

Violation of the weak energy condition in inflating spacetimes

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We argue that many future-eternal inflating spacetimes are likely to violate the weak energy condition. It is possible that such spacetimes may not enforce any of the known averaged conditions either. If this is indeed the case, it may open the door to constructing nonsingular, past-eternal inflating cosmologies. Simple nonsingular models are, however, unsatisfactory, and it is not clear if satisfactory models can be built that solve the problem of the initial singularity. [S0556-2821(97)02714-8]

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I. INTRODUCTION

Inflationary cosmological models [1] are generically future eternal [2–7]. In such models, the universe consists of a number of post-inflationary, thermalized regions embedded in an always inflating background. The thermalized regions grow in time, but the inflating background in which they are embedded grows even faster, and the thermalized regions do not, in general, merge. As a result, there never arrives an instant of time after which the universe is completely thermalized. This scenario is schematically illustrated in Fig. 1.

Quantum fluctuations of the inflaton field ϕ play an essential role in many models of eternal inflation. In such models there is a parameter H (the Hubble parameter, also referred to as the expansion rate), such that the fluctuations of ϕ can be pictured as a “random walk,” or “diffusion,” in which ϕ varies by approximately $\pm H/2\pi$ on the scale H^{-1} (the “horizon scale”) per time H^{-1} (the “Hubble time”). The fluctuations are superimposed on the classical evolution of ϕ determined by its potential $V(\phi)$. Although there is an overall tendency for ϕ to roll down the potential, it will be pushed up occasionally by quantum fluctuations. It is this effect that is responsible for the eternal nature of inflation [8].

These quantum fluctuations of ϕ induce fluctuations of the spacetime geometry, and we expect that the expansion rate will also fluctuate from one horizon-size region to another. The quantum nature of the fluctuations becomes unimportant when the expansion of the Universe stretches their wavelength well beyond the horizon. Hence, one can meaningfully define classical spacetime histories for the scalar field $\phi^{(av)}(x)$ and the metric $g_{\mu\nu}^{(av)}(x)$ averaged (“smeared”) over a scale $\ell > H^{-1}$ [9].

In the rest of this paper the spacetime geometry and the field ϕ will be understood in the averaged sense defined above, and we shall drop the superscript “(av).” In the inflating part of the Universe both the averaged field ϕ and the expansion rate H are expected to be slowly varying functions, i.e., $(\partial_\mu H)^2 \ll H^4$.

The future-eternal nature of inflation suggests that we consider the possibility that inflating spacetimes can also be extended to the infinite past, resulting in a “steady-state” nonsingular cosmological model. This possibility was discussed in the early days of inflation [10] but it was soon realized by Linde [11] and by others [2,12] that the idea could not be implemented in the simplest model in which the inflating universe is described by an exact de Sitter space. It was then proved by one of us [13] that a generic two-dimensional spacetime that was eternally inflating to the future could not be geodesically complete to the past [14]. This paper also gave a plausibility argument that suggested that the two-dimensional result would continue to hold in four spacetime dimensions.

A rigorous four-dimensional proof was subsequently provided by us [15,16], in a theorem that showed that under some natural assumptions about the spacetime geometry, a future-eternal inflationary model cannot be globally extended into the infinite past; i.e., it is not geodesically complete in the past direction. The assumptions that lead to geodesic incompleteness in this result are the following.

(A) The Universe is causally simple [17]. (A theorem with this condition replaced by a condition called the “limited influence condition” was subsequently obtained [18,19].)

(B) The Universe is open. (An extension to certain closed universes was subsequently obtained [20].)

(C) The Universe obeys the “finite past-volume difference condition” [21].

(D) The Universe obeys the null convergence condition.

The main purpose of the present paper is to reexamine the validity of this last condition.

