2 \varepsilon_1^2(u) + 3\sqrt{2} G_- \varepsilon_1^4(u) \\ &= a_4 v^4 - a_3 v^3 + a_2 v^2 - a_1 v + a_0,\end{aligned} \tag{102}$$

and the coefficients are

$$\begin{aligned} a_4 &= S_0 + \sqrt{5} D_0 + 3 G_0, \\ a_2 &= 8 S_0 + 2\sqrt{5} D_0 - 36 G_0, \\ a_0 &= 16 S_0 - 8\sqrt{5} D_0 + 18 G_0, \\ a_3 &= 2\sqrt{15} D_- + 6\sqrt{10} G_-, \\ a_1 &= 8\sqrt{15} D_- - 18\sqrt{10} G_-. \end{aligned} \tag{103}$$

Solving for the partial waves, one finds

$$30S_0=a_0+a_2+6a_4,$$

$$42\sqrt{5}D_0=-2a_0+a_2+24a_4,$$

$$210G_0=a_0-4a_2+16a_4,$$

$$14\sqrt{15}D_-=a_1+3a_3,$$

$$42\sqrt{10}G_-=-a_1+4a_3. \quad (104)$$

Let $\{v_1,v_2,v_3,v_4\}$ be the complex roots of the function $\mathcal{G}_-(v)$, so that

$$a_1=a_4(v_1+v_2+v_3+v_4),$$

$$a_2=a_4(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4),$$

$$a_3=a_4(v_1v_2v_3+v_2v_3v_4+v_3v_4v_1+v_4v_1v_2),$$

$$a_4=a_4(v_1v_2v_3v_4). \quad (105)$$

The partial waves are determined by substituting these into Eqs. (104). Since taking the complex conjugate of all four roots is not a new solution, one can leave one root fixed and take complex conjugates of the remaining three—there are thus $2^3=8$ ways of doing this, and this is the number of ambiguous solutions involving the partial waves with unnatural-parity exchange.

Given a set of the partial waves with unnatural-parity exchange, one can calculate the partial waves D_+ and G_+ and the phase between them via Eqs. (99) and (100):

$$|G_+|^2=|G_-|^2-\frac{2431}{42\sqrt{35}}H(82),$$

$$|D_+|^2=|D_-|^2-\frac{7\sqrt{2}}{\sqrt{3}}H(22)+\frac{221}{4\sqrt{35}}\left[\frac{17}{14}H(82)-\frac{11}{17\sqrt{3}}H(62)\right],$$

$$2\,\mathrm{Re}\{D_+G_+^*\}=2\,\mathrm{Re}\{D_-G_-^*\}+\frac{143}{2\sqrt{70}}\left[\frac{17}{14\sqrt{3}}H(82)-H(62)\right]. \quad (106)$$

From this, one must conclude that the partial waves D_+ and G_+ with natural-parity exchange can be determined uniquely—subject to the conditions that $|D_+|^2$ and $|G_+|^2$ cannot be negative and that the cosine of the angle between D_+ and G_+^* cannot be greater than 1. Therefore, the problem of finding S, D, and G partial waves amounts to exploring in general eight ambiguous solutions among the partial waves with unnatural-parity exchange.

X. EXPERIMENTAL MEASUREMENT OF THE MOMENTS

The partial waves $[\ell]_0$, $[\ell]_-$, and $[\ell]_+$ can be used directly as unknown parameters in the extended maximum-likelihood fits [12]. Because there is an absolute scale in an extended maximum-likelihood fit, one obtains directly the predicted numbers of events for all the partial waves, corrected for finite acceptance and angular distributions. The partial waves in turn give rise to a set of predicted moments $\{H\}$, as given in Tables I(a) and I(b) in the case of the $\pi\eta$ system. But the $H(00)$ is not 1 [(10)] but the total predicted number of events from the fit [see Eq. (39)]; i.e., one should be using the unnormalized moments. One could choose H's as unknowns in the fit, but the two sets of H's should be the same ideally—this affords one an effective way of assessing self-consistency between the moments and the partial waves.