We use conventions in which Einstein’s equation is

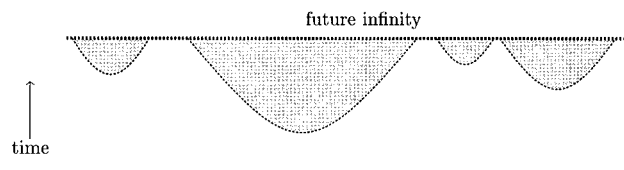


FIG. 1. A schematic representation of an inflating universe. The shaded regions are thermalized regions, where inflation has ended. We live in such a region (i.e., the entire observed Universe lies within a single thermalized region).

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$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1)$$

Under these conventions, the null convergence condition requires that the Ricci tensor $R_{\mu\nu}$ satisfies

$$R_{\mu\nu} N^\mu N^\nu \geq 0 \quad (2)$$

for all null vectors N^μ . This condition is closely related to the weak energy condition, which requires that the energy-momentum tensor $T_{\mu\nu}$ satisfies

$$T_{\mu\nu} V^\mu V^\nu \geq 0 \quad (3)$$

for all timelike vectors V^μ . An observer whose world line has tangent V^μ at a point will measure an energy density of $T_{\mu\nu} V^\mu V^\nu$ at that point. Thus, the weak energy condition means physically that the matter energy density is non-negative when measured by any observer.

In models that obey Einstein's equation (1) a violation of the null convergence condition (2) implies a violation of the weak energy condition (3). To see this, suppose that there is a (say, future-directed) null vector N^μ such that $R_{\mu\nu} N^\mu N^\nu = -\delta < 0$. Einstein's equation (1) implies that $T_{\mu\nu} N^\mu N^\nu = -(8\pi G)^{-1} \delta < 0$. Then the timelike vector given by $V^\mu = N^\mu + \epsilon T^\mu$, where T^μ is a unit, future-directed timelike vector will obey $T_{\mu\nu} V^\mu V^\nu < 0$ for sufficiently small values of ϵ .

Thus, the null convergence condition appears to be a very reasonable requirement on the spacetime geometry. For a perfect-fluid spacetime with energy density ρ and pressure p the weak energy condition (and, therefore, the null convergence condition) holds if $\rho \geq 0$ and $\rho + p \geq 0$. This is satisfied by all known forms of matter. An inflating universe is characterized by a nearly vacuum equation of state, $p \approx -\rho$, and, when the exact equality holds, the null convergence condition (2) is satisfied, but only marginally. This is less unstable than it seems, because all classical deviations from the vacuum equation of state appear to work in the direction of making $\rho + p$ positive rather than negative. For example, the energy-momentum tensor of the inflaton field ϕ is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} (\partial_\sigma \phi)^2 - V(\phi) \right], \quad (4)$$

and we can write

$$R_{\mu\nu} N^\mu N^\nu = 8\pi G T_{\mu\nu} N^\mu N^\nu = 8\pi G (N^\mu \partial_\mu \phi)^2 \geq 0. \quad (5)$$

Moreover, the addition of any ordinary matter with $p > 0$ further tips the balance in the direction of a positive sign for $R_{\mu\nu} N^\mu N^\nu$.

Equation (5) shows us that the null convergence condition is satisfied in inflationary models as long as their dynamics is accurately described by Einstein's classical equation with a scalar field source. The situation is not so clear in the "diffusion" regions of spacetime where the dynamics is dominated by quantum fluctuations of ϕ . The energy-momentum tensor in such regions can be written as

$$T_{\mu\nu} = T_{\mu\nu}[\phi] + T_{\mu\nu}^{(\text{fluct})}, \quad (6)$$

where $T_{\mu\nu}[\phi]$ is constructed from the smeared-over-an-horizon field $\phi(x)$ and $T_{\mu\nu}^{(\text{fluct})}$ is the contribution of short-

wavelength mode of ϕ (with wavelengths $\lambda \lesssim H^{-1}$). Accounting for this contribution in a systematic way remains an interesting unsolved problem. For our purposes it will be sufficient to use the estimate

$$T_{\mu\nu}^{(\text{fluct})} \sim H^4. \quad (7)$$

(This estimate is easily understood if we recall that ϕ fluctuates by $\delta\phi \sim H$ on time and length scales $\delta t \sim \delta \ell \sim H^{-1}$ and that $T_{\mu\nu}$ includes terms quadratic in gradients of ϕ .) In the diffusion region, the smeared field gradients are small (i.e., $|\partial_\mu \phi| \lesssim H^2$), and $T_{\mu\nu} N^\mu N^\nu$ will now contain both a manifestly non-negative term, as in Eq. (5), as well as a non-negligible correction from $T_{\mu\nu}^{(\text{fluct})}$. It is no longer obvious, in this case, whether the null convergence condition (2) is satisfied. In the next section we argue that the condition is indeed violated in the "diffusion" regions of inflationary spacetimes. This violation may open the door to escaping the conclusion of our previous theorems [15,18], and towards constructing past-eternal, nonsingular cosmologies. Violations of the weak energy condition may also allow us to avoid the conclusions of Farhi and Guth [22] whose results appear to forbid the creation of an inflating universe in a laboratory. (Other ways around the results of Farhi and Guth have previously appeared in the literature [23–25].)

The rest of this paper is organized as follows: In Sec. II we discuss how the violation of the weak energy condition arises in inflationary cosmology. In Sec. III we discuss whether a suitable integral convergence condition might hold, even if the pointwise condition does not. Several integral conditions are known to give rise to the focusing effects necessary for results such as our previous theorems [15,18] to go through [26–29]. We argue, however, that even the weakest of the known integral conditions [29] may not hold here. In Sec. IV we discuss the implications of the violation of the weak energy condition for the existence of nonsingular, eternally inflating cosmological models. We construct an explicit class of nonsingular cosmologies, and we discuss why they are unsatisfactory as models of eternal inflation. We also discuss a property that realistic inflationary scenarios might possess that would make all nonsingular models unsuitable as models of eternal inflation. In Sec. V we take stock of the situation: we compare our approach to quantum stress-energy tensors with that of some other authors, and we discuss the models to which our earlier theorems [15,18] might still apply.

II. VIOLATION OF THE WEAK ENERGY CONDITION

We first look at a simple model in which the inflating universe is locally approximated by a Robertson-Walker metric:

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2). \quad (8)$$

The approximation is justified when the scale of the spatial variation of the inflaton field ϕ and of the Hubble parameter H is much greater than H^{-1} . The Hubble parameter is defined by $H \equiv a'/a^2$ (where a prime is a derivative with respect to η) and it obeys

$$H'(\eta) = a^{-3}(aa'' - 2a'^2). \quad (9)$$

Consider a null vector of the form

$$N^\mu = a^{-2}(1, \vec{n}), \quad |\vec{n}| = 1, \quad (10)$$

where the ‘‘normalizing factor’’ a^{-2} is chosen so as to ensure that N^μ is the tangent to an affinely parametrized geodesic (a feature that we will need later). For such a vector, we have

$$R_{\mu\nu}N^\mu N^\nu = -\frac{2}{a^6}(aa'' - 2a'^2). \quad (11)$$

Comparing this with Eq. (9) we see that

$$R_{\mu\nu}N^\mu N^\nu = -\frac{2}{a^3}H' = -\frac{2}{a^2}\dot{H}. \quad (12)$$

An overdot here is a derivative with respect to the proper time t , related to η by $dt = ad\eta$. Thus, in a region where $H' > 0$ the null convergence condition will be violated.