One may determine directly the experimental moments (unnormalized) as follows:

$$H_x(LM)=\sum_i^n D_{M0}^L(\phi_i,\theta_i,0), \quad (107)$$

where the sum is over a given number n of experimental data in a mass bin. But this is given by, from Eq. (9),

$$H_x(LM)=\int d\Omega\,\eta(\Omega)I(\Omega)D_{M0}^L(\phi,\theta,0), \quad (108)$$

where $\eta(\Omega)$ represents the finite acceptance of the apparatus, and it includes software cuts, if any. From Eq. (6), one finds that

$$H_x(LM)=\sum_{L'M'}H(L'M')\Psi_x(LML'M'), \quad (109)$$

where

$$H_x(LML'M')=\left(\frac{2L'+1}{4\pi}\right)\int d\Omega\,\eta(\Omega)D_{M0}^L(\phi,\theta,0)\times D_{M'0}^{L'*}(\phi,\theta,0). \quad (110)$$

Note that the Ψ's have a simple normalization

$$\Psi_x(LML'M')=\delta_{LL'}\delta_{MM'} \quad (111)$$

in the limit $\eta(\Omega)=1$. The integral (110) can be calculated using a sample of ''accepted'' Monte Carlo (MC) events. Let N_x be the number of accepted MC events, out of a total of N raw MC events. Then, the integral is

$$\Psi_x(LML'M')=\left(\frac{2L'+1}{4\pi}\right)\frac{1}{N}\sum_i^{N_x}D_{M0}^L(\phi_i,\theta_i,0)\times D_{M'0}^{L'*}(\phi_i,\theta_i,0). \quad (112)$$

Equation (109) shows that one can predict the experimentally measurable moments (107), given a set $\{H\}$ and Ψ's; this provides a means of assessing the goodness of fit by forming a χ^2 based on the set $\{H_x\}$.

There exists an alternative method of determining Ψ's. For the purpose, one expands the acceptance function $\eta(\Omega)$ in terms of the orthonormal D functions, as follows:

$$\eta(\Omega)=\sum_{LM}(2L+1)\xi(LM)D^{L*}_{M0}(\phi,\theta,0), \qquad (113)$$

where $\xi(LM)$ is given by

$$\xi(LM)=\frac{1}{4\pi}\int d\Omega\,\eta(\Omega)D^{L}_{M0}(\phi,\theta,0), \qquad (114)$$

The complex conjugate is, from the defining formula above,

$$\xi^*(LM)=(-)^M\xi(L-M), \qquad (115)$$

so that the acceptance function can be made explicitly real,

$$\eta(\Omega)=\sum_{LM}(2L+1)\tau(M)\mathrm{Re}\{\xi(LM)D^{L*}_{M0}(\phi,\theta,0)\}, \qquad (116)$$

using the $\tau(M)$ introduced in Eq. (14). A set of $\xi(LM)$ specifies completely the acceptance in the problem. The normalization for the acceptance function has been chosen such that a perfect acceptance is given by $\eta(\Omega)=1$ and $\xi(LM)=\delta_{L0}\delta_{M0}$. The $\xi(LM)$'s can be measured experimentally using the accepted MC events

$$\xi(LM)=\frac{1}{4\pi N}\sum_{i}^{N_x}D^{L}_{M0}(\phi_i,\theta_i,0). \qquad (117)$$

Finally, substituting Eq. (113) into Eq. (110), one finds

$$\Psi_x(LML'M')=\sum_{L''M''}(2L''+1)\xi^*(L''M'')\times(LML''M''|L'M')(L0L''0|L'0). \qquad (118)$$

This formula shows an important aspect of the $\xi(LM)$ technique of representing acceptance. Although Eq. (114) involves a sum in which L and M could be extended to infinity for an arbitrary acceptance, there is a cutoff if the set $\{H\}$ has maxima L_m and M_m [see Eq. (109)]. The formula above demonstrates that $L''\leqslant 2L_m$ and $|M''|\leqslant 2M_m$.