The Hubble parameter H satisfies

$$H^2 = \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right) + O(GH^4), \quad (13)$$

where the last term represents the effect of the subhorizon scale quantum fluctuations we alluded to earlier [see Eq. (7)]. During inflation, we have $\dot{\phi}^2 \ll V(\phi)$, and if the energy scale of inflation is well below the Planck scale, we also have $GH^2 \ll 1$. The magnitude of H is then determined mainly by the inflaton potential $V(\phi)$. In regions of deterministic slow roll,

$$|\dot{\phi}| \approx |V'(\phi)|/3H \gg H^2, \quad (14)$$

and quantum fluctuations play a subdominant role in the dynamics of ϕ . In such regions, Einstein’s equations with energy-momentum tensor for the averaged field are satisfied with good accuracy, and it is easily verified that $\dot{H} \approx -8\pi G\dot{\phi}^2 < 0$. It then follows from Eq. (12) that the weak energy condition is always satisfied in slow-roll regions. On the other hand, in regions where the dynamics is dominated by quantum diffusion of the field ϕ , Eq. (14) does not hold, and we have

$$H^2 = \frac{8\pi G}{3} V(\phi) + O(GH^4).$$

Quantum fluctuations take ϕ up and down the potential $V(\phi)$, and the range of variation of $V(\phi)$ in the diffusion region is typically much greater than H^4 . Hence, in some parts of the diffusion region, H will grow and in other parts it will decrease. The weak energy condition is thus necessarily violated.

To see how this conclusion is affected by inhomogeneities of the spacetime geometry, we consider a more general ansatz for the metric

$$ds^2 = a^2(\vec{x}, \eta)(d\eta^2 - d\vec{x}^2). \quad (15)$$

For an inflating universe with a slowly varying expansion rate $H(\vec{x}, \eta)$, the scale factor has the form

$$a(\vec{x}, \eta) = \left(1 - \int_{\eta_0(\vec{x})}^{\eta} H(\vec{x}, \tilde{\eta}) d\tilde{\eta} \right)^{-1}.$$

With N^μ given by Eq. (10), we have

$$\begin{aligned} R_{\mu\nu}N^\mu N^\nu &= 2a^{-3}n^\mu n^\nu \partial_\mu \partial_\nu \left(\frac{1}{a} \right) \\ &= -2a^{-3}n^\mu n^\nu \partial_\mu \partial_\nu \int_{\eta_0(\vec{x})}^{\eta} H(\vec{x}, \tilde{\eta}) d\tilde{\eta}, \end{aligned} \quad (16)$$

where $n^\mu = (1, \vec{n})$. To analyze the sign of this expression, we note that the scale factor $a(\vec{x}, \eta)$ (and its inverse a^{-1}) may be expected to have many local minima, maxima, and saddle points as a function of \vec{x} at any ‘‘moment’’ η . At such points, $\vec{\nabla}a = 0$ and Eq. (16) can be written as

$$R_{\mu\nu}N^\mu N^\nu = 2a^{-3} \left[-\frac{\partial H}{\partial \eta} + (\vec{n} \cdot \vec{\nabla})^2 \left(\frac{1}{a} \right) \right].$$

At minima of a , the second term in the square brackets is negative, and at saddle points it is negative at least for some directions of \vec{n} . The first term is negative whenever H is increasing with time. In the diffusion region, we do not expect any strong correlations between the spatial dependence of the scale factor [which is determined by the whole prior history of $H(\vec{x}, \eta)$] and the sign of $\partial H/\partial \eta$ (which depends only on the local quantum fluctuation of H). Thus, it appears very likely that in some regions both terms on the right-hand side will be negative and the weak energy condition will be violated.

III. INTEGRAL CONVERGENCE CONDITIONS

The violation of the weak energy condition discussed above is not total: there are regions where the condition is violated, but also regions where it is satisfied. Moreover, the probability for the field ϕ to move down the potential $V(\phi)$ is always greater than that for it to move upward, and the weak energy condition is satisfied when the field rolls down. This suggests that, although there will be regions where the null convergence condition will be locally violated, it may perhaps be satisfied in some averaged sense.