For completeness, a short comment is given on the extended likelihood functions. The likelihood function for finding n events of a given bin with a finite acceptance $\eta(\Omega)$ is defined as a product of the probabilities:

$$\mathcal{L}\propto\left[\frac{\overline{n}^{n}}{n!}e^{-\overline{n}}\right]\prod_{i}^{n}\left[\frac{I(\Omega_i)}{\int I(\Omega)\eta(\Omega)d\Omega}\right], \qquad (119)$$

where the first brackets are the Poisson probability for n events. This is the so-called extended likelihood function, in the sense that the Poisson distribution for n itself is included in the likelihood function. The expectation value $\overline{n}$ for n is given by

$$\overline{n}\propto\int I(\Omega)\eta(\Omega)d\Omega. \qquad (120)$$

The likelihood function $\mathcal{L}$ can now be written, dropping the factors depending on n alone,

$$\mathcal{L}\propto\left[\prod_{i}^{n}I(\Omega_i)\right]\exp\left[-\int I(\Omega)\eta(\Omega)d\Omega\right].$$

The "log" of the likelihood function now has the form

$$\ln\mathcal{L}\propto\sum_{i}^{n}\ln I(\Omega_i)-\int d\Omega\,\eta(\Omega)I(\Omega), \qquad (121)$$

which can be recast in terms of the $\xi(LM)$'s:

$$\ln\mathcal{L}\propto\sum_{i}^{n}\ln I(\Omega_i)-\sum_{LM}(2L+1)H(LM)\xi^*(LM)$$
$$\propto\sum_{i}^{n}\ln I(\Omega_i)-\sum_{LM}(2L+1)\tau(M)H(LM)\mathrm{Re}\,\xi(LM). \qquad (122)$$

$H(LM)$'s may be used directly as parameters in the fit or may be given as functions of the partial waves. It is interesting to note that the $\xi(LM)$'s for $L>L_m$ and $|M|>M_m$ are not needed in the likelihood fit. Note also that only the real part of the $\xi(LM)$'s are used in the fit.

It should be borne in mind that a set of the moments $\{H\}$ may not always be expressed in terms of the partial waves. This is clear if one examines Tables I(a) and I(b). Consider, for example, an angular distribution in which $H(10)$ is the only nonzero moment. But this moment is given by a set of interference terms involving even-odd partial waves. So at least one term cannot be zero—for example, the interference term involving S and P waves. But then neither $H(00)$ nor $H(20)$ can be zero, since both S and P waves are nonzero. One must conclude then that a χ^2 based on the set $\{H_x\}$ may not necessarily be zero identically.

APPENDIX A: DERIVATION OF *e* FUNCTIONS

The d functions are normally given, from Rose [16], as

$$d^{\ell}_{m'm}(\theta)=(-)^{m'-m}[(\ell+m')!\times(\ell-m')!(\ell+m)!(\ell-m)!]^{1/2}\times\sum_{k=k_1}^{k_2}\frac{(-)^k\cos^{2\ell-n}(\theta/2)\sin^n(\theta/2)}{(\ell-m'-k)!(\ell+m-k)!(m'-m+k)!k!},\tag{A1}$$

where $n=m'-m+2k$ and k is a non-negative integer ranging between k_1 and k_2 given by

$$k_1=\max\{0,\ m-m'\},\quad k_2=\min\{\ell-m',\ \ell+m\}.\tag{A2}$$

Note that n is also a non-negative integer. It is seen that the d functions are nonsingular polynomials of order up to 2ℓ in two variables $\cos(\theta/2)$ and $\sin(\theta/2)$; they both range from 0 to 1, as θ goes from 0 to π. From the definition (A1), one finds the following relationships:

$$d^{\ell}_{m'm}(-\theta)=(-)^{m'-m}d^{\ell}_{m'm}(\theta),$$

$$d^{\ell}_{m'm}(\theta)=(-)^{m'-m}d^{\ell}_{mm'}(\theta),$$

$$d^{\ell}_{m'm}(\theta)=(-)^{m'-m}d^{\ell}_{-m'-m}(\theta),$$

$$d^{\ell}_{m'm}(\pi-\theta)=(-)^{\ell+m'}d^{\ell}_{m'-m}(\theta).\tag{A3}$$

The d functions depend in reality on $z=\cos\theta$ only, through

$$x=\sin\frac{\theta}{2}=\sqrt{\frac{1-z}{2}}\quad\text{and}\quad y=\cos\frac{\theta}{2}=\sqrt{\frac{1+z}{2}}.\tag{A4}$$

This assertion is valid as long as θ remains within the range $(0\rightarrow\pi)$. In particular, negative values of θ are not allowed in this scheme. But note that $\theta<0$ is equivalent to

$$\Omega=(-\theta,\phi)\rightarrow\Omega'=(+\theta,\pi+\phi)\tag{A5}$$

and that θ is always positive in practice, since $\cos\theta$ is evaluated as a scalar product of two momenta. With this proviso, then, there should not be any sign of ambiguities in x and y as defined by Eqs. (A4). The d functions in terms of the variables x and y are given in this appendix, as the existing tabulations are rarely given in these variables. For the purpose it is sufficient to consider only those with $m'\geqslant 0$ and $m'\geqslant|m|$ because of the symmetry relations for d [see Eqs. (A3)]. With these restrictions, the summation on k in Eq. (A1) ranges from $k_1=0$ to $k_2=\ell-m'$, and the d functions take on the form

$$d^{\ell}_{m'm}(z)=(-)^{m'-m}[(\ell+m')!\times(\ell-m')!(\ell+m)!(\ell-m)!]^{1/2}x^{m'-m}\times\left[\sum_{k=k_1}^{k_2}\frac{(-)^k x^{2k}y^{2(\ell-m'-k)}}{(\ell-m'-k)!(\ell+m-k)!(m'-m+k)!k!}\right]\times y^{m'+m}.\tag{A6}$$

Note that all the exponents are non-negative. The function in the square brackets is a polynomial in x^2 and in y^2, each of order $\ell-m'$. The d function itself is a polynomial in x of order $2\ell-(m'+m)$ and in y of order $2\ell-(m'-m)$.

The d functions are tabulated here for ℓ up to 3:

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
0	0	0	1
1	1	1	y^2
1	1	0	$-\sqrt{2}xy$
1	1	-1	x^2
1	0	0	y^2-x^2
2	2	2	y^4
2	2	1	$-2xy^3$
2	2	0	$\sqrt{6}x^2y^2$
2	2	-1	$-2x^3y$
2	2	-2	x^4
2	1	1	$(y^2-3x^2)y^2$
2	1	0	$-\sqrt{6}x(y^2-x^2)y$
2	1	-1	$x^2(3y^2-x^2)$
2	0	0	$y^4-4x^2y^2+x^4$

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
3	3	3	y^6
3	3	2	$-\sqrt{6}xy^5$
3	3	1	$\sqrt{15}x^2y^4$
3	3	0	$-2\sqrt{5}x^3y^3$
3	3	-1	$\sqrt{15}x^4y^2$
3	3	-2	$-\sqrt{6}x^5y$
3	3	-3	x^6
3	2	2	$(y^2-5x^2)y^4$
3	2	1	$-\sqrt{10}x(y^2-2x^2)y^3$
3	2	0	$\sqrt{30}x^2(y^2-x^2)y^2$
3	2	-1	$-\sqrt{10}x^3(2y^2-x^2)y$
3	2	-2	$x^4(5y^2-x^2)$
3	1	1	$(y^4-8x^2y^2+6x^4)y^2$
3	1	0	$-2\sqrt{3}x(y^4-3x^2y^2+x^4)y$
3	1	-1	$x^2(6y^4-8x^2y^2+x^4)$
3	0	0	$y^6-9x^2y^4+9x^4y^2-x^6$