One kind of ‘‘averaged’’ condition is an integral convergence condition [26–29]. If we assume that an inflating spacetime is null complete to the past, then a past-directed null geodesic may be expected to cross regions where the weak energy condition is satisfied as well as ones where it may be violated. Thus, it seems reasonable to ask whether an integral null convergence condition along the lines

$$\int R_{\mu\nu}N^\mu N^\nu dp \geq 0 \quad (17)$$

might hold, where the integral is taken along the geodesic, and p is an affine parameter with respect to which the tangent to the geodesic N^μ is defined (i.e., $N^\mu \equiv dx^\mu/dp$). Condition (17) is either required to hold when the integral is taken over the complete, or in some applications half-complete, geodesic (as in the original proposal for such in-

tegral conditions [26]), or is required to “repeatedly hold,” as will happen when the integrated oscillates [29].

One examination, however, it is not clear if Eq. (17) will hold when interpreted in either way. Consider, for example, the metric (8). Affinely parametrized null geodesics for this metric may be obtained from the Lagrangian $\mathcal{L} = g_{\mu\nu}N^\mu N^\nu$. The Euler-Lagrange equations reduce to $d(a^2 N^\mu)/dp = 0$, where p is an affine parameter, and we use the fact that N^μ is null. One solution of this is the null vector in Eq. (10). For this solution, we have

$$dp = -a^2 d\eta = a dt, \quad (18)$$

and using Eq. (12) we have

$$\int R_{\mu\nu}N^\mu N^\nu dp = -2 \int \frac{H'}{a} d\eta = -2 \int \frac{\dot{H}}{a} dt. \quad (19)$$

The presence of a in the denominator makes the behavior of the integral on the right difficult to assess. Without it, Eq. (19) would reduce merely to the difference in the values of H at the end points of integration, and one could try and arrange for this difference to be positive along at least some geodesics. The presence of a means, however, that there will be increasingly larger contributions to the integral as we go to earlier times (assuming that the Universe is expanding) and it is not easy to decide if the contributions of the wrong sign will always be compensated for by those of the right sign. The situation is even more difficult in the case of the more general metric (15). Here one would have to deal with the integral of the complicated expression on the right-hand side of Eq. (16), and it is hard to see that one can argue that this integral will either converge to a non-negative value, or even that it will be “repeatedly non-negative.”

IV. NONSINGULAR COSMOLOGIES

What are the consequences if, in addition to the pointwise violation of the weak energy condition that we have discussed here, a suitable integral condition also fails to hold? One important consequence is that earlier arguments that suggested that the Universe had a “beginning” [15,18] may no longer hold. A crucial ingredient of these arguments is that a congruence of initially converging geodesics comes to a focus. Convergence conditions, either pointwise or suitable integral ones, guarantee focusing. Without such conditions, models can be constructed where focusing does not occur, and in which geodesics can be extended to infinite affine lengths in the past direction.

If a model based on the metric (8) is to be nonsingular, it follows from Eq. (18) that

$$\int_{-\infty}^t a(\tilde{t}) d\tilde{t} \quad (20)$$

must diverge for all t , where t is the proper time used above (defined via $dt = a d\eta$). We must also have $\dot{a} > 0$ (for the Universe to be expanding). Cosmologies with a scale factor

of the form $a(t) \sim (-t)^{-q}$, where $0 < q \leq 1$ (and $t < 0$), satisfy these conditions. Such a scale factor appears, for example, in the “pre-big-bang” stage of the proposed models of string cosmology [30–32]. These models do not, however, qualify as models of “steady-state” inflation. The Riemann tensor in such models decreases as $R^{\mu\nu}{}_{\sigma\tau} \propto t^{-2}$ when $t \rightarrow -\infty$, indicating that the spacetime is asymptotically flat in the past direction. The Hubble parameter H also vanishes as $t \rightarrow -\infty$. This behavior is very different from the quasiexponential expansion with $H \approx \text{const}$ that is characteristic of inflation at later times. Since the idea behind a steady-state model, and its chief attraction, is that the Universe is in more or less the same state at all times, models with very different behavior at early and late times are not viable as models of steady-state inflation.