The d functions for $\ell=4$ are given below:

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
4	4	4	y^8
4	4	3	$-2+\sqrt{2}xy^7$
4	4	2	$2\sqrt{7}x^2y^6$
4	4	1	$-2\sqrt{14}x^3y^5$
4	4	0	$\sqrt{70}x^4y^4$
4	4	-1	$-2\sqrt{14}x^5y^3$
4	4	-2	$2\sqrt{7}x^6y^2$
4	4	-3	$-2\sqrt{2}x^7y$
4	4	-4	x^8

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
4	3	3	$(y^2-7x^2)y^6$
4	3	2	$-\sqrt{14}x(y^2-3x^2)y^5$
4	3	1	$\sqrt{7}x^2(3y^2-5x^2)y^4$
4	3	0	$-2\sqrt{35}x^3(y^2-x^2)y^3$
4	3	-1	$\sqrt{7}x^4(5y^2-3x^2)y^2$
4	3	-2	$-\sqrt{14}x^5(3y^2-x^2)y$
4	3	-3	$x^6(7y^2-x^2)$
4	2	2	$(y^4-12x^2y^2+15x^4)y^4$
4	2	1	$-\sqrt{2}x(3y^4-15x^2y^2+10x^4)y^3$
4	2	0	$\sqrt{10}x^2(3y^4-8x^2y^2+3x^4)y^2$
4	2	-1	$-\sqrt{2}x^3(10y^4-15x^2y^2+3x^4)y$
4	2	-2	$x^4(15y^4-12x^2y^2+x^4)$
4	1	1	$(y^6-15x^2y^4+30x^4y^2-10x^6)y^2$
4	1	0	$-2\sqrt{5}x(y^6-6x^2y^4+6x^4y^2-x^6)y$
4	1	-1	$x^2(10y^6-30x^2y^4+15x^4y^2-x^6)$
4	0	0	$y^8-16x^2y^6+36x^4y^4-16x^6y^2+x^8$

Next an alternative expression for the function $d^{\ell}_{m'm}(\theta)$ is given as a ratio of two polynomials in a single variable. One finds, collecting the terms of exponent n in Eq. (A1) into a single term,

$$d^{\ell}_{m'm}(\theta)=\left(\cos\frac{\theta}{2}\right)^{2\ell} e^{\ell}_{m'm}\left(\tan\frac{\theta}{2}\right), \tag{A7}$$

where the e function is now a polynomial of order up to 2ℓ in a single variable $u=\tan(\theta/2)$ given by

$$\begin{aligned} e^{\ell}_{m'm}(u) &= (-)^{m'-m}u^{m'-m}[(\ell+m')!(\ell-m')! \\ &\quad \times(\ell+m)!(\ell-m)!]^{1/2} \\ &\quad \times\sum_{k=k_1}^{k_2}\frac{(-)^k u^{2k}}{(\ell-m'-k)!(\ell+m-k)!(m'-m+k)!k!}. \end{aligned} \tag{A8}$$

One may re-express the d functions in terms of the variable u only, as follows:

$$d^{\ell}_{m'm}(u)=\frac{e^{\ell}_{m'm}(u)}{(1+u^2)^{\ell}}. \tag{A9}$$

It is seen that this function remains finite, when $u\to\infty$ as $\theta\to\pi$.