Another example of a geodesically complete cosmology is de Sitter spacetime,

$$ds^2 = dt^2 - a^2(t) d\Omega_3^2, \quad (21)$$

where

$$a(t) = H_0^{-1} \cosh(H_0 t). \quad (22)$$

For $t \gg H_0^{-1}$, the expansion rate H is approximately equal to the constant value H_0 , and we have a canonical model of inflation. This model, however, describes a contracting universe for $t < 0$. Thermalized regions in such a universe would rapidly merge and fill the entire space [13]. The universe would then collapse to a singularity and would never make it to the expanding stage. A further problem with a contracting universe is that it is extremely unstable. The growth of perturbations by gravitational instability is slower in an expanding universe than that in a flat spacetime, but in a contracting universe the growth of perturbations accelerates. Hence, a contracting universe will rapidly reach a grossly inhomogeneous state from which it is not likely to recover.

An inflating spacetime is not, of course, exactly de Sitter spacetime, but is expected to be locally close to de Sitter spacetime. That is, for any spacetime point P there is a neighborhood of proper extent $\sim H^{-1}$ where the metric can be brought to de Sitter form with only small deviations from the exact de Sitter metric. It has been argued by one of us [13] that such a spacetime is necessarily contracting in the past, implying that steady-state inflation is impossible in such a model. That argument involved assumptions on the form of the Riemann tensor. We provide here a new version of the argument based on the Ricci tensor.

Consider a congruence of timelike geodesics, past directed from some point p . Let the proper time t along these geodesics be zero at p , let V^μ be the tangent to the geodesics with respect to t , and let $\theta \equiv D_\mu V^\mu$ be the divergence of the congruence. If the congruence is shear-free, as is the case in de Sitter space for congruences that start without shear, or in general two-dimensional spacetimes, we have

$$\frac{d\theta}{dt} = -\frac{1}{n-1} \theta^2 - R_{\mu\nu} V^\mu V^\nu, \quad (23)$$

where n is the spacetime dimension. (This equation is a trivial extension of the standard four-dimensional geodesic focusing equation [33].) Assume now that the Ricci tensor obeys

$$R_{\mu\nu}V^\mu V^\nu < -\frac{\delta^2}{n-1} < 0 \quad (24)$$

for all unit timelike vectors V^μ . In other words, assume that the strong energy condition is everywhere violated by at least a minimum amount. The strong energy condition requires that $R_{\mu\nu}V^\mu V^\nu \geq 0$ for all timelike vectors V^μ , and this condition is violated in all models of inflation that have been considered. In fact, we have argued elsewhere [16] that a violation of this condition is necessary if a spacetime is to be considered ‘‘inflating.’’

Combining Eqs. (23) and (24) we get

$$\frac{d\theta}{dt} > \frac{1}{n-1}(\delta^2 - \theta^2). \quad (25)$$

Integrating this expression from some negative value of t (i.e., a point to the past of p on the congruence) to zero, and using the fact that $\theta \rightarrow -\infty$ as $t \rightarrow 0^-$, we get

$$\theta < \left[\coth\left(\frac{t}{n-1}\delta\right) \right] \delta. \quad (26)$$

Since $\coth x < -1$ for $x < 0$ Eq. (26) means that the past-directed timelike geodesics from p continue to diverge into the infinite past by an amount $\theta < -\delta$. Compare this with flat space, where the geodesics also continue to diverge, but where $\theta \sim 1/t$. This faster-than-flat-space divergence suggests that the universe is contracting at early times. The objections raised above then apply here as well.

Although suggestive, this argument falls short of a proof. Congruences of geodesics in realistic spacetimes will not remain shear-free, and the effect of shear needs to be taken into account. The global structure of locally de Sitter spacetimes remains an interesting open problem.

V. DISCUSSION

We have shown here that the weak energy condition generically will be violated in inflating spacetimes. Violations of the weak energy condition have been discussed by several other authors (see, for example, Flanagan and Wald [34] and references cited therein). Previous work on the question has focused on the expectation value of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$ and this approach has yielded limits on the violation of the weak energy condition. In particular, Ford and Roman [35,36] have investigated quantum states of free scalar and electromagnetic fields in a flat spacetime for which $\langle T_{00} \rangle < 0$ in some region of spacetime. They have shown that although such states can be constructed, the magnitude of the negative energy density and the time interval during which it occurs are limited by inequalities that have the appearance of uncertainty principle inequalities. Pfenning and Ford have obtained extensions of these ‘‘quantum in-

equalities’’ to some curved spacetimes [37,38]. Flanagan and Wald [34] have shown that an integral form of the weak energy condition is satisfied for an appropriately smeared $\langle T_{\mu\nu} \rangle$ in the case of a free, massless scalar field in a nearly flat spacetime.

Unfortunately, these results cannot be used to restrict the violations of the weak energy condition of the type discussed in this paper. One obvious reason is that the theorems proved so far are restricted to free fields and a special class of spacetimes, which usually does not include locally de Sitter spaces. (Pfenning and Ford have recently obtained restrictions on violations of the weak energy condition in de Sitter spacetime [39], but their results apply to a limited class of world lines.) A more basic reason, however, is that all these results are concerned with the expectation value $\langle T_{\mu\nu} \rangle$, while we are interested in the fluctuations of $T_{\mu\nu}$. The expectation value $\langle T_{\mu\nu}(x) \rangle$ can be thought of as a result of averaging the observed value of $T_{\mu\nu}$ at a point x in an ensemble of identical universes. In some of these universes the inflaton field will fluctuate ‘‘up the hill,’’ and the weak energy condition will be violated, while in others it will go ‘‘down the hill,’’ and the condition will be satisfied. Since the probability to go down is always greater than probability to go up, we expect that on average the weak energy condition will be satisfied; i.e., we expect that

$$\langle T_{\mu\nu}(x) \rangle N^\mu N^\nu \geq 0. \quad (27)$$

The violation of the weak energy condition, as well as the eternal character of inflation, both disappear when the field ϕ and its energy-momentum tensor are replaced by their expectation values, since both effects are due to relatively rare quantum fluctuations of the field ϕ .

It is important to know if we can reasonably expect the energy conditions, or suitable integral versions, to be satisfied, because that will determine whether the singularity theorems, and other results of classical general relativity, will continue to hold (see Hawking and Ellis [33] for a review of these classical results and further references). If previous singularity theorems that were aimed at inflationary cosmology [15,20,18] do not apply to some models, we may then have the possibility of constructing a ‘‘steady-state’’ eternally inflating universe, without a beginning and without an end. The issue of whether or not the Universe is past eternal has been discussed several times in the literature by Linde and his collaborators [39]. They have argued that even when individual geodesics are past incomplete, it may still be possible to view the Universe as infinitely old [40]. What we have done here is to point to a possibility, although a faint one, of constructing fully nonsingular models. Such models, if they indeed exist, would be geodesically past complete.

It must be noted that there is a class of inflationary models to which our previous theorems do continue to apply. These are models of ‘‘open-universe’’ inflation [41–44] where the universe consists of post-inflationary ‘‘bubbles’’ embedded in a metastable false vacuum state. Quantum diffusion of the inflation field does not occur here, and in the false vacuum the Ricci tensor is proportional to the metric. In this case the

null convergence condition is satisfied pointwise, and the models must possess initial singularities.

In other models of inflation, we have shown here that there is a possibility for nonsingular models to exist, based on the violation of the weak energy condition that occurs in these models. Whether realistic models of this type can be constructed, however, remains open. The discussion of Sec. IV suggests that the construction of such models may be difficult, if not impossible.

Note added. After we submitted this paper for publication we learned of other work that also discusses the weak energy condition in cosmology. Kar [45] and Kar and Sahdev [46] have looked at wormholes in a cosmological background and have discussed the weak energy condition using a relation equivalent to Eq. (11) of the present paper. Fukuyama *et al.*

[47] have discussed the violation of the weak energy condition in a specific model in which the Hubble parameter oscillates in time.

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