It is clear that all the symmetry relations for the d functions apply to the e functions as well. Thus, one has

$$\begin{aligned} e^{\ell}_{m'm}(-u)&=(-)^{m'-m}e^{\ell}_{m'm}(u), \\ e^{\ell}_{m'm}(u)&=(-)^{m'-m}e^{\ell}_{mm'}(u), \\ e^{\ell}_{m'm}(u)&=(-)^{m'-m}e^{\ell}_{-m'-m}(u), \\ e^{\ell}_{m'm}(1/u)&=(-)^{\ell-m'}e^{\ell}_{m'-m}(u). \end{aligned} \tag{A10}$$

The e functions are tabulated here for ℓ up to 3:

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
0	0	0	1
1	1	1	1
1	1	0	$-\sqrt{2}u$
1	1	-1	u^2
1	0	0	$1-u^2$
2	2	2	1
2	2	1	$-2u$
2	2	0	$\sqrt{6}u^2$
2	2	-1	$-2u^3$
2	2	-2	u^4
2	1	1	$1-3u^2$
2	1	0	$-\sqrt{6}u(1-u^2)$
2	1	-1	$u^2(3-u^2)$
2	0	0	$1-4u^2+u^4$

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
3	3	3	1
3	3	2	$-\sqrt{6}u$
3	3	1	$\sqrt{15}u^2$
3	3	0	$-2\sqrt{5}u^3$
3	3	-1	$\sqrt{15}u^4$
3	3	-2	$-\sqrt{6}u^5$
3	3	-3	u^6
3	2	2	$1-5u^2$
3	2	1	$-\sqrt{10}u(1-2u^2)$
3	2	0	$\sqrt{30}u^2(1-u^2)$
3	2	-1	$-\sqrt{10}u^3(2-u^2)$
3	2	-2	$u^4(5-u^2)$
3	1	1	$1-8u^2+6u^4$
3	1	0	$-2\sqrt{3}u(1-3u^2+u^4)$
3	1	-1	$u^2(6-8u^2+u^4)$
3	0	0	$1-9u^2+9u^4-u^6$

The e functions for $\ell=4$ are given below:

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
4	4	4	1
4	4	3	$-2\sqrt{2}u$
4	4	2	$2\sqrt{7}u^2$
4	4	1	$-2\sqrt{14}u^3$
4	4	0	$\sqrt{70}u^4$
4	4	-1	$-2\sqrt{14}u^5$
4	4	-2	$2\sqrt{7}u^6$
4	4	-3	$-2\sqrt{2}u^7$
4	4	-4	u^8

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
4	3	3	$1-7u^2$
4	3	2	$-\sqrt{14}u(1-3u^2)$
4	3	1	$\sqrt{7}u^2(3-5u^2)$
4	3	0	$-2\sqrt{35}u^3(1-u^2)$
4	3	-1	$\sqrt{7}u^4(5-3u^2)$
4	3	-2	$-\sqrt{14}u^5(3-u^2)$
4	3	-3	$u^6(7-u^2)$
4	2	2	$1-12u^2+15u^4$
4	2	1	$-\sqrt{2}u(3-15u^2+10u^4)$
4	2	0	$\sqrt{10}u^2(3-8u^2+3u^4)$
4	2	-1	$-\sqrt{2}u^3(10-15u^2+3u^4)$
4	2	-2	$u^4(15-12u^2+u^4)$
4	1	1	$1-15u^2+30u^4-10u^6$
4	1	0	$-2\sqrt{5}u(1-6u^2+6u^4-u^6)$
4	1	-1	$u^2(10-30u^2+15u^4-u^6)$
4	0	0	$1-16u^2+36u^4-16u^6+u^8$

APPENDIX B: DERIVATION OF ε FUNCTIONS

Consider a system of two identical pseudoscalars, so that it consists of even ℓ's only. Let ℓ_m be the maximum even partial wave present at a given mass bin. The purpose of this appendix is to show that the function

$$\mathcal{G}_-(v)=\frac{1}{u^{\ell_m}}G_-(u)=\frac{1}{u^{\ell_m}}(1+u^2)^{\ell_m}[h_0(u)+\sqrt{2}h_-(u)] \tag{B1}$$

is a polynomial of order ℓ_m in the variable $v=1/u-u$. From Eqs. (60), one sees that this problem reduces to transforming the function

$$\frac{1}{u^{\ell_m}}(1+u^2)^{\ell_m-\ell}e^{\ell}_{m0}(u)=\left(\frac{1}{u}+u\right)^{\ell_m-\ell}\varepsilon^{\ell}_m(u) \tag{B2}$$

into a new function of a single variable v. The function ε above is given by Eq. (61), where ℓ is even $(0\leqslant\ell\leqslant\ell_m)$ and $m=0$ or $m=1$ $(m\leqslant\ell)$.

The u dependence in Eq. (B2), including that in ε, can be expressed through a function

$$(n;u)\equiv w_n=\frac{1}{u^n}+(-u)^n, \tag{B3}$$

where n is an arbitrary integer $(\geqslant 0)$. It is *defined* that $w_0=1$ (not 2). This function can be transformed into a rational function of v only, by noting that, from Eq. (59),

$$\frac{1}{u}=\left(\frac{v}{2}\right)\pm\sqrt{1+\left(\frac{v}{2}\right)^2},$$

$$-u=\left(\frac{v}{2}\right)\mp\sqrt{1+\left(\frac{v}{2}\right)^2}. \tag{B4}$$

Substituting these into Eq. (B3), one finds

$$w_n=\tau(n)\sum_{k=0}^{k_0}\frac{n!}{(n-2k)!(2k)!}\left(\frac{v}{2}\right)^{n-2k}\left[1+\left(\frac{v}{2}\right)^2\right]^k, \tag{B5}$$

where $k_0=n/2$ if n is even $(\geqslant 0)$ and $k_0=(n-1)/2$ if n is odd $(\geqslant 1)$. If n is even, w_n is a polynomial of order $n/2$ in v^2. If n is odd, then w_n is a product of v and a polynomial of order $(n-1)/2$ in v^2.

In general, $u^{\ell-m}\varepsilon^{\ell}_m(u)$ is a polynomial of order $\ell-m$ in u^2 and is given by

$$\varepsilon^{\ell}_m(u)=(-)^m\ell![(\ell+m)!(\ell-m)!]^{1/2}\frac{1}{u^{\ell-m}}\times\sum_{k=0}^{\ell-m}\frac{(-)^k u^{2k}}{(\ell-m-k)!(\ell-k)!(m+k)!k!}. \tag{B6}$$

The key observation is that the denominators in the sum remain invariant under the interchange of k by $\ell-m-k$, so that

$$\varepsilon^{\ell}_m(u)=(-)^m\ell![(\ell+m)!(\ell-m)!]^{1/2}\times\sum_{k=0}^{k_m}\frac{(-)^k}{(\ell-m-k)!(\ell-k)!(m+k)!k!}\times w(\ell-m-2k;u). \tag{B7}$$

$k_m=(\ell-m-1)/2$ for odd $\ell-m\geqslant 1$ and $k_m=(\ell-m)/2$ for even $\ell-m\geqslant 0$. Expanding the expression inside the brackets in Eq. (B5), one finds

$$\varepsilon^{\ell}_{m}(u)=(-)^{m}\ell![(\ell+m)!(\ell-m)!]^{1/2}\times\sum_{k=0}^{k_m}\frac{(-)^{k}\tau(\ell-m-2k)}{(\ell-m-k)!(\ell-k)!(m+k)!k!}\times\sum_{i=0}^{i_m}\frac{(\ell-m-2k)!}{(\ell-m-2k-2i)!(2i)!}\sum_{j=0}^{i}\frac{i!}{(i-j)!j!}\times\left(\frac{v}{2}\right)^{\ell-m-2k-2i+2j}, \tag{B8}$$

where $i_m=(\ell-m-2k)/2$ for even $(\ell-m-2k)$ and $i_m=(\ell-m-2k-1)/2$ for odd $(\ell-m-2k)$. So $\varepsilon^{\ell}_{0}(u)$ is a polynomial of order $\ell/2$ in v^2 and $\varepsilon^{\ell}_{1}(u)$ is proportional to v times a polynomial of order $\ell/2-1$ in v^2. The next task is to note that, for even n,

$$\left(\frac{1}{u}+u\right)^{n}=\sum_{k=0}^{n/2}\frac{n!}{(n-k)!k!}\,w(n-2k;u), \tag{B9}$$

which is a polynomial of order $n/2$ in v^2. Once again expanding the expression inside the brackets in Eq. (B5), one sees that

$$\left(\frac{1}{u}+u\right)^{n}=\sum_{k=0}^{n/2}\frac{n!\,\tau(n-2k)}{(n-k)!k!}\sum_{i=0}^{i_0}\frac{(n-2k)!}{(n-2k-2i)!(2i)!}\times\sum_{j=0}^{i}\frac{i!}{(i-j)!j!}\left(\frac{v}{2}\right)^{n-2k-2i+2j}, \tag{B10}$$

where $i_0=(n-2k)/2$ for even $(n-2k)$ and $i_0=(n-2k-1)/2$ for odd $(n-2k)$. Formulas (B7) and (B9) have now been expressed explicitly as polynomials in $v/2$ only.

It is helpful to write down the w_n's explicitly for a few of the practically important values of n:

$$\begin{aligned}
w_0&=1 \ \text{ by definition},\\
w_1&=v,\\
w_2&=v^2+2,\\
w_3&=v(v^2+3),\\
w_4&=v^4+4v^2+2,\\
w_5&=v(v^4+5v^2+5),\\
w_6&=v^6+6v^4+9v^2+2.
\end{aligned} \tag{B11}$$

One is now ready to express the relevant functions in terms of v. For example, one finds that

$$\begin{aligned}
\left(\frac{1}{u}+u\right)^{2}&=w_2+2=v^2+4,\\
\left(\frac{1}{u}+u\right)^{4}&=w_4+4w_2+6=v^4+8v^2+16,\\
\left(\frac{1}{u}+u\right)^{6}&=w_6+6w_4+15w_2+20=v^6+12v^4+48v^2+64.
\end{aligned} \tag{B12}$$

The ε functions have the following expressions in terms of w_n and v:

$$\begin{aligned}
\varepsilon^{2}_{0}(u)&=w_2-4=v^2-2,\\
\varepsilon^{2}_{1}(u)&=-\sqrt{6}w_1=-\sqrt{6}v,\\
\varepsilon^{4}_{0}(u)&=w_4-16w_2+36=v^4-12v^2+6,\\
\varepsilon^{4}_{1}(u)&=-2\sqrt{5}(w_3-6w_1)=-2\sqrt{5}v(v^2-3),\\
\varepsilon^{6}_{0}(u)&=w_6-36w_4+225w_2-400=v^6-30v^4+90v^2-20,\\
\varepsilon^{6}_{1}(u)&=-\sqrt{42}(w_5-15w_3+50w_1)\\
&=-\sqrt{42}v(v^4-10v^2+10).
\end{aligned} \tag{B13}$$

It is instructive to compare the polynomials $\varepsilon^{\ell}_{m}(u)$ as functions of v in Eqs. (B13) with the functions $e^{\ell}_{m0}(u)$ given at the end of Appendix A. One sees that the polynomials in v have little resemblance to the e functions expressed as polynomials in u. Although the polynomials $\varepsilon^{\ell}_{m}(u)$ have been derived from the e functions [see Eq. (B2)] and hence they are ultimately related to the familiar d functions via Eqs. (A7), the polynomials $\varepsilon^{\ell}_{m}(u)$ have been transformed beyond recognition as functions in v (to the best of this authors's knowledge, such polynomials in v have never been encountered so far in physics—at least, and most definitely, in the field of hadron spectroscopy). See Eq. (B8) for an explicit expression of $\varepsilon^{\ell}_{m}(u)$ as a function of v.

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